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STOCHASTIC CONVEX PROGRAMMING: RELATIVELY COMPLETE RECOURSE AND INDUCED FEASIBILITY*

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Abstract. The basic dual problem and extended dual problem associated with a two-stage stochastic program are shown to be equivalent, if the program is strictly feasible and satisfies a condition generalizing, in a sense, the condition of relatively complete recourse in stochastic linear programming. Combined with earlier results, this yields the fact that, under the same assumptions, solutions to the program can be characterized in terms of saddle points of the basic Lagrangian. A couple of examples are used to illustrate the salient points of the theory. The last section contains a review of the principal implications of the results of this paper combined with those of three preceding papers also devoted to stochastic convex programs.

1. Introduction. This is the fourth in a series of papers [1], [2], [3] devoted to the following two-stage model in stochastic programming. Let C_1 and C_2 be nonempty, closed convex sets in R^{n_1} and R^{n_2}, respectively, and let (S, Σ, σ) be a probability space. Let f_{1i} be a finite convex function on R^{n_1} for $i = 0, 1, \cdots, m_1$, and let $f_{2i}(s, \cdot, \cdot)$ be a finite convex function on $R^{n_1} \times R^{n_2}$ for $i = 0, 1, \cdots, m_2$ and $s \in S$. The problem is to minimize

$$f_{10}(x_1) + \int_S f_{20}(s, x_1, x_2(s))\sigma(ds) \tag{1.1}$$

over all $x_1 \in R^{n_1}$ and $x_2 \in \mathscr{L}^\infty_{n2} = \mathscr{L}^\infty(S, \Sigma, \sigma; R^{n_2})$ (the Lebesgue space of equivalence classes) satisfying

$$x_1 \in C_1 \quad \text{and} \quad f_{1i}(x_1) \leqq 0 \quad \text{for } i = 1, \cdots, m_1, \tag{1.2}$$

and almost surely

$$x_2(s) \in C_2 \quad \text{and} \quad f_{2i}(s, x_1, x_2(s)) \leqq 0 \quad \text{for } i = 1, \cdots, m_2. \tag{1.3}$$

It is assumed that $f_{2i}(s, x_1, x_2)$ is measurable in s for each $(x_1, x_2) \in R^{n_1} \times R^{n_2}$, in fact summable if $i = 0$ and bounded if $i = 1, \cdots, m_2$. (From this it follows that for each $x_1 \in R^{n_1}$ and $x_2 \in \mathscr{L}^\infty_{n2}$, $f_{2i}(s, x_1, x_2(s))$ is measurable in s, summable if $i = 0$ and essentially bounded if $i = 1, \cdots, m_2$.)

The *basic Lagrangian* function introduced for this problem in [1] is defined on the product of the sets

$$X_0 = \{(x_1, x_2) \in R^{n_1} \times \mathscr{L}^\infty_{n2} | x_1 \in C_1 \text{ and almost surely } x_2(s) \in C_2\}, \tag{1.4}$$

$$Y_0 = \{(y_1, y_2) \in R^{m_1} \times \mathscr{L}^1_{m2} | y_1 \geqq 0 \text{ and almost surely } y_2(s) \geqq 0\}, \tag{1.5}$$

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by the formula

$$L(x_1, x_2, y_1, y_2) = f_{10}(x_1) + \sum_{i=1}^{m_1} y_{1i} f_{1i}(x_1) \tag{1.6}$$
$$+ \int_S \left[f_{20}(s, x_1, x_2(s)) + \sum_{i=1}^{m_2} y_{2i}(s) f_{2i}(s, x_1, x_2(s))\right]\sigma(ds).$$

The given problem can be identified with

P minimize $f(x_1, x_2)$ over all $(x_1, x_2) \in X_0$, where

$$f(x_1, x_2) = \sup_{(y_1, y_2) \in Y_0} L(x_1, x_2, y_1, y_2).$$

The *basic dual problem* is

D maximize $g(y_1, y_2)$ over all $(y_1, y_2) \in Y_0$, where

$$g(y_1, y_2) = \inf_{(x_1, x_2) \in X_0} L(x_1, x_2, y_1, y_2).$$

The relationship between P and D was studied in [1], and it was shown in particular that

$$\min \mathrm{P} = \sup \mathrm{D} \quad \text{if } C_1 \text{ and } C_2 \text{ are bounded.} \tag{1.7}$$

In cases where actually $\min \mathrm{P} = \max \mathrm{D}$, a pair $(\bar{x}_1, \bar{x}_2)$ solves P if and only if there exists $(\bar{y}_1, \bar{y}_2) \in Y_0$ such that $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2)$ is a saddle point of the Lagrangian. This saddle point property was reduced in [2] to a certain set of Kuhn–Tucker conditions involving a function $p \in \mathscr{L}^1_{n_1}$ which essentially associates prices with the constraint that x_1 must be chosen before the observation of s. The pairs $(\bar{y}_1, \bar{y}_2)$ are, of course, solutions to D.

To apply this basic duality theory at its fullest, one needs a simple criterion for the relation $\inf \mathrm{P} = \max \mathrm{D}$. But the latter does not hold in general, even if P is *strictly feasible* in the sense that for some $\varepsilon > 0$ the constraints (1.2) and (almost surely) (1.3) can be satisfied with $-\varepsilon$ in place of 0.

The goal of this paper is to obtain such a criterion in supplementing strict feasibility by a condition on the availability of second-stage recourse. The technique is to analyze the so-called induced constraints in the first stage in terms of the "extended duality" developed in [3]. The extended duality adjoins to the Lagrangian additional terms involving "singular" linear functionals on $\mathscr{L}^\infty_1$. It is interesting that, despite reliance on such esoteric objects in the proof, our main result on basic duality makes no mention of them in its statement.

Let K_1 be the set of all $x_1 \in R^{n_1}$ satisfying the first-stage constraints (1.2) and let K_2 be the set of all $x_1 \in R^{n_1}$ such that there exists an $x_2 \in \mathscr{L}^\infty_{n_2}$ satisfying the second-stage constraints (1.3) almost surely. It is evident that K_2 is convex. According to [1, proof of Thm. 1], we have $x_1 \in K_2$ if for the set

$$\Gamma(s, x_1) = \{x_2 \in C_2 \mid f_{2i}(s, x_1, x_2) \leqq 0,\ i = 1, \cdots, m_2\}, \tag{1.8}$$

there is a bounded region B with $\Gamma(s, x_1) \cap B \neq \emptyset$ almost surely.

We shall call K_2 the *induced feasible set* for the first stage of P, as opposed to the *explicit constraint set* K_1.

Still another set is of interest in this connection. Let us say that a function $\theta \in \mathcal{L}_1^\infty(S, \Sigma, \sigma)$ is *singularly nonpositive*, if for every $\varepsilon > 0$, there exists a (measurable) set $T \subset S$, comprised of a finite number of atoms with respect to σ (or empty), such that $\theta(s) \leqq \varepsilon$ for almost every $s \in S \setminus T$. The reason for this terminology will become clear in the next section. The *singularly induced feasible set* K_2° is defined as the set of all $x_1 \in R^{n_1}$ such that there exists an $x_2 \in \mathcal{L}_{n_2}^\infty$ with $x_2(s) \in C_2$ almost surely and $f_{2i}(\cdot, x_1, x_2(\cdot))$ singularly nonpositive for $i = 1, \cdots, m_2$. Like K_1 and K_2, the set K_2° is convex. Obviously

$$K_2 \subset K_2^\circ, \tag{1.9}$$

but in general the sets are not equal. The relations between these two sets is investigated further in § 4.

The main result is the following. (ri C denotes the relative interior of a set C, i.e., the interior of C relative to the smallest affine set containing C [10, § 6].)

THEOREM 1. *Suppose that* P *is strictly feasible and* ri $K_1 \subset K_2^\circ$. *Then*

$$\inf \mathrm{P} = \max \mathrm{D}, \tag{1.10}$$

so that solutions to P *and* D *correspond to saddle points of the basic Lagrangian L.*

In the last section (§ 4) of this paper we pursue the implications of this result and the significance of the hypothesis ri $K_1 \subset K_2^\circ$. We note, however, that this hypothesis is automatically satisfied whenever

$$K_2 \supset K_1. \tag{1.11}$$

Stochastic programs satisfying this last condition are known as stochastic programs with *relatively complete recourse*. Strictly speaking, this is the version of that condition for the class of stochastic programs under consideration here.

This is not an unusual property for stochastic programs. In fact, we might expect that for many stochastic programs arising from specific applications a stronger property will actually be satisfied, namely, the so-called *complete recourse* condition, which requires that for all $x_1 \in R^{n_1}$, there exists $x_2 \in \mathcal{L}_{n_2}^\infty$ satisfying the second stage constraints (1.3), or equivalently that $K_2 = R^{n_1}$; this implies that for all x_1, $\Gamma(s, x_1) \neq \emptyset$ almost surely.

The seminal papers on stochastic programming of G. Dantzig [4] and Beale [5] consider only stochastic programs with complete recourse. This restriction is not artificial, since the applications envisaged by these authors fall in this class. Actually, Beale's model [5, § 5] and one of the problems motivating Dantzig's work, described in [6], belong to an even more restrictive class, known as stochastic programs with *simple recourse*, which has received considerable attention (cf. [7] for a survey). Roughly speaking, for simple recourse the recourse decision is ~~simply~~ a way to record the "state of the system" after a first stage decision x_1 has been selected and a particular element s of S has been observed.

The term "*complete*" was first utilized by G. Dantzig in [4]. The more detailed classification sketched out above was introduced in [8]. Interest in the class of stochastic programs with relatively complete recourse—but not necessarily complete recourse—stems from theoretical considerations, but also from the observation made in § 4 of [8] that some important allocation problems arising in agricultural economics and formulated by G. Tintner [9] are indeed members of this class and not of the more restrictive class of stochastic programs with complete recourse. Independently of the implications resulting from the theory developed here, stochastic programs with relatively complete recourse are also of interest from a computational viewpoint, since they usually possess special structures which can be exploited in the solution procedure; see, for example, [8, §§ 2 and 4].

2. Singular multipliers and induced feasibility. As in [3], we denote by Y_0° the set of all $y^\circ=(y_1^\circ, \cdots, y_{m_2}^\circ)$ such that y_i° is a nonnegative singular linear functional on $\mathscr{L}_1^\infty$. The latter means that y_i° is a continuous linear functional with $y_i^\circ(c)\geqq 0$ for every nonnegative $c\in\mathscr{L}_1^\infty$, and there exists an increasing sequence of measurable sets S_k with $\bigcup_{k=1}^\infty S_k=S$, such that $y_i^\circ(c)=0$ if $c(s)=0$ almost surely for $s\notin S_k$.

The *extended Lagrangian* associated with P is the function $L^{\sim}$ on $X_0\times(Y_0\times Y_0^\circ)$ defined by

$$L^{\sim}(x_1, x_2, y_1, y_2, y^\circ)=L(x_1, x_2, y_1, y_2)+\sum_{i=1}^{m_2} y_i^\circ(f_{2i}(\cdot, x_1, x_2(\cdot))). \tag{2.1}$$

The *extended dual problem* is

$\tilde{\mathrm{D}}$ maximize $\tilde{g}(y_1, y_2, y^\circ)$ over all $(y_1, y_2, y^\circ)\in Y_0\times Y_0^\circ$, where

$$\tilde{g}(y_1, y_2, y^\circ)=\inf_{(x_1,x_2)\in X} L^{\sim}(x_1, x_2, y_1, y_2, y^\circ).$$

We have

$$\tilde{g}(y_1, y_2, 0)\equiv g(y_1, y_2), \tag{2.2}$$

so that D can be regarded as a "subproblem" of $\tilde{\mathrm{D}}$.

It was shown in [3] that strict feasibility in P implies inf P = max $\tilde{\mathrm{D}}$. We shall demonstrate in the next section that, in some cases, solving $\tilde{\mathrm{D}}$ is equivalent to solving D, and this will yield Theorem 1. The present section paves the way to this argument by developing a representation of the singularly induced feasible set K_2° in terms of the singular component of $L^{\sim}$ in (2.1). This representation, in the theorem which follows, explains the name we have given to K_2°.

THEOREM 2. *One has $x_1\in K_2^\circ$ if and only if there exists $x_2\in\mathscr{L}_{n_2}^\infty$ such that $x_2(s)\in C_2$ almost surely and*

$$\sum_{i=1}^{m_2} y_i^\circ(f_{2i}(\cdot, x_1, x_2(\cdot)))\leqq 0 \quad \textit{for all } y^\circ\in Y_0^\circ. \tag{2.3}$$

Proof. Clearly, the theorem will be proved if we establish that a function $\theta\in\mathscr{L}_1^\infty$ is singularly nonpositive if and only if $b^\circ(\theta)\leqq 0$ for every nonnegative singular functional b°.

Suppose first that θ is singularly nonpositive, and let b° be a nonnegative singular functional with an associated sequence of sets S_k, as per definition. Let $\varepsilon>0$. Then there exists $T\subset S$, consisting of a finite number of atoms, such that $\theta(s)\leqq\varepsilon$ almost surely outside of T. Since $S_k\uparrow S$, we have $\sigma(S_k)\uparrow 1$. Hence for some k sufficiently large we have $S_k\supset T$ (except possibly for a subset of T of measure zero), implying that $b^\circ(\theta)$ depends only on the restriction of θ to $S \setminus T$. Let e be the function in $\mathcal{L}_1^\infty$ with $e(s) \equiv 1$. Then $b^\circ(\theta) \leqq b^\circ(\varepsilon e) = \varepsilon b^\circ(e)$, because b° is nonnegative and $\theta(s) \leqq \varepsilon e(s)$ almost surely on $S \setminus T$. This is true for arbitrary $\varepsilon > 0$, so we conclude $b^\circ(\theta) \leqq 0$.

Assume now that the function $\theta \in \mathcal{L}_1^\infty$ is not singularly nonpositive. Thus for a certain $\varepsilon > 0$ the set

$$T = \{s \in S \mid \theta(s) > \varepsilon\}$$

is *not* comprised of a finite number of atoms (up to a set of measure zero). We shall construct a nonnegative singular functional b° such that $b^\circ(\theta) \geqq \varepsilon$. The assumed property of T implies the existence of a decreasing sequence of measurable sets $T_k \subset T$ such that $\sigma(T_k) > 0$ for all k and $\sigma(T_{k+1}) \leqq \frac{1}{2}\sigma(T_k)$. Then

$$0 = \lim_{k \to \infty} \sigma(T_k) = \sigma\Big(\bigcap_{k=1}^{\infty} T_k\Big).$$

Deleting the null set $T_\infty = \bigcap_{k=1}^{\infty} T_k$ from each set in the sequence, if necessary, we can suppose that $\bigcap_{k=1}^{\infty} T_k = \varnothing$. For each k, let b_k be the nonnegative linear functional on $\mathcal{L}_1^\infty$ defined by

$$b_k(c) = \frac{1}{\sigma(T_k)} \int_{T_k} c(s)\sigma(ds). \tag{2.4}$$

Observe that

$$b_k(e) = \|b_k\| = 1 \quad \text{for all } k, \tag{2.5}$$

where, as above, $e(s) \equiv 1$. The set $\{b_k \mid k = 1, 2, \cdots\}$ is thus bounded in the dual space $(\mathcal{L}_1^\infty)^*$ and hence has an accumulation point in the weak* topology. Let b° denote any such point. Then b° is again nonnegative, and $b^\circ(e) = 1$ by (2.5). Moreover, b° is singular: setting $S_k = S \setminus T_k$, we have $S = \bigcup_{k=1}^{\infty} S_k$, and for $l \geqq k$ the functional b_l has $b_l(c) = 0$ for all $c \in \mathcal{L}_1^\infty$ vanishing almost surely outside of S_k; thus $b^\circ(c) = 0$ for all $c \in \mathcal{L}_1^\infty$ vanishing almost surely outside of S_k. In particular, for $c(s) = \max\{\theta(s) - \varepsilon e(s), 0\} - [\theta(s) - \varepsilon e(s)]$ we have $c(s) = 0$ for all $s \in T$, and hence $b^\circ(c) = 0$. Therefore

$$b^\circ(\theta) - \varepsilon = b^\circ(\theta - \varepsilon e) = b^\circ(\max\{\theta - \varepsilon e, 0\}) \geqq 0,$$

and the proof is finished.

3. Equivalence of D and $\tilde{\mathbf{D}}$. We consider now, as in the extended Kuhn–Tucker conditions in [3], the function l on $R^{n_1} \times Y_0^\circ$ defined by

$$l(x_1, y^\circ) = \inf\left\{\sum_{i=1}^{m_2} y_i^\circ(f_{2i}(\cdot\,, x_1, x_2(\cdot))) \mid x_2 \in \mathcal{L}_{n_2}^\infty,\ x_2(s) \in C_2 \text{ a.s.}\right\}. \tag{3.1}$$

This is convex in x_1, concave in y°, and nowhere $+\infty$. Let

$$K_2^{\circ\circ} = \{x_1 \in R^{n_1} \mid l(x_1, y^\circ) \leqq 0 \text{ for all } y^\circ \in Y_0^\circ\}. \tag{3.2}$$

This is a closed convex set in R^{n_1}. (Each of the functions $l(\cdot\,, y^\circ)$ for $y^\circ \in Y_0^\circ$.

convex on R^{n_1} and nowhere $+\infty$, is continuous.) Moreover

$$K_2^\circ \subset K_2^{\circ\circ}, \tag{3.3}$$

in view of Theorem 2.

THEOREM 3. *Suppose there exists at least one $x_1 \in C_1$ with $f_{1i}(x_1) < 0$ for $i = 1, \cdots, m_2$, and that every such x_1 which is also in* ri C_1 *belongs to $K_2^{\circ\circ}$. Then the dual problems* D *and* $\tilde{\mathrm{D}}$ *are equivalent, in the sense that for every $(y_1, y_2, y^\circ) \in Y_0 \times Y_0^\circ$ there exists y_1' such that $(y_1', y_2) \in Y_0$ and*

$$\tilde{g}(y_1, y_2, y^\circ) \leqq \tilde{g}(y_1', y_2, 0) = g(y_1', y_2). \tag{3.4}$$

Proof. Let $(y_1, y_2, y^\circ) \in Y_0 \times Y_0^\circ$. We assume $\tilde{g}(y_1, y_2, y^\circ)$ is not $-\infty$ (and hence is finite), since otherwise the conclusion of the theorem is trivial. In this case we have the following formula:

$$\tilde{g}(y_1, y_2, y^\circ) = \inf_{(x_1, x_2) \in X_0} \{L(x_1, x_2, y_1, y_2) + l(x_1, y^\circ)\}. \tag{3.5}$$

To see this, fix $(y_1, y_2, y^\circ) \in Y_0 \times Y_0^\circ$, and observe that for all $x_1 \in C_1$ we have that

$$\inf_{x_2 \in D} \left\{ \int_S L_2(s, x_1, x_2(s), y_2(s))\sigma(ds) + \sum_{i=1}^{m_2} y_i^\circ(f_{2i}(\cdot, x_1, x_2(\cdot))) \right\}$$

$$= \inf_{x_2 \in D} \int_S L_2(s, x_1, x_2(s), y_2(s))\sigma(ds) + \inf_{x_2 \in D} y_i^\circ(f_{2i}(\cdot, x_1, x_2(\cdot)));$$

where

$$\mathscr{D} = \{x_2 \in \mathscr{L}_{n_2}^\infty \mid x_2(s) \in C_2 \text{ almost surely}\}.$$

Since the inequality $\geqq$ certainly holds, equality will follow if we show that for arbitrary $x_2' \in \mathscr{D}$, $x_2'' \in \mathscr{D}$ and $\varepsilon > 0$, there exists $x_2 \in \mathscr{D}$ such that

$$\begin{aligned} &\int_S L_2(s, x_1, x_2(s), y_2(s))\sigma(ds) + \sum_{i=1}^{m_2} y_i^\circ(f_{2i}(\cdot, x_1, x_2(\cdot))) \\ &\leqq \int_S L_2(s, x_1, x_2''(s), y_2(s))\sigma(ds) + \sum_{i=1}^{m_2} y_i^\circ(f_{2i}(\cdot, x_1, x_2'(\cdot))) + \varepsilon. \end{aligned} \tag{3.6}$$

Now to each singular functional y_i°, there correspond an increasing sequence of measurable sets S_{ik} with $\bigcup S_{ik} = S$, such that $y_i^\circ(a) = 0$ if for some k, the function $a \in \mathscr{L}_1^\infty$ vanishes a.e. outside S_{ik}. The latter property implies that $y_i^\circ(b) = y_i^\circ(b')$ if b and b' agree almost everywhere outside of S_{ik}. Now for each index k define

$$x_s^k(s) = \begin{cases} x_s''(s) & \text{if } s \in S_{ik} \quad \text{for } i = 1, \cdots, m_2, \\ x_2'(s) & \text{for all other } s. \end{cases}$$

For each k, the function $x_2^k \in \mathscr{D}$ and

$$f_{2i}(s, x_1, x_2^k(s)) = f_{2i}(s, x_1, x_2'(s)) \quad \text{if } s \notin S_{ik}$$

so that

$$\sum_{i=1}^{m_2} y_i^\circ(f_{2i}(\cdot, x_1, x_2^k(\cdot))) = \sum_{i=1}^{m_2} y_i^\circ(f_{2i}(\cdot, x_1, x_2'(\cdot))).$$

On the other hand, since $\lim_{k\to+\infty} \sigma(S \setminus S_{ik}) = 0$, we get that

$$\lim_{k\to\infty} \int_S L_2(s, x_1, x_2^k(s), y_2(s))\sigma(ds) = \int_S L_2(s, x_1, x_2''(s), y_2(s))\sigma(ds).$$

From the two preceding equalities, it follows that (3.6) holds for $x_2 = x_2^k$ if k is sufficiently large, which in turn directly yields (3.5).

Now, define the functions h and k on R^{n_1} by

$$(3.7)\qquad \begin{aligned} h(x_1) &= \begin{cases} \inf\{L(x_1, x_2, y_1, y_2) | x_2 \in \mathcal{L}^\infty_{n_2}, x_2(s) \in C_2 \text{ a.s.}\} & \text{if } x_1 \in C_1, \\ +\infty & \text{if } x_1 \notin C_1, \end{cases} \\ k(x_1) &= -l(x_1, y^\circ). \end{aligned}$$

Then h is a convex function, not identically $+\infty$, while k is a concave function, nowhere $-\infty$, and

$$(3.8)\qquad \tilde{g}(y_1, y_2, y^\circ) = \inf_{x_1 \in R^{n_1}} \{h(x_1) - k(x_1)\}.$$

The finiteness of $\tilde{g}(y_1, y_2, y^\circ)$ implies k cannot be identically $+\infty$, and hence k is finite everywhere; furthermore h cannot have the value $-\infty$ and hence is proper. Fenchel's duality theorem [10, Thm. 31.1] is thus applicable to (3.8), and we obtain

$$(3.9)\qquad \tilde{g}(y_1, y_2, y^\circ) = \max_{x_1^* \in R^{n_1}} \{k^*(x_1^*) - h^*(x_1^*)\},$$

where the conjugate functions k^* and h^* are defined by

$$(3.10)\qquad h^*(x_1^*) = \sup_{x_1 \in R^{n_1}} \{x_1 \cdot x_1^* - h(x_1)\},$$

and

$$(3.11)\qquad k^*(x_1^*) = \inf_{x_1 \in R^{n_1}} \{x_1 \cdot x_1^* - k(x_1)\}.$$

Fix x_1^* for which the maximum in (3.9) is attained. Then

$$(3.12)\qquad -h^*(x_1^*) = \tilde{g}(y_1, y_2, y_0) - k^*(x_1^*),$$

and therefore by formula (3.10),

$$(3.13)\qquad h(x_1) - x_1 \cdot x_1^* \geqq \tilde{g}(y_1, y_2, y^\circ) - k^*(x_1^*) \quad \text{for all } x_1 \in R^{n_1}.$$

Also from the definition of k and by formula (3.11),

$$(3.14)\qquad l(x_1, y^\circ) + x_1 \cdot x_1^* \geqq k^*(x_1^*) \quad \text{for all } x_1 \in R^{n_1}.$$

The latter implies that $x_1 \cdot x_1^* \geqq k^*(x_1^*)$ if $l(x_1, y^\circ) \leqq 0$, and thus, in particular, if $x_1 \in K_2^{\circ\circ}$. Our hypothesis then yields that $x_1 \cdot x_1^* \geqq k^*(x_1^*)$ for all x_1 in the set

$$(3.15)\qquad K_1' = \{x_1 \in \text{ri } C_1 | f_{1i}(x_1) < 0, i = 1, \cdots, m_1\}.$$

Define

$$K_1'' = \{x_1 \in C_1 | f_{1i}(x_1) < 0,\ i = 1, \cdots, m_1\}. \tag{3.16}$$

By hypothesis, K_1'' is nonempty. From this (and the finiteness, hence continuity, of the convex functions f_{1i}) it follows that $K_1' = \text{ri } K_1''$, while on the other hand,

$$\text{cl } K_1'' = \{x_1 \in C_1 | f_{1i}(x_1) \leqq 0,\ i = 1, \cdots, m_1\} = K_1. \tag{3.17}$$

Hence K_1 is in fact the closure of the set K_1', where the inequality $x_1 \cdot x_1^* \geqq k^*(x_1^*)$ holds, so that

$$k^*(x_1^*) \leqq \inf_{x_1 \in K_1} x_1 \cdot x_1^*. \tag{3.18}$$

The right side of (3.18) represents an ordinary convex program which, by our hypothesis, is strictly feasible. In consequence, there exist multipliers $\bar{y}_{1i} \geqq 0$, $i = 1, \cdots, m_1$, such that

$$k^*(x_1^*) \leqq \inf_{x_1 \in C_1} \left\{x_1 \cdot x_1^* + \sum_{i=1}^{m_1} \bar{y}_{1i} f_{1i}(x_1)\right\}.$$

The latter is better expressed, for our purposes, as

$$\sum_{i=1}^{m_1} \bar{y}_{1i} f_{1i}(x_1) \geqq k^*(x_1^*) - x_1 \cdot x_1^* \quad \text{for all } x_1 \in C_1. \tag{3.19}$$

Combining this inequality with (3.13) and reverting to the definition (3.7) of h, we see that

$$L(x_1, x_2, y_1, y_2) + \sum_{i=1}^{m_1} \bar{y}_{1i} f_{1i}(x_1) \geqq \tilde{g}(y_1, y_2, y^\circ) \quad \text{for all } (x_1, x_2) \in X_0. \tag{3.20}$$

But the left side of (3.20) is $L(x_1, x_2, y_1 + \bar{y}_1, y_2)$. Therefore, setting $y_1' = y_1 + \bar{y}_1$ we have $(y_1', y_2) \in Y_0$ and

$$\tilde{g}(y_1, y_2, y^\circ) \leqq \inf_{(x_1, x_2) \in X_0} L(x_1, x_2, y_1', y_2) = g(y_1'; y_2),$$

which is the desired relation.

Proof of Theorem 1. Since P is strictly feasible, we know that inf P = max $\tilde{\text{D}}$ [3, Thm. 2], and also that the set K_1'', as defined in (3.16), is nonempty. But then, as in the proof above, the set K_1' in (3.15) is ri K_1'' while cl $K_1'' = K_1$. Therefore

$$\text{ri } K_1 = \text{ri}(\text{cl } K_1'') = \text{ri } K_1'' = K_1'.$$

Our assumption that ri $K_1 \subset K_2^\circ$ then gives us, by way of (3.3), that $K_1' \subset K_2^{\circ\circ}$. Thus the hypothesis of Theorem 3 is fulfilled, yielding the conclusion that max $\tilde{\text{D}}$ = max D.

4. Analysis of induced feasibility. We turn now to investigating further the relations between the induced feasible set K_2, the singularly induced feasible set K_2° and a related set K_2'', which consists of all vectors $x_1 \in R^{n_1}$ such that for almost all $s \in S$ there exists a vector $x_2 \in C_2 \subset R^{n_2}$ such that

$$f_{2i}(s, x_1, x_2) \leqq 0 \quad \text{for } i = 1, \cdots, m_2. \tag{4.1}$$

We shall call K_2^σ the σ-induced feasible set. It is evident that

$$K_2^\sigma \supset K_2.$$

One can view K_2^σ as the set of all (first-stage) decisions x_1 with which we can associate at least one feasible recourse decision for almost any "observed value" of s in S. In order for x_1 to be also in K_2, one must be able to string these recourse decisions together so as to form an essentially bounded measurable function of s.

The singularly induced feasible set K_2° is not so easily amenable to physical interpretation. However, the main results do not refer to K_2 but to the larger set K_2^σ, or even (in Theorem 3) to a still larger set $K_2^{\sigma\circ}$. At least in part, this is due to technical reasons which we examine in this section. We concentrate our attention on two "extreme" cases: at one end the *discrete case*, where the support of the random variable consists of a *finite* number of atoms, and at the other end the *nonatomic case*, where the probability space contains *no* atoms. (This latter case includes the one of $S \subset R^N$, N finite, and σ absolutely continuous with respect to Lebesgue measure). These two situations seem to cover nearly all applications of practical interest. By abuse of language we shall refer to (S, Σ, σ) as being a discrete or nonatomic probability space in the respective cases.

Recall that for $s \in S$ and $x_1 \in R^{n_1}$ one has

$$\Gamma(s, x_1) = \{x_2 \in C_2 | f_{2i}(s, x_1, x_2) \leqq 0 \text{ for } i = 1, \cdots, m_2\}. \tag{4.2}$$

As already pointed out in [1, Proof of Thm. 1], the multifunction

$$s \mapsto \Gamma(s, x_1)$$

is measurable. This follows from [11, Corollary 4.3], since for fixed x_1 the functions

$$(s, x_2) \mapsto f_{2i}(s, x_1, x_2) \quad \text{for } i = 1, \cdots, m_2$$

are normal convex integrands [12, Lemma 2]. Thus for each $x_1 \in R^{n_1}$, the set

$$\omega(x_1) = \{s \in S | \Gamma(s, x_1) \neq \varnothing\} \tag{4.3}$$

is a measurable set. Moreover if $x_1 \in K_2^\sigma$, then $\omega(x_1)$ is a set of measure 1, i.e., $\sigma[\omega(x_1)] = 1$. We also define

$$\omega^{-1}(s) = \{x_1 \in R^{n_1} | \Gamma(s, x_1) \neq \varnothing\}, \tag{4.4}$$

which is clearly a convex set. With this notation we have that

$$K_2^\sigma = \{x_1 \in R^{n_1} | \sigma[\omega(x_1)] = 1\}. \tag{4.5}$$

PROPOSITION. *Suppose that for all s in S, $\omega^{-1}(s)$ is closed. Then the σ-induced feasible set K_2^σ is closed and convex.*

Proof. It suffices to show that the σ-induced feasible set can be written as

$$K_2^\sigma = \bigcap_{s \in S'} \omega^{-1}(s), \tag{4.6}$$

where S' is some subset of S of measure 1. The proposition is clearly true if $K_2^\sigma = \varnothing$. Assume otherwise and let D be a countable dense subset of K_2^σ. Such a set exists, since K_2^σ is a subset of the separable metric space R^{n_1}. Take S'

$=\bigcap_{x_1\in D}\omega(x_1)$. Clearly $\sigma(S')=1$ and $K_2^\sigma\supset\bigcap_{s\in S'}\omega^{-1}(s)$. Now for all $s\in S'$, we also have that $\omega^{-1}(s)\supset D$ and thus $\omega^{-1}(s)\supset K_2^\sigma$ since $\omega^{-1}(s)$ is closed, i.e., $K_2^\sigma\subset\bigcap_{s\in S'}\omega^{-1}(s)$.

COROLLARY A. *Suppose that C_2 is compact. Then K_2^σ is closed and convex.*

Proof. In this case, $\omega^{-1}(s)$ is closed for every $s\in S$, since C_2 is compact and the functions $f_{2i}(s,\cdot,\cdot)$ are lower semicontinuous.

COROLLARY B ([13, Thm. 3.5]). *Suppose that C_2 is polyhedral and that for $i=1,\cdots,m_2$ and all $s\in S$ the functions $(x_1,x_2)\mapsto f_{2i}(s,x_1,x_2)$ are affine. Then K_2^σ is closed and convex.*

Proof. For each fixed s, the set

$$W(s)=\{(x_1,x_2)|f_{2i}(s,x_1,x_2)\leqq 0\quad\text{for } i=1,\cdots,m_2,\ x_1\in R^{n_1},\ x_2\in C_2\}$$

is a polyhedral convex set, and its projection in the x_2-coordinates is $\omega^{-1}(s)$. Thus $\omega^{-1}(s)$ is polyhedral convex and consequently closed.

With some additional assumptions, it is also possible to show that $K_2^\sigma=\bigcap_{s\in S}\omega^{-1}(s)$. This essentially requires embedding S in a topological space (with S then the support of σ) and subjecting the maps $s\mapsto f_{2i}(s,x_1,x_2)$ to continuity conditions (cf. [14, Thm. 2]).

The following two theorems establish the relations between the various induced feasible sets in the discrete and nonatomic cases.

THEOREM 4. *Suppose that (S,Σ,σ) is a discrete probability space. Then*

$$R^{n_1}=K_2^\circ=K_2^{\circ\circ}\supset K_2=K_2^\sigma. \tag{4.7}$$

Proof. When (S,Σ,σ) is a discrete probability space, every function in $\mathscr{L}^\infty$ is singularly nonpositive, since the criterion for singular nonpositivity allows us to ignore a finite number of atoms; thus $K_2^\circ=R^{n_1}$. The first string of equalities now follows from the known inclusions $K_2^\circ\subset K_2^{\circ\circ}\subset R^{n_1}$. The equality of $K_2=K_2^\sigma$ is a direct consequence of the definition of these sets when the underlying probability space is discrete.

THEOREM 5. *Suppose that (S,Σ,σ) is a nonatomic probability space. Then*

$$K_2=K_2^\circ. \tag{4.8}$$

Moreover, if to every $x_1\in K_2^\sigma$ there corresponds a bounded region $B\subset R^{n_2}$ such that for almost all s, $\Gamma(s,x_1)\cap B\neq\varnothing$, then

$$K_2^\sigma=K_2=K_2^\circ. \tag{4.9}$$

Proof. When (S,Σ,σ) is nonatomic, a function in $\mathscr{L}^\infty$ is singularly nonpositive if and only if it is nonpositive. This yields (4.8). We have already observed that, in general, $K_2^\sigma\supset K_2$. Thus to prove (4.9) it only remains to show inclusion in the other direction. Fix $x_1\in K_2^\sigma$. The multifunction $s\mapsto\Gamma(s,x_1)$ is closed-convex-valued and measurable, and thus the multifunction $s\mapsto\Gamma(s,x_1)\cap\operatorname{cl}B$ is compact-convex-valued and measurable. Furthermore, by assumption, $\Gamma(s,x_1)\cap\operatorname{cl}B$ is almost surely nonempty. Thus there exists a measurable selector $\bar{x}_2$ with $\bar{x}_2(s)\in\Gamma(s,x_1)\cap\operatorname{cl}B$ for almost all s[12, Cor. 1.1]. Since B is bounded, $\bar{x}_2$ is in $\mathscr{L}_{n_2}^\infty$; hence $x_1\in K_2$ and consequently $K_2^\sigma\subset K_2$.

These two theorems have immediate implications as to the class of dual variables we need to consider in obtaining an inf-max duality theorem.

COROLLARY 4. *Suppose that* P *is strictly feasible and* (S, Σ, σ) *is a discrete probability space (finitely many points). Then*

$$\inf \mathrm{P} = \max \mathrm{D}. \tag{4.10}$$

Proof. Theorems 4 and 1.

COROLLARY 5A. *Suppose that* (S, Σ, σ) *is a nonatomic probability space. Then* $x_1 \in K_2$ *if and only if there exists* $x_2 \in \mathscr{L}^\infty_{n_2}$ *such that* $x_2(s) \in C_2$ *almost surely and*

$$\sum_{i=1}^{m_2} y_i^\circ(f_{2i}(\cdot, x_1, x_2(\cdot))) \leqq 0 \quad \textit{for all } y^\circ \in Y_0^\circ. \tag{4.11}$$

Proof. Theorems 5 and 2.

COROLLARY 5B. *Suppose that* P *is strictly feasible,* (S, Σ, σ) *is a nonatomic probability space, and to each* $x_1 \in K_2^\circ$ *there corresponds a bounded region* B *with* $\Gamma(s, x_1) \cap B \neq \emptyset$ *almost surely. Suppose also that* $\omega^{-1}(s)$ *is closed for all* $s \in S$. *Then* $\operatorname{ri} K_1 \subset K_2^\circ$ *if and only if* P *is a stochastic program with relatively complete recourse, in which case*

$$\inf \mathrm{P} = \max \mathrm{D}.$$

Proof. Theorem 5 with the Proposition ~~above~~ and Theorem 1.

COROLLARY 5C. *Suppose that* P *is a stochastic program with relatively complete recourse, strictly feasible with* C_2 *compact and* (S, Σ, σ) *is nonatomic. Then*

$$\inf \mathrm{P} = \max \mathrm{D}.$$

Proof. Corollary 5B with Corollary A of the above Proposition.

One of the implications of Corollaries 5B and 5C is that under those assumptions K_2 and K_2° are closed.

Corollaries 5A and 5B assert that when (S, Σ, σ) is nonatomic, the "singular multipliers" result from the presence of induced constraints. The singular multipliers y_2° appearing in the extended Kuhn–Tucker conditions [3] correspond—figuratively speaking—to a singular subset T of S which determines the critical points in S. These multipliers can not be $\mathscr{L}^1$ functions, since these critical points have mass 0, yet they do play a crucial role in the optimization problem.

On the other hand, if (S, Σ, σ) is discrete, Corollary 4 indicates that we never need to use "singular multipliers" to obtain the strong form of the duality result. Thus the basic Kuhn–Tucker conditions [2] are in fact *necessary* and sufficient, assuming strict feasibility. This does not mean that we can ignore the induced constraints, but more simply that the multipliers associated to these constraints will be represented by $\mathscr{L}^1$ functions on the probability space. (In the discrete case the dual of $\mathscr{L}^\infty_{n_2}$ *is* $\mathscr{L}^1_{n_2}$.) We illustrate this by a couple of examples.

Example 1. Find $x_1 \in R^{n_1}$, $x_2 \in \mathscr{L}^\infty_1$ such that

$$x_1 \geqq 0,$$

$$x_2(s) \geqq 0 \quad \text{and} \quad s - x_1 + x_2(s) \leqq 0 \quad \text{for almost all } s,$$

?

This is dominated by Thm 1 already. Presumably the statement should have been: Suppose P strictly feas with C_2 compact & the prob space nonatomic. Then $\operatorname{ri} K_1 \subset K_2^\circ \Leftrightarrow$ P has rel complete recourse, in which case and one has the minimum of the expression

$$2x_1 - \frac{1}{n}\sum_{s\in S} x_2(s),$$

where $S=\{s=(k-1)/n,\ k=1,\cdots,n\}$ with $\sigma(s)=1/n$. There are no first-stage constraints; $C_1=\{x_1 | x_1 \geqq 0\}$. The induced feasible set is

$$K_2=\{x_1 | x_1 \geqq 1\},$$

whereas $K_2^\circ = R$ (Theorem 4). From Corollary 4A we know that the basic Kuhn–Tucker conditions are necessary *and* sufficient for this problem. From the differentiable form of these conditions with C_1 and C_2 the nonnegative orthants, we obtain using [2, Cor. B] that a pair $((\bar{x}_1, \bar{x}_2), \bar{y}_2) \in (R\times \mathscr{L}_1^\infty)\times \mathscr{L}_1^1$ determines optimal solutions to the program (4.8), $\cdots$, (4.10) and its dual if there exists a function $\rho \in \mathscr{L}_1^1$ satisfying:

(a) $\bar{x}_1 \geqq 0$;

(b) $\bar{x}_2(s)\geqq 0$, $\bar{y}_2(s)\geqq 0$, $s-\bar{x}_1+\bar{x}_2(s)\leqq 0$, $\bar{y}_2(s)[s-\bar{x}_1+\bar{x}_2(s)]=0$ for all $s\in S$;

($c_\oplus$) $2 \geqq (1/n)\sum_{s\in S}\rho(s)$ and $2x_1 = (\bar{x}_1/n)\sum_{s\in S}\rho(s)$;

($d_\oplus$) $\rho(s)=-\bar{y}_2(s)$, $\bar{y}_2(s)\geqq 1$ and $\bar{x}_2(s)[-1+\bar{y}_2(s)]=0$ for all $s\in S$.

One verifies easily that the values

$$\bar{x}_1 = 1, \qquad \bar{x}_2(s) = 1-s \quad \text{for } s=\frac{k}{n-1}, \quad k=0,1,\cdots,n-1,$$

and

$$\bar{y}_2(s) = -\rho(s) = 1 \quad \text{for } s=\frac{k}{n}, \quad k=0,1,\cdots,n-2, \quad \bar{y}_2(1) = -\rho(1) = n+1$$

satisfy the above conditions. It is striking that the "price" $y_2(s)$ associated with the constraint

$$s - x_1 + x_2(s) \leqq 0$$

is much larger when $s=1$ than when $s<1$.

Example 2. We consider the same problem as in Example 1, except that the probability space is now nonatomic. Specifically: S is the interval $[0,1]$ and σ is the Lebesgue measure. As before, the induced feasible set is

$$K_2=\{x_1 | x_1 \geqq 1\}.$$

This is also the singularly induced feasible set K_2° (Theorem 5), and as can be verified, it is also the set $K_2^{\circ\circ}$ defined by (3.2) and utilized in Theorem 3. Corollary 5A directs us to use in this case the extended Kuhn–Tucker conditions [3, § 5]. Thus, we have that a pair $((\bar{x}_1, \bar{x}_2), (\bar{y}_2, \bar{y}^\circ)) \in (R\times\mathscr{L}_1^\infty)\times(\mathscr{L}_1^\infty \times \mathscr{F}_1)$ determines optimal solutions to program (4.8), $\cdots$, (4.10) and its extended dual (with s uniform on $[0,1]$) if there exists $\rho \geqq \in \mathscr{L}_1^1$ satisfying

(a) $\bar{x}_1 \geqq 0$;

(b) $\bar{x}_2(s)\geqq 0$, $\bar{y}_1(s)\geqq 0$, $s-\bar{x}_1+\bar{x}_2(s)\leqq 0$, $\bar{y}_2(s)[s-\bar{x}_1+\bar{x}_2(s)]=0$ for $s\in[0,1]$;

(c°) $\bar{x}_1$ minimizes $(2x_1+\int\rho(s)\sigma(ds)+l(x_1,\bar{y}^\circ))$ subject to $x_1\geqq 0$;

($d_\oplus$) $\rho(s)=-\bar{y}_2(s)$, $\bar{y}_2(s)\geqq 1$ and $\bar{x}_2(s)[-1+\bar{y}_2(s)]=0$ for $s\in[0,1]$;

(e) $\bar{y}^\circ \geqq 0$, $\bar{y}^\circ(\cdot - \bar{x}_1 + \bar{x}_2(\cdot)) = 0$ and $0 = \inf \{\bar{y}^\circ(\cdot - \bar{x}_1 + \bar{x}_2(\cdot)) | x_2 \in \mathscr{L}_1^\infty$ $\cdot([0, 1], \Sigma, \sigma)$, $x_2(s) \geqq 0$ almost surely$\}$.

Conditions (a), (b) and ($d_\oplus$) are the same as before, but this time a term involving the singular multipliers $l(x_1, y^\circ)$ appears in (c°), and these multipliers must satisfy the condition (e). The functional $\bar{y}^\circ$ is a continuous linear functional on $\mathscr{L}_1^\infty$ and can be expressed as an integral with respect to a purely finitely additive measure ν on S. Let ν be the measure on S which assigns measure 1 to a set A if A is (Lebesgue) measurable and 1 is a point of density of A; otherwise the measure of A is 0. (Such a measure can be generated on the Borel field by a construction similar to the one used in the proof of Theorem 4.1 of [16] starting by simply specifying $\nu(B) = 0$ for every set B of Lebesgue measure 0 and $\nu(B) = 1$ if B is (relatively) open in [0, 1] and contains 1). One can verify that the values

$$\bar{x}_1 = 1, \quad \bar{x}_2(s) = 1 - s \quad \text{for } s \in [0, 1]$$

and

$$\bar{y}_2(s) = -\rho(s) = 1 \quad \text{for } s \in [0, 1] \quad \text{and} \quad \bar{y}^\circ(\cdot) = \int \cdot\, \nu(ds)$$

satisfy the above conditions.

The solutions to the problems in Examples 1 and 2 resemble each other in many ways, except for the presence in the case of Example 2 of the singular function $\bar{y}^\circ$, and on the other hand the "jump" in the $\bar{y}_2$ multiplier when $s = 1$ in the case of Example 1. In fact, if we allow n to go to $+\infty$ in Example 1, it is clear that $\bar{y}_2(1)$ also tends toward $+\infty$. In other words, in the limit there will be an "infinite" price associated with the second-stage constraint when $s = 1$. We know from the derivation in Example 2 that this unusual behavior at $s = 1$ is due to the presence of induced constraints. The relations between these two examples give an illustration of the content of Theorem 1 of [3].

One can also view Theorem 1 as an enticement to introduce the induced constraints explicitly among the first-stage constraints (1.2). If this is done, *every* stochastic program becomes a stochastic program with relatively complete recourse and Theorem 1 becomes applicable to *every* stochastic program.

This, however, requires the actual determination of these induced constraints. The general theory of optimization indicates that merely a finite number of these will be sufficient to represent the binding constraints at the minimum. But this is only of relative comfort since, in general, the constraints in question are not especially easy to identify. Practically, we expect that the appropriate constraints will be generated as needed. By this it is meant that the algorithm builder will use some test to verify if a given $x_1 \in K_1$ is or is not a member of K_2, and in the latter case he will generate certain induced constraints—to be added to the constraints determining K_1—which would "cut out" that particular x_1. This procedure is already used for stochastic *linear* programming [15, § 5], although in that case fairly complete and concrete characterizations of the induced feasible set K_2 are known [15, § 4].

We conclude this paper by illustrating the effect on the dual variables of introducing the induced constraints as first-stage constraints in the case of the examples appearing above.

Example 1′. Same as Example 1, except that the induced constraint

$$x_1 \geqq 1$$

is now explicitly introduced as a first-stage constraint. The same Kuhn–Tucker conditions yield optimality criteria, except that (a) must be changed to

(a′) $\bar{x}_1 \geqq 0, 1-\bar{x}_1 \leqq 0, \bar{y}_1 \geqq 0, (1-\bar{x}_1)\bar{y}_1 = 0$.

With this modification, it can be seen that the following yield optimal solution to P and its dual:

$$\bar{x}_1 = 1, \quad \bar{x}_2(s) = 1-s \quad \text{for } s \in S$$

and

$$\bar{y}_1 = 1, \quad \bar{y}_2(s) = -\rho(s) = 1 \quad \text{for } s \in S.$$

The "curious" behavior of $\bar{y}_2(s)$ at $s = 1$ in Example 1 has now disappeared.

Example 2′. Same as Example 2 except that the induced constraint is explicitly introduced as a first-stage constraint. The new problem 2′ is now a stochastic program with relatively complete recourse. We can thus turn to the *basic* Kuhn–Tucker conditions to obtain optimality criteria. They are (a′) as above, (b) and ($d_\oplus$) as in Example 2, but from [2, Cor. B] we also have

(c′) $2+\int\rho(s)\sigma(ds) \geqq 0$ and $x_1[2+\int\rho(s)\sigma(ds)] = 0$.

This shows that the values

$$x_1 = 1, \quad \bar{x}_2(s) = 1-s \quad \text{for } s \in [0, 1]$$

and

$$\bar{y}_1 = 1, \quad \bar{y}_2(s) = -\rho(s) = 1 \quad \text{for } s \in [0, 1]$$

yield optimal solutions to Example 2′ and its dual. Observe that the $(\bar{y}_2, \bar{y}^\circ)$ solution obtained in Example 2 is actually an optimal solution to the extended dual $\tilde{\mathrm{D}}$ of Example 2′, but so is the solution obtained here (with $y^\circ = 0$), giving us a concrete illustration of Theorem 3.

If in P the set C_1 is replaced by $C_1 \cap K_2$ (or $C_1 \cap K_2^\circ$, or $C_1 \cap K_2^{\circ\circ}$), then every problem so generated is also a stochastic program with relatively complete recourse. But this time the relation between the dual variables associated with the original problem and those of the new problem is no longer as easy to establish.

Finally, we observe that from the proofs of Theorems 1 and 3 it follows that we could actually use the larger set $K_2^{\circ\circ}$ in place of K_2°. This gives a more general result, but $K_2^{\circ\circ}$ is at the same time "less concrete". We have not succeeded in proving any more intimate relationship between $K_2^{\circ\circ}$ and K_2° than the inclusion

$$K_2^{\circ\circ} \supset K_2^\circ,$$

except in the discrete case, when evidently equality holds.

5. Conclusion. The objective of [1], [2], [3] and this paper is to develop necessary and sufficient optimality conditions for stochastic convex programs. The model chosen P (see §1) demands that the recourse (or second-stage) decision as a function of the random elements be measurable (an inconsequential restriction) and essentially bounded. This last condition is a definite restriction, in general, (not if the second-stage feasibility region is bounded [1, Thm. 2]) but it is not a significant restriction [1, Thm. 1] since the main concern is not with the existence of optimal solutions. The approach is through general duality theory: we first embed the original problem in a class of perturbed problems (the natural choice turns out to be to perturb the constraints by elements of $R^{m_1} \times \mathscr{L}^{\infty}_{m_2}$), then set up a *Lagrangian* L associated with the system of perturbations and finally from L derive a dual problem D. Saddle points of L are characterized by the so-called Kuhn–Tucker conditions. These Kuhn–Tucker conditions always provide sufficient optimality conditions for P; moreover they become also necessary if it can be shown that inf P = max D (and not just inf P = sup D). To guarantee the existence of optimal solutions to D, the standard requirement is to demand that P satisfies a constraint qualification (e.g., strict feasibility).

This is precisely what happens [3, Thm. 2] if the space associated with perturbations is sufficiently "large", viz., if the multiplier space is selected to be $R^{m_1} \times (\mathscr{L}^{\infty}_{m_2})^*$. The extended Kuhn–Tucker conditions [3, § 5] are then necessary and sufficient. The choice of $R^{m_1} \times (\mathscr{L}^{\infty}_{m_2})^*$ as the multiplier space is however rather unsatisfactory since calculations involving elements of $(\mathscr{L}^{\infty}_{m_2})^*$ are generally unmanageable unless one can handle "separately" the singular part and the $\mathscr{L}^1$-part of every such $(\mathscr{L}^{\infty}_{m_2})^*$ multiplier.

This paper shows that the singular parts of the optimal multipliers correspond basically to the induced constraints (Theorem 2), more precisely to the singularly induced feasibility set. Consequently, if there are no induced constraints (relatively complete recourse) or, more generally, if the induced constraints do not determine binding constraints at the optimum, we may restrict the multiplier space to $R^{m_1} \times \mathscr{L}^1_{m_2}$ and still obtain the necessity of the Kuhn–Tucker conditions (Theorem 1). Note also that every stochastic program can be transformed into a stochastic program with relatively complete recourse by the inclusion of the induced constraints in the first-stage constraints. In this case the basic duality theory [1, § 4] is applicable, and the necessary and sufficient conditions for optimality are given by the (basic) Kuhn–Tucker conditions [2] involving only $\mathscr{L}^1$-functions as multipliers.

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