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CONVEX ALGEBRA AND DUALITY IN DYNAMIC MODELS OF PRODUCTION

R. T. ROCKAFELLAR*

University of Washington, Seattle, Washington, USA

1. Introduction

The well known economic model of von Neumann, as generalized by Gale, consists of a convex cone G in $R^n \times R^n$. Membership of (y, z) in G is interpreted as meaning that the economic state represented by the vector y can be transformed in a unit time period into the state represented by z. The cone G can be regarded as the graph of a point-to-set mapping T.

More than ten years ago, the author became interested in the idea of studying such mappings T as generalizations of linear transformations (whose graphs are certain subspaces, rather than general convex cones). It turned out that analogues of surprisingly many things in linear algebra could be developed, most of which seemed to have an economic interpretation. For example, the analogue of the formula $T(y) \cdot p = y \cdot T^*(p)$, relating a linear transformation with its adjoint, turned out to be an abstract form of the duality theorem for linear programming problems. Similarly, the study of inverse mappings corresponded to minimax theory, while "eigenvalues" corresponded to growth rates in the von Neumann models. Furthermore, all this could be extended to inhomogeneous models, where point-to-set mappings were replaced by "convex bifunctions". A very broad duality theory of convex programming could thereby be obtained. In this way, many diverse topics of importance in economics and optimization theory could be incorporated into a coherent framework which might appropriately be called "convex algebra".

The subject being so huge, and much of the supporting material on convex functions not yet being available in the literature, only a special portion of "convex algebra" was written up initially for publications by

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Rockafellar [1965, 1967]; this was in 1965. When the book Convex Analysis (Rockafellar [1969]) was written in 1967, a more general form of the theory was presented. Many results, however, remain unpublished.

In the intervening years, the idea of studying point-to-set mappings whose graphs are cones as analogues of linear transformations was hit upon independently by Rubinov [1968], although from a more special point of view. The ideas in Rockafellar [1965, 1967] and Rubinov [1968] were subsequently developed by Rubinov in a series of papers by Rubinov [1969a, 1969b, 1970], directed at economic applications. Most recently, Rubinov and Makarov [1970] in their excellent joint paper have demonstrated the power of this approach in the analysis of dynamic models, growth rates, turnpike theorems and so forth. Many extensions of results in Rockafellar [1967] are presented in Makarov and Rubinov [1970], as well as, of course, many things not even treated in Rockafellar [1967]. (The book by Rockafellar [1969] was not available to Makarov and Rubinov when their paper was written.) On the other hand, Makarov and Rubinov [1970] do not try to convey the broader notions of "convex algebra" in a general setting, where more can still be said that is potentially of economic interest.

The aim of this paper is to describe these broader notions, especially as far as they concern point-to-set mappings whose graphs are *polyhedral* convex cones. Such mappings correspond to the original "finitely generated" model of von Neumann. For this case we present a number of theorems which have not previously appeared in print anywhere. In the last section, some of the relationship with the more general theory of "convex bifunctions" is indicated. Use is made of some new results in the calculus of variations (Rockafellar [1971]) in discussing dynamic models over continuous time. These results should also find other economic applications.

A number of questions are raised which do not yet have answers.

2. Definition of a polyhedral convex process

Recall that a set $C \subset R^n$ is said to be *polyhedral convex* if it can be expressed as the intersection of a finite collection of closed half-spaces, or equivalently (according to a fundamental theorem), if there exist vectors $c_1, \ldots, c_s, d_1, \ldots, d_r$ such that C is the set of all vectors of the form

$$\lambda_1 c_1 + \ldots + \lambda_s c_s + \mu_1 d_1 + \ldots + \mu_r d_r,$$

with $\lambda_k \geqslant 0$, $\mu_k \geqslant 0$, $\mu_1 + \ldots + \mu_r = 1$. It is said to be a *polyhedral convex cone*, if in the first property the half-spaces have the origin on their boundary, or equivalently, if in the second property the μ_k and d_k can be omitted. (Thus by our definition, a polyhedral convex cone always contains the origin.)

We define a *polyhedral convex process* T: $R^m \to R^n$ to be a multifunction (multivalued or set-valued mapping, or correspondence) whose graph

$$G(T) = \{(y, z)|\ z \in T(y)\} \subset R^m \times R^n$$

is a polyhedral convex cone. Obviously every polyhedral convex cone G in $R^m \times R^n$ is the graph of a polyhedral convex process T, where $T(y)$ is the set of all z such that $(y, z) \in G$.

Example 1 (Von Neumann Model for Economy). Let A and B be matrices of dimensions $s \times n$, and define T: $R^n \to R^n$ by

$$T(y) = \{z|\ \exists\, x \in R^s_+,\ y = xA \in R^n_+,\ z = xB \in R^n_+\}.$$

Here R^n_+ denotes the non-negative orthant. We do not assume that the matrices A and B are non-negative. Thus activities of trade and disposal may be represented, and there is no loss of generality in writing $y = xA$ and $z = xB$, instead of $y \geqslant xA$ and $z \leqslant xB$. Observe that

$$G(T) = (R^n_+ \times R^n_+) \cap \Big\{\sum_{k=1}^{s} x_k(a_k, b_k)|\ x_k \in R',\ x_k \geqslant 0\Big\},$$

where a_k and b_k are the k-th rows of A and B. The two sets in the intersection are polyhedral convex cones, and therefore $G(T)$ is itself a polyhedral convex cone. In other words, T is a polyhedral convex process. It follows, incidentally, that by choosing different set of vector pairs (a'_k, b'_k) for $k = 1, \ldots, r$ if necessary, with $a'_k \geqslant 0$ and $b'_k \geqslant 0$, we can get the expression

$$G(T) = \Big\{\sum_{k=1}^{r} x'_k(a'_k, b'_k)|\ x'_k \in R',\ x'_k \geqslant 0\Big\}.$$

This remains true, except for the non-negativity of (a'_k, b'_k), if $R^n_+ \times R^n_+$ is replaced by an arbitrary polyhedral convex cone. In this way, one sees that virtually every polyhedral convex process can be interpreted as representing a generalized von Neumann model. Then $T(y)$ denotes the set of states z (vectors whose components denote quantities of goods or resources) that can be produced in a certain time period from the state y. The pairs (a'_k, b'_k) represent basic processes, and the coefficients x'_k intensities.

Example 2 (Linear Programming). Let A be an $m \times n$ matrix, and define T: $R^m \to R^n$ by

$$T(y) = \{z|\ z \geqslant 0,\ Az \leqslant y\}.$$

Then $T(y)$ is the set of feasible solutions to a standard linear programming problem which depends on the parameter vector y. The vectors $y = (y_1, \ldots, y_m)$ and $z = (z_1, \ldots, z_n)$ satisfy $z \in T(y)$, if and only if they satisfy a certain finite system of homogeneous linear inequalities:

$$-y_i + \sum_{j=1}^{n} a_{ij} z_j \leqslant 0 \quad \text{for} \quad i = 1, \ldots, m,$$

$$z_j \geqslant 0 \quad \text{for} \quad j = 1, \ldots, n.$$

Therefore, $G(T)$ is a polyhedral convex cone, and T is a polyhedral convex process.

Example 3 (Linear Algebra). Let T be a linear transformation from R^m to R^n, i.e. let H be an $n \times m$ matrix, and for each $y \in R^m$ let $T(y)$ be the set consisting of the single element Hy. (In fact we shall not distinguish between a set containing a single element and the element itself, but simply write $T(y) = Hy$.) The graph of T is then a *subspace* of $R^m \times R^n$, consisting of the pairs (y, z) which satisfy a system of linear equations of the form

$$\sum_{i=1}^{m} h_{ji} y_i - z_j = 0, \quad j = 1, \ldots, n.$$

In particular, since a subspace is a polyhedral convex cone, T is a polyhedral process.

This example is especially important as a mathematical guide to the development of the general theory of polyhedral convex processes, because it shows us what analogies to look for in the familiar context of linear algebra.

Example 4 (Inverse Process). Let $T\colon R^m \to R^n$ be any polyhedral convex process, and define $T^{-1}\colon R^n \to R^m$ by

$$T^{-1}(z) = \{y \mid z \in T(y)\}.$$

Obviously T^{-1} is another polyhedral convex process, since its graph is obtained just by reversing the pairs in the graph of T, and $(T^{-1})^{-1} = T$. In the general economic interpretation in Example 1, $T^{-1}(z)$ is the set of states y from which the state z can be produced.

Convex processes which are *not necessarily polyhedral* are defined as multifunctions whose graphs are arbitrary convex cones in $R^m \times R^n$ containing $(0, 0)$. One may also consider inhomogeneous multifunctions whose graphs are just convex sets, or objects still more general, called convex bifunctions.

These cases will be discussed later. However, our purpose at present is to explain certain fundamental ideas, avoiding a prolonged battle with technical complications. The context of polyhedral convex process is well suited for this. The "polyhedral" property, as a kind of finiteness assumption, ensures in particular that various sets remain closed under our manipulations, so that peculiar behavior along boundaries does not arise.

3. Some elementary properties

It is easy to see that the following relations are equivalent to the property of a multifunction $T\colon R^m \to R^n$ that its graph be a convex cone containing the origin:

$$T(y+y') \supset T(y)+T(y'),$$
$$T(\lambda y) = \lambda T(y) \quad \text{for} \quad \lambda > 0,$$
$$0 \in T(0).$$

It follows then that

$$T\Big(\sum_{k=1}^{s} \lambda_k y_k\Big) \supset \sum_{k=1}^{s} \lambda_k T(y_k) \quad \text{for} \quad \lambda_k \geqslant 0.$$

Thus a polyhedral convex process is a multifunction satisfying these laws as well as a certain condition about how it can be generated finitely.

For a polyhedral convex process $T\colon R^m \to R^n$, each of the sets $T(y)$ is polyhedral convex. Indeed, $T(y)$ corresponds for fixed y to the intersection of the polyhedral convex cone $G(T)$ with the translated subspace of $R^m \times R^n$ consisting of the pairs of the form (y, z), $z \in R^n$. The latter is in particular a polyhedral convex set, and the intersection of two polyhedral convex sets is polyhedral convex.

More generally, if $C \subset R^m$ is any polyhedral convex set, then the set

$$T(C) = \bigcup \{T(y) \mid y \in C\}$$

is polyhedral convex. This set is obtained by intersecting $G(T)$ with $C \times R^n$ and then taking the image under the projection $(y, z) \to z$. (The latter operations preserve polyhedral convexity.) Applying this fact to the inverse process T^{-1}, one sees that, for any polyhedral convex set $D \subset R^n$, the set

$$T^{-1}(D) = \{y \mid T(y) \cap D \neq \varnothing\}$$

is polyhedral convex. It is also true, but not as easy to prove, that the sets

$$\bigcap \{T(y) \mid y \in C\} \quad \text{and} \quad \{y \mid T(y) \supset D\}$$

are polyhedral convex.

We define the *effective domain* $D(T)$ and *effective range* $R(T)$ by

$$D(T) = \{y|\ T(y) \neq \emptyset\},$$
$$R(T) = \{z|\ \exists y, z \in T(y)\}.$$

These sets are the projections of the polyhedral convex cone $G(T) \subset R^m \times R^n$ on R^m and R^n, respectively, and hence they are polyhedral convex cones. Clearly

$$D(T^{-1}) = R(T) \quad \text{and} \quad R(T^{-1}) = D(T).$$

The set $T(0)$ is also a polyhedral convex cone, corresponding to the intersection of $G(T)$ with $\{0\} \times R^n$. Note that

$$T(y) = T(y+0) \supset T(y) + T(0) \quad \text{for all } y.$$

Thus $T(0)$ gives "directions" in which $T(y)$ is unbounded. As a matter of fact, for each $y \in D(T)$ we have the following property: the half-line $\{z+\lambda w|\ \lambda \geqslant 0\}$ is contained in $T(y)$ for all $z \in T(y)$, if and only if $w \in T(0)$. In other words, $T(0)$ is the so-called *recession cone* of $T(y)$ for every $y \in D(T)$. This property is easily derived by representing $G(T)$ as an intersection of closed half-spaces. It is well known that a closed convex set is bounded if and only if it contains no half-lines. Thus the sets $T(y)$ are all bounded if and only if $T(0) = \{0\}$.

Using the properties of $T(0)$, we can demonstrate that if T is a polyhedral convex process with $D(T) = R^m$ and with the set $T(y)$ bounded for some $\bar{y}$, then T must be a linear transformation. Namely, $T(0)$ must consist of just the zero vector, since otherwise $T(y)$ would contain a half-line. The relation

$$T(y) + T(-y) \subset T(y-y) = T(0)$$

implies then that, for every y, $T(y)$ consists of a single element, and $T(-y) = -T(y)$. The laws stated at the beginning of this section reduce now to

$$T(y+y') = T(y) + T(y'),$$
$$T(\lambda y) = \lambda T(y) \quad \text{for all } \lambda,$$

and therefore T is a linear transformation.

The set $T^{-1}(0)$ is also a polyhedral convex cone, and for each $z \in R(T)$ it is, according to the above, the recession cone of the set $T^{-1}(z)$. It can also be characterized in terms of the growth properties of T itself: $w \in T^{-1}(0)$ if and only if $T(y+w) \supset T(y)$ for every y.

4. Algebraic operations of combinations

We now describe some operations, parallel to those in linear algebra, for constructing new polyhedral convex processes from given ones.

Let T and T' be polyhedral convex processes from R^m to R^n. We define $T+T'$ by

$$(T+T')(y) = T(y)+T'(y).$$

Thus $(T+T')(y)$ is the set of all sums $z+z'$ as z ranges over $T(y)$ and z' ranges over $T'(y)$. Similarly, for an arbitrary real number λ we define λT by

$$(\lambda T)(y) = \lambda T(y).$$

THEOREM 1: $T+T'$ and λT are again polyhedral convex processes.

Proof: The set $G(T+T')$ may be constructed from $G(T)$ and $G(T')$ by forming the following convex cones in $R^m \times R^m \times R^1$:

$$K = \{(y, 0, z)|\ (y, z) \in G(T)\},$$

$$K' = \{(0, y', z')|\ (y', z') \in G(T')\},$$

$$K'' = \{(y, y', w)|\ y = y'\}.$$

The set $(K+K') \cap K''$ is another polyhedral convex cone, and its image under the projection $(y, y', w) \to (y, w)$ is $G(T+T')$. Therefore $G(T+T')$ is a polyhedral convex cone. The proof for λT is simple, and we omit it.

This theorem implies that an arbitrary linear combination $\lambda_1 T_1 + \ldots \ldots + \lambda_r T_r$ of polyhedral convex processes is a polyhedral convex process. For example, if T: $R^n \to R^n$ we can form $T-\lambda I$, where I is the identity linear transformation, and study the cone $(T-\lambda I)^{-1}(0)$. This consists of the vectors y satisfying the "eigenvector" condition $\lambda y \in T(y)$, which is of obvious importance in the study of growth rates in the von Neumann model.

Two further operations may be defined by

$$(T \wedge T')(y) = T(y) \cap T'(y),$$
$$(T \vee T')(y) = \bigcup \{T(u)+T'(u')|\ u+u' = y\}.$$

THEOREM 2: $T \wedge T'$ and $T \vee T'$ are again polyhedral convex processes. In fact, they are the greatest lower bound and least upper bound, respectively, of T and T' with respect to the natural partial ordering $\supset$, where $T_1 \supset T_2$ means that $T_1(y) \supset T_2(y)$ for every y.

Proof: This is immediate from the relations $G(T \wedge T') = G(T) \cap G(T')$ and $G(T \vee T') = G(T)+G(T')$.

An especially interesting operation is that of multiplication. Given polyhedral convex processes $T\colon R^m \to R^n$ and $S\colon R^n \to R^r$ we define $ST\colon R^m \to R^r$ by

$$(ST)(y) = S\big(T(y)\big) = \bigcup\{S(z) |\ z \in T(y)\}.$$

THEOREM 3: ST is a polyhedral convex process.

Proof: To construct $G(ST)$ from $G(S)$ and $G(T)$, we form the following polyhedral cones in $R^m \times R^n \times R^n \times R^r$:

$$K = \{(y, z, 0, 0) |\ (y, z) \in G(T)\},$$
$$K' = \{(0, 0, z', w) |\ (z', w) \in G(S)\},$$
$$K'' = \{(y, z, z', w) |\ z = z'\}.$$

The set $(K+K') \cap K''$ is then a polyhedral convex cone, and its image under the projection $(y, z, z', w) \to (y, w)$ is $G(ST)$. Hence $G(ST)$ is a polyhedral convex cone.

For a polyhedral convex process $T\colon R^n \to R^n$, we can use this operation of multiplication to form powers T^k. In the general von Neumann model, $T^k(y)$ may be interpreted as the set of states of the economy which can be produced in k time periods.

The operations of addition and multiplication reduce to the usual ones of linear algebra, if the multifunctions are linear transformations. However, they do not satisfy laws as powerful as those of linear algebra. In particular, the "distributive law" is weakened to a pair of "distributive inequalities",

$$S(T_1+T_2) \supset ST_1+ST_2,$$
$$(S_1+S_2)T \subset S_1 T+S_2 T.$$

This is a serious obstacle to translating many classical results, in a formal algebraic manner, into the theory of polyhedral convex processes. On the other hand, it will be seen that some fundamental laws involving the adjoint and inverse operations can be translated. These laws then become charged with a new and deeper meaning, far beyond what they had in linear algebra.

5. The adjoint operation and orientations

The *polar* of a polyhedral convex cone $K \subset R^k$ is the set

$$K^0 = \{w \in R^k |\ \forall y \in K, w \cdot y \leqslant 0\}.$$

It is well known that K^0 is another polyhedral convex cone whose polar is in turn K; thus $K^{00} = K$.

Introducing a slight modification into the polarity operation, we obtain a duality correspondence for convex processes. Let T: $R^m \to R^n$ be any polyhedral convex process. We define the *adjoint* process T^*: $R^n \to R^m$ by

$$T^*(p) = \{q|\ y \cdot q \geqslant z \cdot p,\ \ z \in T(y),\ \forall y\}.$$

Obviously we have

$$G(T^*) = \{(p, q)|\ (-q, p) \in G(T)^0\}.$$

Since the polar cone $G(T)^0$ is polyhedral, T is another polyhedral convex process.

If T is actually a linear transformation, then $G(T)$ is a linear subspace of $R^m \times R^n$, so that when the polar is taken the inequality can be replaced by equality. Thus T turns out to be the usual adjoint linear transformation, corresponding to the transpose matrix. This example shows that the adjoint operation is a natural generalization of the one in linear algebra. It also explains why we do not simply take the polar of $G(T)$, but make a change of sign in one argument.

In the von Neumann economic model, where T transforms states of goods, T^* should be interpreted as transforming states of prices. It assigns to each price vector p for outputs the price vectors q for inputs which have the following property: no matter how the goods are transformed, the total input value will be at least as much as the total output value. In other words, T^* is a pricing or accounting mechanism which attributes value to the factors of production.

We want the law $(T^*)^* = T$ to hold, but when we apply the preceding definition of adjoint to T^* we run into a difficulty: the inequality in the definition is in an awkward direction. The question is raised of whether we should consider two different adjoint operations, depending on the direction of the inequality, with rules about when to apply which one. There is a better way of handling this difficulty, where each process is assigned an "orientation", indicating how the adjoint operation is to manifest itself. These orientations turn out to be useful in other respects as well, and they assume a richer character as the theory progresses.

We imagine a polyhedral convex process T as having been assigned either a "maximizing orientation" or a "minimizing orientation". Of course, there is nothing intrinsic in T which determines which orientation to assign, any more then there is an intrinsic left-handedness or right-handedness to a mathematical model of space. Here we simply have a convenient mathematical device for keeping track of the signs in certain formulas. Nevertheless, in applications it is usually clear what the orientation should be. For instance, as will be even more apparent later, it is natural to consider a production process as max-oriented and a pricing process as min-oriented.

The above definition of T^* corresponds to T *max-oriented.* If T is *min-oriented,* we define T^* instead with the opposite inequality. In either case, we provide T^* with the orientation opposite to that of T. Thus the adjoint operation assigns to each oriented process another *oriented* process. In this way, we obtain from the polarity law $K^{oo} = K$ for polyhedral convex cones the desired law

$$T^{**} = T.$$

The inverse operation must also now be brought into a relationship with orientations. We regard it, like the adjoint operation, as *orientation-reversing*. Then we have the rule

$$(T^*)^{-1} = (T^{-1})^*.$$

The latter process is again from R^m to R^n, and it has the same orientation as T.

All the other algebraic operations are regarded as *orientation-preserving* (except scalar multiplication by a negative number), and we only use them to combine processes whose orientation is the same. In other words, the sum of two max-oriented processes is taken to be max-oriented, etc. There is then a remarkable result:

THEOREM 4: The adjoint operation obeys the laws:

$$(T_1 + T_2)^* = T_1^* + T_2^*, \quad (ST)^* = T^* S^*,$$
$$(T_1 \wedge T_2)^* = T_1^* \vee T_2^*, \quad (T_1 \vee T_2)^* = T_1^* \wedge T_2^*,$$
$$(\lambda T)^* = \lambda T^* \quad \text{for} \quad \lambda > 0.$$

Proof: In a formal way, this can be established by applying the polarity operation to the convex cones used in the constructions of the graphs in the preceding section. However, this procedure, even if satisfactory from a mathematical point of view, is rather mystifying. The real explanation of the relations will not be seen until later, after support functions are introduced.

Theorem 4 yields further rules for processes $T\colon R^n \to R^n$, such as

$$(T^k)^* = (T^*)^k,$$
$$(T - \lambda I)^* = T^* - \lambda I.$$

These will take on significance below in the discussion of models of economic growth over discrete time.

The polarity relationship between the cones $G(T)$ and $G(T^*)$ yields polarity relationships among domain and range cones, etc. For example, supposing that T is max-oriented, it is easy to check that the polar of $D(T^*)$ is $T(0)$, and hence $D(T^*)$ is in turn the polar of $T(0)$. In this way we determine the following relations, where the double arrow means that the cones are polar to each other (for T max-oriented):

$$T(0) \leftrightarrow D(T^*), \qquad D(T) \leftrightarrow -T^*(0),$$
$$R(T) \leftrightarrow (T^*)^{-1}(0), \quad R(T^*) \leftrightarrow -T^{-1}(0).$$

6. Support functions and dual programming problems

The linear transformations $T\colon R^m \to R^n$ correspond classically to bilinear real-valued functions K on $R^m \times R^n$, and this correspondence can be used for the definition of the adjoint transformation:

$$K(y, p) = T(y) \cdot p = y \cdot T^*(p).$$

For polyhedral convex processes, there is a similar correspondence, and it has an astonishing property. The analogous formula relating T and T^* is an abstract form of the duality theorem for linear programming problems.

Let $T\colon R^m \to R^n$ be a max-oriented polyhedral convex process. For each $y \in R^m$ and $p \in R^n$, we define

$$\langle T(y), p\rangle = \langle p, T(y)\rangle = \sup\{p \cdot z \mid z \in T(y)\}.$$

(If T were min-oriented, the supremum would be replaced by infimum.) Here the supremum is $-\infty$ by convention if $T(y) = \varnothing$. Since $T(y)$ is a polyhedral convex set and the function $z \to p \cdot z$ is linear, the supremum is the optimal value in an *abstract linear programming problem*, and if finite it is attained at some point z. If $T(y) \neq \varnothing$, then the recession cone of $T(y)$ is $T(0)$, so that the supremum is $+\infty$, unless $p \cdot w \leqslant 0$ for all $w \in T(0)$. Conversely, if $p \cdot w \leqslant 0$ for all $w \in T(0)$, then the supremum is not $+\infty$. This can be shown directly from the prop rties of recession cones of polyhedral convex sets, but it is also immediate from the theorem below. Thus, recalling from the end of the last section that the polar of $T(0)$ is $D(T^*)$, we see that

$$\langle T(y), p\rangle = \begin{cases} \text{finite number} & \text{if} \quad y \in D(T), p \in D(T^*), \\ +\infty & \text{if} \quad y \in D(T),\ p \notin D(T^*), \\ -\infty & \text{if} \quad y \notin D(T). \end{cases}$$

We call the function $K(y, p) = \langle T(y), p \rangle$ the *support function* of T.

Observe that T is completely determined by K, since $K(y, p)$ gives for each y and p the supporting half-space, if any, to the (closed) convex set $T(y)$ in the direction of p. Specifically,

$$T(y) = \{z \mid p \in R^n, p \cdot z \leqslant K(y, p)\}.$$

Thus we have a one-to-one correspondence between the class of (max-oriented) polyhedral convex processes from R^m to R^n and a certain class of extended-real-valued functions on $R^m \times R^n$. This generalizes the classical correspondence for linear transformations, since if T is linear (hence single-valued) we have $\langle T(y), p \rangle = T(y) \cdot p$.

How can the functions K which arise this way be characterized? For each such K there is a cone $C \times D$, such that K is finite on $C \times D$ but infinite (in a fixed manner) elsewhere. Moreover, for elements of $C \times D$ we have the following properties, which are easily seen from the definition of K and the basic laws for convex processes:

$$K(y, p+p') \leqslant K(y, p) + K(y, p'),$$

$$K(y+y', p) \geqslant K(y, p) + K(y', p),$$

$$K(\lambda y, p) = \lambda K(y, p) = K(y, \lambda p) \quad \text{for } \lambda > 0.$$

Thus $K(y, p)$ is a positively homogeneous, convex function of $p \in D$ for fixed $y \in C$, and a positively homogeneous, concave function of $y \in C$ for fixed $p \in D$. A further fact, not so apparent, is that K is continuous relative to $C \times D$, which of course is a closed set. Conversely, if K has all these properties, and T is defined as in the preceding paragraph, it can be shown that T is a convex process with closed graph, such that $\langle T(y), p \rangle = K(y, p)$. The proof is contained in Rockafellar [1969]. We do not know how to formulate (neatly) the property of K which corresponds to T being polyhedral.

For the adjoint T^* of a max-oriented polyhedral convex process T, we have by definition

$$\langle y, T^*(p) \rangle = \langle T^*(p), y \rangle = \inf\{q \cdot y \mid q \in T^*(p)\}.$$

This again denotes the optimal value in an abstract linear programming problem. By reasoning similar to the above, we see that

$$\langle y, T^*(p) \rangle = \begin{cases} \text{finite number} & \text{if } y \in D(T),\ p \in D(T^*), \\ -\infty & \text{if } y \notin D(T),\ p \in D(T^*), \\ +\infty & \text{if } p \notin D(T^*). \end{cases}$$

Can it be true that $\langle T(y), p\rangle$ and $\langle y, T^*(p)\rangle$ are the same? These quantities certainly do not agree if $y \notin D(T)$ and $p \notin D(T^*)$, but they agree in all other cases. This is the *abstract form of the duality theorem for linear programming problems*:

THEOREM 5: If either $y \in D(T)$ or $p \in D(T^*)$, one has

$$\langle T(y), p\rangle = \langle y, T^*(p)\rangle.$$

In the exceptional case, the two quantities are oppositely infinite.

Proof: Since T is polyhedral, we can represent it in terms of an $s\times m$ matrix A and an $s\times n$ matrix B:

$$G(T) = \{(xA, xB)|\ x \in R^s_+\},$$
$$T(y) = \{xB|\ x \geqslant 0,\ xA = y\}.$$

Then, calculating from the definitions, we have

$$G(T^*) = \{(p, q)|\ \forall\, x \in R^s_+,\ x(Bp - Aq) \geqslant 0\}$$

so that

$$T^*(p) = \{q|\ Aq \leqslant Bp\}.$$

Thus we have for fixed vectors y and $r = Bp$:

$$\langle T(y), p\rangle = \sup\{x\cdot r|\ x \geqslant 0,\ xA = y\},$$
$$\langle y, T^*(p)\rangle = \inf\{q\cdot y|\ Aq \leqslant r\}.$$

These extrema belong to a pair of linear programming problems dual to each other in the usual sense, and hence they are equal unless oppositely infinite. (The latter happens in the case where neither problem has a feasible solution.)

Note that this proof shows the *equivalence* of Theorem 5 and the duality theorem for linear programming, since we could also start from arbitrary matrices A and B and use them to define T.

From Theorem 5 and the fact that processes can be characterized in terms of their support functions, we can see "why" the law $(T_1+T_2)^* = T_1^*+T_2^*$ is valid. Evidently,

$$D(T_1+T_2) = D(T_1) \cap D(T_2),$$

and for y in this set we have

$$\langle (T_1+T_2)(y), p\rangle = \sup\{p\cdot(z_1+z_2)|\ z_i \in T_i(y)\}$$
$$= \langle T_1(y), p\rangle + \langle T_2(y), p\rangle.$$

The law follows by passing to adjoints on both sides of this equation.

The law $(ST)^* = T^* S^*$ is more complicated. Formally, we would like to argue that

$$\langle ST(y), r\rangle = \langle T(y), S^*(r)\rangle = \langle y, T^* S^*(r)\rangle,$$

and hence T^*S^* must agree with $(ST)^*$. However, the middle "inner product" in this chain is undefined; it invoves two polyhedral convex sets $T(y)$ and $S^*(r)$ in R^n. Is there a reasonable way to define it so that the argument works? We shall show in the next section that there is.

7. Minimax theory and the inverse operation

Let T: $R^m \to R^n$ be a max-oriented polyhedral convex process. Then T^{-1}: $R^n \to R^m$ is by our convention a min-oriented polyhedral convex process, so that

$$\langle q, T^{-1}(z)\rangle = \inf\{q \cdot y \mid y \in T^{-1}(z)\},$$
$$\langle (T^{-1})^*(q), z\rangle = \sup\{p \cdot z \mid p \in (T^{-1})^*(q)\},$$

where $(T^{-1})^* = (T^*)^{-1}$. These two extrema agree, unless oppositely infinite, according to Theorem 5. It is not hard to see that they can be expressed in terms of T by

$$\langle q, T^{-1}(z)\rangle = \inf_y \sup_p \{q \cdot y + p \cdot z - \langle T(y), p\rangle\},$$
$$\langle (T^{-1})^*(q), z\rangle = \sup_p \inf_y \{q \cdot y + p \cdot z - \langle T(y), p\rangle\},$$

where $\langle T(y), p\rangle$ can be replaced equivalently by $\langle y, T^*(p)\rangle$. By symmetry we then have, since $(T^{-1})^{-1} = T$,

$$\langle T(y), p\rangle = \sup_z \inf_q \{q \cdot y + p \cdot z - \langle q, T^{-1}(z)\rangle\},$$
$$\langle y, T^*(p)\rangle = \inf_q \sup_z \{q \cdot y + p \cdot z - \langle q, T^{-1}(z)\rangle\},$$

where $\langle q, T^{-1}(z)\rangle$ can be replaced by $\langle (T^*)^{-1}(q), z\rangle$.

In the general language of convex analysis, this is an example of an equivalence class of concave-convex functions of (y, p) and an equivalence class of concave-convex functions of (q, z) which are *conjugate* to each other (see Rockafellar [1969]). From the minimax formulas, we find that, just as the adjoint operation for polyhedral convex processes corresponds to the duality theory for linear programming, the inverse operation corresponds to the minimax theory (saddle-point characterization of optimal solutions). Indeed, one need only substitute the formulas for $\langle T(y), p\rangle$ and $\langle y, T^*(p)\rangle$ into Theorem 5.

As an illustration, let us return to Example 2, with T max-oriented of the form

$$T(y) = \{z|\ z \geqslant 0,\ Az \leqslant y\}.$$

It is clear that

$$T^{-1}(z) = \begin{cases} \{y|\ y \geqslant Az\} & \text{if} \quad z \geqslant 0, \\ \emptyset & \text{if} \quad z \ngeqslant 0, \end{cases}$$

and consequently

$$\langle q, T^{-1}(z)\rangle = \begin{cases} qAz & \text{if} \quad q \geqslant 0, z \geqslant 0, \\ -\infty & \text{if} \quad q \ngeqslant 0,\ z \geqslant 0, \\ +\infty & \text{if} \quad z \ngeqslant 0. \end{cases}$$

From this it is easy to determine the process $(T^{-1})^* = (T^*)^{-1}$ and hence T^*. We obtain

$$T^*(p) = \{q|\ q \geqslant 0,\ A^*q \geqslant p\},$$

where A^* is the transpose matrix. Theorem 5 asserts that

$$\sup\{p \cdot z|\ z \geqslant 0,\ Az \leqslant y\} = \inf\{q \cdot y|\ q \geqslant 0,\ A^*q \leqslant p\},$$

which is one of the usual forms of the duality theorem for linear programming. The formulas for $\langle T(y), p\rangle$ and $\langle y, T^*(p)\rangle$ in terms of $\langle q, T^{-1}(z)\rangle$ express the extremum as the minimax of

$$L(q, z) = q \cdot y + p \cdot z - q \cdot Az$$

relative to $q \geqslant 0$ and $z \geqslant 0$. This is the standard minimax theorem in linear programming.

A deeper minimax theory, still "polyhedral" in character, may be built on a generalized notion of inner product $\langle C, D\rangle$, where C and D are sets. As with processes, let us imagine subsets of R^n which equipped with either a "maximizing orientation" or a "minimizing orientation". If C and D are both max-oriented, we define

$$\langle C, D\rangle = \sup\{z \cdot p|\ z \in C, p \in D\}.$$

Analogously if both are min-oriented. If C is max-oriented and D is min-oriented, there are the two possible quantities

$$\sup\{\inf\{z \cdot p|\ p \in D\}|z \in C\},$$
$$\inf\{\sup\{z \cdot p|\ z \in C\}|p \in D\}.$$

If these are equal, we denote the common value by $\langle C, D\rangle$. Otherwise $\langle C, D\rangle$ is undefined. According to the minimax theorem of Wolfe, if C and D are polyhedral convex sets, $\langle C, D\rangle$ is defined unless the "supinf" and the "infsup" are oppositely infinite. (And then a saddle-point exists.) Thus in particular, $\langle C, D\rangle$ is always defined if C and D are nonempty polyhedral convex sets, one of which is bounded.

In the special case where D consists of a single element p, we simply write $\langle C, p\rangle$, etc. As a function of p, this is called the *support function* of the (oriented) set C. All this agrees with our earlier terminology and notation for processes, if we interpret images $T(y)$ under a max-oriented process as being max-oriented, and so on.

Using the above definition one can prove a generalization of Theorem 5: the equation

$$\langle T(C), D\rangle = \langle C, T^*(D)\rangle$$

holds, if the left side is defined and not $-\infty$, or if the right side is defined and not $+\infty$. Here T is any max-oriented polyhedral convex process, C is a max-oriented polyhedral convex set, and D is a min-oriented polyhedral convex set.

This result justifies in particular the "inner product" argument for the proof of $(ST)^* = T^*S^*$. It may be interpreted as a duality theorem for polyhedral convex programming problems:

$$\sup\{\langle z, D\rangle \mid T^{-1}(z) \text{ meets } C\} = \inf\{\langle C, q\rangle \mid (T^*)^{-1}(q) \text{ meets } D\}.$$

The saddle-point theorem for these problems involves the convex-concave function

$$L(q, z) = \langle C, q\rangle + \langle z, D\rangle - \langle q, T^{-1}(z)\rangle.$$

8. Optimizing trajectories in discrete time

We consider now in more detail the case of a polyhedral convex process T from R^n to R^n. For such a process, a sequence of powers can be formed: $T, T^2, T^3, \ldots$, and questions arise concerning the behavior of this sequence. Here, as already pointed out, $T^k(y)$ can be interpreted as the set of states into which a state y can be transformed in k units of time. If T represents the transformation of goods, the adjoint sequence $T^*, T^{*2}, T^{*3}, \ldots$, can be interpreted in terms of the transformation of prices. For economic models of production, there are natural restrictions which may be placed on T, and these will be described in the next section. However, for the time being we deal with an arbitrary T, taken to be max-oriented. The advantage of this approach technically is that it preserves the symmetry between T and T^*, making it obvious that each definition and result is also valid in a dual form.

By a *trajectory* for T^k from a to b, we shall mean a finite sequence $y_0, y_1, \ldots, y_k$ such that $y_0 = a$, $y_k = b$, and $y_i \in T(y_{i-1})$ for $i = 1, \ldots, k$. Thus, by definition, a trajectory for T^k from a to b exists if and only if $b \in T^k(a)$. Trajectories for T^{*k} are defined in the same way. If $p_0, p_1, \ldots, p_k$ is such a trajectory, then

$$y_0 \cdot p_k \geqslant y_1 \cdot p_{k-1} \geqslant \ldots \geqslant y_k \cdot p_0 .$$

This general inequality is immediate from the fact that $y_i \in T(y_{i-1})$ and $p_i \in T^*(p_{i-1})$ for all i. (Recall the definition of T^*.) The chain of inequalities represents a sort of one-sided law of conservation of value, in the economic setting.

We shall say that the trajectory $y_0, y_1, \ldots, y_k$ for T^k is an *optimizing trajectory* if there exists a trajectory $p_0, p_1, \ldots, p_k$ for T^{*k} such that $p_0 \neq 0$ and

$$y_0 \cdot p_k = y_1 \cdot p_{k-1} = \ldots = y_k \cdot p_0 .$$

Then y_i is for $i = 1, \ldots, k$ a point of $T(y_{i-1})$ for which the maximum of the linear function $y \rightarrow y \cdot p_{k-1}$ is attained. In other words, $y_0, y_1, \ldots, y_k$ arises by solving a certain sequence of optimization problems.

We shall say that $y_0, y_1, \ldots, y_k$ is a *strictly optimizing* trajectory if, in addition, a trajectory $p_0, p_1 \ldots, p_k$ can be found such that y_i is the unique point of $T(y_{i-1})$ for which the maximum of the linear function $y \rightarrow y \cdot p_{k-1}$ is attained. Then there can be no other trajectory for T^k with the same end points. Such a trajectory can be interpreted economically as the unique response of production to a succession of price states.

Optimizing trajectories $p_0, p_1, \ldots, p_k$ correspond similarly to a sequence of minimization problems, the linear function $p \rightarrow p \cdot y_{k-i}$ being minimized over the set $T^*(p_{i-1})$ to get p_i. Obviously, if $y_0, y_1, \ldots, y_k$ is an optimizing trajectory for T^k with $y_0 \neq 0$, and $p_0, p_1, \ldots, p_k$ is the associated trajectory for T^{*k}, then the latter is optimizing, and *vice versa.*

The following theorems characterize the points attainable by optimizing trajectories for T^k. Analogous results hold for T^{*k}.

THEOREM 6. Let T: $R^n \rightarrow R^n$ be any polyhedral convex process. There exists an optimizing trajectory for T^k from the point a to the point b, if and only if b belongs to the boundary of $T(a)$. Moreover, then every trajectory for T^k from a to b is optimizing.

Proof: If $y_0, y_1, \ldots, y_k$ is an optimizing trajectory from a to b then $b \in T^k(a)$, and for certain vectors $p_0 \neq 0$ and $p_k \in T^{*k}(p_0)$ we have $a \cdot p_k = b \cdot p_0$. Since $T^{*k} = T^{k*}$, the inequality $a \cdot p_k \geqslant y \cdot p_0$ is valid for all

$y \in T^k(a)$. Thus the nonzero linear function $y \to y \cdot p_0$ attains its maximum over $T^k(a)$ at b, implying that b is a boundary point. Conversely, let b be any boundary point of $T^k(a)$, and let $y_0, y_1, \ldots, y_k$ be any trajectory for T^k from a to b. Since $T^k(a)$ is convex, there is a supporting hyperplane to $T^k(a)$ at b. Thus there is a $c \neq 0$ such that the linear function $y \to y \cdot c$ attains its maximum over $T^k(a)$ at b. Then, applying Theorem 5, we have

$$b \cdot c = \langle T^k(a), c \rangle = \langle a, T^{k*}(c) \rangle = \langle a, T^{*k}(c) \rangle .$$

Hence there exists $d \in T^{*k}(c)$ such that $a \cdot d = b \cdot c$. Choosing any trajectory $p_0, p_1, \ldots, p_k$ from c to d, we have

$$a \cdot d = y_0 \cdot p_k \geqslant y_1 \cdot p_{k-1} \geqslant \ldots \geqslant y_k \cdot p_0 = b \cdot c .$$

The equality of the first and last terms in the chain implies equality throughout, so that $y_0, y_1, \ldots, y_k$ is an optimizing strategy.

THEOREM 7. Let $T\colon R^n \to R^n$ be any polyhedral convex process. If there exists a strictly optimizing strategy for T^k from a to b, then b is an extreme point of $T^k(a)$.

Proof: If $y_0, y_1, \ldots, y_k$ is a strictly optimizing trajectory from a to b, then, as we have seen in the preceding proof, b maximizes the linear function $y \to y \cdot p_0$ over the polyhedral convex set $T^k(a)$. The additional property now of the trajectory $p_0, p_1, \ldots, p_k$ is that y_i uniquely maximizes the linear function $y \to y \cdot p_{k-i}$ over $T(y_{i-1})$. This implies that, if $y_0', y_1', \ldots, y_k'$ is any trajectory for T^k starting from a, and different from $y_0, y_1, \ldots, y_k$, at least one of the inequalities in the chain

$$y_0' \cdot p_k \geqslant y_1' \cdot p_{k-1} \geqslant \ldots \geqslant y_k' \cdot p_0$$

must be strict. Therefore b is the unique point of $T^k(a)$ for which the linear function $y \to y \cdot p_0$ attains its maximum, so that b must be an extreme point.

The converse of Theorem 7 is not valid without further restrictions, but what these restrictions should be is an open question. Here are two examples which illustrate difficulties.

Counterexample 1. For $y = (y^1, y^2) \in R^2$, define

$$T(y) = \begin{cases} \{z = (z^1, z^2) \mid z^1 = y^1, |z^2| \leqslant y^1\} & \text{if} \quad y^1 \geqslant 0, y^2 = 0 \\ \varnothing & \text{in all other cases.} \end{cases}$$

Then $b = (1, 1)$ is an extreme point of $T^2(a)$, where $a = (1, 0)$. However the only trajectory for T^2 from a to b is $y_0 = a$, $y_1 = a$, $y_2 = b$, and y_1 is not an extreme point of $T(y_0)$. Thus this trajectory can not be strictly optimizing.

Counterexample 2. For $y \in R^1$, define

$$T(y) = \begin{cases} \{z \in R^1 \mid 0 \leqslant z \leqslant y\} & \text{if} \quad y \geqslant 0, \\ \varnothing & \text{if} \quad y \ngeqslant 0. \end{cases}$$

Then $b = 0$ is an extreme point of $T^2(a) = [0, 1]$, where $a = 1$. If y_0, y_1, y_2 is an optimizing trajectory from a to b with associated p_0, p_1, p_2, then $p_0 < 0$ and

$$a \cdot p_2 = y_1 \cdot p_1 = b \cdot p_0 = 0.$$

In particular this implies $p_1 \leqslant 0$, since $y_1 \cdot p_1 = \langle T(a), p_1 \rangle$ and $T(a) = [0, 1]$. On the other hand, we have

$$T^*(p) = \{q \mid q \geqslant 0, q \geqslant p\},$$

and therefore

$$p_1 \in T^*(p_0) = [0, +\infty).$$

Thus $p_1 = 0$, and the maximum of the function $y \rightarrow y \cdot p_1$ over $T(a)$ is not attained uniquely at y_1.

Despite these counterexamples, some kind of converse of Theorem 7 ought to be true, and it would be interesting to know what it is.

9. "Eigenvalue" theory

In the study of the von Neumann model in Example 1, much attention is devoted to the vectors y and numbers λ with the property that $\lambda y \in T(y)$. These correspond to states of the economy which are self-reproducing in a certain sense, and they are closely related to the potential growth rate of the economy.

The most natural assumptions on T in this economic context are that

(1) $D(T) = R^n_+$ and $R(T) \subset R^n_+$,

(2) $T(0) = \{0\}$, i.e. all the sets $T(y)$ are bounded.

(T is a polyhedral convex process, max-oriented.) The properties of T^* which are equivalent to these are:

(1*) $T^*(0) = R^n_+$ and $(T^*)^{-1}(0) \supset -R^n_+$,

(2*) $D(T^*) = R^n$.

The non-negativity of the vector pairs (y, z) in the graph of T corresponds to the notion that the components of these vectors represent material quantities. Models can be imagined in which "debts" in such quantities are temporarily allowed, so that negative numbers occur, but this is partly just a matter of how the unit time period is conceived. There does not seem to be any actual loss of generality in assuming that, although debts of this type may be incurred within the time period, even just after the beginning or just before the end, they must all be made up in the same time period. In other words, the time period represents an accounting interval at the end of which one again looks only at the real (physical) disposition of the material quantities.

The assumption that $D(T)$ is all of R^n_+, not just a part of it, means that every possible state, consisting of a configuration of material quantities, has some successor state. The successor state need not be obtained only through "production", but also through such activities as "storage", "decay", "disposal", etc. It is hard to conceive of a model where, on the contrary, there are states $y \geqslant 0$ with no possible successors, so that "the world must come to an end".

The assumption that $T(0) = \{0\}$ means simply that material quantities can not be created from nothing. The fact that this implies the boundedness of $T(y)$ follows from the fact that $T(y)$ is for each $y \in D(T)$ a nonempty polyhedral convex set with recession cone $T(0)$.

Virtually all the study of the relation $\lambda y \in T(y)$ from an economic point of view has been under, not only the preceding assumptions (except for polyhedral convexity), but also the following, which is much more open to question:

(3) If $z \in T(y)$, $y' \geqslant y$, $z \geqslant z' \geqslant 0$, then $z' \in T(y')$.

This says that free disposal is possible of quantities in surplus. Difficulties arise as soon as we think of the disposal as a physical process applied to material quantities, since such a process might have to compete for resources, and at the very least it would require time proportional to the amount of material to be "annihilated". Without free disposal, on the other hand, there is the complication that some products may be undesirable, so that negative prices may play an essential role.

A process T satisfying (1), (2) and (3) is a "monotone process of concave type" in the sense of Rockafellar [1969]. We shall not address ourselves here to the "eigenvalue" properties of such processes of more familiar type, but describe instead some results which hold for polyhedral convex processes in general.

Let T: $R^n \to R^n$ be an arbitrary polyhedral convex process, and let λ be an arbitrary real number. The set of all vectors y satisfying $\lambda y \in T(y)$ is then a certain polyhedral convex cone in R^n, namely the cone $(T - \lambda I)^{-1}(0)$.

If T is actually a linear transformation, this cone is $\{0\}$ for all but a finite set of values of λ, where it is a nontrivial subspace. However, in general there will be many values of λ for which the cone $(T-\lambda I)^{-1}(0)$ is nontrivial; and interest centers more on the question of whether, for certain more special values of λ, this cone undergoes some abrupt change. In this connection we have a strong result, whose proof must be omitted for lack of space.

THEOREM 8: Let $T: R^n \to R^n$ be any polyhedral convex process and let B be the unit ball of R^n. Then the mapping $\lambda \to (T-\lambda I)^{-1}(0) \cap B$ is upper semicontinuous everywhere, and it is lower semicontinuous except at a finite number of values $\lambda_1, \ldots, \lambda_m$ (maybe none).

The exceptional numbers λ_i may be called the *critical values* of T. If T is a linear transformation, they are precisely the eigenvalues of T. But if T is not a linear transformation, the number of such values is not necessarily bounded by the dimension n, and it may be arbitrary high. One can distinguish between "right" critical values, "left" critical values and "two-sided" critical values, depending on the kind of discontinuity. Are the critical values for T^* the same?

Motivated by the standard economic results, we can define λ to be an *equilibrium* value for T if there exist vectors y and p such that $\lambda y \in T(y)$, $\lambda p \in T^*(p)$, and $y \cdot p \neq 0$. The general economic meaning of this is not so clear in the case where $y \cdot p < 0$, for example. However, at least the following can be shown: every equilibrium value is a critical value. Thus T has only finitely many equilibrium values (maybe none). Again, this number need not be bounded by the dimension n. The polyhedral convexity of T is essential here, as in Theorem 8.

Still another class of special values of λ can be investigated. Let us define λ to be a *pseudo-eigenvalue* of T if there exists a vector $y \neq 0$ such that λy is an extreme point of $T(y)$. If T is a linear transformation, the pseudo-eigenvalues are precisely the eigenvalues of T. It can be shown that an arbitrary polyhedral convex process T has only finitely many pseudo-eigenvalues, but everything else about these values, including their relationship to critical values and equilibrium values, remains a mystery.

Almost nothing is known about the role of such "eigenvalue" considerations in determining the behavior of the sequence $T, T^2, T^3, \ldots$, other than in the linear case or the "traditional" economic case. One would be especially interested to have some generalization of turnpike theory, relating "eigenvalues" to the optimizing trajectories considered in the preceding section.

10. Dynamical models in continuous time

In economic models involving transformations over discrete time (with constant technology), the set of states z into which a given state y can be transformed in k units of time is $T^k(y)$, where T^k is the k-th power of a certain convex process $T = T^1$. What is the natural analogue of this for continuous time?

Let us denote by $T^{(\tau)}(y)$ the set of states into which y can be transformed in τ units of time, where τ is an arbitrary nonnegative real number. Then the following law, resembling the rule for manipulating integral powers, should hold:

$$T^{(\tau+\sigma)} = T^{(\tau)}T^{(\sigma)} \quad (= T^{(\sigma)}T^{(\tau)}) \quad \text{for all} \quad \tau \geqslant 0, \sigma \geqslant 0.$$

This follows from the fact that everything produced in $\tau + \sigma$ units of time corresponds to something produced in τ units of time which is then transformed further in σ additional units of time. Thus the natural dynamical model in continuous time consists of a *one-parameter semigroup* of convex processes, i.e. a homomorphism $\tau \to T^{(\tau)}$ from the semigroup of nonnegative real numbers under addition to the semigroup of convex processes under multiplication.

We have purposely omitted the word "polyhedral" here, since it is by no means clear whether this is appropriate for "continuous" models. Such models necessarily involve topological limits of various sorts, and these are unlikely to preserve the property that a process is "finitely generated". A more reasonable context is that of the algebra of *closed convex processes*, i.e. multifunctions whose graphs are arbitrary nonempty closed convex cones (see Rockafellar [1969]). For economic analysis, assumptions like (1) and (2) (or even (3)) in the last section may be called for. As a mathematical simplification, it might be assumed that the semigroup consists of polyhedral convex processes, so that a number of bothersome technical questions concerning domains, closures and adjoints may be avoided. However, the advantages of this quickly begin to wear thin.

It is one thing to speak of a semigroup of convex processes and another to know whether such semigroups even exist. The following heuristic idea is therefore valuable in suggesting a simple way in which many such semigroups may be generated.

Let us consider an economic model in which transformations over continuous time are described in terms of the instantaneous rates of change which are feasible. Specifically, let $S\colon R^n \to R^n$ be a convex process with the interpretation: $z = (z^1, \ldots, z^n)$ belongs to $S(y)$, where $y = (y^1, \ldots, y^n)$, if and only if it is possible in the state y to arrange production so that the i-th good is being produced at the (instantaneous) rate of z^i units per unit of time. We can then get something corresponding approximately to $T(\tau)(y_0)$ as follows. Divide the time interval $[0, \tau]$ into k subintervals of length τ/k. Over the first subinterval, we can transform y into approximately $y_1 = y_0 + (\tau/k)z_1$, where z_1 is some element of $S(y)$. Over the second subinterval, we can transform y_1 into approximately $y_2 = y_1 + + (\tau/k)z_2$, where $z_2 \in S(y_1)$, and so forth. Note that

$$y_i \in \big(I + (\tau/k)S\big)(y_{i-1}), \quad i = 1, \ldots, k.$$

Thus, over the sequence of k subintervals covering a total of τ units of time, we can approximately transform y_0 into elements of the set

$$\big(I + (\tau/k)S\big)^k(y_0).$$

This indicates the possible definition

$$T^{(\tau)} = \lim_{k \to \infty} \big(I + (\tau/k)S\big)^k \equiv e^{\tau S}$$

as a way to generate a one-parameter semigroup from S. The limit could be taken perhaps in the sense of the convergence of the graphs of the processes, as subsets of $R^n \times R^n$. It would be valuable to know whether this approach can be formalized. The process S would be the "infinitesimal generator" of the semigroup, and in some sense it ought to be true that

$$S = \frac{d}{d\tau} T^{(\tau)} \quad \text{at} \quad \tau = 0.$$

An important converse question would also need to be asked: what conditions on semigroup $\{T^{(\tau)}, \tau \geqslant 0\}$ imply that it arises in such a way from a "rate" process S?

Observe that, even in the most ordinary economic setting, one would not want to impose the restriction that $R(S) \subset R^n_+$, although the assumption $D(S) = R^n_+$ remains reasonable. The assumption that the set $S(y)$ is bounded for each y seems completely appropriate economically, since arbitrarily large instantaneous rates of change are not physically possible. It should be remarked, however, that just such unbounded rates are implicit in the continuous analogue of models with "free disposal".

While the approach described above has not been worked out, another approach is available which can indeed be shown to generate a semigroup from S. This approach is based on newly established results on problems of convex type in the calculus of variations.

THEOREM 9. Let $S: R^n \to R^n$ be a closed convex process with $S(0) = \{0\}$. For each $\tau > 0$ and $a \in R^n$, let $T^{(\tau)}(a)$ denote the set of all $b \in R^n$ such that there exists an absolutely continuous curve $y(t)$, $0 \leqslant t \leqslant \tau$, with $y(0) = a$, $y(\tau) = b$, and $\dot{y}(t) \in S(y(t))$ almost everywhere ($\dot{y} = dy/dt$). Then $T^{(\tau)}$ is a closed convex process with $T^{(\tau)}(0) = \{0\}$, and the semigroup identity

$$T^{(\tau+\sigma)} = T^{(\tau)}T^{(\sigma)}$$

is satisfied. Furthermore, taking S and $T^{(\tau)}$ to be max-oriented, the adjoint processes $T^{(\tau)*}$ likewise satisfy the semigroup identity, and they can be described as follows. One has $d \in T^{(\tau)*}(c)$ if and only if, for every $\varepsilon > 0$, there exists an absolutely continuous curve $p(t)$, $0 \leqslant t \leqslant \tau$, such that $\|p(0)-c\| < \varepsilon$, $\|p(\tau)-d\| < \varepsilon$, *and* $\dot{p}(t) \in S^*(t)$ almost everywhere.

This is an easy consequence of Corollary 2 to Theorem 1 in Rockafellar [1971].

Results in this area of the calculus of variations which have not yet been published yield a stronger characterization of the adjoint semigroup. Assuming that $D(S) = R^n_+$, we have $d \in T^{(\tau)*}(c)$ if and only if there is a curve $p(t)$, $0 \leqslant t \leqslant \tau$, of bounded variation, such that $p(0) = c$, $p(\tau) = d$, $\dot{p}(t) \in S^*(p(t))$ almost everywhere, and the "singular" part of p is non-decreasing. The latter means that if we write $p(t) = p_0(t)+p_1(t)$, where p_0 is absolutely continuous and $\dot{p}_1(t) = 0$ almost everywhere, then $p_1(t) \leqslant p_1(t')$ in the sense of the ordering cone R^n_+ for $t \leqslant t'$. In particular, then, if $p(t)$ has a jump at $t = \bar{t}$, we must have $p(\bar{t}_+) \geqslant p(\bar{t}_-)$.

These facts can be used to generalize to continuous time models the characterization of optimizing strategies given in Section 8.

Even though Theorem 9 shows how to generate certain semigroups of convex processes, it says nothing about the reversal of this procedure, that is, how to get S from $T^{(\tau)}$.

Of course generalizations are possible to the case of technologies which, at least in a limited sense, are able to change in time. Such models have been treated in terms of convex processes by Makarov and Rubinov [1970]. There is always the difficulty in such models, however, that the kind of change one is the most interested in, unexpected change through the discovery of new methods of production and even new goods and products, is the hardest to represent mathematically.

11. Inhomogeneous models

As a generalization of the notion of a convex process, one may consider multifunctions T such that the graph $G(T)$ is a convex set in $R^n \to R^n$, but not necessarily a convex cone. Some things may be proved about such multifunctions, but most of the duality theory falls away, or at least fails to take on a symmetric form. The reason is that, while duality correspondences for defining natural adjoints exist in the class of convex cones (polarity) and in the class of convex functions (conjugacy), there is no such appropriate correspondence for the class of general convex sets. The analogous correspondence there associates with a convex set, not another set, but a convex function, its so-called support function.

Actually, multifunctions whose graphs are arbitrary convex sets are not so great an advance in generality over convex processes as might be thought. In economics, they usually arise because vectors of goods are expressed in reduced units, for instance quantities per capita or per standard unit of labour, rather than in physical units. In this event, homogeneity is restored simply by re-introducing whatever variable was used in the reduction to ratios.

From the purely mathematical standpoint, this conversion to the homogeneous case can always be carried out as follows. (There may be other ways as well.) Given a multifunction T: $R^m \to R^n$ whose graph is a convex set, we define $\hat{T}$: $R^{m+1} \to R^{n+1}$ by

$$\hat{T}(\hat{y}) = \hat{T}(y_0, y) = \begin{cases} \{\hat{z} = (z_0, z) | \; z_0 = y_0, z \in y_0 T(y_0^{-1}, y)\} & \text{if } y_0 > 0, \\ \{\hat{z} = (0, 0)\} & \text{if } y_0 = 0, \\ \emptyset & \text{if } y_0 < 0. \end{cases}$$

(We might want to pass instead to the multifunction whose graph is the closure of $G(\hat{T})$.) It is easy to check that $\hat{T}$ is a convex process, i.e. $G(\hat{T})$ is a convex cone, and that $z \in T(y)$ if and only if $(1, z) \in \hat{T}(1, y)$. Moreover, if $R^m = R^n$, trajectories for T^k from a to b correspond presisely to trajectories for $\hat{T}^k$ from $\hat{a} = (1, a)$ to $\hat{b} = (1, b)$, and so forth. In this way, practically every question about T can be translated into something about $\hat{T}$, to which the "homogeneous" theory can be applied.

Nevertheless, there are some cases where the conversion just described is not convenient. In these cases, if duality is at all of interest, we propose that, to achieve symmetry, one should move conceptually in the opposite direction. Convex subsets of $R^m \times R^n$ should be identified with their indicators, certain convex functions. One should work in the context then of so-called convex "bifunctions", for which a natural analogue of the algebra of convex processes is available.

To explain the idea of a bifunction, let us consider an economic model of the familiar sort where one has a convex subset G of $R^m \times R^n$ and a function $u(y, z)$ defined for $(y, z) \in G$. Here G may be interpreted as usual as the graph of a multifunction T which transforms states y into states z. On the other hand, $u(y, z)$ may be interpreted as the *utility* associated with z when z is obtained from y; in this event it is natural to assume u is concave. An alternative interpretation is that $u(y, z)$ is the *cost* of z when z is obtained from y; then u should be convex.

Looking at this model from a slightly different angle, parallel to the notion of "multifunction", we may conceive of it in terms of a mapping F which assigns to each $y \in R^m$, not a set, but the pair consisting of the set $T(y)$ and the function $z \to u(y, z)$ on $T(y)$. This is what we mean by a "bifunction". For technical purposes, it is convenient to represent the pair $(T(y), u(y, \cdot))$ by a single function which is extended-real-valued namely, (assuming u is convex) the function of z which has the value $u(y, z)$ if $z \in T(y)$, but $+\infty$ if $z \notin T(y)$. (If u is concave, $+\infty$ is replaced by $-\infty$.) This function on R^n, which is assigned to the vector $y \in R^m$, is denoted by Fy.

This leads us to the general definition that a *convex bifunction* $F\colon R^m \to R^n$ is a mapping which assigns to each $y \in R^m$ a function Fy on R^n in such a way that the value $(Fy)(z)$ is convex as a function of y and z jointly. Concave bifunctions are defined analogously.

A convex process $T\colon R^m \to R^n$ may be identified with a certain convex bifunction F, its *indicator*, defined by $(Fy)(z) = 0$ if $z \in T(y)$, $(Fy)(z) = +\infty$ if $z \notin T(y)$.

An example of convex bifunction in convex programming is the following. Let $f_i\colon R^n \to R^1$ be a convex function for $i = 0, 1, \ldots, m$, and for $y = (y_1, \ldots, y_m) \in R^m$ define

$$(Fy)(z) = \begin{cases} f_0(z) & \text{if} \quad f_i(z) \leqslant y_i \quad \text{for} \quad i = 1, \ldots, m, \\ +\infty & \text{in all other cases.} \end{cases}$$

Then Fy is the essential objective function in the problem of minimizing $f_0(z)$ subject to the constraints $f_i(z) \leqslant y_i$, $i = 1, \ldots, m$.

Adjoints of convex bifunctions are defined by means of the *conjugacy* correspondence for convex functions. We can not describe the theory of conjugacy here, but the full details, including everything we are mentioning about bifunctions, may be found in Rockafellar [1969]. The basic idea is that, if f is a convex function on R^n, the function f^* on R^n defined by

$$f^*(p) = \sup\{z \cdot p - f(z) |\ z \in R^n\}$$

is called the *conjugate* of f. It is another convex function, and under mild assumptions on f the conjugate of f^* is in turn f:

$$f(z) = \sup\{z \cdot p - f^*(p) |\ p \in R^n\}.$$

If f is the indicator of a convex cone, then f^* is the indicator of the polar cone.

For a convex bifunction F: $R^m \to R^n$, the adjoint F^*: $R^n \to R^m$ is defined by

$$(F^*p)\,(q) = \inf_{y,\,z}\{(Fy)\,(z) - z \cdot p + y \cdot q\}.$$

(If F is a concave bifunction, inf is replaced by sup.) Then F^* is a concave bifunction, and under mild assumptions $F^{**} = F$. If F is the indicator of a convex process T, then F^* is the indicator of the adjoint process T^*. (Max-oriented processes are associated with convex indicator bifunctions, min-oriented processes with concave indicator bifunctions.) In this sense, the adjoint operation for bifunctions generalizes the one for convex processes.

The "inner product" theory is generalized by defining for a convex bifunction F

$$\langle Fy, p\rangle = \sup_z\{z \cdot p - (Fy)\,(z)\} = (Fy)^*(p).$$

(For a concave bifunction, sup is replaced by inf.) If F is the indicator of a convex process T, we have $\langle Fy, p\rangle = \langle T(y), p\rangle$. Under mild assumptions, it is true that

$$\langle Fy, p\rangle = \langle y, F^*p\rangle.$$

This is the *abstract duality theorem for convex programming*. It equates the supremum of an extended-real-valued concave function on R^n with the infimum of a certain extended-real-valued convex function on R^m.

As a function of y and p, $\langle Fy, p\rangle$ is concave-convex. Moreover, it can be shown in a precise way that essentially every concave-convex function on $R^m \times R^n$ corresponds in this way to a convex bifunction F: $R^m \to R^n$. By considering the concave-convex function corresponding to the inverse bifunction F^{-1}, where

$$(F^{-1}z)\,(y) = -(Fy)\,(z),$$

one obtains the Lagrange multiplier theory of convex programming, and in fact the most general minimax theory for games of concave-convex type.

Operations for convex processes may also be generalized to bifunctions. We mention only one: multiplication. If $F\colon R^m \to R^n$ and $G\colon R^n \to R^r$ are convex bifunctions, we define $GF\colon R^m \to R^r$ by

$$(GFy)(w) = \inf_z \{(Fy)(z) + (Gz)(w)\}.$$

It can be shown that GF is another convex bifunction, and "usually" $(GF)^* = F^*G^*$. For $F\colon R^n \to R^n$, powers F^k may be considered, and these are actually just what one studies (in effect) in many of the economic models alluded to at the beginning of this section. One-parameter semi-groups of bifunctions satisfying

$$F^{(\tau+\sigma)} = F^{(\tau)}F^{(\sigma)} \quad \text{for} \quad \tau \geqslant 0, \sigma \geqslant 0,$$

also assume an importance. These correspond to general "convex" problems in the calculus of variations of the type investigated in Rockafellar [1971].

The possible implications of all this new theory of bifunctions for economics are yet to be worked out.

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