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DUALITY IN OPTIMAL CONTROL

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For many kinds of optimization problems, convexity properties are very important, and when they are present in a thorough form they lead to an interesting kind of duality. This duality is sometimes useful in methods of computation, but it also has theoretical applications, such as in the analysis of economic models where dual variables can be interpreted as prices. The study of duality, even though it may pertain to a special subclass of problems often aids in the general development of a subject by suggesting alternative ways of looking at things.

In the classical calculus of variations, convexity and duality first enter the picture in the correspondence between Lagrangian and Hamiltonian functions and in the way this is connected with necessary conditions and the existence of solutions. Expressed in terms of the Hamiltonian, the optimality conditions for an arc x pair it with an "adjoint" arc p. The pairing carries over to problems of optimal control via the maximum principle. Duality theory in this context aims at uncovering and analyzing cases where p happens to solve a dual problem for which x is in turn the adjoint arc. But although this is the principal motivation, a number of side issues have to be explored along the way, and these suggest new approaches even to problems where duality is not at stake.

1. Implicit constraints

The effects that the aim of developing duality can have on one's point of view

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are seen immediately even in the formulation of the problem. Ordinarily, an optimal control problem for an arc x involves systems of constraints of various types. If the objective is to pass to a dual problem of similar type for an arc p, a means must be found for dualizing the constraint structure. The more details that are built into the model, the more there is to dualize, and by the time every possibility is covered in a symmetric fashion the framework may be impossibly cumbersome. It is here that the idea of representing constraints abstractly by infinite penalties has its origin.

To introduce the idea in a more elementary setting, consider first the problem of minimizing a function $F_0(z)$ over all $z \in C \subset R^N$, where F_0 is a real-valued function. The set C could be described by conditions of various kinds, for instance as the set of points satisfying equations or inequalities, but at the moment we need not be concerned with that. The point is that the problem can be represented notationally in terms of minimizing a certain *extended*-real-valued function F over the *whole* space R^N, namely

$$F(z) = \begin{cases} F_0(z) & \text{if } z \in C, \\ +\infty & \text{if } z \notin C. \end{cases} \tag{1}$$

Indeed, if $C \neq \emptyset$ the only points of interest in minimizing F are those in C, where F agrees with F_0. The case where $C = \emptyset$ (that is the problem has no "feasible solutions") corresponds to $\min F = +\infty$.

What functions $F : R^N \to \overline{R}$ (where $\overline{R} = R \cup \{\pm\infty\}$) are of the form (1) for some *nonempty* C and real-valued F_0? They are, of course, the ones such that $F(z) > -\infty$ for all $z \in R^N$ and $F(z) < \infty$ for at least one $z \in R^N$. Such a function on R^N will be termed "*proper*".

Although topological properties of F clearly must be essential in any discussion of minimization, continuity would generally be too much to ask for, if for no other reason than because jumps to $+\infty$ are allowed at the boundary of C. A more appropriate concept is *lower semicontinuity* (l.s.c.), where the level sets of the form $\{z \in R^N \mid F(z) \leq \alpha\}$ are all required to be closed, or *inf-compactness*, where the sets in question are compact. Inf-compactness implies that F attains its minimum. Note that F is inf-compact in particular if it is of the form (1) with C compact and F_0 continuous relative to C. But F can also be l.s.c., or even inf-compact, without its effective domain $C = \{z \in R^N \mid F(z) < \infty\}$ necessarily being closed. An example in one dimension is

$$F(z) = \begin{cases} \sec z & \text{if } -\pi/2 < z < \pi/2 , \\ +\infty & \text{otherwise.} \end{cases}$$

Geometrically, lower semicontinuity is equivalent to the closedness of the *epigraph* of F , which is the set

$$\text{epi } F = \{(z, \alpha) \in R^N \times R \mid \alpha \geq F(z)\} .$$

The projection of this set on R^N is C , but of course the projection of a closed set is not always closed, as the example shows.

These observations may be summarized by saying that the constrained minimization problems in R^N which are "reasonable" can be identified abstractly with the functions $F : R^N \mapsto \overline{R}$ which are proper and lower semicontinuous. The constraints are implicit in the condition $F(z) < \infty$.

2. Representation of a control example

A typical problem in optimal control might have the form: minimize

$$\int_{t_0}^{t_1} f_0(t, x(t), u(t))dt + l_0(x(t_0), x(t_1)) \tag{2}$$

subject to

$$\begin{gathered} \dot{x}(t) = f(t, x(t), u(t)) , \quad u(t) \in U(t) , \\ x(t) \in X(t) , \quad (x(t_0), x(t_1)) \in E , \end{gathered} \tag{3}$$

where $X(t) \subset R^n$, $U(t) \subset R^m$ and $E \subset R^n \times R^n$ (these sets may be given by explicit constraints), and x and u range over certain function spaces X and U over the fixed interval $[t_0, t_1]$. Setting aside temporarily the issue of measurability with respect to t , let us see how the problem could be represented using the idea of implicit constraints as above, but in a somewhat more subtle fashion. For (t, x, v, u) in $[t_0, t_1] \times R^n \times R^n \times R^m$, define

$$K(t, x, v, u) = \begin{cases} f_0(t, x, u) & \text{if } x \in X(t) , \; u \in U(t) \text{ and } f(t, x, u) = v , \\ +\infty & \text{otherwise,} \end{cases} \tag{4}$$

and for (x_0, x_1) in $R^n \times R^n$ define

$$(5)\qquad l(x_0, x_1) = \begin{cases} l_0(x_0, x_1) & \text{if } (x_0, x_1) \in E , \\ +\infty & \text{otherwise.} \end{cases}$$

It will be argued that the stated problem can be identified with that of minimizing the functional

$$(6)\qquad J(x, u) = \int_{t_0}^{t_1} K(t, x(t), \dot{x}(t), u(t))dt + l(x(t_0), x(t_1))$$

over all $x \in X$ and $u \in U$. Certain conventions must, however be adopted in the definition of J.

One source of difficulty in the definition is that the expression $k(t) = K(t, x(t), \dot{x}(t), u(t))$ needs to be measurable in t, and this will be discussed below. But even if it is measurable, it might not be summable (finitely integrable) in the usual sense. Of course, if $k(t) \geq \beta(t)$ for a summable function β the integral has a well defined classical value which is either finite or $+\infty$. Likewise, if $k(t) \leq \alpha(t)$ for a summable function α the integral is either finite or $-\infty$. The only truly ambiguous case is the one where neither of these alternatives holds, and then we adopt in (6) the convention that the integral is $+\infty$ (if the need ever arises). This convention is equivalent to saying that in the formula $\int k = \int k^+ + \int k^-$, where k^+ and k^- are the positive and negative parts of k, the case $\infty - \infty$, if it occurs, should be resolved as $+\infty$. The latter rule is also the one we adopt in (6) if the integral is $-\infty$ but $l(x(t_0), x(t_1)) = +\infty$.

Under these conventions, it is clear that

$$(7)\qquad J(x, u) < \infty \Rightarrow \begin{cases} K(t, x(t), \dot{x}(t), u(t)) < \infty & \text{almost everywhere in } t , \\ l(x(t_0), x(t_1)) < \infty , \end{cases}$$

and hence the constraints (3) are satisfied (for almost every $t \in [t_0, t_1]$, still assuming measurability). Moreover $J(x, u)$ then reduces to the expression (2), so the problem is represented as claimed.

The approach we shall follow is to treat control problems in the framework of minimizing functionals of the form (6) for K and l of an appropriate general class. The interval $[t_0, t_1]$ will be fixed, but this is not an important restriction, since problems with variable time intervals can usually be recast in this form by a change of parameters. A fixed time interval is needed partly in order that the function spaces X and U over which the minimization takes place have a linear structure, as is prerequisite to the discussion of convexity. In fact, X will be taken to be the space of all absolutely continuous functions and U the space of all Lebesgue measurable functions.

3. Measurability

One of the tasks before us is to delineate a good class of functions K to use in (6). An essential property is that the Lebesgue measurability of the integrand should follow from that of $x(t)$, $\dot{x}(t)$ and $u(t)$. But to be useful, the conditions on K must be readily verifiable in terms of natural assumptions on the underlying data, for instance on f_0, f, X and U in the case of K given by (4). Furthermore, the conditions must be technically robust, in the sense of being easy to handle and preserved under the constructions and transformations that the theory will require.

Fortunately there is a simple and natural answer to the question of what conditions to impose. It has developed in recent years in close relation to the theory of measurable selections and is centered on the notion of a "normal integrand". An exposition in detail may be found in [29], and we shall limit ourselves here to quoting a few pertinent facts.

To save notation, the interval $[t_0, t_1]$ will be denoted by T. A *normal integrand* on $T \times R^N$ is a function $F : T \times R^N \mapsto \overline{R}$ such that $F(t, z)$ is lower semicontinuous in z for fixed t and measurable in (t, z) with respect to the σ-algebra generated by products of Lebesgue sets in T and Borel sets in R^N. The latter property implies in particular that $F(t, z(t))$ is Lebesgue measurable in t when $z(t)$ is. (This would be false for $F(t, z)$ merely Lebesgue measurable in (t, z). It would be true of course for $F(t, z)$ Borel measurable in (t, z), but Borel measurability turns out not to be preserved by some of the operations we will need to perform.) In particular, if $F(t, z) \equiv F_0(z)$, where F_0 is lower semicontinuous, then F is normal.

A normal integrand F is *proper* if $F(t, z)$ is a proper function of z (in the sense of §1) for every $t \in T$. Such an integrand may be construed as representing the kind of structure inherent in a "reasonable" constrained minimization problem, but with "measurable" dependence on the parameter t.

A *Carathéodory integrand* is a finite function F on $[0, T] \times R^N$ such that $F(t, z)$ is continuous in z and Lebesgue measurable in t. This is a classical notion, of which the present one may be viewed as a natural "one-sided" extension. It can be shown that F is a Carathéodory integrand if and only if both F and $-F$ are proper normal integrands. The pointwise supremum of a countable family of Carathéodory integrands is normal, although not necessarily finite or continuous everywhere.

The connection with measurable multifunctions is very important. A *multifunction*

$\Gamma : T \to R^N$ assigns to each $t \in T$ a set $\Gamma(t) \subset R^N$ (possibly empty), and it is *closed-valued* if $\Gamma(t)$ is always closed. A closed-valued multifunction is said to be *measurable* if for every closed $C \subset R^N$ the set

$$\Gamma^{-1}(C) = \{t \in T \mid \Gamma(t) \cap C \neq \emptyset\}$$

is Lebesgue measurable. If Γ is single-valued $(\Gamma(t)$ is a singleton for every $t)$, this reduces to the usual concept for functions.

The main fact is that Γ is a closed-valued measurable multifunction if and only if it has a Castaing representation, that is the set $D = \{t \in T \mid \Gamma(t) \neq \emptyset\}$ is Lebesgue measurable and there is a countable collection $\{z_i\}_{i \in I}$ of Lebesgue measurable functions $z_i : D \mapsto R^N$ such that

$$\Gamma(t) = \mathrm{cl}\{z_i(t) \mid i \in I\} \quad \text{for every} \quad t \in D .$$

As a corollary, one has a fundamental theorem on *measurable selections*: if $\Gamma : T \to R^N$ is closed-valued and measurable, then the set D above is Lebesgue measurable and there is a Lebesgue measurable function $z : D \to R^N$ such that $z(t) \in \Gamma(t)$ for all $t \in D$. (This is not the most general selection theorem, but it covers a vast number of applications; for a survey of selection theory, see [32].)

It happens that a function $F : T \times R^N \to \overline{R}$ is a normal integrand if and only if its epigraph multifunction

$$t \mapsto \mathrm{epi}\, F(t, \cdot) = \{(z, \alpha) \in R^{N+1} \mid \alpha \geq F(t, z)\}$$

is closed-valued and measurable. (This property is used as the *definition* of normality in the general theory where T is replaced by an arbitrary measurable space.) On the other hand, a multifunction $\Gamma : T \mapsto R^N$ is closed-valued and measurable if and only if its *indicator integrand*

$$F(t, z) = \begin{cases} 0 & \text{if } z \in \Gamma(t) , \\ \infty & \text{if } z \notin \Gamma(t) , \end{cases} \tag{8}$$

is normal.

Normality has been established for all integrands of the general form

$$F(t, z) = \begin{cases} F_0(t, z) & \text{if } F_i(t, z) \leq c_i(t) \text{ for all } i \in I , \\ +\infty & \text{otherwise,} \end{cases}$$

where I is a countable (or finite or empty) index set, c_i is Lebesgue measurable, and F_0 and each F_i is a normal integrand (for example, a Carathéodory integrand).

Taking $F_0 \equiv 0$, one gets an indicator as in (8) and can conclude that a certain multifunction described by explicit constraints is measurable. For further examples and details, see [29].

4. Control model

Some basic assumptions that will remain in force may now be stated.

ASSUMPTION 1. *K is a proper normal integrand on $T \times (R^n \times R^n \times R^m)$.*

ASSUMPTION 2. *l is a proper lower semicontinuous function on $R^n \times R^n$.*

Assumption 1 implies in particular that $K(t, x(t), v(t), u(t))$ is Lebesgue measurable in t when $x(t)$, $v(t)$ and $u(t)$ are. Let A be the space of absolutely continuous functions $x : T \to R^n$, and let L be the space of Lebesgue measurable functions $u : T \to R^m$. For $x \in A$ the derivative $\dot{x}(t)$ exists almost everywhere and is Lebesgue measurable. Hence for every $x \in A$ and $u \in L$ the functional J in (6) is well-defined under the conventions for $\pm\infty$ explained in §1. The problem to be studied is

(Q) $$\text{minimize } J(x, u) \text{ over all } x \in A, \ u \in L .$$

For this problem, (7) holds, and this means in terms of the sets

(9) $$\begin{aligned} D(t, x, u) &\triangleq \{v \in R^n \mid K(t, x, v, u) < \infty\} , \\ U(t, x) &\triangleq \{u \in R^m \mid D(t, x, u) \neq \emptyset\} , \\ X(t) &\triangleq \{x \in R^n \mid U(t, x) \neq \emptyset\} , \\ E &\triangleq \left\{(x_0, x_1) \in R^n \times R^n \mid l(x_0, x_1) < \infty\right\} \end{aligned}$$

that one has the implicit constraints

(10) $$\begin{aligned} &\dot{x}(t) \in D(t, x(t), u(t)) && \text{almost everywhere,} \\ &u(t) \in U(t, x(t)) && \text{almost everywhere,} \\ &x(t) \in X(t) && \text{almost everywhere,} \\ &(x(t_0), x(t_1)) \in E . \end{aligned}$$

If these are not satisfied by any $x \in A$ and $u \in L$, then the minimum in (Q) is attained but is $+\infty$. Of course, x is interpreted as the *state trajectory* for a system being modelled, and u is the control.

In the example in §2, what assumptions suffice for the corresponding K and l to fit the conditions above for (Q)? If E is a nonempty closed set and l_0 is continuous (finite) on E , then l is certainly proper and lower semicontinuous. If the multifunctions $t \mapsto X(t)$ and $t \mapsto U(t)$ are nonempty-closed-valued and measurable, and if $f_0(t, x, u)$ and $f(t, x, u)$ are continuous in (x, u) and Lebesgue measurable in t , then K in (4) is a proper normal integrand. The latter follows from the normality criteria furnished in §3 and the elementary fact that the sum of proper normal integrands is normal. (The equation $v - f(t, x, u) = 0$ can be expressed by a finite number of constraints $f_i(t, x, v, u) \le 0$ with f_i a Carathéodory integrand.)

The optimal control problem (Q) is said to be of *convex* type if $K(t, x, v, u)$ is convex in (x, v, u) and $l(x_0, x_1)$ is convex in (x_0, x_1) . (A function $F : R^N \to \overline{R}$ is *convex* if its epigraph is a convex set, or equivalently, if the inequality $F((1-\lambda)z_0+\lambda z_1) \le (1-\lambda)F(z_0) + \lambda F(z_1)$ holds for all $z_0 \in R^N$, $z_1 \in R^N$ and $\lambda \in (0, 1)$ under the obvious conventions for manipulating $\pm\infty$ and, if necessary, the special rule $\infty - \infty = \infty$.) If (Q) is of convex type, then J is a convex functional on the space $A \times L$, as can easily be verified. This case will be especially important for the theory of duality.

A problem of convex type that will serve nicely to illustrate the theory at several stages is

$$(Q_0) \quad \begin{aligned} &\text{minimize} \int_T f(t, C(t)x(t))dt + \int_T g(t, u(t))dt + l(x(t_0), x(t_1)) \\ &\text{subject to } \dot{x}(t) = A(t)x(t) + B(t)u(t) \text{ almost everywhere,} \end{aligned}$$

where f and g are *convex*, proper, normal integrands (that is the functions $f(t, \cdot)$ and $g(t, \cdot)$ are convex - we are never interested in convexity with respect to t), l is convex, proper, lower semicontinuous, and the elements of the matrices $A(t)$, $B(t)$ and $C(t)$ depend Lebesgue measurably on t . This corresponds to

$$(11) \qquad K(t, x, v, u) = \begin{cases} f(t, C(t)x) + g(t, u) & \text{if } v = A(t)x + B(t)u \ , \\ +\infty & \text{otherwise.} \end{cases}$$

It is not hard to show that K is a convex normal integrand; to ensure that K is proper, we assume for simplicity that $f(t, 0) < \infty$ for all t . The vector $y(t) = C(t)x(t)$ might be interpreted in some cases as the "observation" associated with the state $x(t)$.

One special case we shall refer to is

$$(12) \qquad f(t, y) \equiv 0 \ , \quad g(t, u) = \begin{cases} 0 & \text{if } \|u\| \le 1 \ , \\ \infty & \text{if } \|u\| > 1 \ , \end{cases}$$

where $\|\cdot\|$ denotes an arbitrary norm on R^m . Then (Q_0) consists of minimizing

$l(x(t_0), x(t_1))$ subject to $\|u(t)\| \le 1$ for almost every t and $\dot{x} = Ax + Bu$. Another case is

(13) $$f(t, y) = \tfrac{1}{2}y\cdot S(t)y\ , \quad g(t, u) = \tfrac{1}{2}u\cdot R(t)u\ ,$$

where $S(t)$ and $R(t)$ are positive semidefinite matrices depending Lebesgue measurably on t. (Then f and g are Carathéodory integrands.) Note that the first integrand in (Q_0) is then $\tfrac{1}{2}x(t)\cdot Q(t)x(t)$ where $Q = C^*SC$, C^* being the transpose of C. Any positive *semidefinite* (symmetric) Q can be written in this form for some C and positive *definite* S (which are elementary to construct without resorting to eigenvectors or the like).

For the boundary function l, a simple case where it is lower semicontinuous proper convex is

(14) $$l(x_0, x_1) = \begin{cases} 0 & \text{if } x_0 = a_0\ ,\ x_1 = a_1\ , \\ \infty & \text{if } x_0 \ne a_0 \text{ or } x_1 \ne a_1\ , \end{cases}$$

where a_0 and a_1 are two given points in R^n. This corresponds to the implicit fixed endpoint constraint $x(t_0) = a_0$, $x(t_1) = a_1$. A case involving endpoints which are not fixed, yet mutually related, is

(15) $$l(x_0, x_1) = \begin{cases} 0 & \text{if } x_0 = x_1\ , \\ \infty & \text{if } x_0 \ne x_1\ . \end{cases}$$

Then $x(t_0)$ can be arbitrary, but $x(t_1) = x(t_0)$. A mixed example is

(16) $$l(x_0, x_1) = \begin{cases} \tfrac{1}{2}|x_1 - a_1|^2 & \text{if } x_0 \in E_0\ , \\ \infty & \text{otherwise}, \end{cases}$$

where E_0 is a nonempty closed convex set (reducing perhaps to a single point a_0) and a_1 is a given point in R^n. Then $x(t_0)$ must lie in E_0.

5. Reduced problem

For some purposes, it is useful to know that the problem (Q) can be reduced to another form where the control u does not appear explicitly. This is a good approach in proving the existence of solutions and in drawing parallels with the classical calculus of variations. Also, much of the general duality theory applies mainly to the state trajectory $x(t)$ and an adjoint trajectory $p(t)$, although in special cases like (Q_0) it will turn out that there are natural dual controls $w(t)$ to single out for association with $p(t)$.

Starting from the fact that the optimal value in (Q) can be expressed as

$$\inf(Q) = \inf_{x\in A}\left\{l\left(x(t_0),\, x(t_1)\right) + \inf_{u\in L}\int_T K\left(t,\, x(t),\, \dot{x}(t),\, u(t)\right)dt\right\}, \tag{17}$$

we are led to ask whether the minimization over $u \in L$ can be executed simply by choosing for each t a point $u(t) \in \Gamma(t)$, where

$$\Gamma(t) = \arg\min K\left(t,\, x(t),\, \dot{x}(t),\, \cdot\right). \tag{18}$$

Of course, for this to be true the minimizing set $\Gamma(t)$ must be nonempty for almost every t, but there is also an important question of measurability. How do we know we can select $u(t) \in \Gamma(t)$ in such a way that the function u belongs to the space L? More generally, apart from whether the minimum is attained, there is the question of conditions under which the equation

$$\inf_{u\in L}\int_T F\left(t,\, u(t)\right)dt = \int_T \left[\inf_{u\in R^n} F(t,\, u)\right]dt \tag{19}$$

is valid, specifically when $F\left(t,\, u(t)\right) = K\left(t,\, x(t),\, \dot{x}(t),\, u(t)\right)$.

It is demonstrated in [29, §3] that (19) is true for any normal integrand F, the function

$$t \mapsto \inf_{u\in R^m} F(t,\, u)$$

and the multifunction

$$t \mapsto \arg\min_{u\in R^m} F(t,\, u)$$

always being measurable. To the extent that a measurable multifunction is nonempty-valued, it has a measurable selection, as noted in §3 above. The chain of facts needed here is completed by the result in [29] that for $F(t,\, \cdot) = K\left(t,\, x(t),\, v(t),\, \cdot\right)$, the normality of F follows from that of K and the measurability of $x(t)$ and $v(t)$. (In the case of $v(t) = \dot{x}(t)$, there is a minor difficulty with the fact that $\dot{x}(t)$ may be undefined on a certain set of measure zero. This technicality can be handled by supplying an arbitrary definition over that set or by passing to a subset of T of full measure. It causes no real trouble and, for simplicity of exposition, it will be ignored wherever it crops up.)

It follows that for every $x \in A$ the functional

(20) $$\Phi(x) = \int_T L(t, x(t), \dot{x}(t))dt + l(x(t_0), x(t_1))$$

is well defined, where

(21) $$L(t, x, v) = \inf_{u \in R^m} K(t, x, v, u)$$

and moreover

(22) $$\inf_{u \in L} J(x, u) = \Phi(x) ,$$

where the infimum (if not $-\infty$) is attained by u if and only if u is a measurable selection (almost everywhere) for the multifunction (18). The *reduced problem* associated with (Q) is

(P) $$\text{minimize } \Phi(x) \text{ over all } x \in A ,$$

and L is called the *Lagrangian*. The main conclusion is thus the following

REDUCTION THEOREM. *It is always true that* $\inf(Q) = \inf(P)$. *A pair* $(x, u) \in A \times L$ *solves* (Q) *if and only if* x *solves* (P) *and* u *is a measurable selection (almost everywhere) for the multifunction* (18). *In particular, such a selection always exists if* $K(t, x, v, u)$ *is* inf-*compact in* $u \in R^m$ *for every* (t, x, v) *in* $T \times R^n \times R^n$.

This result demonstrates that one can focus all attention temporarily on x , if this is convenient, and pull the control u out of the hat at the last moment. Note that K is not uniquely determined by L , and indeed, the reduced problem (P) may arise from many different control problems (Q), corresponding to different ways of parameterizing the dynamics. In particular, any problem of the form (P) can be regarded as a problem (Q) where u does not actually appear (the control space is zero-dimensional). There is interest therefore in working directly with L , without reference to any particular K , and the basic properties assumed for L must be specified directly. It is obvious that these should be as follows.

ASSUMPTION 3. *L is a proper normal integrand on* $T \times (R^n \times R^n)$.

If L arises from a normal integrand K as in (21), then L is normal if $L(t, x, v)$ is lower semicontinuous in (x, v) . This is shown by [29, Proposition 2R]. One criterion under which the proper normality of L is just a consequence of the proper normality of K is a sort of *uniform* inf-compactness of $K(t, x, v, u)$ in u : for each $t \in T$, $\alpha \in R$, and bounded set $B \subset R^n \times R^n$, the set

$$\{u \in R^m \mid \exists(x, v) \in B \text{ with } K(t, x, v, u) \leq \alpha\}$$

is bounded.

The problem (P) is said to be of *convex* type if $L(t, x, v)$ is convex in (x, v) and $l(x_0, x_1)$ is convex in (x_0, x_1). Then Φ is a convex functional on A. The convexity of K implies the convexity of L in (21), so (P) is of convex type when (Q) is of convex type.

This holds in particular for the convex control problem (Q_0), where

$$L(t, x, v) = F(t, C(t)x) + \inf_u \{g(t, u) \mid B(t)u = v - A(t)x\} . \tag{23}$$

Formula (23) uses the convention that the infimum of an empty set of real numbers is $+\infty$. The lower semicontinuity of L in (x, v) (and hence normality) follows in this case from something simpler than the "uniform inf-compactness" condition just mentioned. It suffices to have $g(t, u)$ inf-compact in u for each t.

6. Hamiltonian Function

Associated with the Lagrangian L on $T \times R^n \times R^n$ is another function H on $T \times R^n \times R^n$ which will be called the *Hamiltonian* for (P). It is defined by

$$H(t, x, p) = \sup_{v \in R^n} \{p \cdot v - L(t, x, v)\} . \tag{24}$$

The Hamiltonian plays an extremely important role in many phases of variational theory, and the correspondence between Hamiltonians and Lagrangians furnishes a preliminary case of the kind of duality we aim at exploring more deeply.

Some insight into the definition of H and its classical ramifications can be gained by seeing how the formula might be applied if $L(t, x, v)$ happened to be differentiable in v. Setting the gradient of the expression to be maximized with respect to v equal to 0, one obtains the condition $p = \nabla_v L(t, x, v)$ as necessary for v to give the maximum for a particular choice of t, x and p. Suppose this can be solved for v as a function: $v = V(t, x, p)$. Then

$$H(t, x, p) = p \cdot V(t, x, p) - L(t, x, V(t, x, p)) .$$

This procedure for passing from a function of v to one of p is called the *Legendre transformation*, and it is the one used in defining the Hamiltonian in the classical calculus of variations. However, it is unsatisfactory in several respects even in that framework: very strong assumptions are needed to ensure that $V(t, x, p)$ is well defined even in a local sense, and there are many technical troubles caused by the vagueness of what the true domain of H is, and the extent to which the transformation is invertible. To put this approach in a truly rigorous and suitable global form, it would be necessary to assume that $L(t, x, v)$ was not only differentiable everywhere in v, but strictly convex and subject to a certain global growth condition (coercivity). Such restrictions would be severe and, of course, would exclude most of the cases we are interested in here.

Fortunately, there is a modern alternative to the Legendre transformation which has the vigor and generality we desire. It was introduced by Fenchel [15] in 1949 and has since become a fundamental tool in convex analysis (see [20]). For any function $F : R^N \mapsto \overline{R}$, the *Fenchel transform* of F is the function $F^* : R^N \mapsto \overline{R}$ defined by

$$F^*(w) = \sup_{z \in R^N} \{w \cdot z - F(z)\} .$$

The Fenchel transform of F^* is in turn

$$F^{**}(z) = \sup_{w \in R^N} \{w \cdot z - F^*(w)\} .$$

It turns out that F^* and F^{**} are always convex and lower semicontinuous, and F^{**} is the *closed convex hull* of F in the following sense: if F majorizes at least one affine (linear-plus-a-constant) function, then the epigraph of F^{**} is the smallest closed convex set containing the epigraph of F; otherwise $F^{**} \equiv -\infty$. In fact if F is lower semicontinuous proper convex, then so is F^*, and $F^{**} = F$. The functions F and F^* are then said to be *conjugate* to each other. (It is also true that $F^{**} = F$ when $F \equiv +\infty$; then $F^* \equiv -\infty$.) One always has $F^{***} = F^*$, so F^* and F^{**} are always conjugate to each other.

Geometrically, the conjugate F^* of a lower semicontinuous proper convex function F amounts to a dual description of the epigraph of F as the intersection of a collection of nonvertical closed half-spaces in R^{N+1}.

These facts can be applied at once to the definition of the Hamiltonian. The formula expresses $H(t, x, \cdot)$ as the Fenchel transform of $L(t, x, \cdot)$. Therefore

$$\sup_{p \in R^n} \{p \cdot v - H(t, x, v)\} = \tilde{L}(t, x, v) ,$$

where $\tilde{L}$ is defined by taking the closed convex hull of $L(t, x, v)$ in v (in the special sense above) for each t, x. The Hamiltonian associated with $\tilde{L}$ is again H. The following result is then obtained from Assumption 3 and other facts of convex analysis.

HAMILTONIAN/LAGRANGIAN THEOREM. *The Hamiltonian* $H(t, x, p)$ *is always lower semicontinuous convex in* p *, and the inverse formula*

$$L(t, x, v) = \sup_{p \in R^n} \{p \cdot v - H(t, x, p)\}$$

holds if and only if the Lagrangian $L(t, x, v)$ *is convex in* v *. In the latter case, the stronger property that* L *is convex in* (x, v) *is equivalent to* H *also being concave in* x *.*

In particular, there is a *one-to-one* correspondence between Lagrangians L which are proper normal integrands, convex in the v argument, and certain functions H. Every property of such a function L is therefore dual, in principle, to some property of the associated H, and the theorem illustrates this in the case of the property of joint convexity in x and v.

When L arises from a control problem (Q) as in (21), the Hamiltonian can be expressed directly in terms of K by

$$H(t, x, p) = \sup_{\substack{v \in R^n \\ u \in R^m}} \{p \cdot v - K(t, x, v, u)\} . \tag{26}$$

Thus for the control example in §2 the Hamiltonian is

$$H(t, x, p) = \begin{cases} \sup_{u \in U(t)} \{p \cdot f(t, x, u) - f_0(t, x, u)\} & \text{if } x \in X(t) , \\ -\infty & \text{if } x \notin X(t) . \end{cases} \tag{27}$$

(Note the coefficient -1 for f_0. In much of the literature on optimal control, a variable coefficient p_0 is allowed, although necessary conditions are derived showing that p_0 must be constant and can be taken as either -1 or 0.) For the convex model (Q_0) where K is given by (11), the Hamiltonian is

$$H(t, x, p) = p \cdot A(t)x - f(t, C(t)x) + g^*(t, B^*(t)p) \quad \text{with } \infty - \infty = -\infty , \tag{28}$$

where $g^*(t, \cdot)$ is the convex function conjugate to $g(t, \cdot)$ for each t. (The fact that the convention $\infty - \infty = -\infty$ is needed in (28), rather than $\infty - \infty = \infty$, should serve as a warning that such conventions must be tied to specific situations and not taken for granted.)

Formulas (27) and (28) illustrate the general fact that

$$H(t, x, p) = -\infty \Leftrightarrow x \notin X(t) , \tag{29}$$

where $X(t)$ is the implicit state constraint set in (P),

$$X(t) = \{x \in R^n \mid \exists v \in R^n \text{ with } L(t, x, v) < \infty\} . \tag{30}$$

7. Existence of Solutions

We shall come in due course to the importance of the Hamiltonian in conditions for optimality, but a few comments about its role in existence theory may now be in order. To prove the existence of a solution to (P), one needs to establish some kind of inf-compactness, or at least lower semicontinuity property of the functional Φ on the space A. Several things are involved in this, but one minimal requirement is that L should be *coercive* in v: for each (t, x), the function $L(t, x, \cdot)$ ought to be bounded below and have

$$\liminf_{|v|\to\infty} L(t, x, v)/|v| = \infty .$$

Equivalent to such coercivity is the property that for each t, x, p there exist $\beta \in R$ such that

$$L(t, x, v) \geq p.v - \beta \quad \text{for all} \quad v \in R^n .$$

But the latter inequality is equivalent by (24) to $H(t, x, p) \leq \beta$. Therefore $L(t, x, v)$ is coercive in v for each (t, x) if and only if $H(t, x, p) < \infty$ for all (t, x, p) .

The classical existence theorems, such as those of Tonelli and Nagumo, require coercivity of L in v which is uniform in x . A similar requirement appears, in effect, in modern treatments of optimal control problems such as in Cesari [6], although the results are expressed in terms of a detailed constraint structure, rather than the framework of extended-real-valued Lagrangians. Matters can be kept simpler by passing to a formulation in terms of H , and in this way a broader class of existence theorems can be obtained. Olech [19] was one of the first to approach the subject from this direction, although he did not define the Hamiltonian as such.

The *Hamiltonian upper boundedness condition* is satisfied if for each $p \in R^n$ and $\beta \in R$ there is a summable function $\theta : T \mapsto R$ such that

$$H(t, x, p) < \theta(t) \quad \text{for all} \quad t \in T \quad \text{when} \quad |x| \leq \beta .$$

In particular, then H is less than $+\infty$ everywhere. To state the main consequence of this property, we need to introduce the Banach space C , consisting of all *continuous* R^n-valued functions over T , and its norm

$$\|x\|_{\mathsf{C}} = \max_{t \in T} |x(t)| .$$

The space A of absolutely continuous functions is, of course, contained in C , and is a Banach space itself under the norm

$$\|x\|_{\mathsf{A}} = |x(0)| + \int_T |\dot{x}(t)| dt . \tag{31}$$

INF-COMPACTNESS THEOREM. *Suppose that the Hamiltonian upper boundedness condition is satisfied and* $L(t, x, v)$ *is convex in* v . *Then for all real numbers* α *and* β *the set*

$$\{x \in \mathsf{A} \mid \Phi(x) \leq \alpha,\ \|x\|_{\mathsf{C}} \leq \beta\}$$

is compact, both in the weak topology of A *and the norm topology of* C .

This is proved in [27]. It leads immediately to a result on the existence of solutions to (P) in the case where the abstract state constraint set $X(t)$ (see (30), (9), (10)) is contained for all t in a fixed bounded region of R^n . How to obtain the existence of solutions in other cases is largely a matter of finding additional growth conditions on H and L which ensure that the level sets of Φ are bounded in the norm of C, and we shall not go into it here (see [27]).

The convexity condition in the theorem deserves more elaboration, however, since it is the first place in the theory that convexity appears in an essential way, and it seems related to the Lagrangian/Hamiltonian duality. A surprising fact of functional analysis, stemming from Liapunov's theorem on the convexity of the range of a vector-valued measure is that an integral functional of the form

$$I(v) = \int_T F(t, v(t))dt \ , \quad v \in L^1(T, R^n) \ ,$$

can hardly be weakly lower semicontinuous without being convex at the same time. Indeed, if one tries to take the weak closure of the epigraph of I one generally gets the epigraph of the corresponding integral functional for $F^{**}(t, \cdot)$, the convexification of $F(t, \cdot)$ described in §6 (see [29, §3] for a proof).

For functionals of the form Φ the situation is somewhat less clear, but convexity of $L(t, x, v)$ in v is crucial in much the same way. For instance, it can be shown under the Hamiltonian upper boundedness condition that any bounded sequence $\{x_k\}_{k=1}^{\infty}$ in A which is "asymptotically minimizing" for Φ (in a certain sense that will not be described here) has a subsequence converging in both the weak topology of A and the norm topology of C to an arc $x \in A$ which minimizes, not Φ, but the corresponding problem with L replaced by its convexification $\tilde{L}$ in the v argument (as defined in §6). This is called the *relaxed problem* $(\tilde{P})$, and $\tilde{L}$ is the *relaxed Lagrangian*.

The meaning of these facts is that, without the convexity of L in v, there is little motivation for studying (P), since it is likely to amount to a problem of minimizing something not possessed of a reasonable continuity property. One should look instead at $(\tilde{P})$ and its interpretation in whatever application may be at hand, since even from a computational point of view the best one could usually hope for is to generate a sequence $\{x_k\}_{k=1}^{\infty}$ converging to a solution to $(\tilde{P})$.

Other facts lend their weight to this point of view. For instance, the Weierstrass necessary condition for optimality in classical problems comes close to saying that a solution to (P) must be a solution to $(\tilde{P})$ along which the two Lagrangians L and $\tilde{L}$ happen to agree. Results of the latter sort have in fact been established for problems of optimal control under certain conditions; *cf.* Clarke [9], Warga [33].

Much can be said, therefore, in favor of compartmentalizing the theory into the study of (P) under the assumption of convexity in v on the one hand, and the study of the relationship between (P) and $(\tilde{P})$ without the assumption on the other. The second part, called *relaxation theory*, encompasses such important topics as "bang-bang" controls, as well as facts of the sort already mentioned. Whatever the merits of this philosophy, we shall follow it here in looking henceforth only at problems which are already "relaxed".

ASSUMPTION 4. *$L(t, x, v)$ is convex in v for every t, x, or in other words, $L = \tilde{L}$.*

Of course, in the main case we shall be concerned with, L will actually be convex jointly in x and v. But Assumption 4 will facilitate comparisons and conjectures having to do with more general problems.

8. Optimality conditions

One of the classical conditions for optimality of x in (P), whose necessity can be proved under certain assumptions when L and l are differentiable, is the Euler-Lagrange equation

$$\frac{d}{dt}\left[\nabla_v L(t, x(t), \dot{x}(t))\right] = \nabla_x L(t, x(t), \dot{x}(t)) .$$

This can also be expressed by asserting that for a certain function $p(t)$ one has

$$(\dot{p}(t), p(t)) = \nabla L(t, x(t), \dot{x}(t)) ,$$

where ∇L denotes the gradient of L with respect to (x, v). (As a general notational rule, we ignore t in the symbolism for gradients, conjugates, and so on, of integrands.) The corresponding condition for endpoints has the form

$$(p(t_0), -p(t_1)) = \nabla l(x(t_0), x(t_1)) . \tag{32}$$

The key to generalizing such equations to the nondifferentiable case dictated by the present model is an appropriate substitute for the notion of "gradient".

Such a notion is well known in the case of *convex* functions. If F is convex on R^N, the *subgradient* set $\partial F(z)$ is defined to consist of all $w \in R^N$ with the property that

$$F(z') \geq F(z) + w\cdot(z'-z) \quad \text{for all} \quad z' \in R^N . \tag{33}$$

If $F(z)$ is finite, this means that the graph of the affine function of z' on the right side of (33) is a supporting hyperplane to the epigraph of F at $(z, F(z))$. (If $F(z) = -\infty$, or if $F \equiv +\infty$, the condition is satisfied by every w, but if $F(z) = +\infty$ and $F \not\equiv +\infty$, it is not satisfied by any w.)

The theory of subgradients is presented in [20], and only a few basic facts will be cited here. The set $\partial F(z)$ is always closed and convex (possibly empty), and it reduces to a *single* element w if and only if F is differentiable at z (in which event $w = \nabla F(z)$). In the case of a lower semicontinuous proper convex function and its conjugate, satisfying

(34) $$F(z) + F^*(w) \geq z \cdot w \quad \text{for all } z, w ,$$

by the definition of conjugacy, there is the important, symmetric equivalence

(35) $$w \in \partial F(z) \Leftrightarrow F(z) + F^*(w) = z \cdot w \Leftrightarrow z \in \partial F^*(w) .$$

A special case worthy of note is the *indicator* of a nonempty closed convex set C :

(36) $$F(z) = \begin{cases} 0 & \text{if } z \in C , \\ \infty & \text{if } z \notin C . \end{cases}$$

Then

(37) $$\partial F(z) = N_C(z) = \textit{normal cone to } C \textit{ at } z ,$$

where

(38) $$N_C(z) = \begin{cases} \{w \in R^N \mid w \cdot (z' - z) \leq 0 \quad \text{for all } z' \in C\} & \text{if } z \in C , \\ \emptyset & \text{if } z \notin C . \end{cases}$$

For problems of convex type, we can work with the subgradient sets $\partial L(t, x, v)$ and $\partial l(x_0, x_1)$ in $R^N \times R^N$. The *Euler-Lagrange condition* is then

(39) $$(\dot{p}(t), p(t)) \in \partial L(t, x(t), \dot{x}(t)) \text{ , almost everywhere,}$$

and the *transversality condition* is

(40) $$(p(t_0), -p(t_1)) \in \partial l(x(t_0), x(t_1)) .$$

We are interested in the functions $x \in A$ which satisfy these for some $p \in A$, which is then said to be *adjoint* to x . (The adjoint arc is not necessarily unique.)

Just what these conditions, first introduced in [21], have to do with optimality in the problem (P) will be the subject of much discussion below. Before getting into that, however, we would like to mention that the definition of $\partial F(z)$ has been extended by Clarke [10], [16], to the case of arbitrary proper lower semicontinuous functions F in such a way as to coincide with the set above when F is convex and with the singleton $\{\nabla F(z)\}$ at points where F is strongly differentiable (not necessarily convex). Moreover $\partial F(z)$ is still always a closed convex set. The Euler-Lagrange condition (39) and transversality condition (40) are therefore well-defined for (P) even without any convexity assumptions. Indeed, Clarke has shown they are necessary for optimality in a number of cases [11], [13]. This more general theory falls outside of our target area of duality and will therefore not be outlined here.

Our discussion of necessity and sufficiency for optimality will be limited mainly to the convex case, where there is a reversal of the situation often encountered in variational theory: the sufficiency is the easy part.

SUFFICIENCY THEOREM. *If* (P) *is of convex type and* $x \in A$ *satisfies the Euler-Lagrange condition for* L *and transversality condition for* l *with adjoint* $p \in A$ *, then* x *furnishes the minimum in* (P).

The argument is so short and simple it will be given in full. Suppose (39) and (40) hold, and let x' be an arbitrary element of A (the prime has nothing to do with derivatives). From the definition of subgradients, we have

$$L\big(t, x'(t), \dot{x}'(t)\big) \geq L\big(t, x(t), \dot{x}(t)\big) + \dot{p}(t)\big(x'(t)-x(t)\big) + p(t)\big(\dot{x}'(t)-\dot{x}(t)\big)$$

for almost every t and

$$l\big(x'(t_0), x'(t_1)\big) \geq l\big(x(t_0), x(t_1)\big) + p(t_0)\big(x'(t_0)-x(t_0)\big) - p(t_1)\big(x'(t_1)-x(t_1)\big) .$$

Integrating the first inequality over $[t_0, t_1] = T$ and adding the second, we obtain

$$\Phi(x') \geq \Phi(x) + \int_{t_0}^{t_1} \frac{d}{dt}\,[p\cdot(x'-x)]dt - [p\cdot(x'-x)]_{t_0}^{t_1} ,$$

where the terms in $p\cdot(x'-x)$ cancel each other.

The necessity of the conditions requires stronger assumptions, as we shall see in §13, and certain extensions have to be made in order to handle the case where the state constraint $x(t) \in X(t)$ becomes effective.

For the moment we turn instead to the question of what the conditions mean for specific cases, such as the control problem (Q_0) in §4. One thing of great practical importance in this respect is that quite a "calculus" exists for determining the subgradients of convex functions which, like L and l , are likely to be given in terms of various other functions, sets, constraints, operations, and so on (see [20], [26]).

Suppose L comes via (21) from a function $K(t, x, v, u)$ which is convex in (x, v, u) . It is known that then

(41) $[(r, p) \in \partial L(t, x, v)$ and $u \in \arg\min K(t, x, v, \cdot)]$

$$\Longleftrightarrow (r, p, 0) \in \partial K(t, x, v, u)$$

(*cf.* [26, Theorem 24 (a)]). Now suppose further that K has the form (11), so that L is given by (23), and that $g(t, \cdot)$ is inf-compact for each t , so that L is normal (as noted at the end of §5). The "arg min" set is then always nonempty, so the calculation of ∂L is reduced by (41) to that of ∂K . Assuming for each t that $f(t, \cdot)$ is finite on a neighborhood of 0 (so as to handle the case where the range space for $C(t)$ might not be all of R^n), one can show by the subgradient calculus that $(r, p, 0) \in \partial K(t, x, v, u)$ if and only if

$$v = A(t)x + B(t)u \text{ and } B^*(t)p \in \partial g(t, u) ,$$

$$\exists w \in \partial f\big(t, C(t)x\big) \text{ with } r = -A^*(t)p + C^*(t)w ,$$

where the asterisk denotes the transpose of a matrix. Using (34), one can write the condition $B^*(t)p \in \partial g(t, u)$ in the dual form $u \in \partial g^*(t, B^*(t)p)$, where g^* is the conjugate integrand.

An application of facts about measurable selections [29] then leads to the conclusion that $x \in A$ and $p \in A$ satisfy the Euler-Lagrange condition for L in this case if and only if there exist functions $u \in L$ and $w \in L$ such that (for almost every t)

$$\text{(42)} \quad \begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \quad \text{with} \quad u(t) \in \partial g^*(t, B^*(t)p(t)) \ , \\ \dot{p}(t) &= -A^*(t)p(t) + C^*(t)w(t) \quad \text{with} \quad w(t) \in \partial f(t, C(t)x(t)) \ . \end{aligned}$$

This is interesting because of the appearance of a dual dynamical system with explicit controls $w(t)$, a property that is not readily captured for general convex K , and because of the complete symmetry in x and p . A dual problem of optimal control in p will be described in the next section.

If f and g have the quadratic form in (13), the control conditions in (42) take the form

$$w(t) = S(t)C(t)x(t) \quad \text{and} \quad u(t) = R(t)^{-1}B^*(t)p(t) \ .$$

In the case of (12), they become

$$w(t) \equiv 0 \quad \text{and} \quad u(t) \in \arg\max_{\|z\| \le 1} p(t) \cdot z \ .$$

If l is differentiable, the transversality condition is just (32). In the fixed endpoint case where l is given by (14), one has

$$\partial l(x_0, x_1) = \begin{cases} R^n \times R^n & \text{if} \quad (x_0, x_1) = (a_0, a_1) \ , \\ \emptyset & \text{if} \quad (x_0, x_1) \ne (a_0, a_1) \ , \end{cases}$$

so the condition reduces merely to the constraints $x(t_0) = a_0$ and $x(t_1) = a_1$, with nothing required of $p(t_0)$ and $p(t_1)$. For (15) it becomes

$$x(t_0) = x(t_1) \quad \text{and} \quad p(t_0) = p(t_1) \ ,$$

while for (16) one gets

$$p(t_0) \ \text{normal to} \ E_0 \ \text{at} \ x(t_0) \ , \quad -p(t_1) = Wx(t_1) + c \ .$$

("Normal" means "belonging to the normal cone" N_{E_0} defined in (38).)

These examples illustrate that a wide spectrum of conditions is covered by the subgradient notation. A similar calculus exists for generalized gradients in the sense of Clarke [10], [16], but it typically involves chains of inclusions rather than equivalences. Fortunately the inclusions are in the direction one needs for the derivation of necessary conditions for optimality.

9. Dual problem

The equivalent ways of writing a subgradient relation in terms of a convex function or its conjugate, as in (35), suggest a dual form for the optimality conditions for problems of convex type:

$$(43)\quad \begin{aligned} (r, p) \in \partial L(t, x, v) &\Leftrightarrow (x, v) \in \partial L^*(t, r, p) , \\ (p_0, -p_1) \in \partial l(x_0, x_1) &\Leftrightarrow (x_0, x_1) \in \partial l^*(p_0, -p_1) . \end{aligned}$$

Here the conjugate functions $L^*(t, \cdot, \cdot)$ and l^* , like $L(t, \cdot, \cdot)$ and l , are lower semicontinuous proper convex, and in fact L^* is again a *normal integrand* [21], [29] (something which might not have been true if a different measurability property had been incorporated in the definition of "normality").

Symmetry is not quite present in (43), so let us introduce the functions

$$(44)\quad \begin{aligned} M(t, p, r) &= L^*(t, r, p) = \sup_{x,v} \{r\cdot x+p\cdot v-L(t, x, v)\} , \\ m(p_0, p_1) &= l^*(p_0, -p_1) = \sup_{x_0,x_1} \{p_0\cdot x_0-p_1\cdot x_1-l(x_0, x_1)\} , \end{aligned}$$

so that reciprocally

$$(45)\quad \begin{aligned} L(t, x, v) &= M^*(t, v, x) = \sup_{p,r} \{r\cdot x+p\cdot v-M(t, p, r)\} , \\ l(x_0, x_1) &= m^*(x_0, -x_1) = \sup_{p_0,p_1} \{p_0\cdot x_0-p_1\cdot x_1-m(p_0, p_1)\} , \end{aligned}$$

and the equivalences (43) become

$$\begin{aligned} (r, p) \in \partial L(t, x, v) &\Leftrightarrow (v, x) \in \partial M(t, p, r) , \\ (p_0, -p_1) \in \partial l(x_0, x_1) &\Leftrightarrow (x_0, -x_1) \in \partial m(p_0, p_1) . \end{aligned}$$

For arcs $x \in A$ and $p \in A$, one therefore has

$$\begin{aligned} (\dot{p}(t), p(t)) \in \partial L(t, x(t), \dot{x}(t)) &\Leftrightarrow (\dot{x}(t), x(t)) \in \partial M(t, p(t), \dot{p}(t)) , \\ (p(t_0), -p(t_1)) \in \partial l(x(t_0), x(t_1)) &\Leftrightarrow (x(t_0), -x(t_1)) \in \partial m(p(t_0), p(t_1)) . \end{aligned}$$

It is appropriate to call M the *dual Lagrangian* and m the *dual boundary function*. They satisfy the same conditions as do L and l for problems of convex type. Thus the functional

$$\Psi(p) = \int_T M(t, p(t), \dot{p}(t))dt + m(p(t_0), p(t_1))$$

is likewise well defined for all $p \in A$ and convex. The problem

(P*) $\qquad$ minimize $\Psi(p)$ over all $p \in A$

is the *dual* of (P) and is again of convex type. The theorem in §8 is therefore applicable and says that p solves (P*) if p satisfies the Euler-Lagrange condition for M and transversality condition for m in terms of some $x \in A$. An interesting connection between (P) and (P*) is then apparent from (46).

DUALITY THEOREM 1. *When* (P) *is of convex type, the following are equivalent for* $x \in A$ *and* $p \in A$*:*

(a) x *satisfies the Euler-Lagrange condition for* L *and transversality condition for* l *with adjoint* p*;*

(b) p *satisfies the Euler-Lagrange condition for* M *and transversality condition for* m *with adjoint* x*.*

Thus the sufficient conditions for (P) *also furnish a solution to* (P*) *and conversely.*

Because of the equivalence of *(a)* and *(b)* we shall simply say in the convex case that x and p satisfy the (sufficient) *optimality conditions* when these properties are present.

In the case where (P) is the reduced problem for the convex control problem (Q_0),

$$\text{minimize} \int_T f\big(t, C(t)x(t)\big)dt + \int_T g\big(t, u(t)\big)dt + l\big(x(t_0), x(t_1)\big)$$

(46)

$$\text{subject to } \dot{x}(t) = A(t)x(t) + B(t)u(t) \ ,$$

the dual has a similar structure. Assume, as was done in §7 in specializing the optimality conditions to this setting, that for each t,

(47)

$$f(t, \cdot) \text{ is finite on a neighborhood of } 0 \ ,$$
$$g(t, \cdot) \text{ is inf-compact.}$$

These two properties happen to be dual to each other with respect to conjugate convex functions [20, §§8, 13], so (47) is equivalent to:

(48)

$$f^*(t, \cdot) \text{ is inf-compact,}$$
$$g^*(t, \cdot) \text{ is finite on a neighborhood of } 0 \ .$$

When the expression

$$L(t, x, v) = f\big(t, C(t)x\big) + \min_u \{g(t, u) \mid B(t)u = v - A(t)x\}$$

is inserted in (44), one obtains with the help of one of the standard formulas for conjugates (*cf.* [20, p. 142]) that

(49) $$M(t, p, r) = g^*\left(t, B^*(t)p\right) + \min_{w} \{f^*(t, w) \mid C^*(t)w = r + A^*(t)p\} .$$

Thus (P^*) is the reduced problem for a certain control problem like (Q_0):

(50) $$\begin{aligned} &\text{minimize} \quad \int_T g^*\left(t, B^*(t)p(t)\right)dt + \int_T f^*\left(t, w(t)\right)dt + m\left(p(t_0), p(t_1)\right) , \\ &\qquad\qquad \text{subject to} \quad \dot{p}(t) = -A^*(t)p(t) + C^*(t)w(t) . \end{aligned}$$

Note that the dual dynamical system is the same one seen earlier in the optimality conditions (42).

Conjugate functions are not always easy to express in a more direct form, even with the machinery in [20] and [26, §9], but this is possible in many important cases. For example, if f and g have the quadratic form in (13) with $S(t)$ and $R(t)$ positive definite, one has

$$f^*(t, w) = \tfrac{1}{2}w \cdot S(t)^{-1}w, \quad g^*(t, q) = \tfrac{1}{2}q \cdot R(t)^{-1}q .$$

If they have the form (12), then

$$f^*(t, w) = \begin{cases} 0 & \text{if } w = 0 , \\ \infty & \text{if } w \neq 0 , \end{cases} \qquad g^*(t, q) = \|q\|_* ,$$

where $\|\cdot\|_*$ is the norm dual to $\|\cdot\|$. Then $w(t)$ is implicitly constrained to vanish in (50), and everything about it drops out of the problem. The same would be true in other problems with $f(t, y) \equiv 0$. Thus for a problem of the form

$$\begin{aligned} &\text{minimize} \quad \int_T g\left(t, u(t)\right)dt + l\left(x(t_0), x(t_1)\right) , \\ &\text{subject to} \quad \dot{x}(t) = A(t)x(t) + B(t)u(t) , \end{aligned}$$

the dual is

$$\begin{aligned} &\text{minimize} \quad \int_T g^*\left(t, B^*(t)p(t)\right)dt + m\left(p(t_0), p(t_1)\right) , \\ &\qquad\quad \text{subject to} \quad \dot{p}(t) = -A^*(t)p(t) . \end{aligned}$$

What is particularly interesting about this case is that the dual problem turns out to be essentially *finite-dimensional*, since p is uniquely determined by $p(t_0)$.

Another good illustration is the case where

(51) $$f(t, y) = \begin{cases} \alpha|y| - \alpha & \text{if } |y| \geq 1 , \\ 0 & \text{if } |y| \leq 1 , \end{cases} \qquad g(t, u) = \max\{a_1 \cdot u, \ldots, a_N \cdot u\} ,$$

where $|\cdot|$ denotes the Euclidean norm and $a_1, \ldots, a_N$ are vectors in R^n, $\alpha > 0$. Then

$$(52)\quad f^*(t,\,w) = \begin{cases} |w| & \text{if } |w| \le \alpha\,, \\ \infty & \text{if } |w| > \alpha\,, \end{cases} \qquad g^*(t,\,q) = \begin{cases} 0 & \text{if } q \in \mathrm{co}\{a_1,\,\ldots,\,a_N\}\,, \\ \infty & \text{otherwise}, \end{cases}$$

where "co" denotes convex hull. This is instructive because the primal problem (46) has no implicit state constraints or control constraints, but the dual problem (50) does, namely

$$(53)\qquad B^*(t)p(t) \in \mathrm{co}\{a_1,\,\ldots,\,a_N\} \text{ and } |w(t)| \le 1\,, \text{ almost everywhere.}$$

These constraints are determined simply by inspecting where the functions in the dual problem are finite, which underscores the economy and effectiveness of the $+\infty$ notation.

When it comes to the possibilities for l and m, the first example to look at is the one for fixed endpoints $x(t_0) = a_0$, $x(t_1) = a_1$, where l is given by (14). Trivially, m is then linear:

$$m(p(t_0),\,p(t_1)) = p(t_0)\cdot a_0 - p(t_1)\cdot a_1\,.$$

Since m is finite everywhere, no implicit constraints are imposed on the endpoints of p; they are free in the dual problem. If instead l has the form (15), corresponding to the constraint $x(t_0) = x(t_1)$, it turns out that $m = l$, so that the dual problem likewise has the constraint $p(t_0) = p(t_1)$. The example of l in (16) yields

$$(54)\qquad m(p(t_0),\,p(t_1)) = \sigma(p(t_0)) + \tfrac{1}{2}|p(t_1)-a_1|^2 + \tfrac{1}{2}|a_1|^2\,,$$

where σ is the *support function* of the convex set E_0:

$$\sigma(p_0) = \sup\{p_0\cdot x_0 \mid x_0 \in E_0\}\,.$$

If E_0 is a cone, σ is just the indicator of the polar cone E_0^*, and the first term in (54) represents the constraint $p(t_0) \in E_0^*$. For instance, if $E_0 = R^n$ $(x(t_0)$ free$)$ one gets $E_0^* = \{0\}$ and the implicit constraint $p(t_0) = 0$. If $E_0 = R^n_+$ $(x(t_0) \ge 0)$, then $E_0^* = R^n_-$ $(p(t_0) \le 0)$. If E_0 equals a subspace N, then $E_0^* = N^\perp$ (orthogonal complement). If E_0 is the unit ball for a norm $\|\cdot\|$, then $\sigma = \|\cdot\|_*$ (dual norm).

Incidentally, the kind of duality seen in (46) and (50), where explicit controls appear in both problems, can be captured in a slightly broader setting with the expression $f(t,\,C(t)x(t)) + g(t,\,u(t))$ replaced by $h(t,\,C(t)x(t),\,u(t))$. This replaces $f^*(t,\,w(t)) + g^*(t,\,B^*(t)p(t))$ in (50) by $h^*(t,\,w(t),\,B^*(t)p(t))$.

The dual problem (P*) was introduced in [21].

10. Hamiltonian equations

The classical reason for introducing the Hamiltonian function is that the Euler-Lagrange condition for L can, under certain assumptions, be written instead in the form

(55) $$\left(-\dot{p}(t),\ x(t)\right) = \nabla H\left(t,\ x(t),\ p(t)\right) .$$

The same thing can be accomplished in the convex case in terms of subgradients instead of gradients.

Since for problems of convex type $H(t, x, p)$ is concave in x (as well as convex in p), we can speak of the subgradient set $\partial_p H(t, x, p)$ and, with a change of sign, the "supergradient" set $\partial_x H(t, x, p)$. The subgradient set of the function $H(t, \cdot, \cdot)$ at (x, p) is

(56) $$\partial H(t, x, p) = \partial_x H(t, x, p) \times \partial_p H(t, x, p) .$$

The generalized *Hamiltonian equation* (really: Hamiltonian "contingent equation" or "differential inclusion") is

(57) $$\left(-\dot{p}(t),\ \dot{x}(t)\right) \in \partial H\left(t,\ x(t),\ p(t)\right) \text{ , almost everywhere.}$$

The product form in (56) may give a misleading impression, in that it is a special feature which does not carry over to other classes of functions H when the definition of the Hamiltonian equation is extended. An extension is indeed possible, for example to all problems satisfying the Hamiltonian upper boundedness condition in §7. Then for each t the function

$$F : (x, p) \mapsto -H(t, x, p)$$

is lower semicontinuous proper [27, Proposition 4], so that ∂F is well defined in the sense of Clarke [10]: take

$$\partial H(t, x, p) = -\partial[-H](t, x, p) .$$

This definition turns out to give the same result as the one above if H is concave-convex, so (56) is natural for that case. But (56) is often false, although $\partial H(t, x, p)$ is always a closed convex subset of $R^n \times R^n$. (Incidentally, there are problems of convex type for which neither $H(t, x, p)$ nor $-H(t, x, p)$ is a lower semicontinuous proper function of (x, p) ; *cf.* [20, §33]. No general definition of ∂H is presently known which covers this case in convex analysis, having significant consequences below, and all the cases amenable to Clarke's definition.)

THEOREM. *In the convex case, the Hamiltonian equation is equivalent to the Euler-Lagrange condition for* L *(and also the one for* M *) and therefore can be substituted for it in the optimality conditions.*

This follows from a rule relating subgradients and the Fenchel transform [20, Theorem 37.5] which in the present notation takes the form

$$(r, p) \in \partial L(t, x, v) \Longleftrightarrow (-r, v) \in \partial H(t, x, p) . \tag{58}$$

Thus

$$\left(\dot{p}(t), p(t)\right) \in \partial L\left(t, x(t), \dot{x}(t)\right) \Longleftrightarrow \left(-\dot{p}(t), x(t)\right) \in \partial H\left(t, x(t), p(t)\right) . \tag{59}$$

The equivalence also holds in the classical, continuously differentiable case, if L is actually strictly convex and coercive in v (not necessarily convex in (x, v)), or if H is convex in (x, p) (not necessarily differentiable), in which event L is concave in x - a reversal of the properties in the theorem in §6. But it can fail for some of the general cases covered in terms of Clarke's definition. (Then the two conditions in (59) seem to say different things, yet Clarke has established that they are both sometimes necessary for optimality. See [12], [14] for Clarke's necessary conditions in Hamiltonian form and [15] for their applications to get an extremely general "maximum principle".)

It may be wondered why in the convex case, as in the theorem above, equal attention is not paid to the *dual* Hamiltonian H' corresponding to the dual Lagrangian M ,

$$H'(t, p, x) = \sup_{r \in R^n} \{r \cdot x - M(t, p, r)\} . \tag{60}$$

The reason is that

$$H'(t, p, x) = -H(t, x, p) \quad \text{"almost"}.$$

Indeed, if the formula for H in terms of L is used to rewrite the formula for M in terms of L , one obtains

$$M(t, p, r) = \sup_{x \in R^n} \{r \cdot x + H(t, x, p)\} , \tag{61}$$

which says that the Fenchel transform of $F(x) = -H(t, x, p)$ is $F^*(r) = M(t, p, r)$. Then from (60) one has $F^{**}(x) = H'(t, x, p)$, so the study of the relationship between H' and H boils down to the question of the extent to which F^{**} must agree with F . Since $F^{**} = F$ when F is lower semicontinuous and nowhere $-\infty$, we may conclude that $H'(t, p, x) = -H(t, x, p)$ for all (t, x, p) when H is upper semicontinuous in x and nowhere $+\infty$. Actually, for H arising from L which is lower semicontinuous proper convex in (x, v) as here, it can be shown that H is upper semicontinuous in (x, p) if it is nowhere $+\infty$. In general, however, there could be slight discrepancies between H' and H , and what one really has is two concave-convex functions equivalent to each other in a sense known in convex analysis (*cf.* [20, §34]). The Hamiltonian equations for H' and H are equivalent.

The Hamiltonian for the control problem (Q_0), expressed in (28), yields (under (47) or equivalently (48)) the equations

$$\dot{x}(t) \in \left[A(t)x(t)+B(t)\partial g^*\left(t,\ B^*(t)p(t)\right)\right], \text{ almost everywhere},$$

$$\dot{p}(t) \in \left[-A^*(t)p(t)+C^*(t)\partial f\left(t,\ C(t)x(t)\right)\right], \text{ almost everywhere},$$

which can be expanded to (42) through an application of the theory of measurable selections.

The Hamiltonian for the nonlinear control problem in §2, given in (27), may well fail to be concave-convex, yet this is a case where under natural assumptions the Hamiltonian equation is well defined in Clarke's sense. It is interesting to see how the equation relates to the maximum principle. For simplicity and in order to ensure that the reduced Lagrangian $L(t, x, v)$ is a proper normal integrand which is convex in v, as we have been assuming, suppose that

(a) $U(t)$ is compact, convex, nonempty,

(b) $f(t, x, u)$ and $f_0(t, x, u)$ are defined on all of $T \times R^n \times R^m$, measurable in t and differentiable in x,

(c) f, f_0, $\nabla_x f$ and $\nabla_x f_0$ are continuous in (x, u),

(d) f is affine in u (that is, $f(t, x, u) = F(t, x) + G(t, x)u$) and f_0 is convex in u,

(e) $x \in \text{int } X(t)$.

These conditions can be shown to imply

$$(s, v) \in \partial H(t, x, p) \Leftrightarrow \begin{cases} \exists u \in \underset{U(t)}{\arg\max} \left\{f(t, x, \cdot)\cdot p - f_0(t, x, \cdot)\right\} \\ \text{such that } v = f(t, x, u), \\ s = \nabla_x f(t, x, u)p - \nabla_x f_0(t, x, u). \end{cases}$$

With the help of measurable selections, this yields the result that, for $x \in A$ and $p \in A$ with $x(t) \in \text{int } X(t)$ for all t, the Hamiltonian equation is satisfied if and only if there is a measurable function u such that for almost every t,

$$u(t) \in \underset{U(t)}{\arg\max} \left\{f\left(t, x(t), \cdot\right)\cdot p(t) - f_0\left(t, x(t), \cdot\right)\right\},$$

$$\dot{x}(t) = f\left(t, x(t), u(t)\right),$$

$$\dot{p}(t) = -\nabla_x f\left(t, x(t), u(t)\right)p(t) + \nabla_x f_0\left(t, x(t), u(t)\right).$$

This amounts to the "maximum principle" in reduced form. (The case where $x(t)$ might be on the boundary of $X(t)$ is more complicated, see the remarks at the end of §14.)

Note that the coefficient of f_0 is -1 in the "arg max", in contrast to most treatments of optimal control, which allow a variable coefficient $p_0(t)$ and show that it must be constant and can be taken as either -1 or 0. Since the "0" possibility is excluded, the conditions are slightly stronger than usual and require for their necessity slightly stronger assumptions (Clarke's concept of "calmness", *cf.* [13], [14]).

11. Hamiltonian trajectories

The advantage of the Hamiltonian equation over the Euler-Lagrange condition is that it has the form of a generalized ordinary differential equation

(62) $$\dot{z}(t) \in \hat{C}\big(t,\ z(t)\big) \quad \text{almost everywhere}, \quad z(t) = \big(x(t),\ p(t)\big)\ ,$$

where $C(t, z)$ is a closed convex set that depends on t and z in a nice way. (The graph $\Gamma(t)$ of the multifunction $z \mapsto C(t, z)$ is closed, and the multifunction $t \mapsto \Gamma(t)$ is measurable.) Local existence theorems are available for such generalized differential equations, at least under certain conditions of nonemptiness and boundedness of $C(t, z)$ (*cf.* [7]). They can be applied to get trajectories $\big(x(t), p(t)\big)$ for the Hamilton equations that emanate from any initial point $\big(x_0, p_0\big)$ in a neighborhood of which H is Lipschitz continuous with respect to x, p, and satisfies a summability condition in t (*cf.* [22] for the convex case).

When H is concave-convex, H is not only Lipschitz continuous on any open set where it is finite, but actually differentiable there almost everywhere [20, §35], so that $\partial H(t, x, p)$ reduces to a single element (the gradient) except on a special set of measure zero. Then the general Hamiltonian equation (57) is not so far from the classical version (55) as might have been thought from its "contingent" form. As a matter of fact, nonuniqueness of solutions from a given starting point appears, from examples, to be a rather rare phenomenon, although it definitely can occur (see below).

Another property known in the convex case is that if H is finite and independent of t, then $H\big(x(t), p(t)\big)$ is constant along all solutions to the generalized Hamiltonian equation. (This extends a classical result in the differentiable case whose proof is trivial, but the multivalued form of the equation requires a somewhat tricky argument, *cf.* [22].) A nice way of generating simple non-classical examples is thereby provided: take any finite concave-convex function H on $R \times R$ and look at its level curves. The trajectories of the Hamiltonian equation (which exist at least locally for this case, as just remarked) must follow these curves. A rather interesting example to look at in such a light is

$$H(x, p) = \max\{0,\ |p|-1\} - \max\{0,\ |x|-1\}\ ,$$

which corresponds to

$$L(x, v) = \max\{0,\ |x|-1\} + \begin{cases} |v| & \text{if } |v| \le 1\ , \\ \infty & \text{if } |v| > 1\ . \end{cases}$$

The trajectories have corners, and they can branch at certain points.

The assertions about $\partial H(t, x, p)$ being a singleton almost everywhere, and H being constant along Hamiltonian trajectories when H is independent of t, carry over to other cases, for instance the Hamiltonian at the end of the preceding section and all Hamiltonians which are convex in (x, p). But they are not true in all cases where H is merely Lipschitz continuous in x.

Local solutions to the Hamiltonian equation have a certain optimality property when H is concave-convex. Suppose for instance that x and p are absolutely continuous functions which satisfy (57) over the whole interval $T = [t_0, t_1]$ (almost everywhere). Defining $a_0 = x(t_0)$ and $a_1 = x(t_1)$ and taking l to be the indicator of this endpoint pair as in (14), we see that x and p satisfy the transversality condition for l, as well as (by virtue of the theorem of §10) the Euler-Lagrange condition for L. Hence by the sufficiency theorem in §8, x minimizes $\int_T L(t, x(t), \dot{x}(t))dt$ over the class of all arcs having the same endpoints a_0 and a_1. Now the same argument can also be applied relative to any subinterval of T. Thus x is *Lagrange optimal* for L over T, in the sense that on every subinterval I it minimizes the Lagrangian integral on I with respect to the class of all arcs that coincide with x at the beginning and end of I. The same can be argued in terms of p via the duality theorem in §9, and one obtains the following.

THEOREM. *In the convex case, if x and p are absolutely continuous functions satisfying the generalized Hamiltonian equation for t in an interval I, then x is Lagrange optimal for L over I, and p is Lagrange optimal for M over I.*

Another special property in the convex case is that if (x, p) and (x', p') are two Hamiltonian trajectories over I, then the quantity $(x(t)-x'(t))\cdot(p(t)-p'(t))$ is nondecreasing over I [22].

12. Optimal values and perturbations

The close relationship between a problem (P) of convex type and its dual (P*) extends beyond the sharing of sufficient conditions for optimality. There is also a tie between the two optimal values

$$\inf(P) = \inf_{x \in A} \Phi(x) , \quad \inf(P^*) = \inf_{p \in A} \Psi(p) . \tag{63}$$

The study of these values and how they behave under certain "perturbations" of (P) is the route to determining the necessity of the optimality conditions that have been introduced.

A basic inequality can be derived easily from the definition (43) of M and m and the relations (34), (35), that hold for any conjugate pair of convex functions. For arbitrary $x \in A$ and $p \in A$ one has

$$L(t, x(t), \dot{x}(t)) + M(t, p(t), \dot{p}(t)) \geq \dot{p}(t)x(t) + p(t)\dot{x}(t) \tag{64}$$

for almost every $t \in T$, where equality holds if and only if $(\dot{p}(t), p(t)) \in \partial L(t, x(t), \dot{x}(t))$. At the same time

$$l(x(t_0), x(t_1)) + m(p(t_0), p(t_1)) \geq p(t_0)x(t_0) - p(t_1)x(t_1) , \tag{65}$$

where equality holds if and only if $(p(t_0), -p(t_1)) \in \partial l(x(t_0), x(t_1))$. Integrating (64) over the interval T and adding (65), we get

(66) $\Phi(x) + \Psi(p) \geq 0$ *for all* $x \in A$, $p \in A$, *with*

equality $\Leftrightarrow$ x *and* p *satisfy the optimality conditions.*

Or do we? There is a slight flaw in the argument, connected with the extended definition of the integrals of L and M as $\pm\infty$. The inequality (66) is quite valid if the convention $\infty - \infty = \infty$ is used on the left side, but the case $\infty - \infty$ could conceivably arise even when (64) and (65) were true with equality, and then there would be strict inequality in (66) despite the optimality conditions being satisfied.

To get around this, a minor assumption must be added. Let L^∞ and L^1 denote the spaces of R^n-valued functions on T which are essentially bounded, or respectively, summable, and define

$$I_L(x, v) = \int_T L(t, x(t), v(t))dt \quad \text{for} \quad (x, v) \in L^\infty \times L^1 . \tag{67}$$

ASSUMPTION 5. *The functional* I_L *is proper on* $L^\infty \times L^1$ *and bounded below on bounded sets.*

This is satisfied in particular if $\Phi(x) < \infty$ for some $x \in A$ and the Hamiltonian upper boundedness condition holds. Assumption 5 is equivalent in the convex case to the same condition on I_M (hence it is really symmetric in character between (P) and (P*)), and it is also equivalent to:

$$\exists(x, v) \in L^\infty \times L^1 \text{ with } I_L(x, v) < \infty ,$$

and

$$\exists(p, r) \in L^\infty \times L^1 \text{ with } I_M(p, r) < \infty .$$

It implies that $\Phi(x)$ and $\Psi(p)$ are never $-\infty$, so the question of $\infty - \infty$ never arises in (66). An important conclusion can then be drawn by rewriting (66) in the form $\Phi(x) \geq -\Psi(p)$.

DUALITY THEOREM 2. *The inequality* inf(P) ≥ -inf(P*) *holds for problems of convex type. For* min(P) = -min(P*) *to hold with attainment at* $x \in A$ *and* $p \in A$ *respectively, it is necessary and sufficient that* x *and* p *satisfy the optimality* *conditions.*

The dual of a minimization problem is customarily expressed as a maximization problem, and of course

$$-\inf(P^*) = \sup_{p \in A} \{-\Psi(p)\} .$$

Rather than speaking of the maximization of $-\Psi$ in the present case, we prefer to keep the exact symmetry reflected in the optimality conditions.

The theorem yields an important clue about the circumstances in which the optimality conditions, as stated, are *necessary*.

COROLLARY. *Suppose* $\inf(P) = -\min(P^*)$. *Then* $x \in A$ *furnishes the minimum in* (P) *if and only if it satisfies the optimality conditions in association with some* $p \in A$.

The challenge laid down by this result is to find conditions guaranteeing that $\inf(P) = -\min(P^*)$. An approach can be made through the analysis of the functional

$$\varphi(y, a) = \inf_{x \in A} \left\{ \int_T L\big(t, x(t)+y(t), \dot{x}(t)\big)dt + l\big(x(t_0)+a, x(t_1)\big) \quad \text{for} \quad (y, a) \in L^\infty \times R^n \right\} .$$

This gives the optimal value in a problem which is like (P) but depends on y and a as parameters (perturbations); clearly $\varphi(0, 0) = \inf(P)$. It is readily seen that φ is convex when (P) is of convex type. Every continuous linear functional on A can be represented in the form $p \mapsto \langle p, (y, a)\rangle$ with

$$\langle p, (y, a)\rangle = \int_T \dot{p}(t)\cdot y(t)dt + p(0)\cdot a , \tag{68}$$

so the space $L^\infty \times R^n$ can be identified with A^* . Each $p \in A$ also defines a continuous linear functional $(y, a) \mapsto \langle p, (y, a)\rangle$ on $L^\infty \times R^n$, and it turns out that to have $\inf(P) = -\min(P^*)$ with attainment at p , it is necessary and sufficient that $p \in \partial\varphi(0, 0)$, or in other words,

$$\varphi(y, a) \geq \varphi(0, 0) + \langle p, (y, a)\rangle \quad \text{for all} \quad (y, a) \in L^\infty \times R^n .$$

This result provides, on the one hand, an interpretation of what the adjoint arc means for (P) itself: it gives coefficients measuring the differential effects of certain perturbations of (P). In particular, if φ happens to be differentiable at $(0, 0)$, one has $p = \nabla\varphi(0, 0)$ in the sense of the pairing (68).

On the other hand, this result reduces the question of whether $\inf(P) = -\min(P^*)$ to the question of the existence of $p \in A$ such that $p \in \partial\varphi(0, 0)$. Such a subgradient p corresponds to a kind of supporting hyperplane to the convex set in $(L^\infty \times R^n) \times R$ which is the epigraph of φ , and so the existence can presumably be obtained from some separation theorem of convex analysis under conditions on L and

l that imply the epigraph has a nonempty interior whose projection on $L^\infty \times R^n$ contains $(0, 0)$.

But there is a catch. With some effort the interiority can be achieved in terms of the topology of $L^\infty \times R^n$ corresponding to the L^∞-norm, but the space of continuous linear functionals in this topology is $(L^\infty)^* \times R^n \approx A^{**}$, not just A. Thus there is a danger that the supporting hyperplane obtained through separation theory might not be of the form (67), and then it would do no good.

13. Necessity and duality

A crucial restriction must be made to get around the obstacle just explained, and it is dual to the kind of restriction mentioned in §7 in connection with the existence of solutions $x \in A$ for (P).

The *Hamiltonian lower boundedness condition* is satisfied if for each $x \in R^n$ and $\beta \in R$ there is a summable function $\theta : T \mapsto R$ such that

$$H(t, x, p) > \theta(t) \quad \text{for all} \quad t \in T \quad \text{when} \quad |p| \le \beta .$$

For problems of convex type, this is just the Hamiltonian upper boundedness condition on the dual Lagrangian H' discussed in §10, so it is clearly related to the existence of solutions $p \in A$ for (P*). In particular it requires $H > -\infty$ everywhere.

A concave-convex Hamiltonian satisfies the lower boundedness condition if and only if for every $x \in L^\infty$ there exists $v \in L^1$ with $I_L(x, v)$ finite (where I_L is the functional in (67)). It satisfies both the lower and upper boundedness conditions if and only if $H(t, x, p)$ is a finite, summable function of $t \in T$ for each $(x, p) \in R^n \times R^n$. (See [23, §2] for these and other equivalences.)

The Hamiltonian lower boundedness condition implies for problems of convex type that the epigraph of the functional φ in §12 is of finite codimension and has a nonempty interior relative to its affine hull; furthermore, all subgradients of φ must belong to A, not just A^{**}. This was proved in [27]. The only thing left to be desired is a condition implying that $(0, 0)$ is in the projection on $L^\infty \times R^n$ of the epigraph of φ. This amounts to an attainability condition on the implicit constraints imposed by L and l.

The sets C_L and C_l defined by

$$C_L = \left\{ (x_0, x_1) \mid \exists x \in A \text{ with } \int_T L(t, x(t), \dot{x}(t))dt < \infty , \right.$$
$$\left. x(t_0) = x_0 \text{ and } x(t_1) = x_1 \right\} ,$$

$$C_l = \{ (x_0, x_1) \mid l(x_0, x_1) < \infty \}$$

obviously have the property that

$$C_L \cap C_l \neq \emptyset \Longleftrightarrow \exists x \in A \text{ with } \Phi(x) < \infty .$$

The *attainability condition* for (P) is the slightly stronger property that $\mathrm{ri}\, C_L \cap \mathrm{ri}\, C_l \neq \emptyset$, where "ri" denotes the relative interior of a convex set (its interior with respect to its affine hull, see [20, §6]). It is certainly satisfied if C_l is all of $R^n \times R^n$ and $C_L \neq \emptyset$, or if C_l consists of a single point lying in the relative interior of C_L . (In [23] the definition of C_L is a bit different but shown to be equivalent to the one here.) The attainability condition for (P*) is the same thing in terms of M and m .

DUALITY THEOREM 3. *For problems of convex type, the following hold.*

(a) If the attainability condition for (P) *is satisfied and* H *has the lower boundedness property, then* $\inf(P) = -\min(P^*) < \infty$ *, and for* $x \in A$ *to furnish the minimum in* (P) *it is necessary (as well as sufficient) that* x *satisfy the optimality conditions (in association with some* $p \in A$ *).*

(b) If the attainability condition for (P*) *is satisfied and* H *has the upper boundedness property, then* $\min(P) = -\inf(P^*) > -\infty$ *, and for* $p \in A$ *to furnish the minimum in* (P*) *it is necessary (as well as sufficient) that* p *satisfy the optimality conditions (in association with some* $x \in A$ *).*

This is the main theorem of [23]. Note that *(b)* is an existence theorem for (P), just as *(a)* is an existence theorem for (P*).

The attainability condition for (P*) can be translated into a growth condition on the convex functional Φ in (P) (see [23]). A condition on L implying in the autonomous case that the sets C_L and C_M in the attainability conditions are non-empty and project onto all of R^n in either argument, regardless of the choice of the interval T , may be found in [28, p. 151].

The most interesting feature is the duality between the existence of solutions to one problem and the necessity of the optimality conditions in the other. The two are closely connected, for better or for worse. The "worse" aspect is that, while the Hamiltonian lower boundedness condition is welcome enough as a burden en route to the existence of solutions to (P*), it has the unwanted effect of eliminating the possibility of real state constraints in (P). Indeed, such constraints appear in the implicit form $x(t) \in X(t)$ almost everywhere, where

$$X(t) = \{x \in R^n \mid \exists v \in R^n \text{ with } L(t, x, v) < \infty\} ,$$

and the lower boundedness condition implies via (29) that $X(t) = R^n$ for all $t \in T$.

However, the fact that state constraints become involved in this way is quite natural, when one thinks about it. The optimality conditions that have been derived for standard kinds of control problems with state constraints typically include multipliers (dual variables) that can jump at times t when $x(t)$ touches the boundary of the state constraint region. This suggests that an adequate treatment of such problems would involve adjoint arcs p that might not be continuous. Since p is required to be absolutely continuous in problem (P^*) and the optimality conditions, it is no wonder that in order to get the necessity of the optimality conditions we have had to impose a restriction that eliminates state constraints.

Where does this leave us in our desire to have a theory applicable also to problems with state constraints? A fundamental extension of the framework is needed. The optimality conditions must be generalized to admit arcs p in a larger space than A, and the natural choice turns out to be the space of arcs of bounded variation. If duality is still to play a role, the formulation of (P^*) must also be extended to this space. Thus we must decide what $\Psi(p)$ should mean for an arc of bounded variation. But symmetry demands that whatever is done for p should be done for x. Both (P) and (P^*) should therefore be in terms of arcs of bounded variation. The hope is that the extended problems will be just "closures" of the original problems in some sense, and it can be left to the optimality conditions themselves to tell us whether a particular solution arc or adjoint arc must actually be absolutely continuous.

14. Arcs of bounded variation

The treatment of state constraints has led us to the question of how to generalize the Hamiltonian equations and the functional Φ from A to the space B of R^n-valued functions of bounded variation on the interval T. An answer that takes care of both of these needs is found in making the right generalization of ordinary differential (contingent) equations of the form

(69) $$\dot{z}(t) \in C\big(t, z(t)\big) \text{ almost everywhere, } z \in A ,$$

to the case of $z \in B$. The Hamiltonian equation is of such type, and the study of the functional $\int_T L\big(t, x(t), \dot{x}(t)\big)dt$ can be reduced if necessary to the study of (69) for $z(t) = \big(x(t), x_0(t)\big)$ and

(70) $$C\big(t, z(t)\big) = \text{epigraph of } L\big(t, x(t), \cdot\big) .$$

Certain simplifications are possible for problems of convex type, but even in the general case it is reasonable to assume at the very least that $C(t, z)$ is a closed convex set for each $t \in T$ and $z \in R^n$ (possibly empty for z belonging to some "forbidden region"), and furthermore that the graph $\Gamma(t) = \{(z, w) \mid w \in C(t, z)\}$ is closed and depends measurably on t, that is, Γ is a measurable multifunction. (For (70), the convexity of $C(t, z)$ corresponds to Assumption 4 in §7.)

For a nonempty closed convex set C, the ***recession cone*** of C, denoted by 0^+C, is the "limit" of $\lambda C = \{\lambda w \mid w \in C\}$ as $\lambda \to 0^+$ (see [20, §8]). It reduces to $\{0\}$ if and only if C is bounded. The basic idea for extending (69) is the following. Each $z \in B$ corresponds to an R^n-valued Borel measure dz on T, and there always exists a nonnegative Borel measure on T with respect to which both dz and the Lebesgue measure dt are absolutely continuous. The latter can be expressed as $d\tau$ for a real valued function τ on T which is increasing (hence also of bounded variation). If $d\tau$ is absolutely continuous with respect to dt, we can use Radon-Nikodym derivatives to write (69) equivalently as

$$\text{(71)} \qquad \frac{dz}{d\tau}(t) \in \frac{dt}{d\tau}(t) \cdot C\big(t, z(t)\big) \quad \text{almost everywhere} \quad (d\tau)\ ,$$

where $(dt/d\tau)(t) > 0$ almost everywhere $(d\tau)$. If $d\tau$ is not absolutely continuous with respect to dt, this is reflected by having merely $(dt/d\tau)(t) \geq 0$ almost everywhere $(d\tau)$. The generalization consists essentially of adopting (71) as the replacement for (69) in this case with the right side interpreted as $0^+C\big(t, z(t)\big)$ when $(dt/d\tau)(t) = 0$.

What one gets is actually independent of the particular choice of $d\tau$. It is equivalent to augmenting the earlier equation (69) (which still makes sense - the derivative $\dot{z} = dz/dt$ does exist, but unless z is absolutely continuous it will not be the integral of $\dot{z}dt$) by a special condition on the singular part of dz :

$$\text{(72)} \qquad \frac{dz}{d\tau}(t) - \frac{dz}{dt}(t)\,\frac{dt}{d\tau}(t) \in 0^+C\big(t, z(t)\big) \quad \text{almost everywhere} \quad (d\tau)\ .$$

For the generalized "equation" (69) plus (72), the notation

$$dz(t) \in C\big(t, z(t)\big)dt$$

seems appropriate.

But there are some wrinkles to be ironed out. In (72) the left side is measurable with respect to $d\tau$, not just dt, so something other than Lebesgue measurability should apparently be demanded of the multifunction $t \mapsto 0^+C\big(t, z(t)\big)$ as well. The possible jumps in z also cause a problem. Besides $z(t)$, one has the limits $z(t+)$ and $z(t-)$, and there can be a countable infinity of points t at which these might not all agree. At such a point, (72) gives the jump condition

$$z(t+) - z(t-) \in 0^+C\big(t, z(t)\big)\ ,$$

but there is some doubt about whether $z(t)$ is really the correct thing to have on the right side or $z(t+)$ or $z(t-)$ (or both), particularly since we may just want to forget about $z(t)$ itself and identify functions of bounded variation which have the same one-sided limits at each point. Another question concerns what $0^+C(t, z)$ should be when $C(t, z) = \emptyset$ but $C\big(t, z_k\big) \neq \emptyset$ and $0^+C\big(t, z_k\big) \neq \{0\}$ for a sequence of points z_k converging to z.

More work is needed in the general case, but these riddles can be answered in a satisfying manner in the context of the application to the theory of state constraints in problems of convex type, *cf.* [24], [30]. The conditions on the Hamiltonian that replace upper and lower boundedness concern the state constraint set $X(t)$ and the corresponding set

$$P(t) = \{p \in R^n \mid \exists r \in R^n \text{ with } M(t, p, r)\}$$

for the dual problem. These are always convex and have the property that

$$H(t, x, p) = \begin{cases} \text{finite value if } x \in X(t) \text{ , } p \in P(t) \text{ ,} \\ +\infty \qquad \text{if } x \in X(t) \text{ , } p \notin \text{cl } P(t) \text{ ,} \\ -\infty \qquad \text{if and only if } x \notin X(t) \text{ .} \end{cases}$$

The case treated in [30] is the one where $X(t)$ and $P(t)$ have nonempty interiors which depend "continuously" on t, and $H(t, x, p)$ is summable in t over finite intervals during which x and p are in the interiors of $X(t)$ and $P(t)$. In the framework of the development outlined for the proof of theorem in the preceding section, the functional $\varphi(y, a)$ is restricted to $C \times R^n$ instead of $L^\infty \times R^n$, so the dual space can be identified with B.

The extended Hamiltonian equation is in terms of

$$C(t, x, p) = \{(v, r) \mid (-r, v) \in \partial H(t, x, p)\} \text{ ,}$$

and if $X(t)$ and $P(t)$ are closed the singular part (72) reduces to a condition in terms of

$$0^+C(t, x, p) = N_{P(t)}(p) \times N_{X(t)}(x) \text{ ,}$$

where $N_{X(t)}$ and $N_{P(t)}$ are the normal cones defined in (38). Results on duality, existence, and necessary and sufficient conditions are obtained, much like those above. Furthermore, solutions to the extended problems in B can be characterized as limits of minimizing sequences for the original problems in A. See [24], [30], for details.

15. Problems over an infinite horizon

There is considerable interest among mathematical economists in problems of convex type with the interval T unbounded, for example, $T = [0, \infty)$. Typically the Lagrangian is of the form

$$L(t, x, v) = -e^{\rho t}U(x, v) \text{ ,}$$

where U is a concave "utility" function and ρ is the "discount rate".

When $\rho = 0$, the Hamiltonian is independent of t and expressed by

$$H(x, p) = \sup_{v \in R^n} \{p \cdot v + U(x, v)\} .$$

Since H is concave in x and convex in p, it may well have a saddle point $(\bar{x}, \bar{p})$ in the minimax sense:

$$H(x, \bar{p}) \le H(\bar{x}, \bar{p}) \le H(\bar{x}, p) \quad \text{for all } x, p .$$

It has been demonstrated in [25] that if H happens to be *strictly* concave in x and *strictly* concave in p in a neighborhood of $(\bar{x}, \bar{p})$, then $(\bar{x}, \bar{p})$ is also a saddle point for the Hamiltonian equation in the sense that the term "saddle point" is used for dynamical systems. More specifically, in a neighborhood of $(\bar{x}, \bar{p})$ the Hamiltonian trajectories $(x(t), p(t))$ that tend to $(\bar{x}, \bar{p})$ as $t \to +\infty$ make up an n-dimensional manifold K_+ in R^{2n}, while those that tend to $(\bar{x}, \bar{p})$ as $t \to -\infty$ form a similar manifold K_- with $K_+ \cap K_- = \{(\bar{x}, \bar{p})\}$. The trajectories in K_+ have a certain natural optimality property over intervals $[t_0, \infty)$, while those in K_- have such a property for $(-\infty, t_1]$.

These results have been obtained through application of the duality theory described here (without getting involved with state constraints). A kind of extension to the case where $\rho > 0$ is carried out in [31].

OTHER EXTENSIONS OF THE THEORY. The duality between (P) and (P*) has been generalized by Barbu [1], [2], [3], [4], [5], to problems where the states $x(t)$ are not in R^n but an infinite-dimensional Hilbert space. Some applications to systems governed by partial differential equations are thereby covered. For another case corresponding to partial differential equations, namely where the interval T is replaced by a region Ω in R^k and $\dot{x}$ by Dx for some operator D, see the book of Ekeland and Temam [17]. Bismut [6] has applied the duality theory to problems in stochastic optimal control.

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