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Deterministic and Stochastic Optimization Problems of Bolza Type in Discrete Time

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In this paper we consider deterministic and stochastic versions of discrete time analogs of optimization problems of the Bolza type. The functionals are assumed to be convex, but we make no differentiability assumptions and allow for the explicit or implicit presence of constraints both on the state x_t and the increments Δx_t. The deterministic theory serves to set the stage for the stochastic problem. We obtain optimality conditions that are always sufficient and which are also necessary if the given problem satisfies a strict feasibility condition and, in the stochastic case, a bounded recourse condition. This is a new condition that bypasses the uniform boundedness restrictions encountered in earlier work on related problems.

1. INTRODUCTION

In the classical calculus of variations, a problem of Bolza type is one where a functional of the form

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$$I(x) := l(x(t_0), x(t_1)) + \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))\, dt \tag{1.1}$$

is minimized over a space of arcs $x:[t_0, t_1] \to \mathbb{R}^n$ subject to a system of equations and inequality constraints on the endpoint pair $(x(t_0), x(t_1))$ and the triple $(t, x(t), \dot{x}(t))$. This fundamental dynamical model has in recent years been a focus of efforts towards developing a variational theory not so dependent on smoothness assumptions, and in which more light can be shed on phenomena of duality. In this theory, the constraints are represented by allowing l and $L(t, \cdot, \cdot)$ to be extended-real-valued functions on $\mathbb{R}^n \times \mathbb{R}^n$, and optimality conditions are expressed in terms of subgradients; see [2, 13].

Our aim here is to treat the analog of this problem in discrete time, imposing convexity assumptions that lead to a close connection between the optimality conditions we derive and a certain dual problem. After taking care of the deterministic case, which is mainly a matter of applying well-known results in convex analysis to a particular situation, we study the stochastic version of this class of optimization problems. The significant new feature, not present in the functional form (1.1), is a process that models the flow of information. Decisions taken at any time t can only depend on the information collected about past random events, the future being known only in a probabilistic sense. Whereas in the deterministic model the decision maker has at any time total information about past and future costs associated with any plan, in the stochastic model at any time t, the uncertainty about the actual cost of any decision plan can only be mitigated by past observations.

In the deterministic problem in discrete time, we consider in place of an arc $x:[t_0, t_1] \to \mathbb{R}^n$ a vector

$$x := (x_0, x_1, \ldots, x_T) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n := (\mathbb{R}^n)^{T+1}$$

and in place of $\dot{x} := dx/dt$ the difference

$$\Delta x_t := x_t - x_{t-1} \quad \text{for} \quad t = 1, \ldots, T.$$

The problem has the form:

(P_{det}) minimize over all $x = (x_0, x_1, \ldots, x_T) \in (\mathbb{R}^n)^{T+1}$ the function

$$j(x) := l(x_0, x_1) + \sum_{t=1}^{T} L_t(x_{t-1}, \Delta x_t),$$

where l and L_t for $t = 1, \ldots, T$ are functions from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$, none of which is identically $+\infty$. We assume these functions are *lower semicontinuous* and *convex*. Then j, too, is lower semicontinuous and convex with values in $\mathbb{R} \cup \{+\infty\}$; we suppose it is not identically $+\infty$.

It is essential to appreciate the fact that in (P_{det}) there are certain constraints implicit in the condition $j(x) < \infty$, which is prerequisite to a vector x being of interest in the minimization. Letting

$$C := \{(a_0, a_T) \in \mathbb{R}^n \times \mathbb{R}^n \mid l(a_0, a_T) < \infty\}. \tag{1.2}$$

$$F_t(z_t) := \{w_t \in \mathbb{R}^n \mid L_t(z_t, w_t) < \infty\}, \tag{1.3}$$

we can, without loss of generality, restrict attention in (P_{det}) to minimizing $j(x)$ over the set of all $x \in (\mathbb{R}^n)^{T+1}$ which satisfy

$$(x_0, x_T) \in C, \tag{1.4}$$

$$\Delta x_t \in F_t(x_{t-1}) \quad \text{for} \quad t = 1, \ldots, T. \tag{1.5}$$

Conversely, if our starting point is a problem of minimizing a function of the form $j(x)$ over all the vectors x which satisfy such a system of constraints, we can pose this as a problem (P_{det}) simply by (re-)defining l to be $+\infty$ everywhere outside of the set C, and L_t to be $+\infty$ everywhere outside the graph of the multifunction F_t.

Implicit in the dynamical constraint (1.5) is the state constraint

$$x_{t-1} \in Z_t \quad \text{for} \quad t = 1, \ldots, T, \tag{1.6}$$

where

$$Z_t := \{z_t \in \mathbb{R}^n \mid F_t(z_t) \neq \emptyset\}. \tag{1.7}$$

Note that the dynamical constraint could also be put in "control" form simply by introducing a parameterization of the sets $F_t(z_t)$ by a parameter vector u_t ranging over some other set U_t, although we will not concern ourselves with such additional structure here.

The stochastic version of our problem requires an underlying probability space $(\Omega, \mathcal{A}, \mu)$ and a nest $\mathcal{G}$ of σ-fields:

$$\mathcal{G} := \{\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_T\}, \quad \text{where} \quad \mathcal{G}_0 \subset \mathcal{G}_1 \subset \ldots \subset \mathcal{G}_T \subset \mathcal{A}. \tag{1.8}$$

The field $\mathcal{G}_t$ represents information available at time t, and to say that a function $x_t: \Omega \to \mathbb{R}^n$ is $\mathcal{G}_t$-measurable is to say that $x_t(\omega)$ can depend on such information only, not on unobserved details of past events, or on random events still in the future. Accordingly we restrict attention in our decision-making process to the (closed) linear function space

$$\mathcal{N} := \{x = (x_0, x_1, \ldots, x_T) \in \mathcal{L}^\infty(\Omega, \mathcal{A}, \mu; (\mathbb{R}^n)^{T+1}) \mid x_t \text{ is } \mathcal{G}_t\text{-measurable}\}. \tag{1.9}$$

The elements x of this space are said to be *nonanticipative* (with respect to the system $\mathcal{G}$ in (1.8)). The stochastic optimization problem is

(P_{sto}) minimize over all $x = (x_0, x_1, \ldots, x_T) \in \mathcal{N}$ the functional

$$J(x) := l(E\{x_0(\omega)\}, E\{x_T(\omega)\}) + E\left\{\sum_{t=1}^{T} L_t(\omega, x_{t-1}(\omega), \Delta x_t(\omega))\right\}.$$

Here $\Delta x_t = x_t - x_{t-1}$ is $\mathcal{G}_t$-measurable and x_{t-1} is $\mathcal{G}_{t-1}$-measurable.

As in the deterministic case, l and the functions $L_t(\omega, \cdot, \cdot)$ for each $t = 1, \ldots, T$ and $\omega \in \Omega$ are convex and lower semicontinuous from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$, not identically $+\infty$. We assume also that the epigraph of $L_t(\omega, \cdot, \cdot)$ depends $\mathcal{G}_t$-measurably on ω, or in other words, that L_t is a $\mathcal{G}_t$-*normal integrand* on $\Omega \times \mathbb{R}^n \times \mathbb{R}^n$ [12, p. 173]. Among other things, this ensures that whenever $z_t(\omega)$ and $w_t(\omega)$ are $\mathcal{G}_t$-measurable in ω, so is $L_t(\omega, z_t(\omega), w_t(\omega))$ [12, Cor. 2B]. Then, certainly, the term $L_t(\omega, x_{t-1}(\omega), \Delta x_t(\omega))$ is $\mathcal{G}_t$-measurable for any $x \in \mathcal{N}$. Last among our basic assumptions on L_t is the condition that

for every $\rho > 0$ and $\sigma > 0$ there is a *summable* function $\gamma: \Omega \to R$ (i.e., integrable with finite integral $E\{\gamma(\omega)\}$) such that

$$L_t(\omega, z_t, w_t) \geqq \gamma(\omega) \quad \text{a.s. when} \quad |z_t| \leqq \rho, \quad |w_t| \leqq \sigma. \tag{1.10}$$

From this it follows that for any $x \in \mathcal{N}$, each of the terms $L_t(\omega, x_{t-1}(\omega), \Delta x_t(\omega))$ in (P_{sto}) majorizes a summable function of ω and therefore has a well-defined expectation, finite or $+\infty$. Thus J is a well-defined functional on $\mathcal{N}$ with values in $\mathbb{R} \cup \{+\infty\}$. In fact J is convex and lower semicontinuous (with respect to the $\mathcal{L}^\infty$-norm topology on $\mathcal{N}$). We suppose $J(x) < \infty$ for at least one $x \in \mathcal{N}$.

Certain constraints are implicit in the stochastic problem, just as in the deterministic problem, because only the elements x of $\mathcal{N}$ which satisfy $J(x) < \infty$ can be candidates for the minimum of J. Let

$$F_t(\omega, z_t) := \{w_t \in \mathbb{R}^n \mid L_t(\omega, z_t, w_t) < \infty\}, \tag{1.11}$$

$$Z_t(\omega) := \{z_t \in \mathbb{R}^n \mid F_t(\omega, z_t) \neq \emptyset\}. \tag{1.12}$$

Every $x \in \mathcal{N}$ with $J(x) < \infty$ must satisfy (see (1.2))

$$(E\{x_0(\omega)\}, E\{x_T(\omega)\}) \in C, \tag{1.13}$$

$$\Delta x_t(\omega) \in F_t(\omega, x_{t-1}(\omega)) \quad \text{a.s. for} \quad t = 1, \ldots, T; \tag{1.14}$$

and consequently

$$x_{t-1}(\omega) \in Z_t(\omega) \quad \text{a.s. for} \quad t = 1, \ldots, T. \tag{1.15}$$

Thus in (P_{sto}) the minimization could be restricted to those $x \in \mathcal{N}$ that satisfy these constraints, rather than over all of $\mathcal{N}$.

The multifunction $Z_t: \omega \to Z_t(\omega)$ is closed-valued under the bounded recourse condition to be given in Section 4 (Definition 1), and it is then also $\mathcal{G}_t$-measurable by virtue of the $\mathcal{G}_t$-normality of L_t. (Namely, $Z_t(\omega)$ is a certain projection of the epigraph of $L_t(\omega, \cdot, \cdot)$, which depends $\mathcal{G}_t$-measurably on ω; see [12, Cor. 1P] for the measurability of projections of multifunctions.) The need for a stronger measurability property of Z_t is suggested, however, by our implicit constraint in (P_{sto}) that $x_{t-1}(\omega) \in Z_t(\omega)$ almost surely, where

x_{t-1} is $\mathscr{G}_{t-1}$-measurable. Unless Z_t is actually $\mathscr{G}_{t-1}$-measurable, we cannot very realistically work with such a constraint, because otherwise $x_{t-1}(\omega)$ cannot fully respond to all the possible variations in $Z_t(\omega)$. For this reason the assumption of $\mathscr{G}_{t-1}$-measurability of Z_t will enter the theorems formulated in Sections 4 and 5.

We have already mentioned earlier that the information process is a significant feature of the stochastic version (P_{sto}) of our problem. We have modeled it here by an increasing sequence of σ-fields $\mathscr{G}_t$, $t=0,\ldots,T$. Each $\mathscr{G}_t$ represents the field generated by the information-events accessible to the decision maker in time period t. We implicitly assume that there is no loss of information from one time period to the next, since for all t, $\mathscr{G}_{t-1}\subset\mathscr{G}_t$. To gauge the flexibility of this modeling of the information process, it is convenient to introduce the increasing sequence of σ-fields $\mathscr{F}_t\subset\mathscr{A}$, $t=0,\ldots,T$. Each $\mathscr{F}_t$ is the σ-field generated by the random events that occur before or at time t. If at time t we only possess partial information about past occurrences, then $\mathscr{G}_t\subset\mathscr{F}_t$ and we can compute the expected value of the information loss as

$$\inf_{x\in\mathscr{N}} J(x)-\inf_{x\in\mathscr{N}_{\mathscr{F}}} J(x), \tag{1.16}$$

where

$$\mathscr{N}_{\mathscr{F}}:=\{x=(x_0,\ldots,x_T)\,|\,x_t \text{ is } \mathscr{F}_t\text{-measurable}\}. \tag{1.17}$$

The quantity in (1.16) is nonnegative, since $\mathscr{G}_t\subset\mathscr{F}_t$ implies $\mathscr{N}\subset\mathscr{N}_{\mathscr{F}}$. In this case it is instructive to view the restriction of the decision process to $\mathscr{N}$ as the result of a double constraint. First a (strict) nonanticipativity constraint, x_t cannot anticipate any future events, which implies that it needs to be $\mathscr{F}_t$-measurable, and second a (partial) information constraint, x_t can only depend on the information collected about these events, i.e., we need to restrict x_t further to $\mathscr{G}_t$-measurability. The (marginal) prices associated with the constraint $x\in\mathscr{N}\subset\mathscr{L}^\infty$ can be decomposed in two parts corresponding to the strict nonanticipativity and the partial information restrictions.

But the cases of partial or total information are not the only ones covered by our model. In fact, it handles the situation equally well when for all t, $\mathscr{G}_t\supset\mathscr{F}_t$, or when there is no inclusion in one direction or the other. The case $\mathscr{G}_t\supset\mathscr{F}_t$ would model the situation when the decision maker has access to a predictor, whereas in the latter case some events would only be partially observable and others could be predicted to some extent. However, our model does not include the case of information loss (the $\mathscr{G}_t$'s not necessarily increasing), or some situations when there is only partial observation and the $\mathscr{G}_t$'s depend on previous decisions. For further details about information patterns see [3], and for a somewhat different approach, [1].

Concerning the form of the boundary expression $l(E\{x_0(\omega)\},E\{x_T(\omega)\})$ in our functional J, the reader may wonder why we do not aim rather at something like $E\{l(\omega,x_0(\omega)),x_T(\omega))\}$. The answer is that this would not actually add much generality, but would tend to mess up the approach we wish to follow by way of duality. Indeed, the terms $E\{L_0(\omega,x_0(\omega),x_1(\omega)-x_0(\omega))\}$ and $E\{L_T(\omega,x_{T-1}(\omega),x_T(\omega)-x_{T-1}(\omega))\}$ appearing in the formula for J already allow free incorporation of terms of the kind $E\{l_0(\omega,x_0(\omega))\}$ and $E\{l_T(\omega,x_T(\omega)\}$ into the quantity to be minimized.

In our earlier work [14, 15, 16], various technical conditions led us to impose uniform boundedness restrictions on the set of feasible solutions. Such restrictions also appear in the related work of Eisner and Olsen [5, 6], Dynkin [4] and Evstigneev [7, 8]. (They are partially skirted by Hiriart-Urruty [9] because he deals with the nonconvex case and does not seek any duality relations.) Here we go a long way towards removing these boundedness conditions. The *bounded recourse condition*, as defined in Section 3, no longer requires that the set of feasible solutions be uniformly bounded, but—up to an integrability condition—it only requires that the feasible solutions, which at time t pass through a bounded set, can be "boundedly" extended. By this it is meant that there exists a feasible extension of these solutions to time period $t+1$ which is also contained in a bounded set. This condition is essential in the derivation of the necessary conditions. For stochastic problems of the Bolza type, the bounded recourse condition compliments the usual strict feasibility condition required to obtain the existence of dual (co-state) variables. The appropriate strict feasibility conditions, *cf.* Definition 2, are somewhat weaker than those we have used in the past [15, 16] but this must be attributed to the special structure of the problem, in particular to the form of the endpoint conditions.

The restriction of the decision processes to the space of essentially bounded measurable functions is chiefly for technical reasons that have mostly to do with the necessity argument. Actually, it is not difficult to see that the optimality conditions given in Theorem 4 are sufficient for any $\mathscr{L}^p$ space, $p \geq 1$, provided that the integrability condition (1.10) be appropriately strengthened.

The prospect of studying stochastic Bolza problems in continuous time, as limits of sequences of discrete time problems, provides some of the motivation for this study. At this time, however, there are major technical obstacles that need to be overcome in carrying out such a program.

2. OPTIMALITY IN THE DETERMINISTIC PROBLEM

Solutions to problem (P_{det}) will now be characterized by relations analogous to those known for deterministic problems in continuous time, where the functional (1.1) is minimized [1, 11]. These conditions involve *subgradients* of the convex functions l and L_t. Recall that for a convex function $g:\mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$, the *subgradient set* $\partial g(u)$ consists of all the vectors $v \in \mathbb{R}^m$ such that $g(u') \geq g(u) + v \cdot (u'-u)$ for all $u' \in \mathbb{R}^m$. Equivalently,

$$v \in \partial g(u) \Leftrightarrow \inf_{u'} \{g(u') - v \cdot u'\} \text{ is attained at } u' = u. \tag{2.1}$$

See [10] for more on subgradients and their properties.

A key to the optimality condition we shall be looking at is provided by the function $\phi:(\mathbb{R}^n)^{T+1} \to \mathbb{R} \cup \{\pm\infty\}$ defined for $y = (y_0, y_1, \ldots, y_T) \in (\mathbb{R}^n)^{T+1}$ by

$$\phi(y) := \inf\left\{ l(x_0 + y_0, x_T) + \sum_{t=1}^{T} L_t(x_{t-1}, \Delta x_t + y_t) \right\}. \tag{2.2}$$

This function is convex, because l and L_t are convex [10, Section 5]. Note that $\phi(0)$ is the infimum in (P_{det}). We can imagine $\phi(y)$ as the infimum obtained when (P_{det}) is "perturbed" by the parameter vector y.

THEOREM 1 *A sufficient condition for the optimality of x in problem (P_{det}) is the existence of some $p = (p_0, p_1, \ldots, p_T) \in (\mathbb{R}^n)^{T+1}$ such that*

a) $(p_0, -p_T) \in \partial l(x_0, x_T)$,

b) $(\Delta p_t, p_t) \in \partial L_t(x_{t-1}, \Delta x_t)$ *for* $t = 1, \ldots, T$.

Indeed, these relations are satisfied by x and p if and only if x solves (P_{det}) and $p \in \partial\phi(0)$.

In parallel with the continuous time case, it is appropriate to speak of (b) as the *discrete Euler–Lagrange relation* and (a) as the *transversality relation*. The pairing off of components of x and p corresponds to some extent, as will be seen below, to the "integration by parts" rule that

$$x_T \cdot p_T - x_0 \cdot p_0 = \sum_{t=1}^{T} x_{t-1} \cdot \Delta p_t + \sum_{t=1}^{T} p_t \cdot \Delta x_t. \tag{2.3}$$

The value of the observation that (a) and (b) correspond to the subgradient condition $p \in \partial\phi(0)$ is, of course, that the components p_t can be interpreted as describing generalized directional derivatives of the infimum in (P_{det}) with respect to certain perturbations.

Proof of Theorem 1 To say that x solves (P_{det}) and $p \in \partial\phi(0)$ is to say that x gives the infimum in (2.2) for $y = 0$, and $\phi(0) + p \cdot y \leq \phi(y)$ for all $y \in (\mathbb{R}^n)^{T+1}$, or in other words that the infimum of the expression

$$l(x'_0 + y_0, x'_T) + \sum_{t=1}^{T} L_t(x'_{t-1}, \Delta x'_t + y_t) - \sum_{t=0}^{T} p_t \cdot y_t \tag{2.4}$$

over all $x' \in (\mathbb{R}^n)^{T+1}$ and $y \in (\mathbb{R}^n)^{T+1}$ is attained at $x' = x, y = 0$. We must show this holds if and only if (a) and (b) are fulfilled.

A change of variables will do the job. For each choice of vectors a_0, a_T, and z_t, w_t, for $t = 1, \ldots, T$, there exist unique $x' \in (\mathbb{R}^n)^{T+1}$ and $y \in (\mathbb{R}^n)^{T+1}$ such that

$$\begin{aligned} &x'_0 + y_0 = a_0 \quad \text{and} \quad x'_T = a_T, \\ &x'_{t-1} = z_t \quad \text{and} \quad \Delta x'_t + y_t = w_t \quad \text{for} \quad t = 1, \ldots, T. \end{aligned} \tag{2.5}$$

In terms of these we can write (by means of the identity (2.3) for x'):

$$\begin{aligned}\sum_{t=0}^{T} p_t \cdot y_t &= p_0 \cdot (a_0 - x'_0) + \sum_{t=1}^{T} p_t \cdot (w_t - \Delta x'_t)\\ &= p_0 \cdot a_0 + \sum_{t=1}^{T} p_t \cdot w_t - x'_T \cdot p_T + \sum_{t=1}^{T} x'_{t-1} \cdot \Delta p_t \qquad (2.6)\\ &= p_0 \cdot a_0 - p_T \cdot a_T + \sum_{t=1}^{T} [\Delta p_t \cdot z_t + p_t \cdot w_t].\end{aligned}$$

Therefore, the infimum of (2.4) over all x', y, is attained at $x' = x$, $y = 0$, if and only if the infimum of the expression

$$l(a_0, a_T) - p_0 \cdot a_0 + p_T \cdot a_T + \sum_{t=1}^{T} [L_t(z_t, w_t) - \Delta p_t \cdot z_t - p_t \cdot w_t] \quad (2.7)$$

over all a_0, a_T, z_t, w_t, is attained at

$$a_0 = x_0, \quad a_T = x_T, \quad z_t = x_{t-1}, \quad w_t = \Delta x_t.$$

But the latter infimum is facilitated by an independence of arguments: an equivalent assertion is that

$$\inf_{a_0, a_T} \{l(a_0, a_T) - p_0 \cdot a_0 + p_T \cdot a_T\} \quad \text{is attained at} \quad (a_0, a_T) = (x_0, x_T), \qquad (2.8)$$

$$\inf_{z_t, w_t} \{L_t(z_t, w_t) - \Delta p_t \cdot z_t - p_t \cdot w_t\} \quad \text{is attained at} \quad (z_t, w_t) = (x_{t-1}, \Delta x_t).$$

This is exactly what (a) and (b) say about x and p, so Theorem 1 has been proved. □

It is clear from Theorem 1 that whenever (P_{det}) is such that $\partial\phi(0) \neq \emptyset$, the condition that there exist a p satisfying (a) and (b) for a given x is not just sufficient for the optimality of x but also necessary. Any convex function ϕ has $\partial\phi(0) \neq \emptyset$ when

$$0 \in \operatorname{ri}(\operatorname{dom} \phi), \qquad (2.9)$$

where "ri" denotes relative interior (the interior of a convex set relative to its affine hull [10, Section 6]) and

$$\operatorname{dom} \phi := \{y \mid \phi(y) < \infty\}. \qquad (2.10)$$

For the function ϕ at hand, we can reduce (2.9) to a kind of strict feasibility assumption on the constraints in (P_{det}), and this yields the next theorem.

THEOREM 2 *Suppose the constraints in (P_{det}) are such that there is at least one $\bar{x} \in (\mathbb{R}^n)^{T+1}$ with*

$$(\bar{x}_0, \bar{x}_T) \in \operatorname{ri} C, \qquad (2.11)$$

$$\bar{x}_{t-1} \in \operatorname{ri} Z_t \quad \textit{and} \quad \Delta\bar{x}_t \in \operatorname{ri} F_t(\bar{x}_{t-1}) \quad \textit{for} \quad t = 1, \ldots, T. \qquad (2.12)$$

Then for an $x \in (\mathbb{R}^n)^{T+1}$ to be optimal in (P_{det}) it is necessary, as well as sufficient, that there exist a $p \in (\mathbb{R}^n)^{T+1}$ satisfying relations (a) and (b) of Theorem 1.

Proof of Theorem 2 To represent the effective domain (2.10) of ϕ in a manner that will expedite the calculation of its relative interior, we define

$$C_t := \operatorname{dom} L_t = \operatorname{gph} F_t \quad \text{for} \quad t = 1, \ldots, T,$$

$$G := C \times C_1 \times \ldots \times C_T,$$

$$\begin{aligned}A_1(x, y) &:= A_1(x_0, \ldots, x_T, y_0, \ldots, y_T)\\ &:= (x_0 + y_0, x_T, x_0, \Delta x_1 + y_1, x_1, \Delta x_2 + y_2, \ldots, x_{T-1}, \Delta x_T + y_T)\end{aligned}$$

$$A_2(x, y) := A_2(x_0, \ldots, x_T, y_0, \ldots, y_T) := y. \qquad (2.13)$$

Here G is a convex set, A_1 and A_2 are linear transformations. Moreover,

$$y \in \operatorname{dom} \phi \Leftrightarrow \exists\, x \quad \text{with} \quad A_1(x, y) \in G. \qquad (2.14)$$

This tells us that $\operatorname{dom}\phi = A_2(A_1^{-1}(G))$. Then from the calculus of relative interiors of convex sets [10, Section 6] we have

$$\operatorname{ri}(\operatorname{dom}\phi) = A_2(A_1^{-1}(\operatorname{ri} G)),$$

where moreover

$$\operatorname{ri} G = \operatorname{ri} C \times \operatorname{ri} C_1 \times \ldots \times \operatorname{ri} C_T,$$

$$\operatorname{ri} C_t = \operatorname{ri}(\operatorname{gph} F_t) = \{(z_t, w_t) \mid z_t \in \operatorname{ri} Z_t,\ w_t \in \operatorname{ri} F_t(z_t)\}.$$

It follows that

$$0 \in \operatorname{ri}(\operatorname{dom}\phi) \Leftrightarrow \exists\, \bar{x} \quad \text{with} \quad A_1(\bar{x}, y) \in \operatorname{ri} G,$$

and that the latter condition is identical to (2.11) and (2.12). Thus the hypothesis of the theorem is equivalent to (2.9), which as we already know guarantees $\partial\phi(0) \neq \emptyset$ and thereby yields the desired conclusion. □

The next two results clarify and elaborate the strict feasibility assumed in Theorem 2.

PROPOSITION 1. *Let C' be the set of attainable endpoint pairs for the multifunctions $F_1, \ldots, F_T$:*

$$C' := \{(a_0, a_T) \in \mathbb{R}^n \times \mathbb{R}^n \mid \exists\, x \in (\mathbb{R}^n)^{T+1} \tag{2.15}$$

with

$$\Delta x_t \in F_t(x_{t-1}), \qquad t = 1, \ldots, T,$$

and

$$x_0 = a_0, \quad x_T = a_T\}.$$

Then C' is convex, and the hypothesis of Theorem 2 is satisfied if and only if

$$\operatorname{ri} C \cap \operatorname{ri} C' \neq \emptyset. \tag{2.16}$$

Proof. All one needs to do is calculate $\operatorname{ri} C'$ by the method used for $\operatorname{ri}(\operatorname{dom}\phi)$ in the Proof of Theorem 2, and the result falls out. The details will be omitted. □

PROPOSITION 2. *The hypothesis of Theorem 2 is satisfied in particular if for some $\bar{x} \in (\mathbb{R}^n)^{T+1}$, $\varepsilon > 0$ and numbers $\alpha_t \in \mathbb{R}$, for $t = 0, 1, \ldots, T$, one has*

$$l(\bar{x}_0, \bar{x}_T) \leq \alpha_0, \tag{2.17}$$

$$L_t(z_t, w_t) \leq \alpha_t \quad \textit{when} \quad |z_t - \bar{x}_{t-1}| \leq \varepsilon,\ |w_t - \Delta\bar{x}_t| \leq \varepsilon. \tag{2.18}$$

Moreover, in this case any $p \in (\mathbb{R}^n)^{T+1}$ which satisfies conditions (a) and (b) of Theorem 1 for some $x \in (\mathbb{R}^n)^{T+1}$ must have

$$\sum_{t=0}^{T} |p_t| \leq 2\left[\sum_{t=0}^{T} \alpha_t - J(x)\right]\Big/\varepsilon. \tag{2.19}$$

Proof. For any choice of vectors z_t and w_t as in (2.17) for $t = 1, \ldots, T$, consider $z_t' = z_t - \bar{x}_{t-1}$, $w_t' = w_t - \Delta\bar{x}_t$. There exist unique $x \in (\mathbb{R}^n)^{T+1}$ and $y \in (\mathbb{R}^n)^{T+1}$ satisfying

$$\begin{gathered} x_0 + y_0 = \bar{x}_0, \quad x_T = \bar{x}_T, \\ x_{t-1} = \bar{x}_{t-1} + z_t' \quad \text{and} \quad \Delta x_t + y_t = \Delta\bar{x}_t + w_t', \end{gathered} \tag{2.20}$$

and then

$$l(x_0 + y_0, x_T) + \sum_{t=1}^{T} L_t(x_{t-1}, \Delta x_t + y_t) = l(\bar{x}_0, \bar{x}_T) + \sum_{t=1}^{T} L_t(z_t, w_t)$$

and consequently

$$\phi(y) \leq \alpha_0 + \alpha_1 + \ldots + \alpha_T. \tag{2.21}$$

In particular, taking any y such that

$$|y_t| \leq \varepsilon/2 \quad \text{for} \quad t = 0, 1, \ldots, T \tag{2.22}$$

and taking

$$x_t = \bar{x}_t - [(T-t)/T]y_0 \quad \text{for} \quad t = 0, 1, \ldots, T$$

we have (2.20) holding, with

$$z_t' = -[(T-t+1)/T]y_0 \quad \text{and} \quad w_t' = y_t - (1/T)y_0,$$

and consequently

$$|z_t'| \leq |y_0| < \varepsilon \quad \text{and} \quad |w_t'| \leq |y_t| + |y_0| \leq \varepsilon.$$

This tells us that (2.21) is true whenever (2.22) is true. Thus the effective domain (2.10) of ϕ actually includes a neighborhood of 0, so that condition (2.9), which we know from the proof of Theorem 2 to be equivalent to the hypothesis of Theorem 2, is certainly satisfied.

Consider now any p and x satisfying conditions (a) and (b) of Theorem 1. We have by Theorem 1 that $j(x) = \phi(0)$ and $p \in \partial\phi(0)$, so that

$$\phi(y) \geq \phi(0) + p \cdot y = j(x) + \sum_{t=0}^{T} p_t \cdot y_t$$

for all $y \in (\mathbb{R}^n)^{T+1}$ and in particular for all y satisfying (2.22). Since (2.21) holds for such y, we obtain

$$\sum_{t=0}^{T} \alpha_T - j(x) \geq \sum_{t=0}^{T} \sup_{|y_t| \leq \varepsilon/2} p_t \cdot y_t = (\varepsilon/2) \sum_{t=0}^{T} |p_t|,$$

and this is the bound (2.19) that we needed to establish. □

The vectors p appearing in the optimality condition in Theorem 1 can be characterized by a dual variational principle, as is no surprise, inasmuch as we are dealing with a problem in the realm of convex analysis. The duality involves the functions l^* and L_t^* conjugate to l and L_t [10, Section 12]. Let

$$m(b_0, b_T) := \sup_{a_0, a_T} \{a_0 \cdot b_0 - a_T \cdot b_T - l(a_0, a_T)\} = l^*(b_0, -b_T), \quad (2.23)$$

$$M_t(q_t, r_t) := \sup_{z_t, w_t} \{q_t \cdot w_t + r_t \cdot z_t - L_t(z_t, w_t)\} = L_t^*(r_t, q_t). \quad (2.24)$$

Then m and M_t are lower semicontinuous, convex functions from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$ which are not identically $+\infty$, and l and L_t can be recovered from them by the inverse formulas

$$l(a_0, a_T) = \sup_{b_0, b_T} \{a_0 \cdot b_0 - a_T \cdot b_T - m(b_0, b_T)\} = m^*(a_0, -a_T), \quad (2.25)$$

$$L_t(z_t, w_t) = \sup_{q_t, r_t} \{q_t \cdot w_t + r_t \cdot z_t - M_t(q_t, r_t)\} = M_t^*(w_t, z_t). \quad (2.26)$$

The problem we identify as dual to (P_{det}) is

(P^*_{det}) maximize $-k(p)$ over all $p = (p_0, p_1, \ldots, p_T) \in (\mathbb{R}^n)^{T+1}$, where

$$k(p) = m(p_0, p_T) + \sum_{t=1}^{T} M_t(p_t, \Delta p_t).$$

THEOREM 3 *The inequality* $\inf(P_{\text{det}}) \geq \sup(P^*_{\text{det}})$ *always holds. One has* $p \in \partial\phi(0)$ *if and only if actually* $\inf(P_{\text{det}}) = \max(P^*_{\text{det}})$, *and* p *is optimal for* (P^*_{det}).

Proof of Theorem 3 Only a slight extension of the proof of Theorem 1 is needed. The infimum of expression (2.4) over all x', y, is by the definition of ϕ equal to

$$\inf_y \{\phi(y) - p \cdot y\} = -\phi^*(p). \quad (2.27)$$

But the change-of-variable argument in Theorem 1 showed that this was also equal to the infimum of the expression (2.7) over all a_0, a_T, z_t, w_t, which by (2.23) and (2.24) is

$$-m(p_0, p_T) - \sum_{t=1}^{T} M_t(p_t, \Delta p_t) = -k(p).$$

Therefore the latter agrees with (2.27), and for every p we have

$$-k(p)=\inf_y\{\phi(y)-p\cdot y\}\leqq\phi(0)-p\cdot 0=\inf(P_{\det}).$$

Taking the supremum with respect to p, we see that $\sup(P^*_{\det})\leqq\inf(P_{\det})$ in general. Moreover, the equation

$$-k(p)=\sup(P^*_{\det})=\inf(P_{\det})$$

holds if and only if

$$\inf_y\{\phi(y)-p\cdot y\}\text{ is attained at } y=0,$$

which is the condition $p\in\partial\phi(0)$. □

COROLLARY *Under the hypothesis of Theorem 2 (or Proposition 1) one has* $\inf(P_{\det})=\max(P^*_{\det})$.

Proof The hypothesis in question has been shown in the proof of Theorem 2 to be equivalent to condition (2.9), which guarantees that $\partial\phi(0)\neq\emptyset$. □

Remark A strict feasibility condition for $(P^*_{\det})$ can be stated in close parallel to the one for $(P_{\det})$ in Theorem 2. It implies by arguments dual to the ones above that $\min(P_{\det})=\sup(P^*_{\det})$.

3. SUFFICIENT CONDITIONS FOR OPTIMALITY IN THE STOCHASTIC PROBLEM

An optimality condition for (P_{sto}) resembling the one for $(P_{\det})$ in Theorem 1 can be formulated in terms of conditional expectations. For the *conditional expectation given* $\mathscr{G}_t$, we write E^t (for the usual but more cumbersome notation $E^{\mathscr{G}_t}$). This is taken to be a *regular* conditional expectation, i.e., representable as an indefinite integral with respect to a regular conditional probability $\mu^t(\cdot|\cdot)$ on $\mathscr{A}\times\Omega$. There is really no loss in assuming that such regular conditional probabilities exist; in practice we can always take $(\Omega,\mathscr{A},\mu)$ as the range space of certain random variables, with Ω a subset of a finite dimensional space and $\mathscr{A}$ the Borel field on Ω.

Given an $\mathscr{A}$-measurable random variable y, the observable aspects at time t are represented by $E^t y$. We shall be interested in the gain of information that can be achieved from one time period to the next. For these purposes, we introduce the operator

$$E^t_\Delta:=E^t-E^{t-1},\tag{3.1}$$

or in the more standard notation $E^t_\Delta=E^{\mathscr{G}_t}-E^{\mathscr{G}_{t-1}}$. Note that whenever $\mathscr{G}_t=\mathscr{G}_{t-1}$, which means that there is no gain of information from one time period $t-1$ to the next, the E^t_Δ terms can always be dropped. This should be kept in mind when comparing our development for the deterministic and stochastic versions of the problem.

Again a crucial role in the derivation and analysis of optimality conditions will be played by a perturbation function. For

$$y=(y_0,y_1,\ldots,y_T)\in\mathscr{L}^\infty(\Omega,\mathscr{A},\mu;(\mathbb{R}^n)^{T+1})=:\mathscr{L}^\infty\tag{3.2}$$

we define

$$\Phi(y):=\inf_{x\in\mathscr{N}}\Big\{l(E(x_0+y_0),Ex_T)$$

$$+E\sum_{t=1}^T L_t(\omega,x_{t-1}-E^t_\Delta y_{t-1},\Delta x_t+E^t_\Delta y_{t-1}+E^t y_t)\Big\},\tag{3.3}$$

where to keep notation as compact as possible we have suppressed indication of the ω argument of the functions x_t, $E_t y_{t-1}$, etc. (or, as we really should say in dealing with elements of L^∞, *equivalence classes* of functions). The functional Φ is well defined from $\mathscr{L}^\infty$ to $\mathbb{R}\cup\{\pm\infty\}$, and it is convex. In what follows, we will need to speak of its subgradients with respect to the natural pairing between functions $y\in\mathscr{L}^\infty$ and functions

$$p=(p_0,p_1,\ldots,p_T)\in\mathscr{L}^1(\Omega,\mathscr{A},\mu;(\mathbb{R}^n)^{T+1})=:\mathscr{L}^1.\tag{3.4}$$

given by

$$\langle p, y\rangle = E\left\{\sum_{t=0}^{T} p_t(\omega)\cdot y_t(\omega)\right\}. \tag{3.5}$$

The set of subgradients of Φ at y in this sense is

$$\partial\Phi(y) = \{p\in\mathscr{L}^1 \mid \Phi(y') \geqq \Phi(y) + \langle p, y'-y\rangle \quad \text{for all} \quad y'\in\mathscr{L}^\infty\}. \tag{3.6}$$

Subgradients of the functions l and $L_t(\omega,\cdot,\cdot)$ will also enter the conditions below. We write

$$\partial L_t(\omega, z, w) = \text{set of subgradients of } L_t(\omega,\cdot,\cdot) \text{ at } (z, w). \tag{3.7}$$

In other words, despite what the notation ∂L_t might suggest, we do not involve ω in the subdifferentiation.

THEOREM 4 *A sufficient condition for the optimality of $x\in\mathcal{N}$ in problem (P_{sto}) is the existence of some $p\in\mathscr{L}^1$ such that*

a) $(E^0 p_0)(\omega)\equiv b_0$ *and* $p_T(\omega)\equiv b_T$ *for some* $(b_0, -b_T)\in\partial l(Ex_0, Ex_T)$,

b) $((E^t\Delta p_t)(\omega), (E^t p_t)(\omega))\in\partial L_t(\omega, x_{t-1}(\omega), (\Delta x_t)(\omega))$ *a.s.*,

c) p_{t-1} *is* $\mathscr{G}_t$*-measurable for* $t=1,\ldots,T$.

Indeed, these relations are satisfied by $x\in\mathcal{N}$ and $p\in\mathscr{L}^1$ if and only if x solves (P_{sto}) and $p\in\partial\Phi(0)$.

In analogy to the deterministic case, we shall refer to (b) as the *stochastic discrete Euler–Lagrange relation*, (a) as the *transversality relation*. We can view $(p_t, t=0,\ldots,T)$ as a stochastic process of *latent multipliers* whose manifestations in time period t, namely

$$E^t\Delta p_t \quad\text{and}\quad E^t p_t,$$

are the usual multipliers or costate variables related to the decision variables subject to selection in period t, *cf.* condition (b). These vectors $E^t\Delta p_t$ and $E^t p_t$ are $\mathscr{G}_t$-measurable functions of ω and can therefore be calculated from knowledge of their functional form and

the observations made up to period t. Their values for particular ω do not depend on any information which, as far as period t is concerned, lies in the future. The latent multipliers $p_0,\ldots,p_T$, on the other hand, exhibit in (c) a *delayed nonanticipativity*. These optimal dual variables do depend on information to be collected in the next time period, although, by construction, not on the whole future. This property of the latent multipliers is special to the structure of the Bolza model adopted here. Ordinarily one could not expect the latent multipliers to be any better than $\mathscr{G}_T$-measurable for all t [15].

Again it should be noted that in identifying the optimality conditions in Theorem 4 with the subgradient relation $p\in\partial\Phi(0)$ we open the way to interpreting the latent multipliers in terms of directional derivatives of the optimal value in (P_{sto}) with respect to certain perturbations.

Proof of Theorem 4 The argument is patterned after the proof of Theorem 1 but has to contend with complications posed by the different information fields $\mathscr{G}_t$. To say that x solves (P_{sto}) and $p\in\partial\Phi(0)$ is to say that x furnishes the infimum in (3.3) for $y=0$, and $\Phi(0)\langle p, y\rangle \leqq \Phi(y)$ for all $y\in\mathscr{L}^\infty$. This property of $x\in\mathcal{N}$ and $p\in\mathscr{L}^1$ is equivalent to having the infimum of the expression

$$-E\sum_{t=0}^{T} p_t\cdot y_t + l(Ex_0' + Ey_0, Ex_T')$$

$$+E\sum_{t=1}^{T} L_t(\omega, x_{t-1}' - E_\Delta^t y_{t-1}, \Delta x_t' + E_\Delta^t y_{t-1} + E^t y_t) \tag{3.8}$$

over all $x'\in\mathcal{N}$ and $y\in\mathscr{L}^\infty$ be attained at $x'=x$, $y=0$. The theorem can be established by showing that this holds if and only if (a), (b) and (c) are satisfied.

As in the proof of Theorem 1, the trick is to make the right change of variables in order to separate variables in calculating the infimum. For arbitrary

vectors a_0, a_T, in $\mathbb{R}^n$ and functions s_0, s_T, u, and z_t, w_t, v_t, for $t=1,\ldots,T$, all in $\mathscr{L}^\infty(\Omega,\mathscr{A},\mu;\mathbb{R}^n)$ with z_t and w_t both $\mathscr{G}_t$-measurable, $E^t v_t=0$, $E^t u=0$, s_0 and s_T respectively $\mathscr{G}_0$- and $\mathscr{G}_T$-measurable, $Es_0=0$, $Es_T=0$, (3.9) there exist unique $x' \in \mathcal{A}^\infty$ and $y \in \mathcal{L}^\infty$ such that

$$
\begin{aligned}
&Ex_0' + Ey_0 = a_0 \text{ and } Ex_T' = a_T,\\
&x_0' + E^0 y_0 - Ex_0' - Ey_0 = s_0 \text{ and } x_T' - Ex_T' = s_T,\\
&x_{t-1}' - E_\Delta^t y_{t-1} = z_t \text{ for } t = 1, \dots, T, \qquad (3.10)\\
&\Delta x_t' + E_\Delta^t y_{t-1} + E^t y_t = w_t \text{ for } t = 1, \dots, T,\\
&y_{t-1} - E^t y_{t-1} = v_t \text{ for } t = 1, \dots, T \text{ and } y_T - E^T y_T = u.
\end{aligned}
$$

The truth of this assertion may not exactly "meet the eye", but it is not as miserable to verify as one might imagine from the complexity of the system to be solved. Namely, we observe at the outset that (3.10) implies

$$x_T' = s_T + a_T \ (\mathcal{G}_T\text{-measurable}). \qquad (3.11)$$

Next, since z_t is given as $\mathcal{G}_t$-measurable, we see by applying E^{t-1} to both sides of the equation $x_{t-1}' - E_\Delta^t y_{t-1} = z_t$ that the latter holds for a $\mathcal{G}_{t-1}$-measurable x_{t-1}' (as required by the condition $x' \in \mathcal{A}^\infty$) if and only if

$$x_{t-1}' = E^{t-1} z_t \quad \text{for} \quad t = 1, \dots, T, \qquad (3.12)$$

$$E_\Delta^t y_{t-1} = E^{t-1} z_t - z_t \quad \text{for} \quad t = 1, \dots, T. \qquad (3.13)$$

These relations with (3.11) determine a unique $x' \in \mathcal{A}^\infty$ as well as place conditions on y that must be satisfied if the system (3.10) is to be solvable at all. Another implication of (3.10) is that

$$
\begin{aligned}
E^t y_t &= w_t - \Delta x_t' - E_\Delta^t y_{t-1} = w_t - x_t' + x_{t-1}' - E_\Delta^t y_{t-1}\\
&= z_t + w_t - x_t' \quad \text{for} \quad t = 1, \dots, T.
\end{aligned} \qquad (3.14)
$$

For $t = T$ we thereby obtain, since $y_T - E^T y_T = u$, that by (3.11)

$$y_T = u + z_T + w_T - x_T' = z_T + w_T - s_T - a_T + u. \qquad (3.15)$$

From the identity

$$y_{t-1} = (y_{t-1} - E^t y_{t-1}) + E_\Delta^t y_{t-1} + E^{t-1} y_{t-1}, \qquad (3.16)$$

on the other hand, we deduce via the last condition in (3.10), combined with (3.13) and then (3.14), that

$$
\begin{aligned}
y_{t-1} &= v_t + (E^{t-1} z_t - z_t) + z_{t-1} + w_{t-1} - x_{t-1}'\\
&= v_t - z_t + z_{t-1} + w_{t-1} \quad \text{for} \quad t = 2, \dots, T,
\end{aligned} \qquad (3.17)
$$

the last by (3.12).

Finally, from (3.16) for $t = 1$ we obtain by the last condition in (3.10), then (3.13) and the second line of (3.10):

$$
\begin{aligned}
y_0 &= (y_0 - E^1 y_0) + E_\Delta^1 y_0 + E^0 y_0\\
&= v_1 + E^0 z_1 - z_1 + s_0 + Ex_0' + Ey_0 - x_0' = v_1 - z_1 + s_0 + a_0,
\end{aligned} \qquad (3.18)
$$

where in the final equality the initial condition of (3.10) is invoked along with (3.12). Equations (3.15), (3.17) and (3.18) determine a unique $\mathcal{G}_T$-measurable $y \in \mathcal{L}^\infty$ to go with the unique $x' \in \mathcal{A}^\infty$ already determined by (3.11) and (3.12), and this x' and y do satisfy (3.10), as can readily be verified.

Thus in taking the infimum of (3.8) over all $x' \in \mathcal{A}^\infty$ and $\mathcal{G}_T$-measurable $y \in \mathcal{L}^\infty$, we can just as well make the substitutions (3.10) and take the infimum subject to (3.9). Under the substitution we obviously have

$$
\begin{aligned}
&l(Ex_0' + Ey_0, Ex_T') = l(a_0, a_T),\\
&L_t(\omega, x_{t-1}' - E_\Delta^t y_{t-1}, \Delta x_t' - E_\Delta^t y_{t-1} + E^t y_t) = L_t(\omega, z_t, w_t).
\end{aligned} \qquad (3.19)
$$

Furthermore, since (3.10) entails (3.15), (3.17) and (3.18), we have

$$\begin{aligned}\sum_{t=0}^{T} p_t\cdot y_t &= p_0\cdot(v_1-z_1+s_0+a_0)+p_T\cdot(z_T+w_T-s_T-a_T+u)\\ &\quad+\sum_{t=1}^{T-1} p_t\cdot(v_{t+1}-z_{t+1}+z_t+w_t)\\ &= p_0\cdot(s_0+a_0)-p_T\cdot(s_T+a_T)+\sum_{t=1}^{T} p_t\cdot(z_t+w_t) \qquad (3.20)\\ &\quad+\sum_{t=1}^{T} p_{t-1}\cdot(v_t-z_t)+p_T\cdot u\\ &= p_0\cdot(s_0+a_0)-p_T\cdot(s_T+a_T)+\sum_{t=1}^{T}[\Delta p_t\cdot z_t+p_t\cdot w_t]\\ &\quad+\sum_{t=1}^{T} p_{t-1}\cdot v_t+p_T\cdot u.\end{aligned}$$

The conditions on s_0, s_T and z_1 in (3.9) imply also that

$$\begin{aligned}&E\{p_0\cdot(s_0+a_0)-p_T\cdot(s_T+a_T)\}\\ &\quad=(Ep_0)\cdot a_0-(Ep_T)\cdot a_T+E\{(E^0p_0-Ep_0)\cdot s_0-(E^Tp_T-Ep_T)\cdot s_T\},\end{aligned} \tag{3.21}$$

while those on v_t and u give us

$$E\left\{\sum_{t=1}^{T} p_{t-1}\cdot v_t+p_T\cdot u\right\}=E\left\{\sum_{t=1}^{T}(p_{t-1}-E^tp_{t-1})\cdot v_t+(p_T-E^Tp_T)\cdot u\right\}. \tag{3.22}$$

Therefore, when the substitutions (3.10) are made the infimum of (3.8) over all $x'\in\mathcal{N}$ and $\mathcal{G}_T$-measurable $y\in\mathcal{L}^\infty$ is converted into the

infimum of

$$\begin{aligned}&-(Ep_0)\cdot a_0+(Ep_T)\cdot a_T-E\{(E^0p-Ep_0)\cdot s_0-(E^Tp_T-Ep_T)\cdot s_T\}\\ &-E\sum_{t=1}^{T}[\Delta p_t\cdot z_t+p_t\cdot w_t]-E\left\{\sum_{t=1}^{T}(p_{t-1}-E^tp_{t-1})\cdot v_t+(p_T-E^Tp_T)\cdot u\right\}\\ &\qquad+l(a_0,a_T)+E\sum_{t=1}^{T}L_t(\omega,z_t,w_t)\end{aligned}$$

subject to (3.9). What we must show in order to prove the theorem is that (a), (b) and (c) hold for $x\in\mathcal{N}$ and $p\in\mathcal{L}^1$ if and only if this infimum is attained at

$$\begin{gathered}a_0=Ex_0,\quad a_T=Ex_T,\quad z_t=x_{t-1},\quad w_t=\Delta x_t,\quad v_t=u=0,\\ s_0=x_0-Ex_0,\quad s_T=x_T-Ex_T\end{gathered} \tag{3.24}$$

(since these are the relations which imply $x'=x$ and $y=0$ in (3.10)).

We know, of course, that the infimum in (3.23) is not $+\infty$, since the one in (3.8) is not $+\infty$ (due to our assumption in Section 1 that $J(x')<\infty$ in (P_{sto}) for at least one $x'\in\mathcal{N}$). It is possible therefore, to choose the elements in (3.9) in such a manner that the expression in (3.23) is not $+\infty$. The infimum in (3.23) can therefore be decomposed into the sum of the separate terms

$$\inf_{v_t,u} E\left\{\sum_{t=1}^{T}(p_{t-1}-E^tp_{t-1})\cdot v_t+(p_T-E^Tp_T)\cdot u\right\}, \tag{3.25}$$

$$\inf_{s_0,s_T} E\{-(E^0p_0-Ep_0)\cdot s_0+(E^Tp_T-Ep_T)\cdot s_T\}, \tag{3.26}$$

$$\inf_{a_0,a_T}\{l(a_0,a_T)-(Ep_0)\cdot a_0+(Ep_T)\cdot a_T\}, \tag{3.27}$$

$$\sum_{t=1}^{T}\inf_{z_t,w_t} E\{L_t(\omega,z_t,w_t)-E^t\Delta p_t\cdot z_t-E^tp_t\cdot w_t\}, \tag{3.28}$$

none of which can be $+\infty$. In each term, the minimization is subject to the restrictions in (3.9). In (3.25) the infimum is $-\infty$ unless

$$p_{t-1} \equiv E^t p_{t-1} \quad \text{for} \quad t=1,\ldots,T \quad \text{and} \quad p_T \equiv E^T p_T,$$

in which event it is 0 and attained at $v_t = u = 0$; similarly in (3.26), the infimum is $-\infty$ unless

$$E^0 p_0 \equiv E p_0 \quad \text{and} \quad E^T p_T \equiv E p_T,$$

in which event it is 0 and attained at $s_0 = s_T = 0$. Together then, it is impossible for the infima in (3.25) and (3.26) to be attained except when they vanish, in which event they are attained by $v_t = u = 0$, $s_0 = x_0 - Ex_0$, $s_T = x_T - Ex_T$; moreover this is the case if and only if p satisfies condition (c) of the theorem and has $(E^0 p_0) \equiv b_0$ and $p_T(\omega) \equiv b_T$ for some $(b_0, b_T) \in \mathbb{R}^n \times \mathbb{R}^n$. Then $Ep_0 \equiv b_0$ and $Ep_T \equiv b_T$, so the infimum in (3.27) is attained at $a_0 = Ex_0$ and $a_T = Ex_T$ if and only if condition (a) of the theorem holds. Finally, since L_t and the corresponding linear terms are $\mathcal{G}_t$-normal integrands and z_t and w_t can be arbitrary $\mathcal{G}_t$-measurable functions in $\mathcal{L}^1(\Omega, \mathcal{A}, \mu; \mathbb{R}^n)$, the infimum in (3.28) can be taken pointwise [12, Theorem 3A]; it reduces to

$$\sum_{t=1}^{T} E\left\{ \inf_{\substack{z_t' \in \mathbb{R}^n \\ w_t' \in \mathbb{R}^n}} \{ L_t(\omega, z_t', w_t') - E^t(\Delta p_t)(\omega) \cdot z_t' - E^t p_t(\omega) \cdot w_t' \} \right\} \tag{3.29}$$

and is attained by the functions $z_t = x_{t-1}$ and $w_t = \Delta x_t$ if and only if the infima over $\mathbb{R}^n$ in (3.29) for each ω are attained almost surely at $z_t' = x_{t-1}(\omega)$ and $w_t' = (\Delta x_t)(\omega)$. But this property is the one in condition (b). In conclusion, it is true that (a), (b) and (c) hold for an $x \in \mathcal{N}$ and $p \in \mathcal{L}^1$ if and only if the infimum of (3.23) subject to (3.9) is attained at (3.24). □

4. NECESSARY CONDITIONS FOR OPTIMALITY IN THE STOCHASTIC PROBLEM

The question now is how to know when the optimality condition in Theorem 4 is not only sufficient but necessary. From the method

used in the deterministic case, the reader may expect that all we need to do is ensure $\partial\Phi(0) \neq \emptyset$ by means of some finiteness property of Φ on an $\mathcal{L}^\infty$-neighborhood of 0. Matters are not so simple, however. The best that a finiteness property of Φ can give us is the existence of a subgradient with respect to the pairing between $\mathcal{L}^\infty$ and $(\mathcal{L}^\infty)^*$. What we want here are subgradients $p \in \mathcal{L}^1$. A general element of $(\mathcal{L}^\infty)^*$ could have, besides an $\mathcal{L}^1$ component, a "singular" component [17, 11]. To eliminate having to deal with singular components, we must make further assumptions about (P_{sto}). These assumptions will allow us to apply earlier results [15] about $\mathcal{L}^1$ multipliers for the nonanticipativity constraint $x \in \mathcal{N}$ in order to obtain the desired result.

DEFINITION 1. Problem (P_{sto}) will be said to satisfy the *bounded recourse condition* if for $t = 1, \ldots, T$,

a) for every $\rho > 0$ and $\sigma > 0$ there is a summable function $\beta : \Omega \to \mathbb{R}$ such that almost surely with respect to $\omega \in \Omega$,

$$[z_t \in Z_t(\omega) \quad \text{and} \quad |z_t| \leqq \rho, \quad w_t \in F_t(\omega, z_t) \quad \text{and} \quad |w_t| \leqq \sigma]$$
$$\Rightarrow L_t(\omega, z_t, w_t) \leqq \beta(\omega); \tag{4.1}$$

b) for every $\rho > 0$ there is a $\rho' > 0$ such that almost surely with respect to $\omega \in \Omega$,

$$[z_t \in Z_t(\omega) \quad \text{and} \quad |z_t| \leqq \rho] \Rightarrow [\exists\, w_t \in F_t(\omega, z_t) \quad \text{with}$$
$$z_t + w_t \in Z_{t+1}(\omega) \quad \text{and} \quad |z_t + w_t| \leqq \rho']; \tag{4.2}$$

interpret $Z_{T+1}(\omega)$ as all of $\mathbb{R}^n$ for this purpose.

Since $L_t(\omega, \cdot, \cdot)$ is lower semicontinuous, property (a) implies that $F_t(\omega, \cdot)$ is a multifunction with closed graph whose domain $Z_t(\omega)$ is a closed set.

The bounded recourse condition is satisfied in particular if for $t = 1, \ldots, T$ there are bounded sets $B_t \subset \mathbb{R}^n \times \mathbb{R}^n$ and summable functions β_t such that almost surely in ω the graph of the multifunction $F_t(\omega, \cdot)$ is included in B_t, and all of its elements (z_t, w_t) satisfy $L_t(\omega, z_t, w_t) \leqq \beta_t(\omega)$ and $z_t + w_t \in Z_{t+1}(\omega)$. The last requirement can be weakened to the following: for fixed ω, a vector sequence $x_0, x_1, \ldots, x_t$ that satisfies $\Delta x_\tau \in F_\tau(\omega, x_{\tau-1})$ for $\tau = 1, \ldots, t$ can be extended almost surely by $x_{t+1}, \ldots, x_T$ to a sequence that satisfies $\Delta x_\tau \in F_\tau(\omega, x_{\tau-1})$ for $\tau = 1, \ldots, T$. This special case where the bounded recourse condition is satisfied corresponds to the combination of the boundedness and essentially complete recourse conditions used in [16], except that the latter, when applied to the present situation, would also place restrictions on the endpoint set $C = \operatorname{dom} l$.

The bounded recourse condition of Definition 1 is a substantial improvement over such previous conditions, because it makes the theory applicable to evolutionary systems not necessarily modeled with bounded feasible regions, such as stochastic dynamic linear models with only nonnegativity constraints. It can be shown that a multistage stochastic linear programming problem, which can be formulated as a stochastic optimization problem of Bolza type, will satisfy the bounded recourse condition whenever the original problem satisfies the essentially complete recourse condition and the matrices involved satisfy a condition somewhat weaker than full row rank. The feasibility sets need not be bounded, much less uniformly bounded.

Definition 2 Problem (P_{sto}) will be said to satisfy the *interior feasibility condition* if for some $\bar{x} \in \mathcal{N}$, $\varepsilon > 0$, and summable functions $\alpha_t : \Omega \to \mathbb{R}$, one has

$$(E\bar{x}_0, E\bar{x}_T) \in C, \tag{4.3}$$

and for $t = 1, \ldots, T$ almost surely with respect to $\omega \in \Omega$, also

$$z_t \in Z_t(\omega), \quad w_t \in F_t(\omega, z_t) \quad \text{and} \quad L_t(\omega, z_t, w_t) \leqq \alpha_t(\omega) \tag{4.4}$$

whenever

$$|z_t - \bar{x}_{t-1}(\omega)| \leqq \varepsilon, \quad |w_t - \Delta\bar{x}_t(\omega)| \leqq \varepsilon.$$

This is a constraint qualification that corresponds in the deterministic case to the one in Proposition 2, rather than the milder one in Theorem 2.

Theorem 5 *Suppose problem* (P_{sto}) *satisfies the bounded recourse condition and the interior feasibility condition, and the multifunction* Z_t *is* $\mathcal{G}_{t-1}$*-measurable for* $t = 1, \ldots, T$. *Then for* $x \in \mathcal{N}$ *to be optimal in* (P_{sto}), *it is necessary, as well as sufficient, that there exist a* $p \in \mathcal{L}^1$ *satisfying* (a), (b) *and* (c) *of Theorem 4.*

The proof of this theorem relies on a result for multistage stochastic programs first derived in [15]. In particular, Theorem 2 of [15] shows that if the constraint multifunction is nonanticipative, the multipliers associated with the nonanticipativity constraint $(x \in \mathcal{N} \subset \mathcal{L}^\infty)$ can be chosen in $\mathcal{L}^1$, rather than in the dual of $\mathcal{L}^\infty$. (In other words, there is no need to introduce the singular part of the continuous linear functionals defined on $\mathcal{L}^\infty$.) An important consequence of all this is that the optimality conditions can be given a pointwise representation. This is exploited at various stages in the proof. In order to be able to apply these results we need some technical facts that relate the bounded recourse condition to the constraint-nonanticipativity condition as it appears in [15].

Definition 3 A compact-valued multifunction $D : \Omega \to (\mathbb{R}^n)^{T+1}$ will be called *nonanticipative* if for each $t = 0, 1, \ldots, T$ the projection

$$D^t(\omega) := \{(x_0, \ldots, x_t) \mid \exists (x_{t+1}, \ldots, x_T) \text{ with } (x_0, \ldots, x_t, x_{t+1}, \ldots, x_T) \in D(\omega)\} \tag{4.5}$$

depends $\mathcal{G}_t$-measurably on ω.

Proposition 3 *Suppose problem* (P_{sto}) *satisfies the bounded recourse condition, and* Z_t *is* $\mathcal{G}_{t-1}$*-measurable for* $t = 1, \ldots, T$. *Then for arbitrary* $\bar{\rho}_t > 0$, $t = 0, 1, \ldots, T$, *there exist constants* $\rho_t \geqq \bar{\rho}_t$ *such that the compact-valued multifunction* $D : \Omega \to (\mathbb{R}^n)^{T+1}$ *defined by*

$$D(\omega) := \{x = (x_0, \ldots, x_T) \mid |x_t| \leqq \rho_t \ \text{for} \ t = 0, 1, \ldots, T \ \text{and} \ \Delta x_t \in F_t(\omega, x_{t-1}) \ \text{for} \ t = 1, \ldots, T\} \tag{4.6}$$

is nonanticipative. Moreover, there are summable functions $\alpha_t:\Omega\to\mathbb{R}$ *such that almost surely*

$$|L_t(\omega, x_{t-1}, \Delta x_t)| \leqq \alpha_t(\omega) \quad \textit{when} \quad x \in D(\omega). \tag{4.7}$$

Proof. Start with $\rho_0=\bar\rho_0$, and for this as ρ in (b) of Definition 1, choose a corresponding $\rho'=\rho'_0$ such that (4.2) holds for $t=1$. Then almost surely

$$[x_0\in Z_1(\omega) \quad\text{and}\quad |x_0|\leqq\rho_0]\Rightarrow[\exists w_1\in F_1(\omega,x_0) \quad\text{with}$$

$$x_0+w_1\in Z_2(\omega) \quad\text{and}\quad |x_0+w_1|\leqq\rho'_0], \tag{4.8}$$

or in other words, taking $\rho_1=\max\{\bar\rho_1,\rho'_0\}$,

$$[x_0\in Z_1(\omega) \quad\text{and}\quad |x_0|\leqq\rho_0]$$

$$\Rightarrow[\exists x_1\in Z_2(\omega) \quad\text{with}\quad |x_1|\leqq\rho_1 \quad\text{and}\quad \Delta x_1\in F_1(\omega,x_0)]. \tag{4.9}$$

Continue recursively in this manner until for a certain $\rho_T\geqq\bar\rho_T$ we have almost surely

$$[x_{T-1}\in Z_T(\omega) \quad\text{and}\quad |x_{T-1}|\leqq\rho_{T-1}]$$

$$\Rightarrow[\exists x_T\in\mathbb{R}^n \quad\text{with}\quad |x_T|\leqq\rho_T \quad\text{and}\quad \Delta x_T\in F_T(\omega,x_{T-1})]. \tag{4.10}$$

From the chain of implications (4.9), ..., (4.10), we observe that almost surely, starting with any t and $x_t\in Z_{t+1}(\omega)$ with $|x_t|\leqq\rho_t$, we can generate $x_{t+1},\ldots,x_T$ such that

$$|x_\tau|\leqq\rho_\tau \quad\text{and}\quad \Delta x_\tau\in F_\tau(\omega,x_{\tau-1}) \quad\text{for}\quad \tau=t+1,\ldots,T.$$

It follows that the projection (4.5) of $D(\omega)$ can be written

$$D^t(\omega)=D^t_0(\omega)\cap\{(x_0,\ldots,x_t)\,|\,x_t\in Z_{t+1}(\omega)\}, \tag{4.11}$$

where

$$D^t_0(\omega)=\{(x_0,\ldots,x_t)\,||x_\tau|\leqq\rho_\tau \quad\text{for}\quad \tau=0,1,\ldots,t$$

$$\text{and}\quad \Delta x_\tau\in F_\tau(\omega,x_{\tau-1}) \qquad \text{for}\quad \tau=1,\ldots,t\}. \tag{4.12}$$

We need to show that D^t is a $\mathscr{G}_t$-measurable multifunction. Since the multifunction Z_{t+1} is closed-valued and $\mathscr{G}_t$-measurable, so is the multifunction

$$\omega\mapsto\{(x_0,\ldots,x_t)\in(\mathbb{R}^n)^{t+1}\,|\,x_t\in Z_{t+1}(\omega)\}. \tag{4.13}$$

[12, Prop. 1 I]. As for the multifunction D^t_0, let us observe that the relation $\Delta x_\tau\in F_\tau(\omega,x_{\tau-1})$ can be written

$$(x_{\tau-1},\Delta x_\tau)\in C_\tau(\omega),$$

where

$$C_\tau(\omega):=\operatorname{gph}F_\tau(\omega,\cdot)=\operatorname{dom}L_\tau(\omega,\cdot,\cdot).$$

This set is the image of the epigraph of $L_\tau(\omega,\cdot,\cdot)$ under the projection $(z_t,w_t,\alpha)\mapsto(z_t,w_t)$, and it is closed as a consequence of (a) of Definition 1, as noted earlier. Since the epigraph of $L_\tau(\omega,\cdot,\cdot)$ depends $\mathscr{G}_\tau$-measurably on ω, it follows that C_τ likewise depends $\mathscr{G}_\tau$-measurably on ω. The multifunction

$$\omega\mapsto C_1(\omega)\times\ldots\times C_t(\omega)$$

is therefore closed-valued and $\mathscr{G}_t$-measurable [12, Prop. 1 I] (recall that $\mathscr{G}_\tau$-measurability implies $\mathscr{G}_t$-measurability when $\tau\leqq t$).

Moreover, we have

$$D^t_0(\omega)=\{(x_0,\ldots,x_t)\in S\,|\,A(x_0,\ldots,x_t)\in C_1(\omega)\times\ldots\times C_t(\omega)\}$$

where

$$S:=\{(x_0,\ldots,x_t)\,||x_\tau|\leqq\rho_\tau\},$$

$$A:(x_0,\ldots,x_t)\mapsto(x_0,\Delta x_1,x_1,\Delta x_2,\ldots,x_{t-1},\Delta x_t).$$

This implies that D^t_0 is closed-valued and $\mathscr{G}_t$-measurable [12, Cor. 10], and then, since D^t is by (4.11) the intersection of two such multifunctions, we may conclude D^t is itself closed-valued (actually compact-valued) and $\mathcal{G}_t$-measurable [12, Theorem 1 M]. Thus D is nonanticipative as claimed.

Finally, by applying (a) of Definition 1 with ρ and σ large enough, we get the existence of a summable function β such that almost surely

$$x \in D(\omega) \Rightarrow L_t(\omega, x_{t-1}, \Delta x_t) \leqq \beta(\omega).$$

On the other hand, from our basic assumption in Section 1 that (1.10) holds for some summable γ, we get almost surely

$$x \in D(\omega) \Rightarrow L_t(\omega, x_{t-1}, \Delta x_t) \geqq \gamma(\omega).$$

Combining these two inequalities, we obtain the last assertion of Proposition 3. □

Proof of Theorem 5 The first part of our argument will characterize the vectors b_0 and b_T which appear in the optimality condition in Theorem 4. Only later will a function p be determined in its entirety. For each $(a_0, a_T) \in \mathbb{R}^n \times \mathbb{R}^n$, let

$$h(a_0, a_T) = \inf\left\{E \sum_{t=1}^{T} L_t(\omega, x'_{t-1}(\omega), \Delta x'_t(\omega)) \,\middle|\, x' \in \mathcal{N}, Ex_0 = a_0, Ex_T = a_T\right\}. \tag{4.14}$$

The function h is convex from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R} \cup \{\pm\infty\}$, and its effective domain

$$C' := \operatorname{dom} h = \{(a_0, a_T) \mid h(a_0, a_T) < \infty\} \tag{4.15}$$

has nonempty interior under our interior feasibility assumption. Indeed, for the function $\bar{x}$ in this assumption and a function β as in property (a) of the bounded recourse assumption, for ρ and σ sufficiently large, we have

$$L_t(\omega, x'_{t-1}(\omega), \Delta x'_t(\omega)) \leqq \beta(\omega) \quad \text{when} \quad \|x' - \bar{x}\|_\infty \leqq \varepsilon,$$

hence

$$(E\bar{x}_0, E\bar{x}_T) \in C \cap \operatorname{int} C'. \tag{4.16}$$

Furthermore,

$$\begin{aligned} \inf(P_{\text{sto}}) &= \inf\{l(a_0, a_T) + h(a_0, a_T) \mid (a_0, a_T) \in C \cap C'\} \\ &= \inf\{l(a_0, a_T) + h(a_0, a_T) \mid (a_0, a_T) \in \operatorname{ri} C \cap \operatorname{int} C'\} \end{aligned} \tag{4.17}$$

because $C \cap \operatorname{int} C' \neq \emptyset$ by (4.16) [10, Sections 6–7]. Since h is convex, it cannot have the value $-\infty$ anywhere unless it is $-\infty$ identically on the set $\operatorname{int}(\operatorname{dom} h) = \operatorname{int} C'$ [10, Section 7], in which event $\inf(P_{\text{sto}}) = -\infty$ by (4.17). In Theorem 5 we are only concerned with the situation where (P_{sto}) has a solution x, and then $\inf(P_{\text{sto}}) = J(x) \neq -\infty$. Therefore, in what follows we may suppose that

$$h(a_0, a_T) > -\infty \quad \text{for all} \quad (a_0, a_T) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{4.18}$$

Then there is no question of $\infty - \infty$ arising when we form $l + h$ in (4.17), and we have the following criterion for optimality: x solves (P_{sto}) if and only if

$$\inf_{(a_0, a_T) \in \mathbb{R}^n \times \mathbb{R}^n} \{l(a_0, a_T) + h(a_0, a_T)\} \quad \text{is attained at}$$

$$(a_0, a_T) = (Ex_0, Ex_T), \tag{4.19}$$

the infimum in (4.14) for

$$(a_0, a_T) = (Ex_0, Ex_T) \text{ is attained at } x. \tag{4.20}$$

We can characterize (4.19) by means of subgradients: it is equivalent to having $(0, 0) \in \partial(l + h)(Ex_0, Ex_T)$. Since $\operatorname{dom} l \cap \operatorname{int}(\operatorname{dom} h) \neq \emptyset$ by (4.16), we can calculate

$$\partial(l + h)(Ex_0, Ex_T) = \partial l(Ex_0, Ex_T) + \partial h(Ex_0, Ex_T)$$

[10, Section 16]. Hence (4.19) is equivalent to

$$\exists (b_0, -b_T) \in \partial l(Ex_0, Ex_T) \quad \text{with} \quad (-b_0, b_T) \in \partial h(Ex_0, Ex_T), \tag{4.21}$$

where the second relation means

$$\inf_{(a_0,a_T)\in\mathbb{R}^n\times\mathbb{R}^n}\{h(a_0,a_T)+b_0\cdot a_0-b_T\cdot a_T\} \text{ is attained at } (Ex_0, Ex_T). \tag{4.22}$$

But this and (4.20) say together that

$$\inf_{x'\in\mathcal{N}}\left\{b_0\cdot Ex'_0-b_T\cdot Ex'_T+E\sum_{t=1}^{T}L_t(\omega,x'_{t-1}(\omega),\Delta x'_t(\omega))\right\} \tag{4.23}$$

is attained at $x'=x$.

Our task therefore in proving Theorem 5 is to show that if the latter holds for some (b_0, b_T) and x, then there exists $p\in\mathcal{L}^1$ satisfying with these elements the relations (a), (b) and (c) of Theorem 4.

Note that since we are dealing with a convex problem in (4.23), any local solution (with respect to the $\mathcal{L}^\infty$ norm on $\mathcal{N}$) is a global solution. It suffices therefore to restrict attention to $x'\in\mathcal{N}$ satisfying $\|x'\|_\infty\leqq\bar\rho$, say, where $\bar\rho>0$ is sufficiently large in the sense that $\bar\rho>\max\{\|x\|_\infty,\|\bar x\|_\infty\}$, where x is the solution being analyzed and $\bar x$ is the function in our interior feasibility assumption. Applying Proposition 3 with $\bar\rho_t=\bar\rho$ for $t=0,\dots,T$, we may obtain a nonanticipative compact-valued multifunction D satisfying (4.6) and (4.7) for vectors $\tilde x\in(\mathbb{R}^n)^{T+1}$ (we use $\tilde x$ here in place of x, since x has already been used in the present argument to denote a solution to (P_{sto}).) Let

$$f(\omega,\tilde x):=\begin{cases} b_0\cdot\tilde x_0-b_T\cdot\tilde x_T+\sum_{t=1}^{T}L_t(\omega,\tilde x_{t-1},\Delta\tilde x_t) & \text{if } |\tilde x_t|\leqq\rho_t \\ & \text{for } t=0,\dots,T,\\ +\infty \text{ otherwise,} \end{cases}$$

so that by the choice of D we have

$$\operatorname{dom} f(\omega,\cdot)=D(\omega)$$

and also

$$|f(\omega,\tilde x)|\leqq\alpha_0+\alpha_1(\omega)+\ldots+\alpha_T(\omega) \quad\text{for}\quad \tilde x\in D(\omega), \tag{4.25}$$

where

$$\alpha_0=|b_0|\rho_0+|b_T|\rho_T\geqq|b_0\cdot\tilde x_0-b_T\cdot\tilde x_T| \quad\text{for}\quad \tilde x\in D(\omega) \tag{4.26}$$

and the functions $\alpha_1,\dots,\alpha_T$ are summable and satisfy

$$\alpha_t(\omega)\geqq|L_t(\omega,\tilde x_{t-1},\Delta\tilde x_t)| \quad\text{for}\quad \tilde x\in D(\omega). \tag{4.27}$$

Since L_t is a $\mathcal{G}_t$-normal integrand on $\Omega\times\mathbb{R}^n\times\mathbb{R}^n$, hence also $\mathcal{A}$-normal, because $\mathcal{G}_t\subset\mathcal{A}$, it follows from (4.24) that f is an $\mathcal{A}$-normal integrand on $\Omega\times(\mathbb{R}^n)^{T+1}$ [12, Proposition 2M]. For $x'\in\mathcal{N}$ satisfying $\|x'\|_\infty\leqq\bar\rho$, we have $|x'_t(\omega)|\leqq\bar\rho_t$ almost surely, so that

$$f(\omega,x'(\omega))=b_0\cdot x'_0(\omega)-b_T\cdot x'_T(\omega)+\sum_{t=1}^{T}L_t(\omega,x'_{t-1}(\omega),\Delta x'_t(\omega))$$

$$\text{when}\quad \|x'\|_\infty\leqq\bar\rho.$$

Thus, since the solution x in (4.23) satisfies $\|x\|_\infty<\bar\rho$, the assertion (4.23) is equivalent to

$$\inf_{x'\in\mathcal{N}} E\{f(\omega,x'(\omega))\} \quad\text{is attained at}\quad x'=x. \tag{4.28}$$

The equivalence of (4.23) and (4.28) enables us to apply our previous results in [15] to obtain $\mathcal{L}^1$-multipliers for the constraint $x'\in\mathcal{N}$. We have observed that f is an $\mathcal{A}$-normal integrand whose effective domain multifunction D is compact-valued, uniformly bounded, nonanticipative and such that the bounds (4.25), (4.26) and (4.27) hold. We also have available to us a function $\bar x$ satisfying the interior feasibility condition and having $\|\bar x\|_\infty<\bar\rho$. These properties imply that for some $\delta>0$ sufficiently small,

$$\tilde x\in D(\omega) \quad\text{when}\quad |\tilde x-\bar x(\omega)|\leqq\delta,$$

and also

$$L_t(\omega,z_t,w_t)\leqq\alpha_t(\omega) \quad\text{when}\quad |z_t-\bar x_t(\omega)|\leqq\delta,\ |w_t-\Delta\bar x_t(\omega)|\leqq\delta, \tag{4.29}$$

$$|b_0\cdot\bar x_0(\omega)-b_T\cdot\bar x_T(\omega)|\leqq\alpha_0.$$

measurability [12, Cor. 2X]. Thus C and C_t are $\mathscr{A}$-measurable multifunctions in (4.34), and hence so is their Cartesian product [12, Prop. 11] and its inverse image under A [12, Cor. 1Q]. This proves the $\mathscr{A}$-measurability of Γ.

We have established the existence of an $\mathscr{A}$-measurable p' such that (4.35) holds, where $\Gamma(\omega)$ consists of the vectors $\tilde{p}$ satisfying (4.33), and all these are known to obey (4.34). Observing from (4.32) that

$$\partial l^\omega(a_0, a_T) = \{(b_0, -b_T - q_T(\omega))\},$$

$$\partial L_t^\omega(z_t, w_t) = \partial L_t(\omega, z_t, w_t) - (q_{t-1}(\omega), 0),$$

while by (4.26) and (4.27) (since $x(\omega) \in D(\omega)$ almost surely)

$$\begin{aligned} |j^\omega(x(\omega))| &= \left| l^\omega(x_0(\omega), x_T(\omega)) + \sum_{t=1}^{T} L_t^\omega(x_{t-1}(\omega), \Delta x_t(\omega)) \right| \\ &\leqq \sum_{t=0}^{T} |x_t(\omega)| \cdot |q_t(\omega)| + |b_0 \cdot x_0(\omega) - b_T \cdot x_T(\omega)| \\ &\leqq \sum_{t=0}^{T} \rho_t |q_t(\omega)| + \alpha_0 + \alpha_1(\omega) + \ldots + \alpha_T(\omega), \end{aligned}$$

we see that when $p'(\omega)$ is substituted for $\tilde{p}$ in (4.31) and (4.32) we have almost surely

$$(p_0'(\omega), -p_T'(\omega) + q_T(\omega)) = (b_0, -b_T), \tag{4.37}$$

$$(\Delta p_t'(\omega) + q_{t-1}(\omega), p_t'(\omega)) \in \partial L_t(\omega, x_{t-1}(\omega), \Delta x_t(\omega)), \tag{4.38}$$

as well as

$$\sum_{t=0}^{T} |p_t'(\omega)| \leqq 2\left[2\alpha_0 + 2\sum_{t=1}^{T} \alpha_t(\omega) + \sum_{t=0}^{T} \rho_t |q_t(\omega)| \right] \Big/ \delta.$$

This last inequality assures us that $p' \in \mathscr{L}^1$, since $q \in \mathscr{L}^1$ and the functions α_t are summable.

The final stage of the proof has been reached. We set

$$p_t := E^{t+1}(p_t' - q_t) \quad \text{for} \quad t = 0, 1, \ldots, T-1,$$

$$p_T := p_T' - q_T.$$

Then p_{t-1} is $\mathscr{G}^t$-measurable for $t = 1, \ldots, T$, so (c) of Theorem 4 is fulfilled. We also have via (4.30) that

$$E^t p_t = E^t p_t' - E^t q_t = E^t p_t' \quad \text{for} \quad t = 0, 1, \ldots, T. \tag{4.39}$$

Considering this for $t = 0$, we see from (4.37) and the definition $p_T = p_T' - q_T$ that

$$((E^0 p_0)(\omega), -p_T(\omega)) = (b_0, -b_T) \quad \text{a.s.},$$

which is (a) of Theorem 4. Another implication of (4.39) is that

$$\begin{aligned} E^t(\Delta p_t' + q_{t-1}) &= E^t(p_t' - p_{t-1}' + q_{t-1}) = E^t(p_t - p_{t-1} + q_t) \\ &= E^t \Delta p_t + E^t q_t = E^t \Delta p_t. \end{aligned} \tag{4.40}$$

Now the multifunction on the right side of (4.38) is is $\mathscr{G}_t$-measurable, because L_t is a $\mathscr{G}_t$-measurable multifunction, and x_{t-1} and Δx_t are both $\mathscr{G}_t$-measurable [12, Cor. 2X]. Since this multifunction is also convex-valued, we can take the conditional expectation with respect to $\mathscr{G}_t$ on the left side of (4.38) and obtain by (4.39) and (4.40) that

$$((E^t \Delta p_t)(\omega), E^t p_t(\omega)) \in \partial L_t(\omega, x_{t-1}(\omega), \Delta x_t(\omega))$$

almost surely. This is relation (b) of Theorem 4.

In summary, we have constructed a function $p \in \mathscr{L}^1$ satisfying (a), (b) and (c) for the given solution x to (P_{sto}), and this is all we had to do in order to prove Theorem 5. □

5. THE DUAL STOCHASTIC PROBLEM

The function p in the optimality condition in Theorem 4 turns out to solve a certain dual problem, which we now formulate. Define the

We can now apply [15, p. 182] and conclude there is a function $q=(q_0, q_1, \ldots, q_T)$ in $\mathcal{L}^1$ for which

$$E^t q_t = 0 \quad \text{for} \quad t=0, 1, \ldots, T, \tag{4.30}$$

and

$$\inf_{x' \in \mathcal{L}^\infty} [E\{f(\omega, x'(\omega))\} - E\{q(\omega)\cdot x'(\omega)\}] \tag{4.31}$$
is attained at $x = x'$.

Since f is $\mathcal{A}$-normal, this minimization over $\mathcal{L}^\infty$ (rather than $\mathcal{N}$) can be reduced to pointwise minimization [12, Theorem 3A]:

$$\inf_{\tilde{x} \in (\mathbb{R}^n)^{T+1}} \{f(\omega, \tilde{x}) - q(\omega)\cdot\tilde{x}\} \text{ is attained at } \tilde{x} = x(\omega) \quad \text{a.s.}$$

Using the definition (4.24) of f and the fact that $\|x\|_\infty < \bar{\rho}$, so $|x_t(\omega)| < \bar{\rho} \leqq \rho_t$ almost surely, and hence

$$\inf_{\tilde{x} \in (\mathbb{R}^n)^{T+1}} \left\{ b_0 \cdot \tilde{x}_0 - b_T \cdot \tilde{x}_T + \sum_{t=1}^{T} L_t(\omega, \tilde{x}_{t-1}, \Delta\tilde{x}_t) - \sum_{t=1}^{T} q_t(\omega)\cdot\tilde{x}_t \right\}$$
is attained at $\tilde{x} = x(\omega)$ a.s.

But this means that almost surely $x(\omega)$ is a solution to a problem in the deterministic format, depending on ω:

(P^ω_{det}) minimize over all $\tilde{x} \in (\mathbb{R}^n)^{T+1}$ the function

$$j^\omega(\tilde{x}) = l^\omega(\tilde{x}_0, \tilde{x}_T) + \sum_{t=1}^{T} L^\omega_t(\tilde{x}_{t-1}, \Delta\tilde{x}_t),$$

where

$$l^\omega(a_0, a_T) := b_0 \cdot a_0 - (b_T + q_T(\omega))\cdot a_T,$$
$$L^\omega_t(z_t, w_t) := L_t(\omega, z_t, w_t) - q_{t-1}(\omega)\cdot z_t. \tag{4.32}$$

For fixed ω the hypothesis of Proposition 2 is satisfied almost surely for (P^ω_{det}) by $\tilde{x}(\omega)$ in view of (4.29). There does then exist by Theorem 2 a corresponding vector $\tilde{p} \in (\mathbb{R}^n)^{T+1}$ with

$$(\tilde{p}_0, -\tilde{p}_T) \in \partial l^\omega(x_0(\omega), x_T(\omega)),$$
$$(\Delta\tilde{p}_t, \tilde{p}_t) \in \partial L^\omega_t(x_{t-1}(\omega), \Delta x_t(\omega)) \quad \text{for} \quad t=1, \ldots, T, \tag{4.33}$$

and every such vector has

$$\sum_{t=0}^{T} |\tilde{p}_t| \leqq 2[\alpha_0 + \alpha_1(\omega) + \ldots + \alpha_T(\omega) - j^\omega(x(\omega))]/\delta. \tag{4.34}$$

Let $\Gamma(\omega)$ denote the set of all $\tilde{p} \in (\mathbb{R}^n)^{T+1}$ for which (4.33) is fulfilled. We have just seen that almost surely $\Gamma(\omega)$ is nonempty and bounded. We must establish next the existence of an $\mathcal{A}$-measurable function p' such that

$$p'(\omega) \in \Gamma(\omega) \quad \text{a.s.} \tag{4.35}$$

(From such a p' we will subsequently be able to manufacture the desired $p \in \mathcal{L}^1$ satisfying (a), (b) and (c) of Theorem 4 for the b_0, b_T and x at hand.) It suffices to verify that the multifunction $\Gamma:\omega \mapsto \Gamma(\omega)$ is closed-valued and $\mathcal{A}$-measurable [12, Cor. 1C]. We use the representation

$$\Gamma(\omega) = A^{-1}(C(\omega) \times C_1(\omega) \times \ldots \times C_T(\omega)), \tag{4.36}$$

where

$$C(\omega) := \partial l^\omega(x_0(\omega), x_T(\omega)), \quad C_t(\omega) := \partial L^\omega_t(x_{t-1}(\omega), \Delta x_t(\omega)),$$

$$A:(\tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_T) \mapsto (\tilde{p}_0, \tilde{p}_T, \tilde{p}_0, \Delta\tilde{p}_1, \tilde{p}_1, \Delta\tilde{p}_2, \ldots, \tilde{p}_{T-1}, \Delta\tilde{p}_T).$$

The subgradient sets $C(\omega)$ and $C_t(\omega)$ are of course closed, and A is just a linear transformation, so (4.36) implies $\Gamma(\omega)$ is closed. As functions on $\Omega \times \mathbb{R}^n \times \mathbb{R}^n$ the expressions in (4.32) are $\mathcal{A}$-normal and the functions g_t are $\mathcal{A}$-measurable [12, Prop. 2M]. In forming $C(\omega)$ and $C_t(\omega)$, therefore, we are merely putting $\mathcal{A}$-measurable arguments into the subgradient multifunctions associated with certain $\mathcal{A}$-normal integrands, and this operation is known to preserve $\mathcal{A}$-

function m on $\mathbb{R}^n \times \mathbb{R}^n$ as before (*cf.* (2.23)) and let

$$M_t(\omega, q_t, r_t) = \sup_{z_t, w_t} \{q_t \cdot w_t + r_t z_t - L_t(z_t, w_t)\}. \tag{5.1}$$

Then M_t is a $\mathscr{G}_t$-normal integrand [12, Theorem 2K] and

$$L_t(\omega, z_t, w_t) = \sup_{q_t, r_t} \{q_t \cdot w_t + r_t \cdot z_t - M_t(q_t, r_t)\}. \tag{5.2}$$

Let

$$\begin{aligned} \mathscr{P}: = \{p = (p_0, \ldots, p_T) \in \mathscr{L}^1 | p_{t-1} \text{ is } \mathscr{G}_t\text{-measurable} \\ \text{for } t = 1, \ldots, T, \text{ and } E^0 p_0 \text{ and } p_T \text{ are constant}\}. \end{aligned} \tag{5.3}$$

This is a closed subspace of $\mathscr{L}^1$. The problem dual to (P_{sto}) is

(P^*_{sto}) maximize $-K(p)$ over all $p = (p_0, p_1, \ldots, p_T) \in \mathscr{P}$, where

$$K(p) := m(E^0 p_0, p_T) + E\left\{\sum_{t=1}^{T} M_t(\omega, (E^t p_t)(\omega), (E^t \Delta p_t)(\omega))\right\}.$$

The functional K is well-defined from $\mathscr{P}$ to $\mathbb{R} \cup \{+\infty\}$, convex, and lower semicontinuous with respect to the $\mathscr{L}^1$-norm on $\mathscr{P}$.

THEOREM 6 *The inequality* $\inf(P_{sto}) \geq \sup(P^*_{sto})$ *always holds. One has* $p \in \partial\Phi(0)$ *if and only if actually* $\inf(P_{sto}) = \max(P^*_{sto})$, *and* p *is optimal for* (P^*_{sto}).

Proof of Theorem 6 This is a consequence of the proof of Theorem 4, just as Theorem 3 was a consequence of the proof of Theorem 1. The trick is to calculate the conjugate Φ^* from the definition (3.3) of Φ and the formula

$$-\Phi^*(p) = \inf_{y \in \mathscr{L}^\infty} \{\Phi(y) - \langle p, y \rangle\} \quad \text{for} \quad p \in \mathscr{L}^1.$$

The change-of-variables argument in the proof of Theorem 4

demonstrates actually that

$$\Phi^*(p) = \begin{cases} K(p) & \text{if } p \in \mathscr{P}, \\ +\infty & \text{for all other } p \in \mathscr{L}^1. \end{cases}$$

The argument for Theorem 3 then takes over, word for word, and gives the claimed result via Theorem 4. □

COROLLARY *Under the hypothesis of Theorem 5 and the additional assumption that* (P_{sto}) *possesses a solution, one has* $\min(P_{sto}) = \max(P^*_{sto})$.

Proof The assumptions in question imply according to Theorem 5 the existence of a function p satisfying (a), (b), (c) for a solution x to (P_{sto}). Then $p \in \partial\Phi(0)$ by Theorem 4, and the desired conclusion is given by Theorem 6. □

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