ON THE ESSENTIAL BOUNDEDNESS OF SOLUTIONS TO PROBLEMS IN PIECEWISE LINEAR-QUADRATIC OPTIMAL CONTROL

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Dedicated to J-L. Lions on his 60^{th} birthday

Abstract. Primal and dual problems of optimal control with linear, quadratic or piecewise linear-quadratic convex objective are considered in which a linear dynamical system is subjected to linear inequality constraints that could jointly involve states and controls. It is shown that when such constraints, except for the ones on controls only, are represented by penalty terms, and a mild coercivity condition is satisfied, the optimal controls for both problems will be essentially bounded in time. The optimal trajectories will thus be Lipschitzian.

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1. Introduction

In problems of optimal control of ordinary differential equations, the "linear-quadratic" case usually refers to a formulation where the dynamics are linear and the objective is quadratic convex (or affine). A broader formulation, termed "generalized linear-quadratic" in Rockafellar [**1**], allows the integrand in the objective to be *piecewise linear-quadratic* (convex) in the state and control vectors. By definition, a function is piecewise linear-quadratic if its effective domain can be expressed as the union of finitely many polyhedral convex sets, relative to each of which the function is at most quadratic.

The introduction of piecewise linear-quadratic expressions can serve various purposes in mathematical modeling, but one of the most important is the handling of various linear inequality constraints by means of penalty terms. This is especially useful in the case of state constraints or restrictions that make the current set of available controls depend on the current state. Such restrictions, which fall outside the usual patterns of treatment in optimal control, are essential to many applications in economic management and operations research.

The theory of piecewise linear-quadratic optimal control, as developed in [**1**], emphasizes duality in the statement of necessary and sufficient conditions and includes results on the existence of primal and dual solutions. In order to achieve such results, one is obliged to work with control spaces consisting of $\mathcal{L}^1$ functions, at least initially, even though a restriction to $\mathcal{L}^\infty$ would be more natural for most applications. The purpose of the present paper is to demonstrate that the necessary conditions derived in this way imply, under the supporting assumptions, that the optimal controls are $\mathcal{L}^\infty$ functions after all. Thus the introduction of $\mathcal{L}^1$ controls can be seen merely as a technical excursion leading in the end to a solid justification for an $\mathcal{L}^\infty$ framework.

2. Primal and Dual Problems

The basic control problem in [**1**] involves certain ρ expressions defined in general by

$$\rho_{V,Q}(r) = \sup_{v \in V} \{ r \cdot v - \tfrac{1}{2} v \cdot Qv \}, \tag{2.1}$$

where V denotes a nonempty polyhedral convex set in $\mathbb{R}^\ell$, and Q is a symmetric, positive *semi*definite matrix in $\mathbb{R}^{\ell \times \ell}$ (possibly the zero matrix!). The nature of such expressions, the alternative formulas they satisfy, and their roles and special cases, are discussed in detail in [**1**]. Suffice it to say here that $\rho_{V,Q}$ is in general a piecewise linear-quadratic convex function of r which depends in a convenient manner on the choice of V and Q as "parameters". If $0 \in V$, then $\rho_{V,Q}$ is nonnegative everywhere, as usually expected of a "penalty function". Moreover the set of vectors r where $\rho_{V,Q}$ actually vanishes is a certain polyhedral convex cone $K_{V,Q}$ (possibly a subspace or just the zero vector itself).

The primal problem we deal with is the following, where the subscript t refers to time in a fixed interval $[t_0, t_1]$ and the subscript e designates "endpoint elements":

$(\mathcal{P}^1)$ subject to the dynamical system

$$\dot{x}_t = A_t x_t + B_t u_t + b_t \text{ a.e. }, \ quad u_t \in U_t \text{ a.e. },$$

$$x_{t_0} = B_e u_e + b_e, \quad u_e \in U_e,$$

with $u : t \mapsto u_t$ an $\mathcal{L}^1$ function, minimize

$$\int_{t_0}^{t_1} [p_t \cdot u_t + \tfrac{1}{2} u_t P_t u_t - c_t \cdot x_t] dt + [p_e \cdot u_e + \tfrac{1}{2} u_e \cdot P_e u_e - c_e \cdot x_{t_1}]$$

$$+ \int_{t_0}^{t_1} \rho_{V_t, Q_t}(q_t - C_t x_t - D_t u_t) dt + \rho_{V_e, Q_e}(q_e - C_e x_{t_1} - D_e u_e).$$

Here we have written u_t, x_t and $\dot{x}_t$ in place of $U(t)$, $x(t)$ and $\dot{x}(t)$, with $u_t \in \mathbb{R}^k$, $x_t \in \mathbb{R}^n$.

The "endpoint control vector" $u_e \in \mathbb{R}^{k_e}$ represents additions parameters supplementary to the "instantaneous control vectors" u_t, but with respect to which some optimization can also take place. The sets

$$U_t \subset \mathbb{R}^k, \ U_e \subset \mathbb{R}^{k_e}, \ V_t \subset \mathbb{R}^\ell, \ V_e \subset \mathbb{R}^{\ell_e}$$

are assumed to be polyhedral convex and nonempty. The matrices P_t, P_e, Q_t, Q_e, are assumed to be symmetric and positive *semi*definite. Furthermore it is assumed that the elements

$$A_t, \ B_t, \ C_t, \ D_t, \ b_t, \ c_t, \ P_t, \ Q_t, \ p_t, \ q_t, \ U_t, \ V_t \quad (2.2)$$

all depend continuously on $t \in [t_0, t_1]$.

Under these assumptions the choice of u and u_e, where u denotes the $\mathcal{L}^1$ control function $t \mapsto u_t$ as indicated in the statement of $(\mathcal{P})$, determines by way of the dynamical system in $(\mathcal{P})$ a unique corresponding trajectory $x : t \mapsto x_t$, which is absolutely continuous. The associated value of the objective functional is then well defined as a real number of ∞, and it is convex in its dependence on (u, u_e); see [**1**, Thm. 4.2].

The ρ-expressions in $(\mathcal{P}^1)$ can be viewed broadly as *monitoring terms*. They monitor the vector $C_t x_t + D_t u_t$ versus a given q_t, and also $C_e x_{t_1} + D_e u_e$ versus q_e. Certain directions or amounts of deviation may be accepted without cost (namely those corresponding to difference vectors in the polyhedral cones K_{Q_t, V_t} and K_{Q_e, V_e}, in the case where $0 \in V_t$ and

$0 \in V_e$), but others may be penalized (or conceivably even rewarded, in cases other than ones where $0 \in V_t$ and $0 \in V_e$).

The problem proposed in [**1**] as dual to $(\mathcal{P})$ is entirely analogous in form, but with a reversal of roles for the various elements (* denotes transpose):

$(\mathcal{Q}^1)$ Subject to the dynamical system

$$-\dot{y}_t = A_t^* y_t + C_t v_t + c_t \text{ a.e. }, \quad v_t \in V_t \text{ a.e. },$$

$$y_{t_0} = C_e^* v_e + c_e, \quad v_e \in V_e,$$

with $v : t \mapsto v_t$ an $\mathcal{L}^1$ function of t, maximize

$$\int_{t_0}^{t_1} [q_t \cdot v_t - \tfrac{1}{2} v_t \cdot Q_t v_t - b_t \cdot y_t] dt + [q_e \cdot v_e - \tfrac{1}{2} v_e \cdot Q_e v_e - b_e \cdot y_e]$$

$$- \int_{t_0}^{t_1} \rho_{U_t,P_t}(B_t^* y_t + D_t^* v_t - p_t) dt - \rho_{U_e,P_e}(B_e^* y_{t_0} + D_e^* v_e - p_e).$$

Again, the choice of the control function $v : t \mapsto v_t$ and vector v_e uniquely determines the dual state trajectory $y : t \mapsto y_t$, which is absolutely continuous. The objective functional in $(\mathcal{Q}^1)$ is concave in (v, v_e) with values in $\mathbb{R} \cup \{\infty\}$.

The main results in [**1**] about the relationship between $(\mathcal{P}^1)$ and $(\mathcal{Q}^1)$ concern duality. For present purposes we focus only on the strongest case.

Definition 2.1. One says that the *primal finiteness condition* is satisfied if $\rho_{V_t,Q_t} < \infty$ everywhere on $\mathbb{R}^\ell$ and $\rho_{V_e,Q_e} < \infty$ everywhere on $\mathbb{R}^{\ell_e}$. Likewise the *dual finiteness condition* is satisfied if $\rho_{U_t,P_t} < \infty$ everywhere on $\mathbb{R}^k$ and $\rho_{U_e,P_e} < \infty$ everywhere on $\mathbb{R}^{k_e}$.

These conditions mean roughly that the monitoring terms in the two control problems do not involve any implicit constraints which are enforced through infinite penalization. It is demonstrated in [**1**, Prop. 2.4] that the primal finiteness condition holds if and only if

(2.3) $$\operatorname{rc} V_t \cap \operatorname{nl} Q_t = \{0\} \text{ for all } t \in [t_0, t_1] \text{ and } \operatorname{rc} V_e \cap \operatorname{nl} Q_e = \{0\},$$

where "rc" denotes the recession cone of a convex set [**2**, §8] and "nl" denotes the null space of a matrix. Similarly, the dual finiteness condition holds if and only if

(2.4) $$\operatorname{rc} U_t \cap \operatorname{nl} P_t = \{0\} \text{ for all } t \in [t_0, t_1] \text{ and } \operatorname{rc} U_e \cap \operatorname{nl} P_e = \{0\}.$$

Condition (2.3) can appropriately be called the *primal coercivity condition*, and (2.5) the *dual coercivity condition*. Thus instead of the primal and dual finiteness conditions we could speak of imposing the primal finiteness and coercivity conditions, or equivalently of imposing the dual finiteness and coercivity conditions.

Theorem 2.2. *Under the assumption that the primal and dual finiteness conditions both hold, one has*

$$-\infty < \min(\mathcal{P}^1) = \max(\mathcal{Q}^1) < \infty.$$

In other words, optimal controls exist for both problems, and the corresponding objective values are finite and equal.

Proof. This is a special case of [**1**, Thm. 6.3]. □

When the finiteness conditions in Theorem 2.2 are invoked separately, each still guarantees the equality of the optimal values in $(\mathcal{P}^1)$ and $(\mathcal{Q}^1)$ but only yields one of the existence assertions. See the full version of this result in [**1**, Thm. 6.3].

Optimality of primal and dual controls in the context of Theorem 2.2 can be expressed as a "minimaximum" principle. To state this we introduce the quadratic convex-concave functions

$$J_t(u_t, v_t) = p_t \cdot u_t + \tfrac{1}{2} u_t \cdot P_t u_t + q_t \cdot v_t - \tfrac{1}{2} v_t \cdot Q_t v_t - v_t \cdot D_t u_t, \tag{2.5}$$

$$J_e(u_e, v_e) = p_e \cdot u_e + \tfrac{1}{2} u_e \cdot P_e u_e + q_e \cdot v_e - \tfrac{1}{2} v_e \cdot Q_e v_e - v_e \cdot D_e u_e. \tag{2.6}$$

Theorem 2.3. *In the case where* $\min(\mathcal{P}^1) = \max(\mathcal{Q}^1)$, *the following conditions are both necessary and sufficient in order that* $(\bar{u}, \bar{u}_e)$ *be optimal in* $(\mathcal{P}^1)$ *and* $(\bar{v}, \bar{v}_e)$ *be optimal in* $(\mathcal{Q}^1)$. *In terms of the primal trajectory* $\bar{x}$ *corresponding to* $(\bar{u}, \bar{u}_e)$ *and the dual trajectory* $\bar{y}$ *corresponding to* $(\bar{v}, \bar{v}_e)$, *one has that*

$$\begin{gathered} (\bar{u}_t, \bar{v}_t) \text{ is a saddle point of } J_t(u_t, v_t) - u_t \cdot B_t^* \bar{y}_t - v_t \cdot C_t x_t \\ \text{relative to } u_t \in U_t,\ v_t \in V_t \quad \text{(a.e.)}, \end{gathered} \tag{2.7}$$

$$\begin{gathered} (\bar{u}_e, \bar{v}_e) \text{ is a saddle point of } J_e(u_e, v_e) - u_e \cdot B_e^* \bar{y}_{t_0} - v_e \cdot C_e \bar{x}_{t_1} \\ \text{relative to } u_e \in U_e,\ v_e \in V_e \quad , \end{gathered} \tag{2.8}$$

(where $\bar{u} : t \mapsto \bar{u}_t$ *and* $\bar{v} : t \mapsto \bar{v}_t$ *are* $\mathcal{L}^1$ *functions).*

Proof. This combines a global saddlepoint criterion for optimality in [**1**, Thm. 6.2] with a pointwise saddlepoint criterion in [**1**, Thm. 6.5]. □

3. Reduction to Bounded Controls.

Let us denote by $(\mathcal{P}^\infty)$ and $(\mathcal{Q}^\infty)$ the modified forms of the primal problem $(\mathcal{P}^1)$ and dual problem $(\mathcal{Q}^1)$ in which the control functions $u : t \mapsto u_t$ and $v : t \to v_t$ are required to be $\mathcal{L}^\infty$ rather than $\mathcal{L}^1$. The optimal values in these problems satisfy

$$\inf(\mathcal{P}^\infty) \geq \inf(\mathcal{P}^1) \geq \sup(\mathcal{Q}^1) \geq \sup(\mathcal{Q}^\infty) \tag{3.1}$$

in general (the middle inequality holds always by [**1**, Thm. 6.2]). This relationship indicates that the prospect of duality between $(\mathcal{P}^\infty)$ and $(\mathcal{Q}^\infty)$ is less promising than between $(\mathcal{P}^1)$ and $(\mathcal{Q}^1)$, despite the fact that $(\mathcal{P}^\infty)$ and $(\mathcal{Q}^\infty)$ are easier to work with in other respects and more natural as problem formulations for many situations. We demonstrate now, though, that duality between $(\mathcal{P}^\infty)$ and $(\mathcal{Q}^\infty)$ does hold nonetheless in the important case where the primal and dual finiteness conditions are both satisfied.

Theorem 3.1. *Under the primal and dual boundedness conditions, every optimal solution $(\bar{u}, \bar{u}_e)$ to $(\mathcal{P}^1)$ actually has $\bar{u} \in \mathcal{L}^\infty$ and therefore is an optimal solution to $(\mathcal{P}^\infty)$; likewise, every optimal solution $(\bar{v}, \bar{v}_e)$ to $(\mathcal{Q}^1)$ has $\bar{v} \in \mathcal{L}^\infty$ and therefore is an optimal solution to $(\mathcal{Q}^\infty)$. Thus*

$$-\infty < \min(\mathcal{P}^\infty) = \max(\mathcal{Q}^\infty) < \infty.$$

Proof. In view of Theorems 2.2 and 2.3, the task before us is to demonstrate that in the presence of the primal and dual finiteness conditions, the instantaneous saddlepoint condition (2.7) implies $\bar{u} \in \mathcal{L}^\infty$, $\bar{v} \in \mathcal{L}^\infty$. We shall do this in the framework of conjugate convex-concave functions [**2**].

Define the function $\bar{J}_t$ on $\mathbb{R}^k \times \mathbb{R}^\ell$ by

$$\bar{J}_t(u_t, v_t) = \begin{cases} J_t(u_t, v_t) & \text{if } u_t \in U_t,\ v_t \in V_t, \\ -\infty & \text{if } u_t \in U_t,\ v_t \notin V_t, \\ \infty & \text{if } u_t \notin U_t. \end{cases} \tag{3.2}$$

This is a closed saddlefunction in the terminology of [**2**, §34], inasmuch as J_t is a continuous convex-concave function and the sets U_t and V_t are closed and convex. The saddlepoint condition (2.5) is equivalent by definition to the subdifferential relation

$$(B_t^* \bar{y}_t, C_t \bar{x}_t) \in \partial \bar{J}_t(\bar{u}_t, \bar{v}_t) \ \text{ a.e.} \tag{3.3}$$

(see [**2**, p. 374]). By introducing the conjugate

$$\begin{aligned} \bar{J}_t^*(r, s) &= \inf_{v_t \in \mathbb{R}^\ell} \sup_{u_t \in \mathbb{R}^k} \{r \cdot u_t + s \cdot v_t - \bar{J}_t(u_t, v_t)\} \\ &= \inf_{v_t \in V_t} \sup_{u_t \in U_t} \{r \cdot u_t + s \cdot v_t - \bar{J}_t(u_t, v_t)\}, \end{aligned} \tag{3.4}$$

which is another closed saddlefunction [**2**, Cor. 37.1.1], we can write this subdifferential relation as

$$(\bar{u}_t, \bar{v}_t) \in \partial \bar{J}_t^*(B_t^* \bar{y}_t, C_t \bar{x}_t), \quad \text{a.e.} \tag{3.5}$$

by [**2**, Thm. 37.5]. To prove that this implies $\bar{u} \in \mathcal{L}^\infty$ and $\bar{v} \in \mathcal{L}^\infty$, it will suffice to show that the multifunction

$$\Gamma : t \in [t_0, t_1] \mapsto \partial \bar{J}_t^*(B_t^* \bar{y}_t, C_t \bar{x}_t)$$

is uniformly bounded when the primal and dual finiteness conditions are satisfied, i.e. has all of its image sets $\Gamma(t)$ contained in a certain bounded subset of $\mathbb{R}^k \times \mathbb{R}^\ell$. Because $B_t^* \bar{y}_t$ and $C_t \bar{x}_t$ depend continuously on t and therefore themselves remain in a bounded set as t passes through $[t_0, t_1]$, it will suffice to show that the multifunctions

$$\partial \bar{J}_t : \mathbb{R}^k \times \mathbb{R}^\ell \rightrightarrows \mathbb{R}^k \times \mathbb{R}^\ell$$

are uniformly bounded on bounded sets.

The formula specified in (3.4) tells us by way of (2.5) that

$$\begin{aligned} \bar{J}_t^*(r,s) &= \inf_{v_t \in V_t} \sup_{u_t \in U_t} \{[r - p_t]\cdot u_t + [s - q_t]\cdot v_t - \tfrac{1}{2} u_t\cdot P_t u_t + \tfrac{1}{2} v_t \cdot Q_t v_t + v_t \cdot D_t u_t\} \\ &= \inf_{v_t \in V_t} \{[s - q_t]\cdot v_t + \tfrac{1}{2} v_t\cdot Q_t v_t\} + \sup_{u_t \in U_t} \{[r - p_t + D_t^* v_t]\cdot u_t - \tfrac{1}{2} u_t \cdot P_t u_t\} \\ &= \inf_{v_t \in V_t} \{[s - q_t]\cdot v_t + \tfrac{1}{2} v_t Q_t v_t + \rho_{U_t, P_t}(r - p_t + D_t^* v_t)\}. \end{aligned} \tag{3.6}$$

The set V_t, which is nonempty, closed and convex, depends continuously on t according to our assumption (2.2) so there exists by the selection theorem of Michael [**3**] a *continuous* function $\bar{v} : t \mapsto \bar{v}_t$ with $\bar{v} \in V_t$ for all $t \in [t_0, t_1]$. Then

$$\bar{J}_t^*(r,s) \le [s - q_t]\cdot \bar{v}_t + \tfrac{1}{2} \bar{v}_t \cdot Q_t \bar{v}_t + \rho_{u_t, P_t}(r - p_t + D_t^* \bar{v}_t) \text{ for all } (r,s) \in \mathbb{R}^k \times \mathbb{R}^\ell. \tag{3.7}$$

Using the fundamental inequality "inf sup $\ge$ sup inf" on the formula in (3.4), we can obtain similarly that

$$\bar{J}_t^*(r,s) \ge [r - p_t]\cdot \bar{u}_t - \tfrac{1}{2} \bar{u}_t \cdot P_t \bar{u}_t - \rho_{V_t, Q_t}(q_t - D_t \bar{u}_t - s) \text{ for all } (v,s) \in \mathbb{R}^k \times \mathbb{R}^\ell, \tag{3.8}$$

where $\bar{u} : t \mapsto \bar{u}_t$ is a *continuous* function having $\bar{u}_t \in U_t$ for all $t \in [t_0, t_1]$. We now invoke the primal and dual finiteness conditions: These say that the estimates in (3.7) and (3.8) are finite. In fact they imply by [**1**, Prop. 4.1] that $\rho_{U_t, P_t}(r)$ is continuous jointly in t and r, and $\rho_{V_t, Q_t}(s)$ is continuous jointly in t and s. (The continuous dependence of P_t, Q_t,

U_t and V_t on t in (2.2) comes in here.) This continuity and that of q_t, Q_t, p_t, P_t, and D_t in (2.2) produces continuity of the right sides of (3.7) and (3.8) with respect to (t, r, s). It follows then that the functions $\bar{J}_t^*$ for $t \in [t_0, t_1]$ are uniformly bounded on any bounded subset of $\mathbb{R}^k \times \mathbb{R}^\ell$.

This boundedness property must be translated next into a uniform boundedness of the subdifferentials $\partial \bar{J}_t^*$. The argument is just a refinement of facts already known in the literature on convex-concave functions. From [**2**, Thm. 35.2] one sees that the boundedness of the values of the functions $\bar{J}_t^*$ implies that these functions are equi-Lipschitzian on any bounded subset of $\mathbb{R}^k \times \mathbb{R}^\ell$. But if $\bar{J}_t^*$ is Lipschitzian with modulus λ on some neighborhood of a point $(\bar{r}, \bar{s})$, the set $\partial \bar{J}_t^*(\bar{r}, \bar{s})$ must be contained in the ball of radius λ around $(\bar{r}, \bar{s})$, as follows simply from the bounds one gets in this case on directional derivatives. This observation leads to our desired conclusion that the sets $\partial \bar{J}_t^*(r, s)$ for $t \in [t_0, t_1]$ are uniformly bounded relative to any bounded set of points (r, s). □

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