Reprinted from JOURNAL OF ECONOMIC THEORY
All Rights Reserved by Academic Press, New York and London
Vol. 12, No. 1, February 1976
Printed in Belgium

A Growth Property in Concave–Convex Hamiltonian Systems

R. T. ROCKAFELLAR*

University of Washington, Seattle, Washington 98195

Received June 29, 1975; revised October 2, 1975

The analysis of the stability of Hamiltonian dynamical systems in various economic models depends on the "curvature" of the Hamiltonian function at a rest point of the system, or equivalently, on growth properties involving gradients or subgradients. The purpose of this note is to establish a general property pointing in particular to a simplification of conditions assumed by Cass and Shell [1].

In this context, a Hamiltonian $H\colon R^n \times R^n \to [-\infty, +\infty]$ is an extended-real-valued function such that $H(k, Q)$ is concave in k and convex in Q. The dynamical system of interest in the case of a constant discount rate ρ is

$$\dot{k} \in \partial_Q H(k, Q),\ -\dot{Q} + \rho Q \in \partial_k H(k, Q) \tag{1}$$

where $\partial_Q H$ and $\partial_k H$ are the subdifferentials [2] with respect to Q and k (or if differentiability is present, the gradients), and a rest point is a pair, (k^*, Q^*) satisfying

$$0 \in \partial_Q H(k^*, Q^*),\ \rho Q^* \in \partial_k H(k^*, Q^*). \tag{2}$$

Particular attention is directed to determining the existence of solutions to (1) satisfying

$$k(0) = \bar{k} \qquad \text{and} \qquad \lim_{t\to\infty} e^{-\rho t}(k(t) - k^*) \cdot (Q(t) - Q^*) = 0, \tag{3}$$

and whether such a solution is stable in the sense of converging to (k^*, Q^*) as $t \to +\infty$.

The author in [3] and [4] developed results on existence and stability on the basis of assuming H was strictly or strongly convex–concave in a neighborhood of (2). Cass and Shell [1] showed under different assumptions, to be considered below, that (1) and (3) imply convergence of

* This research was supported in part by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under AFOSR grant number 72-2269 at the University of Washington.

$k(t)$ to k^*. Brock and Scheinkman [5] obtained convergence of $(k(t), Q(t))$ to (k^*, Q^*) under more differentiability but less convexity. Gaines [6] proved the existence of solutions to (1), (3), using a different approach and different growth assumptions on H than the author.

In the paper of Cass and Shell, a certain global growth property of H is implicit in the conditions they impose on the underlying technology. For instance, it is a consequence of their model that solutions to (1) satisfy a universal bound $\| k(t)\| \leqslant B$. We wish to discuss not this aspect of their work, but their conditions involving the function

$$\Phi(k, Q) = \inf\{-R \cdot (k - k^*) + z \cdot (Q - Q^*) + \rho Q^* \cdot (k - k^*) | \times R \in \partial_k H(k, Q), z \in \partial_Q H(k, 0)\}. \tag{4}$$

(where $\inf \varnothing = +\infty$). Clearly $\Phi(k^*, Q^*) = 0$. It is known that Φ is everywhere nonnegative by virtue of the concavity–convexity of H (see below).

The crucial condition invoked by Cass and Shell can be stated as follows:

(S) For every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\| k - k^* \| > \epsilon \Rightarrow \Phi(k, Q) + \rho(k - k^*) \cdot (Q - Q^*) > \delta.$$

We shall establish that (S) can be expressed equivalently in the apparently weaker forms in Theorem 1 below. For this we need the harmless technical assumption that H is *closed* in the sense of [2, Section 34]. (This is always true for H arising from an economic model. A direct *sufficient* condition is that $H < +\infty$ everywhere and $H(k, Q)$ is upper semicontinuous in k.)

THEOREM 1. *Suppose H is finite on a neighborhood of (k^*, Q^*) and closed.*

(a) *If $\rho = 0$, (S) holds if merely $\Phi(k, Q) > 0$ for all $(k, Q) \neq (k^*, Q^*)$.*

(b) *If $\rho \neq 0$, (S) holds if for every (k', Q') with $k' \cdot Q' \leqslant 0$ the function*

$$\varphi(t) = \Phi(k^* + tk', Q^* + tQ') + \rho t^2 k' \cdot Q'$$

satisfies $\varphi(t) > 0$ for all $t > 0$ and

$$\liminf_{t \to \infty} \varphi(t) > 0.$$

Theorem 1 will be derived from a more general result. Consider an arbitrary multifunction $A\colon R^N \to R^N$ and any pair x^*, y^* such that $y^* \in A(x^*)$. Define

$$\Phi(x) = \inf\{(x - x^*) \cdot (y - y^*) | y \in A(x)\}. \tag{5}$$

The case above corresponds to $x = (k, Q) \in R^n \times R^n$,

$$A(k, Q) = \{(-R, z) \mid R \in \partial_k H(k, Q), z \in \partial_Q H(k, Q)\}, \tag{6}$$

$$x^* = (k^*, Q^*), y^* = (\rho Q^*, 0). \tag{7}$$

One says in general that A is *monotone* if

$$y_i \in A(x_i) \qquad \text{for } i = 0, 1, \text{ implies } (x_1 - x_0) \cdot (y_1 - y_0) \geqslant 0. \tag{8}$$

It is *maximal monotone* if it is monotone and its graph

$$G(A) = \{(x, y) \in R^N \times R^N \mid y \in A(x)\}$$

is not properly contained in the graph of any other monotone $A' : R^N \to R^N$. The *effective domain* of A is

$$D(A) = \{x \mid A(x) \neq \varnothing\}.$$

The connection with monotonicity and the present context is the following.

THEOREM 2 [7]. *If A is given by (6) for a closed concave-convex function H which is finite on a neighborhood of a point (k^*, Q^*), then A is maximal monotone and $(k^*, Q^*) \in \operatorname{int} D(A)$.*

THEOREM 3. *Let $A : R^N \to R^N$ be an arbitrary maximal monotone multifunction and let Φ be defined by (5) for any x^* and y^* satisfying $x^* \in \operatorname{int} D(A)$ and $y^* \in A(x^*)$. Then the* inf *in (5) is always attained, and Φ is an everywhere lower semicontinuous function with*

$$\Phi(x) \geqslant \Phi(x^*) = 0 \qquad \text{for all } x.$$

Moreover, the expression $\Phi(x^* + sx')/s$ is for any x' nondecreasing as a function of $s > 0$.

In particular, the function

$$\theta(s) = (1/s) \min\{\Phi(x) \mid \| x - x^* \| = s\}, \qquad s > 0.$$

is nonnegative, lower semicontinuous, nondecreasing, and

$$\Phi(x) \geqslant \theta(\| x - x^* \|) \| x - x^* \| \qquad \text{for all } x \neq x^*. \tag{10}$$

Proof. Consider two values $s_1 > s_2 > 0$ and any $y_i \in A(x^* + s_i x')$, $i = 1, 2$. The monotonicity of A implies

$$\begin{aligned} 0 &\leqslant [(x^* + s_1 x') - (x^* + s_2 x')] \cdot (y_1 - y_2) \\ &= (s_1 - s_2)\, x' \cdot (y_1 - y_2) \\ &= (s_1 - s_2)[s_1^{-1}((x^* + s_1 x') - x^*) \cdot (y_1 - y^*) \\ &\quad - s_2^{-1}((x^* + s_2 x') - x^*) \cdot (y_2 - y^*), \end{aligned}$$

and consequently, since the y_i are arbitrary,

$$0 \leqslant (s_1 - s_2)[s_1^{-1}\Phi(x^* + s_1x') - s_2^{-1}\Phi(x^* + s_2x')].$$

Thus $s^{-1}\Phi(x^* + sx')$ is nondecreasing in $s > 0$ as claimed.

The maximal monotonicity of A is known [8] to imply that int $D(A)$ is convex and (since int $D(A)$ is nonempty by hypothesis),

$$D(A) \subset \text{cl int } D(A). \tag{11}$$

Furthermore A is compact-valued and upper semicontinuous on int $D(A)$ (cf. [8]). Therefore, Φ is lower semicontinuous on int $D(A)$ and the inf in (5) is attained there.

Consider now a noninterior point $\bar{x}$ of $D(A)$. By (11) and the convexity of int $D(A)$ we have $x^* + s(\bar{x} - x^*) \in \text{int } D(A)$ for $0 \leqslant s < 1$. In view of this and the monotonicity of $\Phi(x^* + sx')/s$ in $s > 0$ for all x', the global lower semicontinuity of Φ will follow if we show that the restriction of Φ to the line

$$L = \{x^* + \lambda(\bar{x} - x^*) \mid -\infty < \lambda < +\infty\}$$

is lower semicontinuous at $\bar{x}$. Let

$$M = \{u \mid u(\bar{x} - x^*) = 0\},$$

i.e., M is the $N - 1$ dimensional subspace of R^N orthogonal to L. Define

$$\begin{aligned} A_0(x) &= M \qquad \text{if } x \in L \\ &= \varnothing \qquad \text{if } x \notin L \\ A_1(x) &= (A + A_0)(x) = A_0(x) + M \qquad \text{if } x \in L \\ &\qquad\qquad\qquad\quad = \varnothing \qquad \text{if } x \notin L. \end{aligned}$$

The multifunction A_0 is trivially maximal monotone. Therefore, A_1 is maximal monotone, because $A_1 = A + A_0$ and $D(A_0) \cap \text{int } D(A) \neq \varnothing$ [9]. In particular $A_1(x)$ is closed for each $x \in L$. Note that since M is $N - 1$ dimensional, $A_1(x)$ actually has a very simple structure:

$$A_1(x) = (A(x) \cap L) + M \qquad \text{for } x \in L$$

(equivalent to a *one-dimensional* maximal monotone multifunction). Also $y^* \in A_1(x^*)$, and the function

$$\Phi_1(x) = \inf\{(x - x^*) \cdot (y - y^*) \mid y \in A_1(x)\} \tag{12}$$

coincides on L with $\Phi(x)$. One sees easily from the "one-dimensional" nature of A_1 that, relative to L, Φ_1 is lower semicontinuous and the inf in (12) is always attained. Therefore, Φ is lower semicontinuous relative to L and (inasmuch as $x - x^* \perp M$ for $x \in L$) the inf in (5) is attained for all $x \in L$. This finishes the argument that Φ is globally lower semicontinuous.

Proof of Theorem 1. (a) is obvious from (10) and the lower semicontinuity of Φ: if $\Phi(x) > 0$ for all $x \neq x^*$, then $\theta(s) > 0$ for all $s > 0$. For (b), we make use merely of the lower semicontinuity of

$$\Psi(k', Q') = \Phi(k^* + k', Q^* + Q') + \rho k' \cdot Q'.$$

We have $\Psi(k', Q')$ positive by hypothesis except at the origin, where it vanishes. The hypothesis that

$$\liminf_{t \to \infty} \Psi(tk', tQ') > 0$$

implies by a simple compactness argument that actually for some $r > 0$ and $\bar{\delta} > 0$

$$\Psi(k', Q') \leqslant \bar{\delta} \Rightarrow \|(k', Q')\| \leqslant r.$$

Therefore, for any $\epsilon > 0$,

$$\inf\{\Psi(k', Q') \mid \| k' \| \geqslant \epsilon\} \geqslant \min\{\bar{\delta}, \delta_\epsilon\} > 0$$

where

$$\delta_\epsilon = \min\{\Psi(k', Q') \mid \| k', Q' \| \leqslant r, \| k' \| \geqslant \epsilon\}.$$

This is the desired conclusion.

References

1. D. Cass and K. Shell, "The Structure of Competitive Dynamical Systems," *J. Econ. Theory Symposium.*
2. R. T. Rockafellar, "Convex Analysis," Princeton Univ. Press, Princeton, N.J., 1970.
3. R. T. Rockafellar, Saddlepoints of Hamiltonian systems in convex problems of Lagrange, *J. Optimization Theory Appl.* **12** (1973), 367–390.
4. R. T. Rockafellar, "Saddlepoints of Hamiltonian Systems in Convex Lagrange Problems having a Positive Discount Rate," *J. Econ. Theory Symposium.*
5. W. Brock and J. Scheinkman, "Global Asymptotic Stability of Optimal Control Systems with Applications to the Theory of Economic Growth," *J. Econ. Theory Symposium.*
6. R. Gaines, Existence of solutions to Hamiltonian dynamical systems of optimal growth, to appear.

7. R. T. Rockafellar, Monotone operators associated with saddle-functions and minimax problems, *in* "Nonlinear Functional Analysis, Part 1," (F. E. Browder (Ed.), pp. 241–250, Proc. of Symposia in Pure Math., Vol. 18, Amer. Math. Soc., Providence, R.I., 1970.
8. R. T. Rockafellar, Local boundedness of nonlinear monotone operators, *Michigan Math. J.* **16** (1969), 397–407.
9. R. T. Rockafellar, Maximality of sums of nonlinear monotone operators, *Trans. Amer. Math. Soc.* **149** (1970), 75–88.

Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium.