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III. CONVEX PROCESSES AND HAMILTONIAN DYNAMICAL SYSTEMS

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Many economists have studied optimal growth models of the form

$$\text{(1)}\qquad \begin{aligned} &\text{maximize} \quad \int_0^\infty e^{-\rho t}\, U(k(t),\, z(t))dt \\ &\text{subject to} \quad k(0) = k_0,\ \dot{k}(t) = z(t) - \gamma k(t), \end{aligned}$$

where k is a vector of capital goods, γ is the rate of depreciation, ρ is the discount rate, and U is a continuous concave utility function defined on a closed convex set D in which the pair (k,z) is constrained to lie. The theory of such problems is plagued by technical difficulties caused by the infinite time interval. The optimality conditions are still not well understood, and there are serious questions about the existence of solutions and even the meaningfulness, in certain cases, of the expression being maximized.

One thing is clear, however. Any trajectory $\bar{k}(t)$ which is worthy of consideration as optimal in (1) would in particular have to have the property that for every finite time interval $[t_0,t_1] \subset [0,\infty)$ one has

$$\text{(2)}\qquad \int_{t_0}^{t_1} e^{-\rho t}\, U(k(t),\dot{k}(t) + \gamma k(t))dt \leqq \int_{t_0}^{t_1} e^{-\rho t}\, U(\bar{k}(t),\dot{\bar{k}}(t) + \gamma\bar{k}(t))dt.$$

(For otherwise, the portion of $\bar{k}$ over $[t_0,t_1]$ could be replaced by k, and this would constitute a definite improvement.) This condition severely limits candidates for optimal paths and allows us to study them in terms of Hamiltonian dynamical systems involving subgradients.

Hamiltonian dynamical systems arise in the optimality conditions for variational problems of the form

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$$\text{(3)} \quad \begin{array}{ll} \text{minimize} & \int_{t_0}^{t_1} L(t,x(t),\dot{x}(t))dt \\ \text{subject to} & x(t_0) \in D_0,\ x(t_1) \in D_1. \end{array}$$

Classically, one always supposed L to a finite, differentiable function, but for the purpose of applications to economic models it is essential that one be able to treat the case where $L(t,.,.)$ is for each t a closed, proper, convex function on $R^n \times R^n$. The theory of problem (3) has been extended in this direction by Rockafellar [1], [2], [3]. The model (1) corresponds with the change of notation $x(t) = e^{\gamma t}k(t)$ to

$$\text{(4)} \qquad L(t,x,v) = -e^{-\rho t}\, U(e^{-\gamma t}x, e^{-\gamma t}v),$$

where U is interpreted as $-\infty$ outside of D.

Of course something must be assumed about the way that L depends on t. The correct condition in general is that L should be a "normal integrand" [1], [4]. This technical property of measurability will not be discussed here, but it is certainly satisfied when L is of the form (4) (under the assumptions already stated) and also when L is independent of t. Concerning the trajectory $x(t)$, one does not have to assume differentiability, but merely absolute continuity; the time derivative $\dot{x}(t)$ then exists for almost every t.

The <u>Hamiltonian</u> associated with L is the function

$$\text{(5)} \qquad H(t,x,p) := \sup_{v \in R^n} \{p.v - L(t,x,v)\}.$$

Thus $H(t,x,.)$ is the convex function conjugate to $L(t,x,.)$, so that L is in turn determined uniquely by H:

$$L(t,x,v) = \sup_{p \in R^n} \{p.v - H(t,x,p)\}.$$

Since $L(t,x,v)$ is not just convex in v but in (x,v), it turns out that $H(t,x,p)$ is not just convex in p but <u>concave</u> in x. The subgradient sets $\partial_x H(t,x,p)$ (concave sense) and $\partial_p H(t,x,p)$ (convex sense) are therefore welldefined [5]. The relation

$$\text{(6)} \qquad \dot{\bar{x}}(t) \in \partial_p H(t,\bar{x}(t),\bar{p}(t)),\ -\dot{\bar{p}}(t) \in \partial_x H(t,\bar{x}(t),\bar{p}(t))$$

is the generalized Hamilton condition. If H were differentiable as in classical mathematics, it would reduce to the equations

$$\dot{\bar{x}}(t) = \nabla_p H(t,\bar{x}(t),\bar{p}(t)), \quad -\dot{\bar{p}}(t) = \nabla_x H(t,\bar{x}(t),\bar{p}(t)).$$

An absolutely continuous trajectory $\bar{x}(t)$ is said to be an extremal for L over an interval I if there is an absolutely continuous $\bar{p}(t)$ (called a co-extremal for $\bar{x}$) such that the Hamiltonian condition (6) holds (for almost every t in I). On the other hand, $\bar{x}$ is said to be piecewise optimal for L over I if for every finite subinterval $[t_0,t_1] \subset I$ one has

$$(7) \qquad \int_{t_0}^{t_1} L(t,x(t),\dot{x}(t))dt \geqq \int_{t_0}^{t_1} L(t,\bar{x}(t),\dot{\bar{x}}(t))dt$$

for all (absolutely continuous) x(t) over $[t_0,t_1]$ such that $x(t_0) = \bar{x}(t_0)$, $x(t_1) = \bar{x}(t_1)$. The main result about these concepts in the present setting is the following.

THEOREM 1[1],[2],[3]. If x is an extremal for L, then x is piecewise optimal for L. If x is piecewise optimal for L and certain "constraint qualifications" are fulfilled, then x is an extremal for L.

The exact nature of the "constraint qualifications" will not be discussed here; see [2], [3]. Basically one needs to know that the pair $(\bar{x}(t_0), \bar{x}(t_1))$ always belongs to the relative interior of the (convex) set of all pairs $(x(t_0),x(t_1))$ corresponding to trajectories for which the integral on the left of (7) is finite, and also that $\bar{x}(t)$ does not touch the boundary of the natural "state constraint set"

$$\{x \in R^n \mid v \in R^n \text{ with } L(t,x,v) < \infty\}.$$

(If the second condition fails, a more general theory must be invoked in which p(t) is not absolutely continuous and may have jumps. The corresponding version of the Hamiltonian equation has been developed in [3]. This is indeed the situation that must be dealt with in economic applications where x(t) is a nonnegative vector of goods, some components of which may well vanish from time to time.)

In economics, the variables p(t) usually have an interpretation as prices of some kind. It is of great interest, therefore, that they have optimality properties relative to a function M dual to L, namely

$$M(t,p,w) := \sup_{(x,v)\in R^{2n}} \{p.v+x.w-L(t,x,v)\},$$

$$L(t,x,v) := \sup_{(p,w)\in R^{2n}} \{p.v+x.w-M(t,p,w)\}.$$

THEOREM 2[1]. *If* x *is an extremal for* L *with co-extremal* p, *then* p *is an extremal for* M *with co-extremal* x, *and hence in particular* p *is piecewise optimal for* M.

For the case of the economic model (4), one obtains

$$(8)\qquad H(t,x,p) = \sup_{v\in R^n} \{e^{-\rho t}U(e^{-\gamma t}x,e^{-\gamma t}v) + p.v\}$$

$$= e^{-\rho t}\, h(e^{-\gamma t}x,e^{\delta t}p)$$

where δ is the interest rate defined by

$$(9)\qquad \delta = \rho + \gamma,$$

and h is the concave-convex function defined by

$$(10)\qquad h(k,q) := \sup_{z\in R^n} \{q.z+U(k,z)\}.$$

The Hamiltonian condition (7) has a rather complicated expression in terms of $\bar{x}(t)$ and $\bar{p}(t)$, but in terms of

$$(11)\qquad \bar{k}(t) := e^{-\gamma t}\bar{x}(t),\ \bar{q}(t) := e^{\delta t}\bar{p}(t),$$

it takes the autonomous form

$$(12)\qquad \begin{aligned} \dot{\bar{k}}(t) &\in \partial_q h(\bar{k}(t),\bar{q}(t))-\gamma\bar{k}(t), \\ -\dot{\bar{q}}(t) &\in \partial_k h(\bar{k}(t),\bar{q}(t))-\delta\bar{q}(t). \end{aligned}$$

It follows from Theorem 1 that *every trajectory* k(t) *satisfying* (12) *has the piecewise optimality property in* (2) (and the converse is "almost" true).

The function dual to L in this model is

$$(13) \qquad M(t,p,w) = \sup_{(x,v)\in R^{2n}} \{p.v+x.w+e^{-\rho t}U(e^{-\gamma t}x,e^{-\gamma t}v)\}$$

$$= e^{-\rho t}V\,(e^{\delta t}p,e^{\delta t}w)$$

where

$$(14) \qquad V(q,s) := \sup_{(k,z)\in D} \{q.z+s.k+U(k,z)\}.$$

According to Theorem 2, the trajectories $q(t)$ appearing in (12) have the piecewise optimality property that for every finite subinterval $[t_0,t_1]$ one has

$$(15) \qquad \int_{t_0}^{t_1} e^{-\rho t}V(q(t),\dot{q}(t)-\delta q(t))dt \geqq \int_{t_0}^{t_1} e^{-\rho t}V(\bar{q}(t),\dot{\bar{q}}(t)-\delta\bar{q}(t))dt$$

for all trajectories $q(t)$ over $[t_0,t_1]$ with $q(t_0) = \bar{q}(t_0)$, $q(t_1) = \bar{q}(t_1)$.

Here q can be interpreted as a vector of dated prices and $r = -s$ as a vector of rents: $\dot{q} = \delta q-r$. Thus $V(q,s)$ represents the maximum rate at which "value" can be created in the economy.

A big advantage in the study of (12) (and more generally (6)) is that this condition is an "ordinary differential equation with multivalued right side". It is known, for example, that a solution $(k(t),q(t))$ exists over an interval $[t_0,t_0+\varepsilon)$ starting from any point $(k(t_0),q(t_0)) = (k_0,q_0)$ interior to the region where h is finite (cf. [6],[7]). For the most part, the solutions turn out to be unique despite the multivaluedness, although branching can sometimes occur.

In the context of the infinite horizon problem (1), a critical question is how to single out, from among the trajectories $k(t)$ with $k(0) = k_0$ that satisfy (12) for some $q(t)$ (and there seems more or less to be one such for each choice of q_0), a trajectory worthy of being deemed "optimal" (or at least "extremal") over the whole interval $[0,\infty)$. No limitations are imposed <u>a priori</u> on the behavior of $k(t)$ as $t\to\infty$ (free end-point problem). Heuristic considerations lead one to believe that there should "usually" be just one trajectory $k(t)$ of the desired type for each k_0 (in a reasonable region) and this seems to suggest a correspondence between k_0 and q_0 whose graph forms a sort of n-dimensional manifold in R^{2n}. The corresponding special trajectories $(k(t),q(t))$ would trace out this manifold.

If so, then in looking at examples of dynamical systems of the form (12) we should readily be able to detect a special n-dimensional manifold that is the natural candidate for expressing "optimality" over $[0,\infty)$. One approach to this question is to try to analyze behavior about a rest point (constant solution) to the system.

A rest point (k^*,q^*) of (12) is characterized by the relations

$$(16)\qquad \begin{aligned} &0 \in \partial_q h(k^*,q^*)-\gamma k^*,\\ &0 \in \partial_k h(k^*,q^*)-\delta q^*. \end{aligned}$$

These are equivalent to the condition that

$$(17)\qquad 0 \in \partial_q \bar{h}(k^*,q^*),\quad 0 \in \partial_k \bar{h}(k^*,q^*),$$

where

$$\bar{h}(k,q) = h(k,q)-\gamma k^*.q-\delta k.q^*,$$

and (17) means that (k^*,q^*) is a <u>minimax saddle point</u> of the function $\bar{h}$ (which, like h, is concave-convex). What might this imply for the behavior of Hamiltonian system (12) around (k^*,q^*)?

If h were actually twice differentiable, it would be possible to write the system in the form $(\dot{k},\dot{q}) = F(k,q)$ and analyze the behavior in terms of the matrix of derivatives of F at (k^*,q^*) in the classical manner of the theory of ordinary differential equations. If h were in fact strongly concave in k and strongly convex in q, the Jacobian of F with respect to k would be negative definite at k^*, while the Jacobian with respect to q would be positive definite. Thus the matrix in question would have n negative and n positive eigenvalues, so that system would have a <u>dynamic saddle point</u> at (k^*,q^*). This means that there would exist (locally) an n-dimensional manifold traced by the solutions $(k(t),q(t))$ that coverage to (k^*,q^*) as $t\to\infty$, as well as another n-dimensional manifold traced by the solutions that diverge form (k^*,q^*) at $t = -\infty$, the two manifolds intersecting only in the point (k^*,q^*) itself.

Karl Shell focused on this idea in his study of economic growth models and was led to conjecture that the picture of dynamic saddle point behavior should generalize somehow to the case where h is <u>not</u> differentiable. Moreover, the trajectories that are "optimal" over $[0,\infty)$ should be the ones converging to (k^*,q^*) as $t\to\infty$. For the economic background, see the articles [8] and [9] of Cass and Shell.

This conjecture was verified by Rockafellar in [10] for the case $\rho = 0$ $(\delta = \gamma)$ with h strictly concave-convex and in [11] for $\rho > 0$ $(\delta > \gamma)$ with h strongly concave-convex. (There is a mistake in the proof of Proposition 2' of [11] which invalidates the assertions made in the article about the complementary manifold of Hamiltonian trajectories diverging from (k^*,q^*) at $t = -\infty$ when $\rho > 0$, but this does not affect the main results, which concern the trajectories converging to (k^*,q^*).) In the case of $\rho = 0$, "optimality" must be interpreted in a certain relative sense. For $\rho > 0$, it is necessary to limit attention in (1) to trajectories $k(t)$ which do not grow at a rate faster than ρ. It must also be supposed that ρ is not too large.

The complications involved in establishing "true" optimality of some sort, and the serious restrictions on the nature of h and ρ that are entailed, bring one to the view that "optimality" over $[0,\infty)$ may not be the natural concept to be aiming at in models like (1). The justification usually given for the infinite horizon is that it enables one to avoid the selection of a particular terminal time τ and the awkward decision about what the levels of goods or prices should be at that time. However, there are other ways of avoiding this dilemma.

For example, one could consider for each time τ the trajectories $k(t)$ that would solve (1) with ∞ replaced by τ (no constraint being imposed on $k(\tau)$) and then see what trajectories these converge to as $\tau\to\infty$. Such limit trajectories would be a natural object of study. They would again be "piecewise optimal", but not necessarily optimal in any sense with respect to the integral (1) over $[0,\infty)$ (which anyway might not be well defined). There is reason to believe that this is the desired class of trajectories that exhibits the dynamic saddle point behavior (approaching a rest point as $t\to\infty$) in the many cases where the Hamiltonian system has such behavior and yet "optimality over $[0,\infty)$" cannot be established.

Convex Processes. The subject of discussion is related more closely than might be supposed to the theory of economic models in which the evolution of the state $X(t)$ (a vector of goods, resources, labor, etc.) is governed by $(X(t),\dot{X}(t)) \in T$, where T is a nonempty closed convex set in $R^N \times R^N$. In such a setting there is no real loss of generality (and considerable advantage) in taking T to be a cone and writing the dynamics in the form

(18) $\dot{X}(t) \in A(X(t))$.

Since the graph of the multifunction A is a closed convex cone containing the origin, A is called a closed convex process. The general theory of convex processes has been developed in [5, §39]; for the special "monotone" and "polyhedral" cases, see [12] and [13], respectively. Convex processes play a large role in the 1973 book of Makarov and Rubinov on economic dynamics (translated 1977 by Springer-Verlag [14]).

If we associate with A the convex function

(19) $$L(X,V) := \begin{cases} 0 \text{ if } V \in A(X), \\ +\infty \text{ if } V \notin A(X), \end{cases}$$

the problem (3) appears rather degenerate. Indeed, one has

(20) $$\int_{t_0}^{t_1} L(X(t),\dot{X}(t))dt = \begin{cases} 0 \text{ if } X \text{ satisfies (18)} \\ +\infty \text{ otherwise} \end{cases} .$$

Nevertheless, the corresponding Hamiltonian system is very interesting. The Hamiltonian function is

(21) $$H(X,P) := \sup_{V \in A(X)} P.V .$$

This is not only concave in X and convex in P but positively homogeneous in each of these variables separately. For each $P \in R^N$, let

(22) $$A^*(P) := \{W \mid W.X \geq P.V, \forall X, V \in A(X)\}.$$

(The multifunction A^* is the closed convex process adjoint to A.) The Hamiltonian condition

(23) $$\dot{\bar{X}}(t) \in \partial_P H(\bar{X}(t),\bar{P}(t)), \; -\dot{\bar{P}}(t) \in \partial_X H(\bar{X}(t),\bar{P}(t)),$$

is then equivalent to

$$(24)\quad \begin{aligned} &\sup \{\bar{P}(t).V \mid V \in A(\bar{X}(t))\} \text{ attained at } \dot{\bar{X}}(t),\\ &\inf \{W.X(t) \mid W \in A^*(\bar{P}(t))\} \text{ attained at } -\dot{\bar{P}}(t). \end{aligned}$$

It can also be written simply as

$$(25)\quad \begin{aligned} &\dot{\bar{X}}(t) \in A(\bar{X}(t)),\ -\dot{\bar{P}}(t) \in A^*(\bar{P}(t)),\\ &\bar{P}(t).\dot{\bar{X}}(t) \equiv -\dot{\bar{P}}(t).\bar{X}(t). \end{aligned}$$

Observe that the last relation is equivalent to

$$(26)\quad \bar{X}(t).\bar{P}(t) \equiv \text{const.}$$

Trajectories $\bar{X}(t)$ which satisfy (25) for some $\bar{P}(t) \not\equiv 0$ are said to be price-supported or competitive. It is remarkable that such trajectories and their "supports" can be generated by solving the ordinary differential "equation" (23) from arbitrary initial points (X_0, P_0) inside the region where H is finite, just as with the Hamiltonian systems discussed earlier [6] ,[7] .

The origin (0,0) is always a rest point of (23), but it usually lies on the boundary of the region where H is finite. A more promising class of points for study is obtained through change of variables. Setting

$$(27)\quad \bar{K}(t) := e^{-\gamma t}\bar{X}(t),\ \bar{Q}(t) := e^{\delta t}\,\bar{P}(t),$$

for arbitrary real numbers γ and δ, one can express the Hamiltonian condition in the form

$$(28)\quad \begin{aligned} &\dot{\bar{K}}(t) \in \partial_P H(\bar{K}(t),\bar{Q}(t)) - \gamma\bar{K}(t),\\ &-\dot{\bar{Q}}(t) \in \partial_X H(\bar{K}(t),\bar{Q}(t)) - \delta\bar{Q}(t), \end{aligned}$$

or equivalently

$$(29)\quad \begin{aligned} &\dot{\bar{K}}(t)+\gamma\bar{K}(t) \in A(\bar{K}(t)),\ -\dot{\bar{Q}}(t)+\delta\bar{Q}(t) \in A^*(\bar{Q}(t)),\\ &\bar{Q}(t).(\dot{\bar{K}}(t)+\gamma\bar{K}(t)) = (-\dot{\bar{Q}}(t)+\delta\bar{Q}(t)).\bar{K}(t). \end{aligned}$$

(This transformation makes use of the homogeneity of H.) A rest point

(K^*, Q^*) of the transformed system is characterized by the relations

$$\gamma K^* \in A(K^*),\ \delta Q^* \in A^*(Q^*),$$

(30)

$$0 = (\delta-\gamma)K^*.Q^* = \rho K^*.Q^*$$

(where (9) is used now as the definition of ρ).
The study of the vectors K^* and Q^* satisfying (3) for various choices of γ and δ amounts to the generalized eigenvalue theory for the process A and its adjoint. In the case of A "monotone", it is closely related to the theory of growth and interest rates for the Gale-Von Neumann model (cf. [12], [13], [14]). Presumably the dynamic system (28) should exhibit a kind of "turnpike" behavior around rest points (K^*,Q^*) in (30) for which $K^*.Q^* \neq 0$ (implying $\delta = \gamma$), or in other words, such that (K^*,Q^*) is a "nondegenerate" minimax saddle point for the concave-convex function

$$\phi_\lambda(K,Q) := H(K,Q) - \lambda K.Q \quad (\lambda = \delta = \gamma).$$

It would be interesting to see this worked out in detail, which has not yet been done. The "turnpike" behavior should correspond to the geometric picture of a dynamic saddle point.

In fact, the theory of the inhomogeneous Hamiltonian system (12) can be recast to fit the mold of a homogeneous system associated with a closed convex process A. Consider a decomposition $R^N = R \times R^n \times R$ with corresponding notation

(31) $\quad X = (x_\ell, x, x_c),\ P = (p_\ell, p, p_c).$

Let the graph of A be the closure of the set of all pairs

$$(X,V) = (x_\ell, x, x_c, v_\ell, v, v_c)$$

such that

(32) $\quad x_\ell > 0,\ v_c \leqq x_\ell U(x/x_\ell, v/x_\ell) + \delta x_c,\quad v_\ell = \gamma x_\ell.$

The graph of the adjoint A^* is then the closure of the set of all pairs

$$(P,S) = (p_\ell, p, p_c, s_\ell, s, s_c)$$

such that

(33) $\quad p_c > 0,\ s_\ell \geqq p_c\, V(p/p_c, -s/p_c) + \gamma p_\ell,\ s_c = \delta p_c,$

where V is the convex function in (14). The corresponding Hamiltonian is

(34) $\quad H(X,P) = x_\ell p_c h(x/x_\ell, p/p_c) + \gamma x_\ell p_\ell + \delta x_c p_c$

$$\text{for } x_\ell > 0,\ p_\ell > 0,$$

where h is given by (10). (For $x_\ell = 0$ or $p_\ell = 0$, the values of H are obtained from (34) by a limit process; for $x_\ell < 0$ or $p_\ell < 0$, the values of H are infinite.)

The dynamical relation $\dot{X} \in A(X)$ reduces under (32) to

(35)
$$x_\ell(t) = \alpha e^{\gamma t} \quad (\alpha > 0)$$
$$\dot{x}_c(t) \leqq \alpha e^{\gamma t}\, U(e^{-\gamma t} x(t)/\alpha, e^{-\gamma t}\dot{x}(t)/\alpha) + \delta x_c(t).$$

The interpretation is that x_ℓ represents a basic factor that grows at a constant rate γ (positive, negative or zero!); the parameter α merely sets the scale and can just as well be chosen as 1. The variable x_c measures "utility satisfaction" and is typically negative; it would grow (more negative) at the rate δ if this tendency were not counteracted by continual inputs of utility dependent on the vectors x/x_ℓ and $\dot{x}/x_\ell$ (quantities of goods per unit of the basic factor). Similarly, the dual dynamical relation $-\dot{P} \in A^*(P)$ reduces under (33) to

(36)
$$p_c(t) = \beta e^{-\delta t} \quad (\beta > 0)$$
$$-\dot{p}_\ell(t) \geqq \beta e^{-\delta t}\, V(e^{\delta t} p(t)/\beta, e^{\delta t}\dot{p}(t)/\beta) + \gamma p_\ell(t).$$

Again β is just a scale parameter that can be taken as 1.

The Hamiltonian system (23) for the function (34) takes on a particularly simple form when expressed equivalently as in (28) in terms of

$$(\bar{k}_\ell(t), \bar{k}(t), \bar{k}_c(t)) = e^{-\gamma t}(\bar{x}_\ell(t), \bar{x}(t), \bar{x}_c(t)),$$

$$(\bar{q}_\ell(t), \bar{q}(t), \bar{q}_c(t)) = e^{\delta t}(\bar{p}_\ell(t), \bar{p}(t), \bar{p}_c(t)),$$

namely:

$$\bar{k}_\ell(t) \equiv \alpha, \ \bar{q}_c(t) \equiv \beta,$$

$$\dot{\bar{k}}(t) \in \alpha\partial_q h(\bar{k}(t)/\alpha, \bar{q}(t)/\beta) - \gamma\bar{k}(t)$$

$$(37) \qquad -\dot{\bar{q}}(t) \in \beta\partial_k h(\bar{k}(t)/\alpha, \bar{q}(t)/\beta) - \delta\bar{q}(t)$$

$$\dot{\bar{k}}_c(t) = \alpha U(\bar{k}(t)/\alpha, [\dot{\bar{k}}(t) + \gamma\bar{k}(t)]/\alpha) + \rho\bar{k}_c(t)$$

$$-\dot{\bar{q}}_\ell(t) = \beta V(\bar{q}(t)/\beta, [\dot{\bar{q}}(t) - \delta\bar{q}(t)]/\beta) - \rho\bar{q}_\ell(t).$$

Taking $\alpha = 1 = \beta$, one can write this as the previous system (12) for h, augmented by the equations (for all $\tau > 0$):

$$(38) \qquad \begin{aligned} \bar{k}_c(\tau) &= e^{\rho\tau}[\bar{k}_c(0) + \textstyle\int_0^\tau e^{-\rho t} U(\bar{k}(t), \dot{\bar{k}}(t) + \gamma\bar{k}(t))dt], \\ \bar{q}_\ell(\tau) &= e^{\rho\tau}[\bar{q}_\ell(0) - \textstyle\int_0^\tau e^{-\rho t} V(\bar{q}(t), \dot{\bar{q}}(t) - \delta\bar{q}(t))dt]. \end{aligned}$$

This demonstrates that the inhomogeneous system (12) can indeed be treated in terms of a special case of the homogeneous system (28). The analysis of rest points carries over at the same time. As a matter of fact, for the convex process A in question, a vector pair

$$(39) \qquad K^* = (1, k^*, k_c^*), \quad Q^* = (q_\ell^*, q^*, 1),$$

is a rest point for (28) (i.e. satisfies (30)) if and only if (k^*, q^*) is a rest point for (12) and (from (37))

$$(40) \qquad \rho k_c^* = -U(k^*, \gamma k^*), \quad \rho q_\ell^* = V(q^*, -\delta q^*).$$

Of course, due to the special way the numbers γ and δ enter the definition of A, they are then unique values for which (30) has a solution $K^* \neq 0$, $Q^* \neq 0$.

It is interesting to note that the rest points (K^*, Q^*) just described necessarily have $K^*.Q^* = 0$, however. Despite this, the analysis of the homogeneous system around (K^*,Q^*) is important, because it corresponds to the inhomogeneous system. Thus one apparently should not, in the general study of (28), limit attention to rest points (30) such that $K^*.Q^* \neq 0$.

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