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SOLVING A NONLINEAR PROGRAMMING PROBLEM BY WAY OF A DUAL PROBLEM (*) (**)

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The uses of duality in linear programming are well known. For instance, it is often advantageous to solve a problem by applying the simplex method to its dual, rather than directly. There are also computational procedures for special classes of problems which invoke duality with respect to subroutines, or to obtain some sort of decomposition. Some of these applications can be extended to convex programming, but in this case it is important to exploit the many possibilities offered by current theory. Different duals and Lagrangians with contrasting properties can be associated with the same primal, and the choice of one of these depends very much on the purpose one has in mind. The same is true in nonconvex programming where, until quite recently, almost no really substantial duality theorems or minimax theorems were known at all.

Our purpose here is to review some of the ideas in this direction, especially a duality scheme related to penalty methods which has important implications for both convex and non convex problems. Much research has been devoted in the last couple of years to this form of duality and corresponding algorithms, but many interesting questions and possibilities remain. This is especially true regarding problems, such as in optimal control, where there is essentially an infinite number of constraints, or in general where the dual elements range over an infinite-dimensional space. Such problems will not be discussed below, but it is hoped that our remarks may serve to stimulate further thinking in the area.

The primal problem we consider is:

$$\text{(P)} \qquad \text{minimize } f_0(x) \text{ over all } x \in X \text{ such that } f_1(x) \leqslant 0, \ldots, f_m(x) \leqslant 0,$$

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where X is a subset of a locally convex linear space E, and f_i is a real-valued function given on X for $i = 0, 1, \ldots, m$. This problem can also be expressed abstractly as that of minimizing the so-called *essential* objective function f over all $x \in E$, where

$$(0.1) \qquad f(x) = \begin{cases} f_0(x) & \text{if } x \text{ is feasible}, \\ +\infty & \text{if } x \text{ is not feasible}. \end{cases}$$

For simplicity, we suppose that (P) has at least one feasible solution, and we make the following *compactness assumption*: for every choice of the real numbers α_i, the set

$$(0.2) \qquad \{x \in X \mid f_0(x) \leqslant \alpha_0,\, f_1(x) \leqslant \alpha_1,\, \ldots,\, f_m(x) \leqslant \alpha_m\}$$

is compact in E. The latter is satisfied in particular, of course, if X is compact and all the functions f_i are lower-semicontinuous.

These assumptions imply, among other things, that the infimum in (P) is finite and attained, or in symbols:

$$(0.3) \qquad -\infty < \min(\mathrm{P}) < +\infty .$$

By the *convex case*, we shall mean the case where X and all the functions f_i are convex. The compactness assumption is then fulfilled if X is closed, every f_i is lower-semicontinuous, and there is a single choice of the α_i's such that the set (0.2) is compact and

$$(0.4) \qquad \{x \in X \mid f_0(x) < \alpha_0,\, f_1(x) < \alpha_1,\, \ldots,\, f_m(x) < \alpha_m\} \neq \emptyset .$$

(If $E = R^n$, one can replace $<$ by $\leqslant$ in (0.4)). This can be established by applying [3, Theorem 2] to the level sets of the function

$$h = \max_{i=0,1,\ldots,m} \{f_i - \alpha_i\} \text{ on } X .$$

1. Ordinary duality.

The ordinary Lagrangian associated with (P) is the function L_0 on $X \times R^m$ defined by

$$(1.1) \qquad L_0(x, y) = \begin{cases} f_0(x) + \sum_{i=1}^{m} y_i f_i(x) & \text{if } y = (y_1, \ldots, y_m) \geqslant 0 , \\ -\infty & \text{if } y \not\geqslant 0 . \end{cases}$$

One has

$$\sup_{y \in R^m} L_0(x, y) = f(x) ,$$

and hence

$$\min_{x \in X} (P) = \min_{y \in R^m} \ \sup L_0(x, y) .$$

This leads one to introduce, as a dual of (P), the problem

(D$_0$) maximize g_0 over R^m, where $g_0(y) = \inf_{x \in X} L_0(x, y)$.

If the functions f_i are affine and $E = R^n$, $X = R^n_+$, so that (P) is a linear programming problem, then (D$_0$) amounts to the familiar (« linear ») dual. In the general case (even without convexity), g_0 is a concave function and upper semicontinuous. Thus (D$_0$) is an abstract concave programming problem whose implicit feasible set is

(1.2) $$\{x \in R^m | g_0(y) > -\infty\} \subset R^m_+ .$$

If (P) is strictly feasible, i.e. there exists

(1.3) $$\tilde{x} \in W \quad \text{with} \quad f_i(\tilde{x}) < 0 \quad \text{for} \quad i = 1, \ldots, m ,$$

then all the level sets of the form

(1.4) $$\{y \in R^m | g_0(y) \geqslant \beta\} , \qquad \beta \in R ,$$

are compact, and one can therefore write max (D$_0$) in place of sup (D$_0$) for the dual optimal value. This follows from the upper-semicontinuity of g_0 and the fact that the set (1.4) is contained in

(1.5) $$\left\{y \in R^m_+ | f_0(\tilde{x}) + \sum_{i=1}^{m} y_i f_i(\tilde{x}) \geqslant \beta\right\}.$$

Convexity in (P) does not really enter the picture decisively until one comes to the main duality theorem in this context:

THEOREM 1: *In the convex case,* min (P) = sup (D$_0$).

For a proof of this result, which depends of course also in our compactness assumption, see [16, Theorems 17', 18' (*e*)]. The significance of Theorem 1 for computational procedures lies essentially in the following well known fact.

THEOREM 2. *Suppose that* $\min(\mathrm{P}) = \sup(\mathrm{D}_0)$. *Then* $(\bar{x}, \bar{y})$ *is a saddle point of* L_0 *on* $X \times R^m$ *if and only if* $\bar{x}$ *solves* (P) *and* $\bar{y}$ *solves* (D_0). *Indeed, if* $\bar{y}$ *solves* (D_0), *then for* $\bar{x}$ *to solve* (P) *it is necessary and sufficient that*:

(a) *the minimum of* $L_0(x, \bar{y})$ *over all* $x \in X$ *is achieved at* $\bar{x}$, *and*

(b) $f_i(\bar{x}) \leqslant 0$ *for* $i = 1, \ldots, m$, *with equality for* i *such that* $\bar{y}_i > 0$.

PROOF: The first assertion is an elementary fact in minimax theory (cf. [4, Theorem 2]). If $\bar{y}$ solves (D_0), then $\bar{y} \geqslant 0$ and

$$(1.6) \qquad \inf_{x \in X} L_0(x, \bar{y}) = \inf_{x \in X} f(x), \text{ where } L_0(x, \bar{y}) \leqslant f(x) \text{ for all } x \in X .$$

In view of this and the definition of f, conditions (a) and (b) are equivalent to f achieving its minimum at $\bar{x}$, or in other words, to $\bar{x}$ being a solution of (P). The theorem is thereby proved.

Theorem 2 furnishes the *dual method of solution of* (P) in its pure form: first determine any $\bar{y}$ solving (D_0), and then determine the elements $\bar{x}$ minimizing $L_0(\cdot, \bar{y})$ over X, discarding the ones which turn out not to satisfy (b) of Theorem 2; these $\bar{x}$'s are then the solutions to (P). (Thus, if there is a *unique* $\bar{x}$ minimizing $L_0(\cdot, \bar{y})$ over X, as is true in particular when f_0 is strictly convex, it must satisfy the constraints and be the unique solution to (P).)

The potential advantages of this procedure lie in the fact that the original problem, with the constraints $f_i(x) \leqslant 0$, is replaced by two optimization problems in which the constraint situation may be much easier to handle. Presumably X is, if not the whole space or an orthant, a fairly elementary kind of set, such as a ball or generalized rectangle. The implicit feasible set (1.2) in (D_0) may also have a simple expression. For example, if X is compact it is just R^m_+.

Even if the essential objective function g_0 cannot be reduced to a more directly convenient formula, there are methods which can be used to solve (D_0). These are largely based on the definition of g_0 as the pointwise infimum of a collection of affine functions of y, restricted to R^m_+. The epigraph

$$(1.7) \qquad G = \{(y, y_{m+1}) \mid y \in R^m, y_{m+1} \in R, y_{m+1} \leqslant g_0(y)\}$$

is given as the set of points in R^{m+1} satisfying a certain infinite system of linear inequalities:

$$(1.8) \qquad f_0(x) + y_1 f_1(x) + \ldots + y_m f_m(x) - y_{m+1} \geqslant 0$$
$$\text{for each } x \in X , \ y_1 \geqslant 0, \ldots, y_m \geqslant 0 .$$

In particular, we may regard (D_0) as the problem of maximizing the value of y_{m+1} over all vectors $(y_1, \ldots, y_m, y_{m+1})$ satisfying (1.8) and (1.9) and try in this context to apply a cutting hyperplane method.

Consider moreover the situation where, for a given $y \in R^m$ and $\varepsilon > 0$, we are able to determine an

(1.10) $\qquad x$ minimizing $L_0(\cdot, y)$ over X to within ε,

or in other words satisfying

$$L_0(x, y) \leqslant g_0(y) + \varepsilon . \tag{1.11}$$

Then from the fact that

$$g_0(y') \leqslant L_0(x, y') \qquad \text{for all } y' \in R^m ,$$

we have the inequality

$$g_0(y') \leqslant [g_0(y) + \varepsilon] + \sum_{i=1}^{m} (y'_i - y_i) f_i(x) \quad \text{for all } y' \in R^m . \tag{1.12}$$

This means by definition that the vector $\big(f_1(x), \ldots, f_m(x)\big)$ is a so-called *ε-subgradient* of g_0 at y [39, § 23]:

$$\big(f_1(x), \ldots, f_m(x)\big) \in \partial_\varepsilon g_0(y) . \tag{1.13}$$

Thus an ε-subgradient of g_0 at a given point y can be calculated by finding an x satisfying (1.10). This is especially interesting in view of recent work of Bertsekas and Mitter [7′] and Lemarechal [27], [28] providing algorithms for maximizing a concave function in terms of ε-subgradients. In the ideal case of $\varepsilon = 0$, one actually calculates, by minimizing $L_0(\cdot, y)$ over X, an element of the subgradient set $\partial g_0(y)$, as well as the value $g_0(y)$. If for each $y \geqslant 0$ the minimum of $L_0(\cdot, y)$ over X is attained at a unique $x = \xi(y)$ (as is true for instance if f_0 is strictly convex and X is compact), then g_0 is finite and differentiable relative to R^m_+ and $\big(f_1(\xi(y)), \ldots, f_m(\xi(y))\big)$ is actually the relative gradient at y (cf. Falk [12]). In other words, (D) amounts to maximizing a differentiable concave function over R^m_+.

However, it is clear that the process of evaluating g_0 and its subgradients (or gradient) at y by minimizing $L_0(\cdot, y)$ over X is computationally expensive. One cannot hope for much practical success with an algorithm which, say, requires in each iteration that g_0 be maximized over some line segment.

The greatest potential for a dual algorithm based on this approach is in situation where, because of special structure, it is unusually easy to carry out the minimization step. The case which has received the most attention is the one where the dual approach leads to a *decomposition* of the original problem. (For a general account, consult Lasdon [26] and Geoffrion [15].) Suppose that

$$(1.14)\qquad \begin{cases} X \;\;= X_1 \times \ldots \times X_N\,, \\ f_i(x) = f_{i1}(x_1) + \ldots + f_{iN}(x_N)\,, \end{cases}$$

where X_j is a subset of a linear space E_j, $x_j \in X_j$ and $f_{ij}\colon X_j \to R$. Then

$$(1.15)\qquad L_0(x, y) = \sum_{j=1}^{N} L_{0j}(x_j, y)\,,$$

where $x \in X$ and

$$(1.16)\qquad L_{0j}(x_j, y) = \begin{cases} f_{0j}(x_j) + \sum_{i=1}^{m} y_i f_{ij}(x_j) & \text{if } y \geqslant 0\,, \\ -\infty & \text{if } y \not\geqslant 0\,. \end{cases}$$

Minimizing $L_0(x, y)$ in $x \in X$ is therefore reduced to minimizing $L_{0j}(x_j, y)$ in $x_j \in X_j$ for $j = 1, \ldots, N$. This reduction can be very advantageous for large-scale problems, and even for problems of moderate scale if the L_{0j} problems can then be solved by special methods (e.g. when multicommodity flow problems are decomposed into single commodity problems).

It is well known that the Dantzig-Wolfe decomposition method corresponds to this approach in the case where the functions f_{ij} are affine and the sets X_j polyhedral. The simplex method is used to minimize each $L_{0j}(\cdot, y)$ on X_j exactly, and this information is incorporated, in effect, into a cutting-plane method for maximizing g_0. Here g_0 is a *polyhedral* concave function, so the algorithm terminates in principle in finitely many iterations. However, computational experience has been disappointing.

Probably this is due mainly to the use of a cutting-hyperplane method. It is conceivable, therefore, that by using another method to maximize g_0 (one which does not entail maximization along line segments), better convergence can be obtained. The most interesting results in this direction have been reported recently by Held, Wolfe and Crowder [18]. These authors have had considerable computational success with highly decomposable problems, using for the maximization of g_0 a slightly modified version of an algorithm developed chiefly by Shor [53] and Polyak [33], [34]. This is remarkable, because the algorithm in question, while having the advantage that one only needs to calculate an arbitrary subgradient at each point, does not have a reputation for fast convergence.

For another algorithm whose applicability to the dual problem in special cases deserves further thought, see Oettli [31].

Some results of interest for nonconvex programming may be found, for example, in Everett [10] and Falk [12]. But applications of ordinary duality in this area are severely restricted for intrinsic reasons, as explained at the beginning of the next section. Another discussion of ordinary duality from the point of view of computational applications has been furnished by Geoffrion [14].

2. General duality.

No matter what procedure is used to maximize g_0, there are certain inherent limitations in any dual method of solution of (P) based on L_0 and (D_0). First of all, there are the difficulties of determining a representation of the implicit feasible set (1.2) in (D_0). Specifically, it is often hard to find a criterion for whether the infimum of $L_0(\cdot, y)$ over X is finite, and whether it is attained. A second difficulty is that, even if $\bar{x}$ minimizes $L_0(\cdot, \bar{y})$, where $\bar{y}$ is a solution to (D_0), it is not sure that $\bar{x}$ is a solution to (P), unless there is only one solution (see Theorem 2). Finally, this form of duality is fully meaningful only in the convex case, since it is only there that one can establish the relation $\min(P) = \sup(D_0)$ with any generality.

To see that there is no real hope of having $\min(P) = \sup(D_0)$ in the nonconvex case, except as a rare « accident », it is helpful to examine the function

$$\varphi_0(u) = \min_{x \in X} F_0(x, u)\,, \qquad u = (u_1, \ldots, u_m) \in R^m\,, \tag{2.1}$$

where

$$F_0(x, u) = \begin{cases} f_0(x) & \text{if } x \in X \text{ and } f_i(x) \leqslant u_i\,, \qquad i = 1, \ldots, m\,, \\ +\infty & \text{otherwise}\,. \end{cases} \tag{2.2}$$

The minimum in (2.1), instead of infimum, is appropriate because of our fundamental compactness assumption. (We use the convention that $+\infty = \min \emptyset$). The following result is known.

THEOREM 3: *The function $\varphi_0 \colon R^m \to R \cup \{+\infty\}$ is lower-semicontinuous, and*

$$\min(P) = \varphi_0(0) \qquad \text{(finite)}\,. \tag{2.3}$$

Moreover, in order that the relation min (P) = max (D_0) *hold, it is necessary and sufficient that there exist a vector* $\bar{y}$ *with*

$$(2.4)\qquad \varphi_0(u) \geqslant \varphi_0(0) - \bar{y}\cdot u \qquad \textit{for all } u \in R^m ,$$

such vectors $\bar{y}$ *being then precisely the solutions of* (D_0).

PROOF: The first statement is immediate from the definitions and our compactness assumption. As for (2.4), this is equivalent to

$$(2.5)\qquad \begin{aligned} \min (P) &= \inf_{u\in R^m} \{\varphi_0(u) + \bar{y}\cdot u\} \\ &= \inf_{u\in R^m} \inf_{x\in X} \{F_0(x, u) + \bar{y}\cdot u\} \\ &= \inf_{x\in R^m} L_0(x, \bar{y}) = g_0(\bar{y}) . \end{aligned}$$

Since in general

$$(2.6)\qquad \min (\mathrm{P}) \geqslant \sup (\mathrm{D}_0) \geqslant g_0(\bar{y}) ,$$

equation (2.5) is equivalent to $\bar{y}$ being a solution to (D_0) with min (P) = = max (D_0).

Note that (2.4) necessitates $\mathrm{co}\,\varphi_0(0) = \varphi_0(0)$, where $\mathrm{co}\,\varphi_0$ is the convex hull of φ_0 (the greatest convex function majorized by φ_0). The special significance of the convex case of (P) is found in the fact that one then has $\mathrm{co}\,\varphi_0 = \varphi_0$, or in other words, φ_0 is convex. Aside from one or two very special examples, no other practical criterion is known which implies $\mathrm{co}\,\varphi_0(0) = \varphi_0(0)$.

However, all these limitations concern only « ordinary duality ». If L_0 and (D_0) are replaced by a different Lagrangian function L and dual problem (D), still associated with (P), it may be possible to have min (P) = max (D) is important nonconvex cases. Even for convex problems, one may achieve properties more advantageous for dual methods of solution.

A general scheme for generating duality in (P) by means of the theory of conjugate functions, as we have expounded in [39] and [46], may be based on specifying any function

$$(2.7)\qquad F: X\times R^M \to R \cup \{+\infty\}$$

such that $F(x, u)$ is lower-semicontinuous in (x, u), convex in u, and

$$(2.8)\qquad F(x, 0) \equiv f(x) .$$

(Here f is the essential objective function (0.1) in (P)). The vectors $u \in R^M$ correspond to perturbations of the original problem (for $u \neq 0$, we have the perturbed essential objective function $F(\cdot, u)$ on X). They need not have anything to do with the vectors u above, and in particular it is not necessary that $M = m$.

Associated with F is the Lagrangian function

$$L\colon X \times R^M \to R \cup \{-\infty\} \tag{2.9}$$

defined by

$$L(x, y) = \inf_{u \in R^M} \{F(x, u) + y \cdot u\} . \tag{2.10}$$

This is always upper-semicontinuous and convex in y, and it is also convex in x in the case where X is convex and F is jointly convex in (x, u).

Moreover, one has

$$F(x, u) = \sup_{y \in R^M} \{L(x, y) - y \cdot u\} , \tag{2.11}$$

and hence in particular

$$\sup_{y \in R^M} L(x, y) = f(x) \quad \text{for all } x \in X . \tag{2.12}$$

Thus, parallel to the situation with ordinary duality, we have

$$\min (P) = \min_{x \in X} \sup_{y \in R^M} L(x, y) .$$

The problem

(D) maximize g over R^M, where $g(y) = \inf_{x \in X} L(x, y)$,

is therefore taken as the dual of (P) corresponding to F and L. Note that (D) consist of maximizing an upper semicontinuous concave function, and one always has

$$\min (\mathrm{P}) \geqslant \sup (\mathrm{D}) .$$

Most of the results stated for ordinary duality carry over to this general duality. Thus:

THEOREM 1′: *In the convex case where X is finite-dimensional and $F(x, u)$ is jointly convex in (x, u), one has* $\min (\mathrm{P}) = \sup (\mathrm{D})$.

PROOF: See [46, Theorems 17′ and 18′ (d)].

THEOREM 2′: *Suppose that* $\min(\mathrm{P}) = \sup(\mathrm{D})$. *Then* $(\bar{x}, \bar{y})$ *is a saddle point of* L *on* $X \times R^M$ *if and only if* $\bar{x}$ *solves* (P) *and* $\bar{y}$ *solves* (D). *Indeed, if* $\bar{y}$ *solves* (D), *then for* $\bar{x}$ *to solve* (P) *it is necessary and sufficient that:*

(a) *the minimum of* $L(x, \bar{y})$ *over all* $x \in X$ *is achieved at* $\bar{x}$, *and*

(b) $f_i(\bar{x}) \leqslant 0$ *for* $i = 1, \ldots, m$, $f(\bar{x}_0) \leqslant L(\bar{x}, \bar{y})$.

PROOF: The argument is the same as for Theorem 2.

A dual method of solution of (P) can be developed from Theorem 2′ just as from Theorem 2. The hope is that, by the right construction of L and (D), condition (b) will become superfluous, and the minimization of $L(\cdot, y)$ and maximization of g will be easier to execute.

We note that gradients, subgradients and ε-subgradients of g can be calculated in principle much as before:

THEOREM 4: *Let* x *minimize* $L(\cdot, y)$ *over* X *to within* ε, *and let* $u \in R^M$ *be a subgradient of* $L(x, \cdot)$ *at* y. *Then* $u \in \partial_\varepsilon g(y)$. *(The subgradients are taken in the sense appropriate for concave functions.)*

PROOF: By definition, we have for all $y' \in R^M$

$$g(y') \leqslant L(x, y') \leqslant L(x, y) + (y' - y) \cdot u \leqslant (g(y) - \varepsilon) + (y' - y) \cdot u \, . \tag{2.14}$$

This says that $u \in \partial_\varepsilon g(y)$.

In the cases of importance, L is presumably a convenient function whose subgradients with respect to y are easily determined. For instance, if the gradient of L with respect to y exists, it is the unique subgradient. But of course, to implement the calculation of ε-subgradients in Theorem 4, it is also necessary to have a practical criterion for whether a given x does minimize $L(\cdot, y)$ over X to within ε. Typically, such a criterion is obtained by specifying for each $(x, y) \in X \times R^m$ with $L(x, y)$ finite a set $S(x, y)$ and a function $l(\cdot\,; x, y)$ on $S(x, y)$ such that

$$L(w, y) \leqslant L(x, y) \text{ implies } w \in S(x, y) \text{ and } l(w; x, y) \leqslant L(w, y) \, . \tag{2.15}$$

Then

$$\inf_{w \in S(x,y)} l(w; x, y) \leqslant \inf_X L(\cdot, y) \leqslant L(x, y) \, . \tag{2.16}$$

Suppose l and S are such that the value

$$k(x, y) = \inf \{ l(w; x, y) \mid w \in S(x, y) \} \tag{2.17}$$

is known directly, and the difference $L(x, y) - k(x, y)$ approaches 0 as x comes closer and closer minimizing $L(\cdot, y)$. We then have a practical criterion of the form:

$$(2.18) \qquad x \text{ minimizes } L(\cdot, y) \text{ to within } \varepsilon \text{ if } L(x, y) - k(x, y) < \varepsilon .$$

For example, if $L(x, y)$ is differentiable and convex in $x \in X \subset R^n$, we can take (generalizing an approach of Bertsekas [6]):

$$(2.19) \qquad S(x, y) = \{w \in R^n \,|\, |w - x| \leqslant \beta\} \qquad \text{where } 0 < \beta \leqslant +\infty ,$$

$$(2.20) \qquad l(w; x, y) = L(x, y) + (w - y)\cdot \nabla_x L(x, y) + \alpha |w - y|^2 , \qquad \text{where } 0 \leqslant \alpha < +\infty .$$

Certainly (2.16) is valid for $\alpha = 0$, but there are also cases of strong convexity where a positive value of α can be used. As for β, it is enough that $\beta \geqslant \operatorname{diam} X$.

We then have

$$(2.21) \qquad L(x, y) - k(x, y) = \begin{cases} |\nabla_x L(x, y)|^2/4\alpha & \text{if } |\nabla_x L(x, y)| < 2\alpha\beta, \\ \beta |\nabla_x L(x, y)| - \alpha\beta^2 & \text{if } |\nabla_x L(x, y)| \geqslant 2\alpha\beta . \end{cases}$$

If either $\beta < +\infty$ or $\alpha > 0$, this furnishes a simple implementation of (2.18) in terms of the available magnitude $|\nabla_x L(x, y)|$. Note that the boundedness of X, making possible $\beta < +\infty$, is hardly any restriction in practical terms, since X can always be replaced by its intersection with some large ball or cube.

Other results relevant to general dual methods of solution may be derived in terms of the function

$$(2.22) \qquad \varphi(u) = \inf_{x \in X} F(x, u) ,$$

using the fact that

$$(2.23) \qquad \begin{aligned} g(y) &= \inf_{x \in X} \inf_{u \in R^M} \{F(x, u) + y \cdot u\} \\ &= \inf_{u \in R^M} \{\varphi(u) + y \cdot u\} . \end{aligned}$$

THEOREM 5: *The level set* $\{y \in R^M | g(y) \geqslant \beta\}$ *is compact for every* $\beta \in R$ *if*

$$(2.24) \qquad 0 \in \operatorname{int} \operatorname{co} \{u \in R^M | \varphi(u) < +\infty\} .$$

(*In this case, therefore* max (D) *can be written in place of* sup (D).)

PROOF: From (2.22) we have $-g(y)=\varphi^*(-y)$, where φ^* is the cojugate of φ (in the convex sense). On other hand, (2.24) implies $0 \in \operatorname{int} \operatorname{dom} \operatorname{co} \varphi$ and hence $0 \in \operatorname{in} \operatorname{dom} \varphi^{**}$, where φ^{**} is the biconjugate of φ and « dom » denotes the effective domain. Since φ^* and φ^{**} are convex functions conjugate to each other, the condition $0 \in \operatorname{int} \operatorname{dom} \varphi^{**}$ implies that the level sets $\{y|\varphi^*(y) \leqslant \alpha\}$ are compact (see [46, Theorem 10]), and this is the desired property.

The condition Theorem 5 can be viewed as a generalization of strict feasibility.

THEOREM 6: *The function* φ *satisfies*

$$\min(\mathrm{P}) = \varphi(0) \qquad (\text{finite}), \tag{2.25}$$

$$\sup(\mathrm{D}) = \varphi^{**}(0). \tag{2.26}$$

If $F(x, u)$ *is convex in* (x, u), *then* φ *is convex. In this convex case with* X *finite-dimensional, or in the case of* X *compact, it is also true that* φ *is lower semicontinuous and nowhere* $-\infty$.

In general, in order that the relation $\min(\mathrm{P}) = \max(\mathrm{D})$ *hold, it is necessary and sufficient that there exist a vector* $\bar{y}$ *with*

$$\varphi(u) \geqslant \varphi(0) - \bar{y}\cdot u \qquad \textit{for all } u \in R^M, \tag{2.27}$$

such vectors $\bar{y}$ *then being precisely the solutions of* (D) .

PROOF: These facts are stated, for example, in the general theory of [46], except for the lower semicontinuity assertions. For X compact, the lower-semicontinuity is obvious from the lower-semicontinuity of F. When F is convex and X is finite-dimensional, the equality in Theorem 1′ gives us $\varphi^{**}(0)=\varphi(0)$. But all the hypotheses remain satisfied if F is translated in the u argument, which corresponds to translating φ so that a different point becomes the origin. Therefore $\varphi^{**}(u)=\varphi(u)$ for every $u \in R^M$, and in particular φ is lower-semicontinuous.

In a moment, a case will be described where (2.27) can generally be satisfied even without convexity, although this is not true with φ_0 in place of φ. There may also be other cases, not yet discovered, and this is a tantalizing prospect for computation applications of duality in nonconvex programming.

General duality can also be used in many instances to construct dual problems explicit enough to be tackled by direct methods of solution. For example, it was shown by Duffin and Peterson [9] that for certain convex problems in exponential programming (the original « geometric programming ») one can substitute a dual problem different from (D_0) which «almost» amounts to maximizing an explicit, differentiable concave function subject to only *linear* constraints. This was extended by Peterson and Ecker [32] to some other special problems. A broad theory of such cases of linearly constrained duals (for nonlinearly constrained primals) has been furnished by Rockafellar [41]. Another versatile scheme, where the primal and dual are linearly constrained but the passage between the two is almost as easy as in linear programming has been presented in [38].

So far, there has been relatively little attempt to take advantage of such explicit dual problems computationally. Besides the straightforward applications, there may be possibilities of dualizing direction-finding schemes in various algorithms, and so forth.

3. Augmented Lagrangians.

We turn now to a particular choice of the Lagrangian function and dual problem for (P) which illustrates rather dramatically what changes can be achieved in the properties of the dual method of solution. This Lagrangian, originally proposed by Rockafellar [40] in 1970, was first studied in detail by Buys in his thesis [8], although an important saddle-point property was derived earlier by Arrow, Gould and Howe [1]. (A related function was arrived at independently by Wierzbicki [54]). Many recent papers have dealt with it as indicated below, especially in connection with the generalization to inequality-constrained problems of the method of multipliers (see the next section). Of course, although we have not mentioned it earlier, equality constraints are also covered in the obvious way by the theory described here.

Letting r denote a positive parameter, we set

$$F_r(x, u) = F_0(x, u) + r|u|^2, \tag{3.1}$$

where F_0 is still the function (2.2) and $|u|$ is the Euclidean norm of $u \in R^m$ $(= R^M$ here). Trivially, F_r is lower-semicontinuous in (x, u), convex in u, and in the convex case of (P) it is convex in (x, u).

The corresponding Lagrangian L_r, obtained from F_r by formula (2.10), is easily calculated to be

$$L_r(x, y) = f_0(x) + \sum_{i=1}^{m} \theta_r(y_i, f_i(x)), \tag{3.2}$$

where

$$\theta_r(y_i, f_i(x)) = \begin{cases} y_i f_i(x) + r f_i(x)^2 & \text{if } f_i(x) \geqslant -y_i/2r, \\ -y_i^2/4r & \text{if } f_i(x) \leqslant -y_i/2r. \end{cases} \tag{3.3}$$

Thus L_r is a *finite* function on $X \times R^m$. From the general theory, we know that $L_r(x, y)$ is concave in $y \in R^m$ and in the convex case of (P) also convex in $x \in X$ (since in this case $F(x, u)$ is convex in (x, u)).

Note that $L_r(x, y)$ is also differentiable everywhere with respect to y, and

$$\frac{\partial L_r}{\partial y_i}(x, y) = \max\{f_i(x), -y_i/2r\}. \tag{3.4}$$

This is to be contrasted with the situation for $L_0(x, y)$, which is not even finite everywhere on $X \times R^m$, much less differentiable. If the functions f_i are all differentiable on X (where $X \subset R^n$), then $L_r(x, y)$ is also differentiable with respect to x on X, and

$$\nabla_x L_r(x, y) = \nabla f_r(x) + \sum_{i=1}^{m} \sigma_r(y_i, f_i(x)) \nabla f_i(x), \tag{3.5}$$

where

$$\sigma_r(y_i, f_i(x)) = \max\{y_i + 2rf_i(x), 0\} = y_i + 2r \frac{\partial L_r}{\partial y_i}(x, y). \tag{3.6}$$

Higher-order differentiability of L_r is also inherited from the f_i's, except along the hypersurfaces given by the equations $y_i + 2rf_i(x) = 0$.

The dual problem corresponding to F_r and L_r is by definition

$$\text{maximize } g_r \text{ over } R^m, \text{ where } g_r(y) = \inf_{x \in X} L_r(x, y). \tag{D$_r$}$$

The special properties of this dual follow mainly from (2.23), which here takes the form

$$g_r(y) = \inf_{u \in R^m} \{\varphi_r(u) + y \cdot u\} = -\varphi_r^*(-y), \tag{3.7}$$

where

$$\varphi_r(u) = \inf_{x \in X} F_r(x, u) = \varphi_0(u) + r|u|^2. \tag{3.8}$$

We observe first an immediate consequence of Theorems 3 and 6, first stated in [42].

THEOREM 7: *In order that the relation* $\min(\mathrm{P}) = \max(\mathrm{D}_r)$ *hold, it is necessary and sufficient that there exist a vector* $\bar{y}$ *with*

$$\varphi_0(u) \geqslant \varphi_0(0) - \bar{y} \cdot u - r|u|^2 \qquad \textit{for all } u \in R^m, \tag{3.9}$$

such vectors $\bar{y}$ *being then precisely the solutions of* (D_r).

COROLLARY: *In the convex case, $\bar{y}$ solves (D_r) if and only if $\bar{y}$ solves (D_0). Thus $(\bar{x}, \bar{y})$ is a saddle-point of L_r if and only if $(\bar{x}, \bar{y})$ is a saddle-point of L_0.*

PROOF: In this case φ_0 is convex, and hence (3.9) is equivalent to (2.4).

In the nonconvex case, there is considerable hope that (3.9) can be satisfied by some $\bar{y}$ for r sufficiently large. Suppose, for example, that φ_0 happens to be continuously twice differentiable in a neighborhood of $u = 0$; a situation where this is true will be described in a moment. The Hessian matrices of φ_r and φ_0 are related by

$$\nabla^2 \varphi_r(u) = \nabla^2 \varphi_0(u) + rI \,, \tag{3.10}$$

and hence $\nabla^2 \varphi_r(u)$ is positive-definite in some convex neighborhood N of $u = 0$ for r sufficiently large. But then φ_r is convex on N and hence satisfies

$$\begin{aligned} \varphi_r(u) &\geqslant \varphi_r(0) + \overline{u} \cdot \nabla \varphi_r(0) \\ &= \varphi_0(0) + \overline{u} \cdot \nabla \varphi_0(0) \qquad \text{for all } u \in N \,. \end{aligned} \tag{3.11}$$

In other words,

$$\varphi_0(u) \geqslant \varphi_0(0) - \bar{y} \cdot u - r|u|^2 \text{ for all } \overline{u} \in N \text{ where } \bar{y} = -\nabla \varphi_0(0) \,. \tag{3.12}$$

If φ_0 is bounded below globally by some quadratic function (as is certainly true for instance if f_0 is bounded below on X, since then φ_0 is bounded below by a constant function), it is easy to see that (3.12) can be transformed into (3.9) by taking r still larger, if necessary. In this event, therefore, we have $\min(\mathrm{P}) = \max(\mathrm{D}_r)$.

For φ_0 to be twice differentiable near 0, it is enough in the finite-dimensional case that (P) have a unique solution $\bar{x}$ at which the *strong form of the second-order sufficient conditions* for a local constrained minimum are satisfied, and that X be a sufficiently small neighborhood of $\bar{x}$. Suppose the functions f_i are thrice differentiable near $\bar{x}$. The conditions in question assert the existence of a vector $\bar{y} \in R^m$ satisfying the usual first-order Kuhn-Tucker conditions and also the following, where $I \subset \{1, \ldots, m\}$ consists of the indices such that $f_i(\bar{x}) = 0$:

(*a*) $\bar{y}_i > 0$ for all $i \in I$, and the vectors $\nabla f_i(\bar{x})$ for $i \in I$ are linearly independent;

(*b*) the Hessian $H = \nabla^2 f_0(\bar{x}) + \sum_{i \in I} \bar{y}_i \nabla^2 f_i(\bar{x})$ satisfies $z \cdot Hz > 0$ for every nonzero vector z with $z \cdot \nabla f_i(\bar{x}) = 0$ for all $i \in I$.

Under these conditions, it is possible using the implicit function theorem to deduce the existence of differentiable functions $x(u)$ and $y(u)$ of u in a neighborhood of 0 in R^m, with $x(0) = \bar{x}$ and $y(0) = \bar{y}$, such that $x(u)$ and $y(u)$ satisfy the corresponding conditions for $f_i - u_i$, $i = 1, \ldots, m$. It follows then that $\varphi_0(u) = f_0(x(u))$ if u is sufficiently near to 0, and if X is a neighborhood of $\bar{x}$ sufficiently small that the local solution $x(u)$ to the perturbed problem (with f_i replaced by $f_i - u_i$) is actually the global solution. The thrice-differentiability of the functions f_i implies that $x(u)$ is twice differentiable, and hence that $\varphi_0(u)$ is twice differentiable in u as desired.

This idea was exploited by Buys [8], although more in a context of « local » duality, like the earlier result of Arrow, Gould and Howe [1] (which did not actually require that $\bar{y}_i > 0$ for all $i \in I$). These authors imposed the restriction that $y \geqslant 0$. More recently, Bertsekas [5] used the strong form of the sufficient conditions in obtaining the duality in a global form highly suited to the analysis of convergence properties of algorithms.

Rockafellar [44] has established a sharper criterion of the above sort for the existence of a $\bar{y}$ satisfying (3.9), not entailing the twice-differentiability of φ_0 near $u = 0$. Recall the *weak form of the second-order sufficient conditions for a local constrained minimum in* (P) *at* $\bar{x}$ [5, p. 30]: there exists $\bar{y}$ satisfying the first-order Kuhn-Tucker conditions such that

$$z \cdot Hz > 0 \qquad \text{for every nonzero } z \in Z\,, \tag{3.13}$$

where H is the Hessian above, and Z is the set of all $z \in R^n$ such that

$$\begin{aligned} z \cdot \nabla f_i(\bar{x}) &= 0 \qquad \text{for all } i \in I \text{ with } y_i > 0\,, \\ z_i \cdot \nabla f_i(\bar{x}) &\leqslant 0 \qquad \text{for all } i \in I \text{ with } y_i = 0\,. \end{aligned} \tag{3.14}$$

Only continuous twice-differentiability of the function f_i is assumed.

Let us say that (P) satisfies the *quadratic growth condition* if for some $r \geqslant 0$ the function

$$L_r(x, 0) = f_0(x) + r \sum_{i=1}^{m} \max{}^2 \{0, f_i(x)\} \tag{3.15}$$

is bounded below on X. (This very mild condition will be illuminated in Theorem 10).

THEOREM 8: *Suppose in the finite-dimensional case that* (P) *has a unique solution* $\bar{x}$ *satisfying the weak form of the second-order sufficient conditions, and that the quadratic growth condition holds.*

Then $\min(\mathrm{P}) = \max(\mathrm{D}_r)$ *for all* r *sufficiently large.*

PROOF: This differs from [44, Theorem 6] only in that there is no mention of $\bar{x}$ being «globally unique in the strong sense». The latter assumption is made superfluous by the blanket compactness assumption in this paper.

These results show the applicability of dual methods of solution of (P) based on (D_r), even in the nonconvex case. In fact, the problems (D_r) have important special characteristics that can be used to advantage in algorithms. The central properties are the following.

THEOREM 9: *For any $\bar{r} \geq 0$ and $r > \bar{r}$, one has*

$$g_r(y) \geq \max_{z \in R^m} \{g_{\bar{r}}(z) - |z-y|^2/4(r-\bar{r})\}, \tag{3.16}$$

and in the convex case this holds as an equation.

PROOF: See [42], [44].

THEOREM 10: *The expression $g_r(y)$ is concave and upper semicontinuous as a function of (y, r), and it is nondecreasing as a function of r. There exists a number ϱ (in general $0 \leq \varrho \leq +\infty$, but $\varrho = 0$ in the convex case or if X is compact), such that g_r is finite on all of R^m if $r \in (\varrho, +\infty)$, while g_r is identically $-\infty$ if $r \in (0, \varrho)$. The quadratic growth condition is satisfied if and only if $\varrho < +\infty$. In the latter case, the functions g_r converge uniformly on all bounded sets to the constant function $g_\infty(y) \equiv \equiv \min(P)$ as $r \to \infty$, and in particular*

$$\sup(D_0) \leq \sup(D_r) \uparrow \min(P) \qquad as\ r \to \infty . \tag{3.17}$$

PROOF: See [44]. Our compactness assumption guarantees that the asymptotic optimal value in (P), which is the limit of $\sup(D_r)$ according to the cited theorem, is the same as min (P). Pointwise convergence of the functions g_1 implies uniform convergence on bounded sets, since these functions are finite and concave for $r > \varrho$ (see [39, §10]).

COROLLARY: *If $\bar{y}$ solves $(D_{\bar{r}})$ and $\max(D_{\bar{r}}) = \min(P)$, then $\bar{y}$ also solves (D_r) for every $r > \bar{r}$.*

The next result should be compared with Theorems 2 and 2′.

THEOREM 2″: *Suppose $\bar{r} \geq 0$ is such that $\min(P) = \sup(D_{\bar{r}})$. Then $(\bar{x}, \bar{y})$ is a saddle-point of $L_{\bar{r}}$ on $X \times R^m$ if and only if $\bar{x}$ solves (P) and $\bar{y}$ solves $(D_{\bar{r}})$. Indeed, if $\bar{y}$ solves $(D_{\bar{r}})$ and $r > \bar{r}$, then for $\bar{x}$ to solve (P) it is necessary and sufficient that $\bar{x}$ minimize $L_r(\cdot, \bar{y})$ over X.*

The first part of this theorem is a special case of Theorem 2′. The second part can be stated much more broadly in terms of a general dual method of solving (P):

THEOREM 11: *Let* $(r_k)_{k=1}^{\infty}$ *be a nondecreasing sequence in* R_+. *Let* $(y_k)_{k=1}^{\infty}$ *be a bounded sequence in* R^m *such that* y_k *maximizes* g_{r_k} *to within* λ_k, *and let* x_k *minimize* $L_{r_k}(\cdot, y_k)$ *on* X *to within* ε_k, *where* $\lim \varepsilon_k = 0$. *Suppose either that* $\lim r_k = +\infty$, *or that* $\lim \lambda_k = 0$ *and* $\lim r_k > \bar{r}$, *where* $\bar{r}$ *is such that* $\min(\mathrm{P}) = \sup(\mathrm{D}_{\bar{r}})$. *Then the sequence* $(x_k)_{k=1}^{\infty}$ *is relatively compact, and all of its cluster points are solutions to* (P).

PROOF: Under either set of assumptions we have

$$\lim g_{r_k}(y_k) = \min(\mathrm{P}) \tag{3.18}$$

by virtue of Theorem 11. Then [44, Theorem 3] is applicable and asserts that the sequence $(x_k)_{k=1}^{\infty}$ is asymptotically minimizing for (P). This implies by our compactness assumption that the sequence is relatively compact, and its cluster points are actually solutions to (P).

The standard quadratic exterior penalty method [13] for solving (P) corresponds to the case of Theorem 11 where $y_k \equiv 0$. Thus the theorem generalizes this method in allowing $(y_k)_{k=1}^{\infty}$ to be any bounded sequence. This extra flexibility makes possible a great improvement in convergence as has been proved in particular by Bertsekas [3], [4] in connection with the « multiplier method » described in the next section.

On the other hand, Theorem 11 also describes cases where r_k remains bounded and the success of the method is achieved entirely through the choice of the multiplier vectors y_k. This is interesting, because the well known numerical difficulties associated with having $r_k \to \infty$ are thereby avoided.

The next results have a bearing on how one might execute the maximization of g_r or minimization of $L_r(\cdot, y)$.

THEOREM 12: *Let* $r > \varrho$, *where* ϱ *is the number described in Theorem* 10. *Then for every compact set* $Y \subset R^m$ *and every* $\alpha \in R$, *the set*

$$\{x \in X \mid \exists y \in Y \text{ with } L_r(x, y) \leqslant \alpha\} \tag{3.19}$$

is compact. Hence in particular, for every $y \in R^m$ *the infimum of* $L_r(\cdot, y)$ *over* X *is attained.*

If in addition (P) *is strictly feasible, then for every* $\beta \in R$ *the level set*

$$\{y \in R^m \mid g_r(y) \geqslant \beta\} \tag{3.20}$$

is compact. Hence in particular. the supremum of g_r *over* R^m *is attained.*

PROOF: Let $\bar{r}$ lie between r and ϱ, and let $\bar{y} \in R^m$ be arbitrary. Then

$$-\infty < g_{\bar{r}}(\bar{y}) = \inf_{u \in R^m} \{\varphi_{\bar{r}}(u) + \bar{y} \cdot u\}, \tag{3.21}$$

so that

$$g_{\bar{r}}(\bar{y}) - \bar{y} \cdot u + (r - \bar{r})|u|^2 \leqslant \varphi_{\bar{r}}(u) + (r - \bar{r})|u|^2 = \varphi_r(u) \tag{3.22}$$

for all $u \in R^m$. It follows that the set

$$T = \{(u, y) \in R^m \times Y | \varphi_r(u) + y \cdot u \leqslant \alpha\} \tag{3.23}$$

is contained in the bounded set

$$\{(u, y) \in R^m \times Y | g_{\bar{r}}(\bar{y}) + (y - \bar{y}) \cdot u + (r - \bar{r})|u|^2 \leqslant \alpha\}. \tag{3.24}$$

But T is also closed, because φ_r is lower-semicontinuous by (3.3) and Theorem 3. Therefore T is compact. Now consider the set

$$S = \{(x, u, y) \in X \times R^m \times Y | F_0(x, u) + r|u|^2 + u \leqslant \alpha\}, \tag{3.25}$$

whose image under the projection $(x, u, y) \to (u, y)$ is contained in T. Our basic compactness assumption asserts that for each $u \in R^m$ and $\alpha_0 \in R$ the set

$$\{x \in X | F_0(x, u) \leqslant \alpha_0\}$$

is compact, and it implies further that F_0 is lower-semicontinuous. Therefore S is closed, and all of its sections

$$S_{(u,y)} = \{x | (x, u, y) \in S\}$$

are compact. This, combined with the fact that

$$((u, y) | S_{(u,y)} \neq \emptyset\} \subset T \text{ (compact)},$$

allows us to conclude that S is compact. The image of S under $(x, u, y) \to x$ is then likewise compact. But this image is the set (3.19) in view of formula (2.10) for L_r and F_r.

The second assertion is obtained from Theorem 5 and the relation

$$\{u \in R^m | \varphi_r(u) < +\infty\} = \{u \in R^m | \exists x \in X \text{ with } f_i(x) \leqslant u_i,\ i = 1, \ldots, m\}. \tag{3.26}$$

This complete the proof of Theorem 12.

The compactness of the level sets of $L_r(\cdot, y)$ in X has previously been observed by Bertsekas [6] in the finite-dimensional, convex case. This property is valuable, of course, in computing ε-subgradients of g_r in the pattern of Theorem 4. If the minimum of $L_r(\cdot, y)$ over X is attained at x, then the vector $\nabla_y L_r(x, y)$ given by (3.4) belongs to $\partial g_r(y)$. If g_r is actually differentiable at y, as is true in the convex according to Theorem 14 below, then $\nabla_y L_r(x, y) = \nabla g_r(y)$.

The arguments proving the compactness in Theorem 12 can be turned into useful estimates, which we now state.

THEOREM 13. *If $r > \bar{r} \geq 0$, then for all $x \in X$, $y \in R^m$ and $\bar{y} \in R^m$, one has*

$$(3.27) \qquad |y - \bar{y} + 2(r - \bar{r}) \nabla_y L(x, y)|^2 \leq \leq |y - \bar{y}|^2 + 4(r - \bar{r})[L_r(x, y) - g_{\bar{r}}(\bar{y})] .$$

On the other hand, if $\tilde{x}$ satisfies the strict feasibility condition (1.3), *then for every $y \in R^m$ and $r \geq 0$ one has*

$$(3.28) \qquad g_r(y) \leq L_r(\tilde{x}, y) \leq f_0(\tilde{x}) + r \sum_{i=1}^{m} f_i(\tilde{x})^2 - \sum_{i=1}^{m} |y_i| \cdot |f_i(\tilde{x})| .$$

PROOF: It is quickly established that the minimum in the formula

$$(3.29) \qquad L_r(x, y) = \inf_{u \in R^m} \{F_0(x, u) + r|u|^2 + y \cdot u\}$$

is attained at $\bar{u} = \nabla_y L_r(x, y)$ (see (3.4)). If α is such that $L_r(x, y) \leq \alpha$, then by the argument of Theorem 12 we have

$$(3.30) \qquad g_{\bar{r}}(\bar{y}) + (y - \bar{y}) \cdot \bar{u} + (r - \bar{r}) |\bar{u}|^2 \leq \alpha ,$$

at least if $\bar{r} > \varrho$. Taking the limit as $\bar{r} \to \varrho$ we see that (3.30) holds also for $\bar{r} = \varrho$. It holds trivially for $\bar{r} < \varrho$, since then $g_{\bar{r}}(\bar{y}) = -\infty$. Hence (3.30) is valid for all $\bar{r}$, $0 \leq \bar{r} < r$. Rewriting this inequality, we obtain

$$(3.31) \qquad |y - \bar{y}|^2 + 4(r - \bar{r})(y - \bar{y}) \cdot \bar{u} + 4(r - \bar{r})^2 |\bar{u}|^2 \leq |y - \bar{y}|^2 + (r - \bar{r})(\alpha - g_{\bar{r}}(\bar{y})) ,$$

from which (3.27) follows by taking $\tilde{x} = L_r(x, y)$. The first inequality in (3.28) holds by the definition of g_r, while the second follows from the fact that the expression θ_r is (3.2) and (3.3) satisfies

$$(3.32)\qquad \begin{aligned}\theta_r(y_i, f_i(\tilde{x})) &= \min\{y_i u_i + r|u_i|^2 \,|\, u_i \geq f_i(\tilde{x})\}\\ &\leq \min\{y_i u_i + r|u_i|^2 \,|\, -f_i(\tilde{x}) \geq u_i \geq f_i(\tilde{x})\}\\ &\leq r|f_i(\tilde{x})|^2 + \min\{y_i u_i \,|\, -f_i(\tilde{x}) \geq u_i \geq f_i(\tilde{x})\}\\ &= r|f_i(\tilde{x})|^2 - |y_i|\,|f_i(\tilde{x})|\,.\end{aligned}$$

COROLLARY: *If $r > s \geq 0$ and x minimizes $L_r(\cdot, y)$ over X to within ε, then*

$$(3.33)\qquad (r-s)|\nabla_y L_r(x, y)|^2 \leq g_r(y) - g_s(y) + \varepsilon \leq \min(\mathrm{P}) - gl(y) + \varepsilon\,.$$

PROOF: We have $L_r(x, y) \leq g_r(y) + \varepsilon$ by hypothesis. Take $\bar{y} = y$ and $\bar{r} = s$ in (3.27).

In the convex case, at least, tests for whether $L_{\bar{r}}(\cdot, y)$ minimizes $L_r(\cdot, y)$ over X to within ε can be set up as described following Theorem 4 (cf. also Bertsekas [6]). It turns out that in this way one can actually calculate the gradient of g_r to within any desired approximation.

THEOREM 14. *In the convex case with $r > 0$, g_r is continuously differentiable throughout R^m, and in fact the gradient mapping has the Lipschitz property*

$$(3.34)\qquad |\nabla g_r(y') - \nabla g_r(y)| \leq |y' - y|/2r\,.$$

Furthermore, one has for all y and $\bar{y}$

$$(3.35)\qquad g_r(y') \geq g_r(y) + (y' - y)\nabla g_r(y) - r|y' - y|^2\,.$$

If x minimizes $L_r(\cdot, y)$ over X to within ε, then

$$(3.36)\qquad |\nabla_y L_r(x, y) - \nabla g_r(y)|^2 \leq 4\varepsilon r\,.$$

PROOF: All these properties are consequences of Theorem 9; cf. [42].

The Lipschitz property in Theorem 14 was first noted explicitly by Martinet [29].

4. The method of multipliers.

The results of the preceding section are not limited to any one scheme for maximizing g_r in the dual problem, aside from the stipulation that it should be efficient with respect to the number of steps where g_r or one of its subgradients must be evaluated. However, an interesting method is suggested by certain of the estimates.

In the convex case, for instance, we can proceed as follows for any fixed $r > 0$: given the vector y_k, let y_{k+1} be chosen to maximize the expression

$$g_r(y_k) + (y' - y_k)\nabla g_r(y_k) - r|y' - y_k|^2$$

in y'; thus

$$y_{k+1} = y_k + 2r\nabla g_r(y_k) . \tag{4.1}$$

Then from (3.35) we have

$$\begin{aligned} g_r(y_{k+1}) &\geqslant g_1(y_k) + (y_{k+1} - y_k)\cdot\nabla g_r(y_k) - r|y_{k+1} - y_k|^2 \\ &= g_r(y_k) + r|\nabla g_r(y_k)|^2 . \end{aligned} \tag{4.2}$$

It has been demonstrated in [43] that the sequence $(y_k)_{k=1}^{\infty}$ converges to a solution of (D_r) (and hence of (D_0) by the corollary to Theorem 7), provided a solution exists. This remains true if the rule (4.1) is replaced by

$$y_{k+1} = y_k + 2r\nabla_y L_r(x_k, y_k) , \tag{4.3}$$

as suggested by (3.36), where x_k minimizes $L_r(\cdot, y_k)$ over X to within ε_k, provided that

$$\sum_{k=1}^{\infty}\sqrt{\varepsilon_k} < +\infty . \tag{4.4}$$

Moreover, Theorem 11 is then applicable, and hence cluster points of $(x_k)_{k=1}^{\infty}$ exist and are solutions to (P).

Martinet [29] has shown that (4.4) is superfluous if (P) is strictly feasible; then one only needs $\varepsilon_k \to 0$. Bertsekas [7] has pointed out that convergence is obtained in a finite number of steps if (P) is linear and $\varepsilon_k = 0$. Both of these authors have also considered replacing the step size $2r$ in (4.3) by a different value. In general, one can study the rule

$$y_{k+1} = y_k + 2s_k\nabla_y L_{r_k}(x_k, y_k) . \tag{4.5}$$

This approach to solving (P) is called the *multiplier method*. It was originally proposed for equality-constrained problems by Hestenes [19] and Powell [37], and somewhat later by Haarhoff and Buys [17], all independently.

There has been a great amount of research on this method recently in the nonconvex case, but we cannot go into the details here. The most notable results are due to Bertsekas, partly in collaboration with Kort [3], [4], [5], [6], [7], [23], [24], [25]. These results besides strengthening and extending the theoretical framework, include excellent theorems on convergence which show that versions of the multiplier method with r_k bounded or even $r_k \to \infty$ are much superior to the usual quadratic penalty method. Convergence properties in the nonlinear inequality-constrained case have also been studied recently by Rupp [51], [52]. See also Buys [8] and Polyak [35], [36].

More research would be valuable especially in connection with how to extend the multiplier method effectively to problems involving infinitely many constraints. Interesting efforts have been made in this direction by Janin [20], [21], [22], Martinet [29], Rupp [47], [48], [49], [50], and Wierzbicki and Hatko [55].

Intriguing questions are also raised by new results of Mangasarian [30]. These concern a Lagrangian which has many of the advantages of L_r and better differentiability properties, but does not quite fit into the duality framework presented here.

Finally, we would like to bring to attention an important respect in which L_r is *worse* than L_0, and that is when (P) is decomposable in the sense described in section 2. It is not true in that case that minimizing $L_r(x, y)$ in x is equivalent to N separate problems in the variables x_j. Work should be done on how to combine the idea behind L_r compatibly with those of decomposition, so as to forge a new tool for the solution of large-scale problems.

Note added in proof. In the two years since this paper was written (March, 1974), there have been a number of new developments, and several important earlier contributions in the Russian literature (by Polyak and Tretyakov) have come to my attention. For more on this, see R. T. Rockafellar, *Augmented Lagrangians and applications of the proximal point algorithm in convex programming*, Math. of Operations Research, **1** (1976).

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