COSMIC CONVERGENCE†

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Abstract. We introduce a new notion of convergence for sets that plays a crucial role in the study of the preservation of set convergence under various operations.

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A comprehensive approximation theory for optimization problems, variational inequalities, linear or nonlinear systems of equations (including differential equations) is ultimately rooted in the convergence theory for sets. This has been vividly demonstrated by the results obtained via epi-convergence (equivalently, Γ-convergence) for constrained optimization problems and the calculus of variations, via graphical convergence (equivalently, G-convergence) for differential equations, in particular for problems involving nonsmooth coefficients (e.g., for conductivity problems involving composite materials).

Although set convergence, introduced by Painlevé (between 1902 and 1905), has a long mathematical history, it is only during the last two decades that it has started to be viewed as a major tool for dealing with approximations in optimization, systems of equations and related objects. One of the major shortcomings of the theory, as developed mostly by point-set topologists (between 1930 and 1970), was the (somewhat arbitrary) restriction of many results to hyperspaces (spaces of sets) consisting of the subsets of compacta. Because epigraphs of functions and graphs of operators are typically unbounded sets, there was a need for a theory that was not specifically aimed at the convergence of bounded sets, and where the continuity of set-valued mappings was not defined to suit only the case when these mappings are bounded. In its present state, the theory no longer makes any restrictions on set types. In particular, the relationship between set convergence and the various topologies one can define on hyperspaces is well understood, and this without any restrictions on the hyperspace type.

However, even now the theory of set convergence has some operational shortcomings. Very little is known about the preservation of convergence under various operations that can be performed on sets, apart from the convex case. In fact, most of what is known for nonconvex sets is "negative," i.e., quite a number of examples can be exhibited which show that set convergence is not preserved under addition, projections, taking convex hulls (or affine hulls, or positive hulls), linear transformations, intersections, and so on. We are going to show that it is possible to obtain conditions that will guarantee convergence under most of these operations, provided that one relies on a somewhat stronger notion of convergence for sets which also takes into account the points "at infinity" (direction points) that can be associated with the limit sets.

Attention will be restricted here to the case where the underlying space is $\mathbb{R}^n$. There are no basic reasons why the theory could not be extended to a more abstract setting, in particular to the case where the underlying space is an infinite-dimensional linear space, but certain properties that we are going to bring to the fore, and exploit repeatedly, cannot be carried over to the infinite-dimensional setting without modification.

We begin with a brief review of the theory of set convergence, at least as far as is needed in the present context. Next, we introduce $\operatorname{csm}\mathbb{R}^n$, the *n-dimensional* cosmic space, as an augmentation of $\mathbb{R}^n$ obtained by adding (abstract) *direction points* to $\mathbb{R}^n$. We then turn to the characterization of set convergence in $\operatorname{csm}\mathbb{R}^n$, and finally obtain a number of results about the preservation of set convergence under various operations that spell out the role played by this extended notion of convergence.

1. SET CONVERGENCE

We assume that the reader is familiar with the basic facts about set convergence; an excellent up-to-date exposition can be found in Aubin and Frankowska [**2**]. In this section, we follow basically the same pattern that we have adopted in [**4**], to which we refer for the detail of the proofs omitted here.

Throughout, we use the following notation: $\mathbb{R}^n$ for n-dimensional Euclidean space, $|\cdot|$ for the Euclidean norm, $d(x,y) := |x-y|$ for the metric on $\mathbb{R}^n$, $\mathbb{B}$ for the unit ball and $\mathbb{B}(x,\eta)$ for the closed ball of center x and radius η. The distance function associated with a set C is denoted by d_C, or when more convenient by $d(\cdot, C)$, with

$$d_C(x) = d(x,C) := \inf\{\, |x-y| \,\big|\, y \in C \,\}.$$

The hyperspace associated with $\mathbb{R}^n$ is

$$\mathrm{sets}(X) := \text{ collection of all subsets of } X.$$

We shall be mostly dealing with sequences, but for operational reasons it is convenient to develop the theory in terms of the Fréchet filter on $\mathbb{N}$ which, in a general index space can be replaced by a filter and its associated grill. Let

$$\begin{aligned}\mathcal{N}_\infty &:= \{\, N \subset \mathbb{N} \,|\, \mathbb{N} \setminus N \text{ finite} \}\\ \mathcal{N}_\infty^\# &:= \{\, N \subset \mathbb{N} \,|\, N \text{ infinite} \} = \{\text{ all subsequences of } \mathbb{N}\,\} \supset \mathcal{N}_\infty.\end{aligned}$$

We write $\lim_{\nu\in N}$ or $\lim_{\nu \xrightarrow[N]{} \infty}$ in the case of convergence of a subsequence designated by the selection of an index set N in $\mathcal{N}_\infty^\#$ or $\mathcal{N}_\infty$.

1.1. Definition. *For a sequence $\{C^\nu\}_{\nu\in\mathbb{N}}$ of subsets of $\mathbb{R}^n$, the inner limit is the set*

$$\liminf_{\nu\to\infty} C^\nu := \{\, x \,|\, \exists\, N \in \mathcal{N}_\infty,\ x^\nu \in C^\nu \ \text{for } \nu \in N,\ \text{with } x^\nu \xrightarrow[N]{} x \,\}$$

while the outer limit is the set

$$\limsup_{\nu\to\infty} C^\nu = \{\, x \,|\, \exists\, N \in \mathcal{N}_\infty^\#,\ x^\nu \in C^\nu \ \text{for } \nu \in N,\ \text{with } x^\nu \xrightarrow[N]{} x \,\}$$

The limit of the sequence exists if the inner and outer limit sets are equal:

$$\lim_{\nu\to\infty} C^\nu := \liminf_{\nu\to\infty} C^\nu = \limsup_{\nu\to\infty} C^\nu. \tag{1–1}$$

It is clear from the inclusion $\mathcal{N}_\infty \subset \mathcal{N}_\infty^\#$ that always $\liminf_\nu C^\nu \subset \limsup_\nu C^\nu$. The limits sets can also be expressed in the following terms,

$$\liminf_{\nu\to\infty} C^\nu = \bigcap_{N\in\mathcal{N}_\infty^\#} \mathrm{cl} \bigcup_{\nu\in N} C^\nu, \qquad \limsup_{\nu\to\infty} C^\nu = \bigcap_{N\in\mathcal{N}_\infty} \mathrm{cl} \bigcup_{\nu\in N} C^\nu. \tag{1–2}$$

From this it follows that limit sets are always closed, and that they only depend on the closure of the sets C^ν, i.e., if

$$\operatorname{cl} C^\nu = \operatorname{cl} D^\nu \quad\Longrightarrow\quad \begin{cases} \liminf_\nu C^\nu &= \liminf_\nu D^\nu \\ \limsup_\nu C^\nu &= \limsup_\nu D^\nu. \end{cases}$$

The sequence $\{C^\nu\}_{\nu\in\mathbb{N}}$ is said to *converge* to a set C when $\lim_\nu C^\nu$ exists and equals C. Set convergence in this sense is known more specifically as *Painlevé-Kuratowski* convergence. In section 3, we are going to consider another kind of set convergence induced by the imbedding of $\mathbb{R}^n$ in the cosmic space $\operatorname{csm}\mathbb{R}^n$, which allows for the possibility of an unbounded sequence of points to converge to a direction point (in the "horizon" of $\mathbb{R}^n$, as described in the next section). Other convergence notions occupy an important place as well, when dealing with subsets of an infinite-dimensional linear space, but in finite dimensions, they all coincide with Painlevé-Kuratowski convergence.

1.2. Theorem. (Salinetti and Wets [**5,** Theorem 2.2]). *For subsets C^ν and C of $\mathbb{R}^n$ with C closed, one has*

(a) $C \subset \liminf_\nu C^\nu$ *if and only if for every $\rho > 0$ and $\varepsilon > 0$ there is an index set $N \in \mathcal{N}_\infty$ such that*

$$C \cap \rho\mathbb{B} \subset C^\nu + \varepsilon\mathbb{B} \ \text{ for all } \nu \in N;$$

(b) $C \supset \limsup_\nu C^\nu$ *if and only if for every $\rho > 0$ and $\varepsilon > 0$ there is an index set $N \in \mathcal{N}_\infty$ such that*

$$C^\nu \cap \rho\mathbb{B} \subset C + \varepsilon\mathbb{B} \ \text{ for all } \nu \in N.$$

Thus, $C = \lim_\nu C^\nu$ if and only if for every $\rho > 0$ and $\varepsilon > 0$ there is an index set $N \in \mathcal{N}_\infty$ such that such that both of these inclusions hold.

Taking $C = \emptyset$ in statement (a) of the theorem, leads to the following characterization of "convergence" to the empty set.

1.3. Corollary. (Salinetti and Wets [**5,** Lemma 2.1]). *The condition $C^\nu \to \emptyset$ (or equivalently,* $\limsup_\nu C^\nu = \emptyset$*) holds for a sequence $\{C^\nu\}_{\nu\in\mathbb{N}}$ in $\mathbb{R}^n$ if and only if for every $\rho > 0$ there is an index set $N \in \mathcal{N}_\infty$ such that $C^\nu \cap \rho\mathbb{B} = \emptyset$ for all $\nu \in N$.*

Instead of "convergence" to the empty set, it will be helpful in the situation described in 1.3 to refer to the sequence $\{C^\nu\}_{\nu\in\mathbb{N}}$ in $\operatorname{sets}(\mathbb{R}^n)$ as *escaping to the horizon.* When we shift in section 3 from the context of subsets of $\mathbb{R}^n$ to that of subsets of the n-dimensional cosmic space $\operatorname{csm}\mathbb{R}^n$, the "horizon" in this terminology will take on its specific meaning (as the set of all "direction" points).

Certain geometric properties are preserved under set convergence. The following are immediate consequences of the definitions of set limits.

1.4. Proposition.

(a) *Monotone sequences always converge. If C^ν is nondecreasing (i.e., $C^{\nu+1} \supset C^\nu$), then* $\lim_\nu C^\nu = \operatorname{cl}\bigcup_{\nu\in\mathbb{N}} C^\nu$*. If C^ν is nonincreasing (i.e., $C^{\nu+1} \subset C^\nu$), then* $\lim_\nu C^\nu = \bigcap_{\nu\in\mathbb{N}} \operatorname{cl} C^\nu$.

(b) *Suppose the sequence* $\{C^\nu\}_{\nu\in\mathbb{N}}$ *consists of convex subsets of* $\mathbb{R}^n$*. Then* $\liminf_\nu C^\nu$ *is convex, and so too, when it exists, is* $\lim_\nu C^\nu$*. (But* $\limsup_\nu C^\nu$ *need not be convex in general.)*

(c) *For a sequence of cones* K^ν *in* $\mathbb{R}^n$*, the inner and outer limits, as well as the limit if it exists, are cones.*

The next result gives a useful characterization of general set convergence from a different angle.

1.5. Theorem. (hit-and-miss criterion, Choquet [**3**]). *For subsets* C^ν *and* C *of* $\mathbb{R}^n$ *with* C *closed, one has*

(a) $C \subset \liminf_\nu C^\nu$ *if and only if for every open set* $O \subset \mathbb{R}^n$ *with* $C \cap O \neq \emptyset$ *there exists* $N \in \mathcal{N}_\infty$ *such that*

$$C^\nu \cap O \neq \emptyset \text{ for all } \nu \in N;$$

(b) $C \supset \limsup_\nu C^\nu$ *if and only if for every compact set* $B \subset \mathbb{R}^n$ *with* $C \cap B = \emptyset$ *there exists* $N \in \mathcal{N}_\infty$ *such that*

$$C^\nu \cap B = \emptyset \text{ for all } \nu \in N;$$

A remarkable feature of set convergence is the existence of convergent subsequences for any sequence $\{C^\nu\}_{\nu\in\mathbb{N}}$ in the space $\mathrm{sets}(\mathbb{R}^n)$.

1.6. Theorem. (Zarankiewicz [**6**]). *Every sequence of sets* C^ν *in* $\mathbb{R}^n$ *that does not escape to the horizon has a subsequence converging to a nonempty set* C *in* $\mathbb{R}^n$*.*

We turn next to a quantification of set convergence in terms of the convergence of distance functions. The role of distance functions has already been glanced at in 1.2(a).

1.7. Theorem. (Choquet [**3**]). *For subsets* C^ν *and* C *of* $\mathbb{R}^n$ *with* C *closed and nonempty, one has* $C^\nu \to C$ *if and only if* $d(x, C^\nu) \to d(x, C)$ *for all* $x \in \mathbb{R}^n$*, in which event the functions* d_{C^ν} *actually converge uniformly to* d_C *on all bounded subsets of* $\mathbb{R}^n$*.*

These properties of the distance function provide the springboard to a description of set convergence in terms of a certain metric. We begin with an intermediate description involving a family of pseudo-metrics. Of course, because set convergence does not distinguish between a set and its closure, no full metric space interpretation of set convergence is possible unless we now focus on

$$\underline{\mathrm{sets}}(\mathbb{R}^n) = \text{ the space of all nonempty, closed subsets of } \mathbb{R}^n, \tag{1–3}$$

rather than $\mathrm{sets}(\mathbb{R}^n)$. The exclusion of the empty set in the definition of $\underline{\mathrm{sets}}(\mathbb{R}^n)$ is in line with the separate emphasis we give to sequences that escape to the horizon.

Two measures of distance between sets turn out to be the most convenient both for theoretical and estimation purposes, they seem to have been first suggested in [**1**]. For every $\rho \in \mathbb{R}_+ = [0, \infty)$ and pair of *nonempty* sets C_1 and C_2, one defines

$$\begin{aligned} d\!l_\rho(C_1, C_2) &:= \max_{|x|\le\rho} \bigl|d_{C_1}(x) - d_{C_2}(x)\bigr|, \\ \hat{d\!l}_\rho(C_1, C_2) &:= \min\{\, \eta \ge 0 \,|\, C_1 \cap \rho\mathbb{B} \subset C_2 + \eta\,\mathbb{B},\ C_2 \cap \rho\mathbb{B} \subset C_1 + \eta\,\mathbb{B} \,\}. \end{aligned} \tag{1–4}$$

Clearly, $d\!l_\rho$ is inspired by the uniformity property in 1.7 while $\hat{d\!l}_\rho$ relates to the one in 1.2. Although we do not insist on applying these expressions only to closed sets, the main interest lies in thinking of $d\!l_\rho$ and $\hat{d\!l}_\rho$ as functions from the product space $\underline{\text{sets}}(\mathbb{R}^n) \times \underline{\text{sets}}(\mathbb{R}^n)$ to $\mathbb{R}_+$.

1.8. Theorem. (Attouch and Wets [**1**]). *For each $\rho \geq 0$, $d\!l_\rho$ is a pseudo-metric on the space $\underline{\text{sets}}(\mathbb{R}^n)$, but $\hat{d\!l}_\rho$ is not. Both families $\{d\!l_\rho\}_{\rho\geq 0}$ and $\{\hat{d\!l}_\rho\}_{\rho\geq 0}$ characterize set convergence: for any $\rho_0 \in \mathbb{R}_+$, one has*

$$\begin{aligned} C^\nu \to C \iff & d\!l_\rho(C^\nu, C) \to 0 \text{ for all } \rho \geq \rho_0 \\ \iff & \hat{d\!l}_\rho(C^\nu, C) \to 0 \text{ for all } \rho \geq \rho_0. \end{aligned}$$

Proof. Theorem 1.7 gives us the characterization of set convergence in terms of $d\!l_\rho$, while 1.2 gives it to us for $\hat{d\!l}_\rho$. For $d\!l_\rho$, the pseudo-metric properties of nonnegativity $d\!l_\rho(C_1, C_2) \in \mathbb{R}_+$, symmetry $d\!l_\rho(C_1, C_2) = d\!l_\rho(C_2, C_1)$, and the triangle inequality $d\!l_\rho(C_1, C_2) \leq d\!l_\rho(C_1, C) + d\!l_\rho(C, C_2)$, are obvious from the definition (1-4) and the inequality

$$|d_{C_1}(x) - d_{C_2}(x)| \leq |d_{C_1}(x) - d_C(x)| + |d_C(x) - d_{C_2}(x)|.$$

The triangle inequality can fail for $\hat{d\!l}_\rho$: take $C_1 = \{1\} \subset \mathbb{R}$, $C_2 = \{-1\}$, $C = \{-6/5, 6/5\}$ and $\rho = 1$. Thus $\hat{d\!l}_\rho$ is not a pseudo-metric. $\square$

The *Pompeiu-Hausdorff distance*, $d\!l_\infty(C_1, C_2) := \sup_{x\in\mathbb{R}^n} \big|d_{C_1}(x) - d_{C_2}(x)\big|$ is a well-known measure of the distance between sets, but $d\!l_\infty(C^\nu, C) \to 0$ would define a kind of convergence more stringent than Painlevé-Kuratowski convergence. The two kinds of convergence are equivalent in some situations, such as for bounded sequences. But convergence with respect to $d\!l_\infty$ is certainly not suitable for sequences involving unbounded sets (such as epigraphs). We must therefore look elsewhere than the Pompeiu-Hausdorff distance for a single metric characterizing ordinary convergence in $\underline{\text{sets}}(\mathbb{R}^n)$. There are many ways that such a metric can be derived from the family of pseudo-metrics $d\!l_\rho$. A convenient expression is

$$d\!l(C_1, C_2) := \int_0^\infty d\!l_\rho(C_1, C_2) e^{-\rho} d\rho. \tag{1–5}$$

This will be called the *set distance* between C_1 and C_2, in contrast to the quantity $d\!l_\rho(C_1, C_2)$ being called the ρ-distance. Note that

$$d\!l(C_1, C_2) \leq d\!l_\infty(C_1, C_2), \tag{1–6}$$

because $d\!l_\rho(C_1, C_2) \leq d\!l_\infty(C_1, C_2)$ for all ρ, and $\int_0^\infty e^{-\rho} d\rho = 1$.

1.9. Lemma. *For any nonempty, closed subsets C_1 and C_2 of $\mathbb{R}^n$ and any $\rho \in \mathbb{R}_+$, one has*

(a) $d\!l(C_1, C_2) \geq (1 - e^{-\rho})\big|d_{C_1}(0) - d_{C_2}(0)\big| + e^{-\rho} d\!l_\rho(C_1, C_2)$,

(b) $\mathbb{d}(C_1,C_2) \le (1-e^{-\rho})\mathbb{d}_\rho(C_1,C_2) + e^{-\rho}\big(\max\{d_{C_1}(0),d_{C_2}(0)\} + \rho + 1\big)$.

Proof. We write

$$\mathbb{d}(C_1,C_2) = \int_0^\rho \mathbb{d}_\tau(C_1,C_2))e^{-\tau}d\tau + \int_\rho^\infty \mathbb{d}_\tau(C_1,C_2)e^{-\tau}d\tau$$

and note from the monotonicity of $\mathbb{d}_\rho(C_1,C_2)$ in ρ that

$$\begin{aligned}
\mathbb{d}_0(C_1,C_2)\int_0^\rho e^{-\tau}d\tau &\le \int_0^\rho \mathbb{d}_\tau(C_1,C_2)e^{-\tau}d\tau \le \mathbb{d}_\rho(C_1,C_2)\int_0^\rho e^{-\tau}d\tau,\\
\mathbb{d}_\rho(C_1,C_2)\int_\rho^\infty e^{-\tau}d\tau &\le \int_\rho^\infty \mathbb{d}_\tau(C_1,C_2)e^{-\tau}d\tau\\
&\le \int_\rho^\infty \big[\max\{d_{C_1}(0),d_{C_2}(0)\}+\tau\big]e^{-\tau}d\tau,
\end{aligned}$$

where the last inequality comes from the fact that for all $\rho \in \mathbb{R}_+$ and any pair of nonempty sets C_1, C_2, $\mathbb{d}_\rho(C_1,C_2) \le \max\{\, d_{C_1}(0), d_{C_2}(0)\,\} + \rho$. The lower estimates calculate to the inequality in (a), and the upper estimates to the one in (b). $\square$

1.10. Theorem. *The expression $\mathbb{d}$ gives a metric on* sets $(\mathbb{R}^n)$, *and this metric characterizes ordinary set convergence: one has*

$$C^\nu \to C \iff \mathbb{d}(C^\nu,C) \to 0.$$

Furthermore, (sets $(\mathbb{R}^n), \mathbb{d}$) *is a complete metric space in which a sequence $\{C^\nu\}_{\nu\in\mathbb{N}}$ escapes to the horizon if and only if for some set C in this space (and then for every such set C) one has $\mathbb{d}(C^\nu,C) \to \infty$.*

Proof. We get $\mathbb{d}(C_1,C_2) \ge 0$, $\mathbb{d}(C_1,C_2) = \mathbb{d}(C_2,C_1)$, and the triangle inequality $\mathbb{d}(C_1,C_2) \le \mathbb{d}(C_1,C) + \mathbb{d}(C,C_2)$ from the corresponding properties of the pseudo-metrics $\mathbb{d}_\rho(C_1,C_2)$. The estimate $\mathbb{d}_\rho(C_1,C_2) \le \max\{\, d_{C_1}(0), d_{C_2}(0)\,\} + \rho$ gives us $\mathbb{d}(C_1,C_2) < \infty$. Since for closed sets C_1 and C_2 the distance functions d_{C_1} and d_{C_2} are continuous and vanish only on these sets, respectively, we have $\big|d_{C_1}(x) - d_{C_2}(x)\big|$ positive on some open set unless $C_1 = C_2$. Thus $\mathbb{d}(C_1,C_2) > 0$ unless $C_1 = C_2$. This proves that $\mathbb{d}$ is a metric.

It is clear from the estimates in 1.9(a) and (b) that $\mathbb{d}(C^\nu,C) \to 0$ if and only if $\mathbb{d}_\rho(C^\nu,C) \to 0$ for every $\rho \ge 0$. In view of theorem 1.7, we know therefore that the metric $\mathbb{d}$ on sets$(\mathbb{R}^n)$ characterizes set convergence.

From 1.3, a sequence $\{C^\nu\}$ in sets $(\mathbb{R}^n)$ escapes to the horizon if and only if it eventually misses every ball $\rho\mathbb{B}$, or equivalently, has $d_{C^\nu}(0) \to \infty$. Since

$$\big|d_{C^\nu}(0) - d_C(0)\big| \le \mathbb{d}(C^\nu,C) \le \max\{\, d_{C^\nu}(0), d_C(0)\,\} + 1,$$

the sequence escapes to the horizon if and only if $\mathbb{d}(C^\nu,C) \to \infty$ for every C (or for just one C). In particular, a Cauchy sequence cannot have any subsequence escaping to the horizon, because in such a sequence the distances $d\!\!l(C^\mu, C^\nu)$ must be bounded, so that $\limsup_\nu d\!\!l(C^\mu, C^\nu) < \infty$. It follows from the compactness property in 1.6 that every Cauchy sequence in $\underline{\text{sets}}(\mathbb{R}^n)$ has a subsequence converging to an element of $\underline{\text{sets}}(\mathbb{R}^n)$, this element then necessarily being the actual limit of the sequence. Thus, the metric space $(\underline{\text{sets}}(\mathbb{R}^n), d\!\!l)$ is complete. □

1.11. Corollary (local compactness)**.** *The metric space* $(\underline{\text{sets}}(\mathbb{R}^n), d\!\!l)$ *has the property that for every* C_0 *and every* $r > 0$ *the ball* $\{\, C \mid d\!\!l(C, C_0) \le r \,\}$ *is compact.*

Proof. This is a consequence of the compactness in 1.6 and the criterion in 1.3 for escape to the horizon. □

2. n-DIMENSIONAL COSMIC SPACE

An important advantage that the extended real line $\overline{\mathbb{R}}$ has over the real line $\mathbb{R}$ is *compactness*: every sequence of elements has a convergent subsequence. This property is achieved by adjoining to $\mathbb{R}$ the special elements ∞ and $-\infty$, which can act as limits for unbounded sequences under special rules. An analogous compactification is possible for $\mathbb{R}^n$ and will now be introduced.

We wish to think of the direction of x, denoted by $\operatorname{dir} x$, as an abstract attribute associated with $x \in \mathbb{R}^n$ under the rule that $\operatorname{dir} x = \operatorname{dir} y$ if $y = \lambda x \neq 0$ for some $\lambda > 0$. The zero vector is to be viewed as having no direction; $\operatorname{dir} 0$ is undefined. Directions correspond thus to equivalence classes under the relation $y = \lambda x \neq 0$ for some $\lambda > 0$.

There is a one-to-one correspondence between the various directions of vectors $x \neq 0$ in $\mathbb{R}^n$ and the *rays* in $\mathbb{R}^n$. Every direction can be *represented* uniquely by a ray, but we shall think of directions themselves as abstract points, called *direction points*, which lie outside of $\mathbb{R}^n$ and form a set called the *horizon* of $\mathbb{R}^n$, denoted by $\operatorname{hor} \mathbb{R}^n$.

In the case of $n = 1$ there are only two direction points, symbolized by ∞ and $-\infty$. By adding these direction points to $\mathbb{R}$ one obtains the extended real line $\overline{\mathbb{R}}$. We follow the same procedure when $n > 1$ by adding all the direction points in $\operatorname{hor} \mathbb{R}^n$ to $\mathbb{R}^n$ to form the extended space

$$\operatorname{csm} \mathbb{R}^n := \mathbb{R}^n \cup \operatorname{hor} \mathbb{R}^n,$$

which will be called the *cosmic closure* of $\mathbb{R}^n$, or *n-dimensional cosmic space.* (Note that $\operatorname{csm} \mathbb{R}^n$ is not the n-fold product $\overline{\mathbb{R}} \times \cdots \times \overline{\mathbb{R}}$.) We need to supply $\operatorname{csm} \mathbb{R}^n$ with geometry and topology so that it can serve as a companion to $\mathbb{R}^n$ in various questions of analysis, just as $\overline{\mathbb{R}}$ serves alongside of $\mathbb{R}$.

There are several ways of viewing the n-dimensional cosmic space $\operatorname{csm} \mathbb{R}^n$ geometrically, all of them leading to the same mathematical structure but having different advantages in different contexts. The simplest and most intuitive perhaps is the *celestial model*, in which $\mathbb{R}^n$ is imagined as shrunk down (e.g., $x \to x/(|x|+1)$) to an n-dimensional open ball of finite radius, and $\operatorname{hor} \mathbb{R}^n$ is identified with the surface of this ball.

A second approach to setting up the structure of analysis in n-dimensional cosmic space utilizes the *ray space model*. Each $x \in \mathbb{R}^n$ is identified (uniquely) with a ray in $\mathbb{R}^n \times \mathbb{R}$, specifically the ray passing through $(x, -1)$. The points in $\operatorname{hor} \mathbb{R}^n$, on the other hand, correspond uniquely to the "horizontal" rays in $\mathbb{R}^n \times \mathbb{R}$, which are the ones lying in the hyperplane $(\mathbb{R}^n, 0)$. This model, although less intuitive than the celestial model and requiring an extra dimension, is superior for some purposes, such as extensions of convexity.

A third approach, which in some respects fits between the other two, uses the fact that each ray in the ray space model in $\mathbb{R}^n \times \mathbb{R}$, whether associated with an ordinary point of $\mathbb{R}^n$ or a direction point in $\operatorname{hor} \mathbb{R}^n$, pierces the closed unit hemisphere

$$\mathbb{H} = \mathbb{H}_n = \left\{ (x, \beta) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; \beta \leq 0,\ |x|^2 + \beta^2 = 1 \right\} \tag{2–1}$$

in a unique point. Thus, the hemisphere $\mathbb{H}$ furnishes an alternative model of $\operatorname{csm} \mathbb{R}^n$ in which the rim of $\mathbb{H}$ represents the horizon of $\mathbb{R}^n$. This will be called the *hemispherical model* of $\operatorname{csm} \mathbb{R}^n$.

Fig. 2–1. The hemispherical model for n-dimensional cosmic space.

2.1. Definition. *A sequence of points* x^ν *in* $\mathbb{R}^n$ *is said to converge to a direction point* $\operatorname{dir} x$, *written* $x^\nu \to \operatorname{dir} x$, *if for some sequence of scalars* $\lambda^\nu \downarrow 0$ *(meaning that* $\lambda^\nu \to 0$ *with* $\lambda^\nu > 0$*) one has* $\lambda^\nu x^\nu \to x$. *Similarly, a sequence of direction points* $\operatorname{dir} x^\nu$ *is said to converge to a direction point* $\operatorname{dir} x$, *written* $\operatorname{dir} x^\nu \to \operatorname{dir} x$, *if for some sequence of scalars* $\lambda^\nu > 0$ *one has* $\lambda^\nu x^\nu \to x$.

A mixed sequence of ordinary points and direction points is said to converge to $\operatorname{dir} x$ *if every subsequence consisting of ordinary points converges to* $\operatorname{dir} x$, *and the same holds for every subsequence consisting of direction points.*

This extension of convergence to allow for direction points as possible limits, in combination with the usual notion of convergence to an ordinary point in $\mathbb{R}^n$, yields the notion of *cosmic convergence* for sequences of points in $\operatorname{csm} \mathbb{R}^n$. It is easily checked that a sequence of points in $\operatorname{csm} \mathbb{R}^n$ converges if and only if the corresponding sequence in the hemisphere $\mathbb{H}$ converges. The compactness of $\mathbb{H}$ in $\mathbb{R}^n \times \mathbb{R}$ then yields the central fact about $\operatorname{csm} \mathbb{R}^n$.

2.2. Theorem. *The cosmic closure* $\operatorname{csm} \mathbb{R}^n$ *of* $\mathbb{R}^n$ *is a compact space: every sequence of points in* $\operatorname{csm} \mathbb{R}^n$ *(whether ordinary points, direction points or some mixture) has a convergent subsequence (in the extended sense of cosmic convergence). In this, the bounded sequences in* $\mathbb{R}^n$ *are characterized as the sequences of ordinary points for which no direction point is a cluster point.*

Sometimes it is useful to quantify cosmic convergence through the *cosmic metric* d^c on $\operatorname{csm} \mathbb{R}^n$, in contrast to the Euclidean metric d on $\mathbb{R}^n$. In this metric the distance between any two points of $\operatorname{csm} \mathbb{R}^n$ is taken to be the geodesic distance between the corresponding points on the hemisphere $\mathbb{H}$, i.e., the angle in radians between the corresponding rays in $\mathbb{R}^n \times \mathbb{R}$. Within $\mathbb{R}^n$ itself, the convergence induced by the cosmic metric is the same as that induced by the Euclidean metric.

For a set $C \subset \mathbb{R}^n$, we must distinguish between $\operatorname{csm} C$, the *cosmic closure* of C in which direction points are allowed as possible limits, and $\operatorname{cl} C$, the ordinary closure in $\mathbb{R}^n$.

The collection of all direction points obtainable as limits of sequences in C will be called the *horizon* of C and denoted by $\operatorname{hor} C$. Thus,

$$\operatorname{csm} C = \operatorname{cl} C \cup \operatorname{hor} C. \tag{2–2}$$

The set $\operatorname{hor} C$ furnishes a precise description of the unboundedness, if any, in C.

However, rather than working directly with $\operatorname{hor} C$ and other subsets of $\operatorname{hor} \mathbb{R}^n$ in developing specific conditions, we shall generally find it more expedient to work with representations of those sets in terms of rays in $\mathbb{R}^n$.

Aside from the *zero* cone $\{0\}$, the cones K in $\mathbb{R}^n$ are characterized as the sets expressible as nonempty unions of rays. The set of direction points represented by the rays in K will be denoted by $\operatorname{dir} K$. There is thus a one-to-one correspondence between cones in $\mathbb{R}^n$ and subsets of the horizon of $\mathbb{R}^n$,

$$K \ \text{(cone in } \mathbb{R}^n\text{)} \qquad \longleftrightarrow \qquad \operatorname{dir} K \ \text{(direction set in } \operatorname{hor} \mathbb{R}^n\text{)},$$

where the cone is closed in $\mathbb{R}^n$ if and only if the direction set is cosmically closed in $\operatorname{csm} \mathbb{R}^n$. Cones K are said in this manner to *represent* sets of direction points. The zero cone corresponds to the empty set of direction points, while the *full* cone $K = \mathbb{R}^n$ gives the entire horizon.

A general subset of $\operatorname{csm} \mathbb{R}^n$ can be expressed *in a unique way* as $C \cup \operatorname{dir} K$ for some set $C \subset \mathbb{R}^n$ and some cone $K \subset \mathbb{R}^n$. We shall say that C gives the *ordinary* part of the set and K the *horizon* part. The corresponding cone in the ray space model is then

$$\{ (\lambda x, -\lambda) \mid x \in C,\ \lambda > 0 \} \cup \{ (x, 0) \mid x \in K \}.$$

Our pattern will typically be to express the properties of $C \cup \operatorname{dir} K$ in terms of the properties of the pair C, K.

2.3. Definition. *For a set $C \subset \mathbb{R}^n$, the horizon cone C^∞ is the closed cone in $\mathbb{R}^n$ representing the direction set* $\operatorname{hor} C$*:*

$$C^\infty = \begin{cases} \{ x \mid \exists x^\nu \in C,\ \lambda^\nu \downarrow 0,\ \text{ with } \lambda^\nu x^\nu \to x \} & \text{when } C \neq \emptyset, \\ \{ 0 \} & \text{when } C = \emptyset. \end{cases}$$

In this notation and terminology, we have $\operatorname{hor} C = \operatorname{dir} C^\infty$ and therefore $\operatorname{csm} C = \operatorname{cl} C \cup \operatorname{dir} C^\infty$ in place of the formula in (2-2). Note that $(\operatorname{cl} C)^\infty = C^\infty$. If C itself happens to be a cone, C^∞ is just $\operatorname{cl} C$.

A subset of $\operatorname{csm} \mathbb{R}^n$, written as $C \cup \operatorname{dir} K$ for a set $C \subset \mathbb{R}^n$ and a cone $K \subset \mathbb{R}^n$, is closed in the cosmic sense if and only if C and K are closed in the ordinary sense and $C^\infty \subset K$. In general, the cosmic closure of $C \cup \operatorname{dir} K$ is given by

$$\operatorname{csm}(C \cup \operatorname{dir} K) = \operatorname{cl} C \cup \operatorname{dir}(C^\infty \cup \operatorname{cl} K).$$

The remainder of this section is devoted to the use of cosmic concepts in the formulation of boundedness criteria.

2.4. Theorem. *A set $C \subset \mathbb{R}^n$ is bounded if and only if its horizon cone is just the zero cone: $C^\infty = \{0\}$.*

Proof. A set is unbounded if and only if it contains an unbounded sequence. Equivalently by the facts in 2.2, a set is bounded if and only if its closure in the sense of cosmic convergence contains no direction points, i.e., $\operatorname{hor} C = \emptyset$. Since C^∞ is the cone representing the points of $\operatorname{hor} C$, this means that $C^\infty = \{0\}$. □

A simple case where horizon cones can readily be determined, and theorem 2.4 then applied, is that of sets defined by systems of linear equations and inequalities. It is not difficult to verify that for a convex polyhedral set C,

$$C = \{\, x \mid \langle a_i, x\rangle \le \alpha_i \text{ for } i \in I_1,\ \langle a_i, x\rangle = \alpha_i \text{ for } i \in I_2 \,\},$$

one has that if $C \neq \emptyset$,

$$C^\infty = \{\, x \mid \langle a_i, x\rangle \le 0 \text{ for } i \in I_1,\ \langle a_i, x\rangle = 0 \text{ for } i \in I_2 \,\}.$$

Thus C is bounded if and only if the linear system that defines C^∞ has only the trivial solution $x = 0$.

One can also develop systematic rules for determining or estimating the horizon cones of various sets defined by s combination of linear and nonlinear constraints. However this would take us to far afield of our main concern. By means of such rules, a “calculus of boundedness” takes shape. The next few results give some of the most useful criteria, which are rather straightforward consequences of the definitions.

2.5. Proposition. *For a linear mapping $L : \mathbb{R}^d \to \mathbb{R}^n$ and a closed set $D \subset \mathbb{R}^d$, a sufficient condition for $L(D)$ to be closed is that $L^{-1}(0) \cap D^\infty = \{0\}$. Under this condition, $L(D^\infty) = L(D)^\infty$. In particular, if D is a closed subset of $\mathbb{R}^n$, C is its image under orthogonal projection onto a linear subspace $M \subset \mathbb{R}^n$, and $M^\perp \cap D^\infty = \{0\}$. Then C is closed, and its horizon cone C^∞ is the orthogonal projection of D^∞ on M.*

Even in the simple setting of the projection, it is easy to find examples where the hypothesis in 2.5 on the horizon cone D^∞ is essential for the conclusions.

2.6. Proposition.

(a) *For sets $C_i \subset \mathbb{R}^{n_i}$, $i = 1, \ldots, m$, one always has $(C_1 \times \cdots \times C_m)^\infty \subset C_1^\infty \times \cdots \times C_m^\infty$, with equality if the sets C_i are convex.*

(b) *For closed sets $C_i \subset \mathbb{R}^n$, $i = 1, \ldots, m$, a sufficient condition for $C_1 + \cdots + C_m$ to be closed is the nonexistence of any combination of vectors $x_i \in C_i^\infty$ such that $x_1 + \cdots + x_m = 0$, except when $x_i = 0$ for all i. Then also $(C_1 + \cdots + C_m)^\infty \subset C_1^\infty + \cdots + C_m^\infty$.*

(c) *For any collection of sets $C_i \subset \mathbb{R}^n$ for $i \in I$, an arbitrary index set, one has*

$$\Big[\bigcap\nolimits_{i\in I} C_i\Big]^\infty \subset \bigcap\nolimits_{i\in I} C_i^\infty, \qquad \Big[\bigcup\nolimits_{i\in I} C_i\Big]^\infty \supset \bigcup\nolimits_{i\in I} C_i^\infty,$$

with equality for the first inclusion if the sets are convex, and with equality for the second inclusion if I is finite.

3. COSMIC CONVERGENCE

In the brief survey of the theory of set limits in section 1, our concern has been exclusively with sequences of sets in $\mathbb{R}^n$. The definitions and most of the results, however, carry over to a more general framework. The definition of inner and outer limits in 1.1 makes sense for subsets of any metric space, for instance. The formulas then remain equivalent to the ones in (1-2), and many of the results, such as the convergence of monotone sequences (1.4a(a)), the closedness of limit sets, and the hit-and-miss criteria (1.5) are still valid with only minor notational adjustments. We want to focus now on set convergence in the cosmic space $\operatorname{csm}\mathbb{R}^n$, which is a metric space under the cosmic metric d^c introduced in section 2 (following 2.2).

For set convergence in $\operatorname{csm}\mathbb{R}^n$, to which we refer as *cosmic* set convergence, one has to bear in mind that the limit of a sequence of points can be either an ordinary point in $\mathbb{R}^n$ or a direction point, i.e., an element of the horizon set $\operatorname{hor}\mathbb{R}^n$ (2.1). To avoid possible confusion in situations where the sets we may be dealing with are actually in $\mathbb{R}^n$, as a subspace of $\operatorname{csm}\mathbb{R}^n$, we signal cosmic set convergence by writing $\xrightarrow{c}$ and c-lim in place of $\to$ and lim.

3.1. Definition. *For a general sequence of sets $C^\nu \cup \operatorname{dir} K^\nu$ in $\operatorname{csm}\mathbb{R}^n$ with each K^ν a cone, the cosmic inner limit is the set*

$$C \cup \operatorname{dir} K = \text{c-lim}\inf\nolimits_\nu \big(C^\nu \cup \operatorname{dir} K^\nu\big)$$

in $\operatorname{csm}\mathbb{R}^n$ consisting of (1) all the ordinary points x obtainable as limits of sequences $\{x^\nu\}_{\nu\in N}$ with $x^\nu \in C^\nu$ for an index set $N \in \mathcal{N}_\infty$, (2) all the direction points $\operatorname{dir} y$ obtainable as limits of such sequences (when unbounded), and (3) all the direction points $\operatorname{dir} y$ obtainable as limits of sequences of direction points $\{\operatorname{dir} y^\nu\}_{\nu\in N}$ selected from the horizon sets $\operatorname{dir} K^\nu$. (The set C gives the ordinary points belonging to the limit, while the cone K represents through $\operatorname{dir} K$ the various direction points belonging to it.) The cosmic outer limit is the set

$$C \cup \operatorname{dir} K = \text{c-lim}\sup\nolimits_\nu \big(C^\nu \cup \operatorname{dir} K^\nu\big)$$

in $\operatorname{csm}\mathbb{R}^n$ obtained in like manner, except with index sets $N \in \mathcal{N}_\infty^\#$. When the cosmic inner and outer limits are the same set $C \cup \operatorname{dir} K$, the cosmic limit is said to exist. This is indicated by

$$C \cup \operatorname{dir} K = \text{c-lim}_\nu \big(C^\nu \cup \operatorname{dir} K^\nu\big) \quad \text{or} \quad C^\nu \cup \operatorname{dir} K^\nu \xrightarrow{c} C \cup \operatorname{dir} K.$$

We are going to look more closely at the relationship between C and K and the sequences of sets C^ν and K^ν, and this will enable us to proceed in a more "computational" manner. For now we focus on the geometry.

A sequence of subsets of $\operatorname{csm}\mathbb{R}^n$ that happens to be in $\mathbb{R}^n$ is identified by having $K^\nu = \{0\}$ for all ν. Thus, for C^ν in $\mathbb{R}^n$ we may, on the one hand, have ordinary convergence $C^\nu \to C$ for some set C in $\mathbb{R}^n$, while in the context of cosmic convergence in $\operatorname{csm}\mathbb{R}^n$ the same sequence may fail to converge, or may be such that $C^\nu \xrightarrow{c} C \cup \operatorname{dir} K$ where the cone K

is not just $\{0\}$. This is because the cosmic inner and outer limits of the sequence $\{C^\nu\}_{\nu\in\mathbb{N}}$ may contain direction points, whereas such points are excluded from the ordinary inner and outer limits.

Indeed, the cosmic outer limit set must contain a direction point unless the sequence is bounded: for an unbounded sequence $\{C^\nu\}_{\nu\in\mathbb{N}}$ one can select for ν in some index set $N \in \mathcal{N}_\infty^\#$ points $x^\nu \in C^\nu$ such that $|x^\nu| \underset{N}{\to} \infty$, and the sequence $\{x^\nu\}_{\nu\in N}$ will then have cluster points, each of which is a direction point belonging to $\text{c-lim sup}_\nu C^\nu$ (2.2). A couple of examples will help to make this clearer.

- For a fixed vector $a \neq 0$ in $\mathbb{R}^n$ and fixed $\delta > 0$, let C^ν be the ball $\mathbb{B}(\nu a, \delta)$. With respect to ordinary convergence we have $C^\nu \to \emptyset$, i.e. the sets C^ν escape to the horizon (1.3), but with respect to cosmic convergence we have $C^\nu \xrightarrow{c} \{\operatorname{dir} a\}$.
- Expand this by taking any compact set $C \neq \emptyset$ in $\mathbb{R}^n$ and defining $C^\nu = C \cup \mathbb{B}(\nu a, \delta)$ for ν even but $C^\nu = C \cup \mathbb{B}(-\nu a, \delta)$ for ν odd. Then $C^\nu \to C$ but the cosmic limit does not exist. One has $\text{c-lim inf}_\nu C^\nu = C$, whereas $\text{c-lim sup}_\nu C^\nu = C \cup \operatorname{dir} K$ for the cone $K = \{\, \lambda a \mid \lambda \in \mathbb{R} \,\}$ (the one-dimensional subspace generated by a).

The interplay between the two forms of convergence of a sequence of sets C^ν in $\mathbb{R}^n$ is one of the main motivations for studying cosmic set convergence. Cases where the cosmic limit is "predictable" from the ordinary limit correspond to a form of convergence in the space $\underline{\text{sets}}(\mathbb{R}^n)$ that is stronger than ordinary set convergence and is associated with a metric $d\!\!l^c$ different from the metric $d\!\!l$ utilized in theorem 1.10. This will be discussed in section 4 under the heading of "paracosmic" convergence.

From the closedness of limits comes the following basic property of cosmic set convergence.

3.2. Proposition. *If $C \cup \operatorname{dir} K$ is the inner or outer cosmic limit of a sequence of sets $C^\nu \cup \operatorname{dir} K^\nu$, where K and K^ν are cones, then $C \cup \operatorname{dir} K$ can depend only on the cosmic closures of the sets $C^\nu \cup \operatorname{dir} K^\nu$, i.e., their closures as subsets of $\operatorname{csm} \mathbb{R}^n$. Furthermore, $C \cup \operatorname{dir} K$ must itself be cosmically closed: C and K must be closed in $\mathbb{R}^n$ with $C^\infty \subset K$.*

These facts can be obtained by elementary direct argument, but they can also be deduced from general principles about the relationship between $\operatorname{csm} \mathbb{R}^n$ and $\mathbb{R}^n$. One can appeal to the isometry between $\operatorname{csm} \mathbb{R}^n$ and its hemispherical model $\mathbb{H}$ in $\mathbb{R}^{n+1}$, recalling that the cosmic metric on $\operatorname{csm} \mathbb{R}^n$ arises from the geodesic metric on $\mathbb{H}$. This approach leads to two handy methods of shifting arguments about cosmic set convergence back to the framework of ordinary set convergence.

3.3. Lemma.

(a) *Inner and outer cosmic limits of sequences of sets in $\operatorname{csm} \mathbb{R}^n$ correspond to the ordinary inner and outer limits of the associated sequences of subsets of the hemisphere $\mathbb{H}$, regarded simply as sequences of bounded subsets of $\mathbb{R}^{n+1}$.*

(b) *Inner and outer cosmic limits of sequences of sets in $\operatorname{csm} \mathbb{R}^n$ correspond to the ordinary inner and outer limits of the sequences of cones in $\mathbb{R}^{n+1}$ that are associated with them in the ray space model for $\operatorname{csm} \mathbb{R}^n$.*

Proof. (a) This is true because a sequence of points in $\mathbb{H}$ converges with respect to the geodesic metric on $\mathbb{H}$ if and only if it converges with respect to the ordinary metric on $\mathbb{R}^{n+1}$.

(b) This is an application to $\mathbb{R}^{n+1}$ of the following fact about the unit sphere $S = \{ x \mid |x| = 1 \}$: for a sequence of cones K^ν, one has $K = \limsup_\nu K^\nu$ if and only if $K \cap S = \limsup_\nu (K^\nu \cap S)$, and similarly for lim inf. This fact is established from the observation that one has $x^\nu \to x$ for nonzero vectors x^ν and x if and only if $x^\nu/|x^\nu| \to x/|x|$ and $|x^\nu| \to |x|$. □

Through these models, many results about ordinary set convergence can be translated to cosmic set convergence. For example, the fact that monotone sequences always converge cosmically is obvious from either model via 1.4(a). The fact that cosmically convergent sequences of convex sets have convex limits follows from 1.4(b). A uniformity result for general cosmic convergence along the lines of 1.2 can be obtained through the hemispherical model in terms of the cosmic metric, and so forth. Rather than make a formal statement of each such result, we shall usually rely on the general principles of translation in lemma 3.3.

As already seen, the chief distinction between ordinary set convergence in $\mathbb{R}^n$ and cosmic set convergence in $\operatorname{csm}\mathbb{R}^n$ lies in the way unbounded sequences of sets C^ν in $\mathbb{R}^n$ are handled. The cosmic limit of such a sequence, if it exists, has to contain at least one direction point. An unbounded sequence that escapes to the horizon can have *only* such points in the limit. The contrast in this respect is especially evident in the compactness result that holds for cosmic set convergence.

3.4. Theorem. *Every sequence of nonempty sets $C^\nu \cup \operatorname{dir} K^\nu$ in $\operatorname{csm}\mathbb{R}^n$ has a subsequence converging cosmically to a nonempty, cosmically closed subset $C \cup \operatorname{dir} K$ in $\operatorname{csm}\mathbb{R}^n$. This is true in particular for an unbounded sequence of sets C^ν in $\mathbb{R}^n$ that escapes to the horizon, the cosmic limit set then being a nonempty subset of $\operatorname{hor}\mathbb{R}^n$.*

Proof. Apply the compactness theorem for ordinary set convergence in 1.6 to the hemispherical setting in 3.3(a). □

The cosmic metric d^c on $\operatorname{csm}\mathbb{R}^n$ gives us a cosmic distance function $d^c(\cdot, C \cup \operatorname{dir} K)$ associated with any subset $C \cup \operatorname{dir} K$. This is slightly more complicated to deal with in notation than the distance functions on $\mathbb{R}^n$ utilized so far, because of the need to distinguish whether an argument is an ordinary point or a direction point, but otherwise the considerations are quite the same. A useful observation is that because $d^c(\cdot, C \cup \operatorname{dir} K)$ is continuous on $\operatorname{csm}\mathbb{R}^n$, its values on the horizon $\operatorname{hor}\mathbb{R}^n$ are obtainable as limits of its values on $\mathbb{R}^n$, and the latter therefore furnish an adequate hold on the cosmic metric aspects of the set $C \cup \operatorname{dir} K$ for many purposes.

3.5. Theorem. *For $C^\nu \cup \operatorname{dir} K^\nu$ and $C \cup \operatorname{dir} K$ as subsets of $\operatorname{csm}\mathbb{R}^n$ with $C \cup \operatorname{dir} K$ cosmically closed, one has that $C^\nu \cup \operatorname{dir} K^\nu \xrightarrow{c} C \cup \operatorname{dir} K$ if and only if the functions $d^c(\cdot, C^\nu \cup \operatorname{dir} K^\nu)$ converge pointwise to $d^c(\cdot, C \cup \operatorname{dir} K)$ on $\operatorname{csm}\mathbb{R}^n$. This is the case if and only if these functions converge uniformly on $\mathbb{R}^n$.*

Proof. This is true on the grounds of the hemispherical interpretation in 3.3(a). We know that the cosmic metric on $\operatorname{csm}\mathbb{R}^n$ corresponds to the geodesic metric on the hemisphere

$\mathbb{H}$, which in turn is equivalent to the ordinary Euclidean metric on $\mathbb{H}$ as a bounded subset of $\mathbb{R}^n$. The convergence facts in 1.7 can be invoked therefore relative to subsets of $\mathbb{H}$. Uniform convergence of $d^c(\cdot, C^\nu \cup \operatorname{dir} K^\nu)$ to $d^c(\cdot, C \cup \operatorname{dir} K)$ on $\operatorname{csm} \mathbb{R}^n$ is equivalent to uniform convergence on $\mathbb{R}^n$, on the basis of the observation made just before the statement of the theorem. ☐

The uniform convergence in this theorem suggests a good way of defining the *cosmic distance* between two sets $C_1 \cup \operatorname{dir} K_1$ and $C_2 \cup \operatorname{dir} K_2$ in $\operatorname{csm} \mathbb{R}^n$, namely by

$$\begin{aligned} &d\!\!l^c\big(C_1 \cup \operatorname{dir} K_1, C_2 \cup \operatorname{dir} K_2 \big) \\ &\qquad := \sup_{x \in \mathbb{R}^n} \big| d^c(x, C_1 \cup \operatorname{dir} K_1) - d^c(x, C_2 \cup \operatorname{dir} K_2) \big|. \end{aligned} \tag{3–1}$$

This formula can be invoked without the sets necessarily being closed in $\operatorname{csm} \mathbb{R}^n$. At present, though, our concern lies with the space

$$\underline{\text{sets}}\,(\operatorname{csm} \mathbb{R}^n) := \text{ the space of all } \textit{nonempty, closed} \text{ subsets of } \operatorname{csm} \mathbb{R}^n. \tag{3–2}$$

3.6. Theorem. *The expression* $d\!\!l^c(C_1 \cup \operatorname{dir} K_1,\, C_2 \cup \operatorname{dir} K_2)$ *is a metric on* $\underline{\text{sets}}\,(\operatorname{csm} \mathbb{R}^n)$, *and this metric characterizes cosmic set convergence: one has*

$$C^\nu \cup \operatorname{dir} K^\nu \xrightarrow{c} C \cup \operatorname{dir} K \iff d\!\!l^c(C^\nu \cup \operatorname{dir} K^\nu, C \cup \operatorname{dir} K) \to 0.$$

In fact $(\underline{\text{sets}}\,(\operatorname{csm} \mathbb{R}^n), d\!\!l^c)$ *is a metric space which is not only complete but compact.*

Proof. The metric properties of $d\!\!l^c$ are apparent from the formula in (3-1) and the observation that for cosmically closed sets $C_1 \cup \operatorname{dir} K_1$ and $C_2 \cup \operatorname{dir} K_2$, where K_1 and K_2 are cones, the distance functions $d^c(\cdot, C_1 \cup \operatorname{dir} K_1)$ and $d^c(\cdot, C_2 \cup \operatorname{dir} K_2)$ cannot coincide on $\mathbb{R}^n$ unless $C_1 = C_2$ and $K_1 = K_2$. The compactness of the metric space is immediate from the compactness theorem 3.4. A compact metric space is also complete. ☐

We have seen that the inner or outer limit of a sequence of sets $C^\nu \cup \operatorname{dir} K^\nu$ in $\operatorname{csm} \mathbb{R}^n$, where each K^ν is a cone, can be written uniquely as $C \cup \operatorname{dir} K$, where K likewise is a cone. This description has already served us in several ways, but for closer analysis it is desirable to have formulas indicating exactly how C and K can be derived from C^ν and K^ν. Such formulas will enable us to utilize properties of cosmic convergence as we wish without leaving the framework of convergence in $\mathbb{R}^n$ itself. The secret lies in translating to sequences of sets the dictum in 2.1 that convergence $x^\nu \to \operatorname{dir} x$, where $x \neq 0$, corresponds to having $\lambda^\nu x^\nu \to x$ for some sequence $\lambda^\nu \downarrow 0$.

3.7. Definition. *For a sequence* $\{C^\nu\}_{\nu \in \mathbb{N}}$ *of subsets of* $\mathbb{R}^n$, *the inner horizon limit, denoted by* $\liminf_\nu^\infty C^\nu$, *and the outer horizon limit, denoted by* $\limsup_\nu^\infty C^\nu$, *are the cones in* $\mathbb{R}^n$ *representing the direction points that belong to the cosmic inner and outer limits* $\text{c-}\liminf_\nu C^\nu$ *and* $\text{c-}\limsup_\nu C^\nu$, *respectively:*

$$\begin{aligned} \liminf\nolimits_\nu^\infty C^\nu &:= \{0\} \cup \{\, x \neq 0 \mid \exists x^\nu \underset{N}{\to} \operatorname{dir} x \ \textit{with}\ N \in \mathcal{N}_\infty,\, x^\nu \in C^\nu \,\} \\ &= \{0\} \cup \{\, x \mid \exists\, \lambda^\nu x^\nu \underset{N}{\to} x \ \textit{with}\ N \in \mathcal{N}_\infty,\, \lambda^\nu \downarrow 0,\, x^\nu \in C^\nu \,\}, \\ \limsup\nolimits_\nu^\infty C^\nu &:= \{0\} \cup \{\, x \neq 0 \mid \exists x^\nu \underset{N}{\to} \operatorname{dir} x \ \textit{with}\ N \in \mathcal{N}_\infty^\#,\, x^\nu \in C^\nu \,\} \\ &= \{0\} \cup \{\, x \mid \exists\, \lambda^\nu x^\nu \underset{N}{\to} x \ \textit{with}\ N \in \mathcal{N}_\infty^\#,\, \lambda^\nu \downarrow 0,\, x^\nu \in C^\nu \,\}. \end{aligned}$$

The horizon limit is said to exist when the inner and outer horizon limits coincide:

$$\lim_{\nu}^{\infty} C^{\nu} := \liminf_{\nu}^{\infty} C^{\nu} = \limsup_{\nu}^{\infty} C^{\nu}.$$

In these formulas, the insistence on the union with $\{0\}$ in the second version in each case is superfluous when $C^{\nu} \neq \emptyset$. On the other hand, for the empty sequence $C^{\nu} \equiv \emptyset$ both the inner and outer horizon limits are just $\{0\}$.

With the device of horizon limits, the formulas for general cosmic limits can be stated as follows.

3.8. Proposition. *For any sequence of sets $C^{\nu} \cup \operatorname{dir} K^{\nu}$ in $\operatorname{csm} \mathbb{R}^n$, where each K^{ν} is a cone, one has*

$$\begin{aligned} \text{c-}\liminf_{\nu} \big(C^{\nu} \cup \operatorname{dir} K^{\nu}\big) &= \big(\liminf_{\nu} C^{\nu}\big) \cup \operatorname{dir}\big(\liminf_{\nu}^{\infty} C^{\nu} \cup \liminf_{\nu} K^{\nu}\big), \\ \text{c-}\limsup_{\nu} \big(C^{\nu} \cup \operatorname{dir} K^{\nu}\big) &= \big(\limsup_{\nu} C^{\nu}\big) \cup \operatorname{dir}\big(\limsup_{\nu}^{\infty} C^{\nu} \cup \limsup_{\nu} K^{\nu}\big). \end{aligned}$$

Thus c-$\lim_{\nu} \big(C^{\nu} \cup \operatorname{dir} K^{\nu}\big)$ *exists if and only if* $\lim_{\nu} C^{\nu}$ *exists and*

$$\liminf_{\nu}^{\infty} C^{\nu} \cup \liminf_{\nu} K^{\nu} = \limsup_{\nu}^{\infty} C^{\nu} \cup \limsup_{\nu} K^{\nu}.$$

Proof. This is hardly more than a reinterpretation of definition 3.1 along the geometric lines in 3.7 $\square$

Note that the existence of the cosmic limit of $C^{\nu} \cup \operatorname{dir} K^{\nu}$ requires neither the existence of the horizon limit of the sets K^{ν} nor the existence of the ordinary limit of the cones K^{ν}, but a sort of joint property, at least in general.

3.9. Proposition. *For any sequence $\{C^{\nu}\}_{\nu \in \mathbb{N}}$ of subsets of $\mathbb{R}^n$, the following properties hold.*

(a) *The horizon limit sets $\liminf_{\nu}^{\infty} C^{\nu}$ and $\limsup_{\nu}^{\infty} C^{\nu}$, as well as $\lim_{\nu}^{\infty} C^{\nu}$ when it exists, are closed and depend only on the sequence $\{\operatorname{cl} C^{\nu}\}_{\nu \in \mathbb{N}}$.*

(b) *Always, $\liminf_{\nu}^{\infty} C^{\nu} \subset \limsup_{\nu}^{\infty} C^{\nu}$.*

(c) *If $C^{\nu} \equiv C$, then $\lim_{\nu}^{\infty} C^{\nu} = C^{\infty}$.*

(d) *Always, $\liminf_{\nu} (C^{\nu})^{\infty} \subset \liminf_{\nu}^{\infty} C^{\nu}$ and $\limsup_{\nu} (C^{\nu})^{\infty} \subset \limsup_{\nu}^{\infty} C^{\nu}$.*

(e) *For cones C^{ν}, $\liminf_{\nu}^{\infty} C^{\nu} = \liminf_{\nu} C^{\nu}$ and $\limsup_{\nu}^{\infty} C^{\nu} = \limsup_{\nu} C^{\nu}$.*

(f) *If $C \subset \liminf_{\nu} C^{\nu}$, then $C^{\infty} \subset \liminf_{\nu}^{\infty} C^{\nu}$. (But the inclusion $C \supset \limsup_{\nu} C^{\nu}$ does not imply that $C^{\infty} \supset \limsup_{\nu}^{\infty} C^{\nu}$.)*

Proof. For the most part, these assertions are direct consequences of the definitions. But their validity can also be established, sometimes more easily, through the interpretations in 3.3. A counterexample showing in (f) that the inclusion $C \supset \limsup_{\nu} C^{\nu}$ does *not* imply that $C^{\infty} \supset \limsup_{\nu}^{\infty} C^{\nu}$ is obtained by taking $C^{\nu} = \{\nu\}$ in $\mathbb{R}$. $\square$

3.10. Proposition. *Let C^{ν} and K^{ν} be subsets of $\mathbb{R}^n$ with each K^{ν} a cone.*

(a) *If the sequence* $\{C^\nu\}_{\nu\in\mathbb{N}}$ *is nondecreasing, then the horizon limit* $\lim_\nu^\infty C^\nu$ *exists. If in addition the sequence* $\{K^\nu\}_{\nu\in\mathbb{N}}$ *is nondecreasing, then the cosmic limit of the sets* $C^\nu \cup \operatorname{dir} K^\nu$ *exists and is given by*

$$\text{c-lim}_\nu \big(C^\nu \cup \operatorname{dir} K^\nu\big) = \big(\operatorname{cl} \cup_\nu C^\nu\big) \cup \operatorname{dir}\big(\lim_\nu^\infty C^\nu \cup (\operatorname{cl} \cup_\nu K^\nu)\big).$$

(b) *If the sequence* $\{C^\nu\}_{\nu\in\mathbb{N}}$ *is nonincreasing, then the horizon limit* $\lim_\nu^\infty C^\nu$ *exists. If in addition the sequence* $\{K^\nu\}_{\nu\in\mathbb{N}}$ *is nonincreasing, then the cosmic limit of the sets* $C^\nu \cup \operatorname{dir} K^\nu$ *exists and is given by*

$$\text{c-lim}_\nu \big(C^\nu \cup \operatorname{dir} K^\nu\big) = \big(\cap_\nu \operatorname{cl} C^\nu\big) \cup \operatorname{dir}\big(\lim_\nu^\infty C^\nu \cup (\cap_\nu \operatorname{cl} K^\nu)\big).$$

3.11. Proposition. *For any sequence of convex sets* C^ν *in* $\mathbb{R}^n$ *the cone* $\liminf_\nu^\infty C^\nu$ *is convex. For cones* K^ν *such that the sets* $C^\nu \cup \operatorname{dir} K^\nu$ *are convex in* $\operatorname{csm} \mathbb{R}^n$*, the cosmic inner limit* $\text{c-lim inf}_\nu\,(C^\nu \cup \operatorname{dir} K^\nu)$ *is likewise convex.*

We now investigate the nature of cosmic limits $C^\nu \cup \operatorname{dir} K^\nu \xrightarrow{c} C \cup \operatorname{dir} K$ in the special case when $K^\nu = \{0\} = K$, so that actually $C^\nu \xrightarrow{c} C$ in $\mathbb{R}^n$. A sequence $\{C^\nu\}_{\nu\in\mathbb{N}}$ in $\mathbb{R}^n$ is *ultimately* (equi-)bounded if there an index set $N \in \mathcal{N}_\infty$ such that the subsequence $\{C^\nu\}_{\nu\in N}$ is bounded, which means the sets C^ν for $\nu \in N$ are all included in some bounded subset B of $\mathbb{R}^n$.

3.12. Proposition. *For nonempty sets* C^ν *and* C *in* $\mathbb{R}^n$*, considered as subsets of* $\operatorname{csm} \mathbb{R}^n$*, one has* $C^\nu \xrightarrow{c} C$ *if and only if* C *is compact,* $C^\nu \to C$ *(ordinary set convergence), and the sequence* $\{C^\nu\}_{\nu\in\mathbb{N}}$ *is ultimately bounded.*

Proof. The condition is sufficient by definition 3.1, because the boundedness excludes any direction points from being involved in the cosmic limit. Taking a cosmic limit under such circumstances is thus the same as taking an ordinary limit. Conversely, if C is the cosmic limit of the sets C^ν, it must coincide with its cosmic closure $\operatorname{csm} C$ by 3.2. This is impossible for a subset of $\mathbb{R}^n$ unless it is compact. Likewise, it is impossible for the cosmic outer limit of the sets C^ν not to contain direction points, unless it is ultimately bounded. Thus the condition is necessary. □

Proposition 3.12 highlights one of the important differences between ordinary set convergence and cosmic set convergence. In the case of ordinary convergence, one can well have $C^\nu \to C$ with C bounded but $\{C^\nu\}_{\nu\in\mathbb{N}}$ not ultimately bounded. A simple example is furnished by the sets $C^\nu = \{0, \nu a\}$ in $\mathbb{R}^n$, where $a \neq 0$. Then $C^\nu \to \{0\}$ but $C^\nu \xrightarrow{c} \{0\} \cup \{\operatorname{dir} a\}$.

To look at this another way, in the metric space $(\,\underline{\text{sets}}(\mathbb{R}^n),\ \mathbb{d}\,)$ of ordinary set convergence (1.10), the subspace consisting of all the nonempty, compact subsets of $\mathbb{R}^n$ is not open, and in fact its complement is dense, i.e., every such set is the limit of a sequence of unbounded sets. In the metric space $(\,\underline{\text{sets}}(\operatorname{csm} \mathbb{R}^n),\ \mathbb{d}^c\,)$ of cosmic set convergence in 3.6, this same subspace is open by virtue of 3.12, while still dense by 3.7(a).

4. PARACOSMIC CONVERGENCE

When a sequence of sets $C^\nu \subset \mathbb{R}^n$ converges in the ordinary sense to a set $C \subset \mathbb{R}^n$, we know that C is closed and depends only on the closures $\operatorname{cl} C^\nu$. When the same sequence is regarded as residing in $\operatorname{csm}\mathbb{R}^n$, it may or may not also have a limit in the cosmic sense, but when it does, the limit must be a cosmically closed subset of $\operatorname{csm}\mathbb{R}^n$ which includes the cosmic closure $\operatorname{csm} C$ and depends only on the cosmic closures $\operatorname{csm} C^\nu$ (3.2). The question of whether the cosmic limit actually coincides with $\operatorname{csm} C$ in a given case is often crucial.

4.1. Definition. *A set $C \subset \mathbb{R}^n$ is the paracosmic limit of a sequence of sets $C^\nu \subset \mathbb{R}^n$, written $C^\nu \xrightarrow{p} C$, if not only $C^\nu \to C$ but also $C^\nu \xrightarrow{c} \operatorname{csm} C$ (or equivalently, $\operatorname{csm} C^\nu \xrightarrow{c} \operatorname{csm} C$).*

Paracosmic convergence is the type of convergence induced on the space $\underline{\text{sets}}(\mathbb{R}^n)$ by identifying it with a certain subspace of $\underline{\text{sets}}(\operatorname{csm}\mathbb{R}^n)$ supplied with cosmic convergence, namely through the pairing of each element of $\underline{\text{sets}}(\mathbb{R}^n)$ with its cosmic closure. From another perspective, this is the convergence we get by considering on $\underline{\text{sets}}(\mathbb{R}^n)$ not the metric $d\!l$, but $d\!l^c$. The latter metric is well defined by (3-1) for all nonempty subsets of $\operatorname{csm}\mathbb{R}^n$, including those that happen to lie just in $\mathbb{R}^n$:

$$d\!l^c\big(C_1, C_2\big) := \sup_{x\in\mathbb{R}^n} \big|\, d^c(x, C_1) - d^c(x, C_2) \,\big|. \tag{4–1}$$

4.2. Proposition. *Each of the following conditions is both necessary and sufficient in order that $C^\nu \xrightarrow{p} C$:*

(a) *$C = \lim_\nu C^\nu$ and $C^\infty = \lim_\nu^\infty C^\nu$;*
(b) *$C = \lim_\nu C^\nu$ and $C^\infty \supset \limsup_\nu^\infty C^\nu$;*
(c) *$d\!l^c(C^\nu, C) \to 0$ (assuming C^ν and C are nonempty and C is closed).*

Proof. In all the situations described, C is closed in $\mathbb{R}^n$, and therefore $\operatorname{csm} C = C \cup \operatorname{dir} C^\infty$. The characterization in (a) is immediate then from 3.8 in the case of $K^\nu = \{0\}$. The characterization in (b) goes a step further in taking advantage of the inclusion in 3.9(f). The condition $d\!l^c(C^\nu, C) \to 0$ is identical to $d\!l^c(\operatorname{csm} C^\nu, \operatorname{csm} C) \to 0$, and this gives us (c) in accordance with the definition of paracosmic convergence in 4.1 and the interpretations preceding the current proposition. □

Paracosmic convergence can be counted on to be present, in the case of (ordinary) set convergence, in some of the most common situations. In other words, the important consequences of paracosmic convergence can be obtained often without extra cost.

4.3. Theorem. *In the cases that follow, ordinary convergence $C^\nu \to C$ for nonempty sets C^ν and C in $\mathbb{R}^n$ always entails the stronger property of paracosmic convergence $C^\nu \xrightarrow{p} C$:*

(a) *the sets C^ν are convex;*
(b) *the sets C^ν are cones;*
(c) *the sequence is ultimately bounded;*
(d) *the sequence is nondecreasing;*

(e) $d\!\!l_\infty(C^\nu, C) \to 0$ *(Pompeiu-Hausdorff convergence).*

Proof. We take the cases in reverse order, invoking each time the criterion in 4.2(b). Case (e) involves having $C^\nu \subset C + \varepsilon^\nu \mathbb{B}$ for a sequence of values $\varepsilon^\nu \downarrow 0$. Then

$$\limsup_\nu^\infty C^\nu \subset \limsup_\nu^\infty (C + \varepsilon^\nu \mathbb{B}) = C^\infty.$$

In case (d), we recognize the specialization of 3.10(a) to $K^\nu = \{0\}$. In (c), we have $\limsup_\nu^\infty C^\nu = \{0\}$ and necessarily also $C^\infty = \{0\}$. Case (b) is covered by 3.9(e).

Finally, for (a) we consider an arbitrary point $x \in \limsup_\nu^\infty C^\nu$ and proceed to demonstrate that $x \in C^\infty$. By definition 3.7, there must exist $N \in \mathcal{N}_\infty^\#$ such that $\lambda^\nu x^\nu \underset{N}{\to} x$ for some choice of $x^\nu \in C^\nu$ and $\lambda^\nu > 0$ with $\lambda^\nu \underset{N}{\to} 0$. Then in particular x belongs to $\liminf_{\nu\in N}^\infty C^\nu$, a cone we may denote by K_N. At the same time we have $C = \lim_{\nu\in N} C^\nu$, so that $C \cup \operatorname{dir} K_N = \text{c-}\liminf_{\nu\in N} C^\nu$ by 3.8. Since C^ν is convex, it follows that $C \cup \operatorname{dir} K_N$ is convex as a subset of $\operatorname{csm} \mathbb{R}^n$ (3.11). Thus, $C + K_N \subset C$. Because $x \in K_N$, this implies that $x_0 + \tau x \in C$ for all $x_0 \in C$ and $\tau \geq 0$, hence $x \in C^\infty$. $\square$

Paracosmic convergence, rather than just ordinary convergence, of sets in $\mathbb{R}^n$ turns out to be a necessary ingredient in obtaining results on the continuity of various operations that can be performed on sets, such as taking convex hulls, sums, and images under mappings.

5. IMAGES AND OPERATIONS

Let us begin with the following simple relations which are immediate consequences of the definition of paracosmic convergence (4.1) and its characterizations provided by 4.2 (with (2.6) used also for the first one of these implications):

$$C_1^\nu \xrightarrow{p} C_1, C_2^\nu \xrightarrow{p} C_2 \quad\Longrightarrow\quad C_1^\nu \cup C_2^\nu \xrightarrow{p} C_1 \cup C_2,$$

$$C_1^\nu \xrightarrow{p} C_1, C_2^\nu \xrightarrow{p} C_2 \text{ and } C_1^\nu \subset C_2^\nu \quad\Longrightarrow\quad C_1 \subset C_2.$$

5.1. Proposition. *Let C_1, C_2, C_1^ν, C_2^ν be subsets of $\mathbb{R}^n$. Then*

$$C_1^\nu \to C_1, C_2^\nu \to C_2 \quad\Longrightarrow\quad C_1^\nu \times C_2^\nu \to C_1 \times C_2.$$

Moreover, if $(C_1 \times C_2)^\infty = C_1^\infty \times C_2^\infty$, then

$$C_1^\nu \xrightarrow{p} C_1, C_2^\nu \xrightarrow{p} C_2 \quad\Longrightarrow\quad C_1^\nu \times C_2^\nu \xrightarrow{p} C_1 \times C_2.$$

Thus, if the sets C_1^ν, C_2^ν are convex, $C_1^\nu \to C_1$, $C_2^\nu \to C_2$ implies $C_1^\nu \times C_2^\nu \xrightarrow{p} C_1 \times C_2$.

The following example shows that, in general, paracosmic convergence is not preserved under taking products: pick $C_1^\nu = C_2^\nu = \{\, 2^k, k < \nu \,\} \cup [2^\nu, \infty)$.

Most of our results about operations will come from this fact about products and a general theorem involving the images (under a mapping S) of convergent sequences.

The *graph* of a set-valued mapping $u \mapsto S(u) \subset \mathbb{R}^n$ defined on $\mathbb{R}^d$ is a subset of $\mathbb{R}^d \times \mathbb{R}^n$, namely

$$\operatorname{gph} S := \big\{\, (u, x) \,\big|\, x \in S(u) \,\big\},$$

rather than a subset of $U \times \operatorname{sets}(X)$. To emphasize this we write $S : U \rightrightarrows X$ instead of $S : U \to \operatorname{sets}(X)$. Note that $S(u)$ is possibly empty; the "effective" domain of S is $\operatorname{dom} S := \{\, u \in \mathbb{R}^d \mid S(u) \neq \emptyset \,\}$.

5.2. Definition. *A set-valued mapping $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ is inner semicontinuous (isc) at u if for all $u^\nu \to u$, one has $\liminf_\nu S(u^\nu) \supset S(u)$; it is outer semicontinuous (osc) at u if $\limsup_\nu S(u^\nu) \subset S(u)$ for all $u^\nu \to u$. It is continuous at u if it is both isc and osc at u. The mapping S is isc, osc or continuous (on $\mathbb{R}^d$) if it is isc, osc or continuous at all $u \in \mathbb{R}^d$.*

A mapping $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ is paracosmically continuous at u if it is continuous at u and whenever $u^\nu \to u$ and $\lim_\nu^\infty S(u^\nu) = S(u)^\infty$. The mapping S is paracosmically continuous if this holds at every $u \in \mathbb{R}^d$.

Since $S(u)^\infty \subset \liminf_\nu^\infty S^\nu(u)$ is guaranteed by $S(u) \subset \liminf_\nu S(u^\nu)$ (3.10(f)), the condition on the horizon limit of the sets $S(u^\nu)$ could thus be replaced by the apparently weaker condition: $\limsup_\nu^\infty S(u^\nu) \subset S^\infty(u)$.

5.3. Proposition. *A set-valued mapping $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ that is continuous at u is also paracosmically continuous at u if there exists a neighborhood U of u on which S is convex- or cone-valued, or if S is locally bounded at u, or if S is monotone increasing or decreasing (with respect to inclusion) for any sequence $u^\nu \to u$, or if $d\!l_\infty(S(u^\nu), S(u)) \to 0$ for all $u^\nu \to u$ (i.e., S is continuous at u with respect to the Pompeiu-Hausdorff metric).*

Proof. For a mapping S, continuity at u implies that $S(u^\nu) \to S(u)$ for any sequence $u^\nu \to u$. It now suffices to observe under the conditions stated that criteria 3.9(a)-(e) actually guarantee paracosmic convergence of the sets $S(u^\nu)$ to $S(u)$ for every $u^\nu \to u$. And this is all that is required for paracosmic continuity (5.2). $\square$

Note that as a consequence of this proposition, if $G : \mathbb{R}^d \to \mathbb{R}^n$ is a continuous function, viewed as a set-valued mapping, it is always paracosmically continuous because it is locally bounded.

Our general result about the convergence of the images $S(C^\nu)$ of a sequence C^ν under a mapping S requires not only an examination of the horizon limits associated with the sequence of sets C^ν, but also how the mapping S will affect them. This brings us to consider a certain positively homogeneous mapping associated with S.

5.4. Definition. *For $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$, one says that S is positively homogeneous if $0 \in S(0)$ and $S(\lambda u) = \lambda S(u)$ for all $\lambda > 0$ and $u \in \mathbb{R}^d$, or in other words, $\operatorname{gph} S$ is a cone in $\mathbb{R}^d \times \mathbb{R}^n$.*

Linear mappings are positively homogeneous, in particular. This concept leads to an extension of the developments on horizon cones (section 2) to nonlinear mappings, single-valued or set-valued, and in the process we reap a generalization of the important result in 2.5 on the horizon cones of image sets. For $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$, the associated *horizon mapping* $S^\infty : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ is specified by

$$\operatorname{gph} S^\infty := (\operatorname{gph} S)^\infty. \tag{5–1}$$

Thus $x \in S^\infty(u)$ if and only there exist sequences of points $u^\nu \in \mathbb{R}^d$, $x^\nu \in S(u^\nu)$, and scalars $\lambda^\nu \downarrow 0$ such that $\lambda^\nu u^\nu \to u$ and $\lambda^\nu x^\nu \to x$. Note that since the graph of S^∞ is a cone in $\mathbb{R}^d \times \mathbb{R}^n$, S^∞ is a positively homogeneous mapping (5.4). Obviously $(S^{-1})^\infty = (S^\infty)^{-1}$.

The horizon mapping S^∞ expresses certain growth properties of S. In the case where S is a linear mapping L, one has $S^\infty = L$ because the set $\operatorname{gph} S = \operatorname{gph} L$ is a linear subspace of $\mathbb{R}^d \times \mathbb{R}^n$. More generally, $S^\infty = \operatorname{cl} S$ whenever S is positively homogeneous.

5.5. Theorem. *A sufficient condition for a mapping $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ to be locally bounded is $S^\infty(0) = \{0\}$. Then the horizon mapping S^∞ is locally bounded as well.*

Proof. Suppose S is not locally bounded. Then for some $u \in \mathbb{R}^d$ we can find $u^\nu \to u$ and $x^\nu \in S(u^\nu)$ with $\{x^\nu\}$ unbounded. Passing to subsequences if necessary, we can assume x^ν tends to some direction point $\operatorname{dir} x$. This means there are scalars $\lambda^\nu \downarrow 0$ such that $\lambda^\nu x^\nu \to x \neq 0$. Then $\lambda^\nu(u^\nu, x^\nu) \to (0, x)$ with $(u^\nu, x^\nu) \in \operatorname{gph} S$, so $(0, x) \in (\operatorname{gph} S)^\infty$. Thus, $S^\infty(0)$ contains the nonzero vector x in violation of the stated condition.

Because $(\operatorname{gph} S)^\infty$ is a closed cone, we have $((\operatorname{gph} S)^\infty)^\infty = (\operatorname{gph} S)^\infty$ and consequently $(S^\infty)^\infty = S^\infty$. Thus, when $S^\infty(0) = \{0\}$, the mapping $T = S^\infty$ has $T^\infty(0) = \{0\}$ and, by the argument already given, is locally bounded. □

The fact that the condition in 5.5 is not necessary for S to be locally bounded, merely sufficient, is seen from simple examples like the mapping $S : \mathbb{R} \to \mathbb{R}$ defined by $S(u) = u^2$. This mapping is locally bounded but has $S^\infty(0) = [0, \infty)$. The condition in 5.5 is useful nevertheless because of the convenient calculus that can be built around it. For example one obtains the following simple criterion for the closedness of images (that we state without proof).

5.6. Theorem. *Let $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ be osc. The set $S(D)$ is closed when D is closed and $(S^\infty)^{-1}(0) \cap D^\infty = \{0\}$ (as is true if either $(S^\infty)^{-1}(0) = \{0\}$ or $D^\infty = \{0\}$). Then $S(D)^\infty \subset S^\infty(D^\infty)$.*

We now have all the tools needed to formulate our main result about the convergence of images.

5.7. Theorem. *Let $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ be a mapping and let C and C^ν be subsets of $\mathbb{R}^d$.*

(a) $\liminf_\nu S(C^\nu) \supset S(C)$ *when* $\liminf_\nu C^\nu \supset C$ *and S is isc on C.*

(b) $\limsup_\nu S(C^\nu) \subset S(C)$ *when* $\limsup_\nu C^\nu \subset C$, *$S$ is osc on C, $(S^\infty)^{-1}(0) \cap C^\infty = \{0\}$ and* $\limsup_\nu^\infty C^\nu \subset C^\infty$. *One further has* $\limsup_\nu^\infty S(C^\nu) \subset S(C)^\infty$ *when, in addition, $(S^\infty)(C^\infty) \subset S(C)^\infty$ and for all $u \in C$ and all $u^\nu \to u$,* $\limsup_\nu^\infty S(u^\nu) \subset S(u)^\infty$ *(i.e., S is "paracosmically outer semicontinuous" on C).*

(c) *Also,* $\limsup_\nu S(C^\nu) \subset S(C)$ *when* $\limsup_\nu C^\nu \subset C$, *$S$ is osc on C, and $(S^\infty)^{-1}(0) = \{0\}$.*

Hence $S(C^\nu) \to S(C)$ when S is continuous on C and either $C^\nu \to C$ and $(S^\infty)^{-1}(0) = \{0\}$ or $C^\nu \xrightarrow{p} C$ and $(S^\infty)^{-1}(0) \cap C^\infty = \{0\}$. One has $S(C^\nu) \xrightarrow{p} S(C)$ if S is paracosmically continuous on C, $C^\nu \xrightarrow{p} C$, $(S^\infty)^{-1}(0) \cap C^\infty = \{0\}$ and $(S^\infty)(C^\infty) \subset S(C)^\infty$.

Proof. Statement (a) is a direct consequence of the definitions of the inner limit (1.1) and of inner semicontinuity (5.2).

To prove (c) we need to show that $x \in S(C)$ whenever there exist $x^\nu \underset{N}{\to} x$ for some $N \in \mathcal{N}_\infty^\#$ with $x^\nu \in S(C^\nu)$ for all $\nu \in N$. Pick $u^\nu \in S^{-1}(x^\nu) \cap C^\nu$. If the sequence $\{u^\nu\}_{\nu \in N}$ clusters to a point u, then $u \in C(\supset \limsup_\nu C^\nu)$ and by outer semicontinuity of S at u, we have that $x \in S(u) \subset S(C)$. Otherwise, the u^ν cluster to a point in the horizon of $\mathbb{R}^n$, say $\operatorname{dir} u$ (with $u \neq 0$). Since $(u^\nu, x^\nu) \in \operatorname{gph} S$, this would imply that $u \in (S^\infty)^{-1}(0)$, but the assumption that $(S^\infty)^{-1}(0) = \{0\}$ does not allow for such a possibility.

Essentially the same argument works for the first statement in (b), except that when $\operatorname{dir} u$ is a cluster point of the u^ν, then $u \in \limsup_\nu^\infty C^\nu \subset C^\infty$. Hence $0 \neq u \in (S^\infty)^{-1}(0) \cap C^\infty$ and this violates the assumption that only 0 belongs to their intersection.

To complete the proof of (b), we need to show that $\limsup_\nu^\infty S(C^\nu) \subset S(C)^\infty$ when $\limsup_\nu^\infty S(u^\nu) \subset S(u)^\infty$ for all $u \in C$ and all $u^\nu \to u$, i.e., $x \in S(C)^\infty$ whenever there exist $x^\nu \to \operatorname{dir} x$ for some $N \in \mathcal{N}_\infty^\#$ with $x^\nu \in S(C^\nu)$ for all $\nu \in N$. Pick $u^\nu \in S^{-1}(x^\nu) \cap C^\nu$. If the u^ν cluster to $u \in \mathbb{R}^n$, then $u \in C (\supset \limsup C^\nu)$ and since by assumption $\limsup_\nu^\infty S(u^\nu) \subset S(u)^\infty$, it follows that $x \in S(u)^\infty \subset S(C)^\infty$. Otherwise, there exist $N_0 \subset N$, $N_0 \in \mathcal{N}_\infty^\#$, $u \neq 0$ such that $u^\nu \underset{N_0}{\to} \operatorname{dir} u$; note that then $u \in C^\infty \supset \limsup_\nu^\infty C^\nu$. Since $\{(u^\nu, x^\nu)\}_{\nu \in N_0} \subset \operatorname{gph} S$ and $|x^\nu| \uparrow \infty$, $|u^\nu| \uparrow \infty$, there exists $\lambda^\nu \downarrow 0$, $\nu \in N_0$ such that $\lambda^\nu(u^\nu, x^\nu)$ clusters to a point of the type $(\alpha u, \beta x) \neq (0,0)$ with $\alpha \geq 0$, $\beta \geq 0$. If $\beta = 0$, then $0 \neq u \in (S^\infty)^{-1}(0) \cap C^\infty$ and that is ruled out by the assumption that $(S^\infty)^{-1}(0) \cap C^\infty = \{0\}$. Thus $\beta > 0$, and $x \in (S^\infty)(\alpha\beta^{-1}u) \subset (S^\infty)(C^\infty) \subset S(C)^\infty$ where the inclusion comes from the last assumption in (b).

The two remaining statements are just a rephrasing of the consequences of (a) and (b) when limits or paracosmic limits exists, making use of 3.9(f). □

Applying the theorem in the case $G : \mathbb{R}^d \to \mathbb{R}^n$ continuous and $(G^\infty)^{-1}(0) \cap C^\infty = 0$ yields $G(C^\nu) \to G(C)$ if $C^\nu \to C$. If actually $C^\nu \xrightarrow{p} C$, then $G(C^\nu) \xrightarrow{p} G(C)$ provided that also $G^\infty(C^\infty) \subset G(C)^\infty$.

5.8. Corollary. *Let C^ν, $C \subset \mathbb{R}^d$ and let $H : \mathbb{R}^d \to \mathbb{R}^n$ be an affine mapping, $H(x) := Ax + a$ for A a matrix in $\mathbb{R}^{n \times d}$ and a vector $a \in \mathbb{R}^n$. If $C^\nu \to C$ and A is invertible, then $H(C^\nu) \to H(C)$. If $C^\nu \xrightarrow{p} C$ and $\{ x \mid Ax = 0 \} \cap C^\infty = \{0\}$, then $H(C^\nu) \xrightarrow{p} H(C)$.*

Proof. When viewed as set-valued mappings, affine mappings are not only continuous but also paracosmically continuous (proposition 5.3), and $H^\infty(x) = Ax$. In general, $H^\infty(C^\infty) \subset H(C)^\infty$. There now remains only to apply the theorem. □

Except for the special case when the affine mapping is invertible, not that as soon as we impose the conditions suggested by the theorem to obtain (ordinary) convergence of the images, we actually end up with the paracosmic convergence of the images.

5.9. Corollary. *Let M be a linear subspace of $\mathbb{R}^n$ with proj_M the orthogonal projection on M, and $C^\nu \to C \subset \mathbb{R}^n$. If $M^\perp \cap C^\infty = \{0\}$, then $\operatorname{proj}_M(C^\nu) \xrightarrow{p} \operatorname{proj}_M(C)$.*

Proof. This is the special case of 5.8 when $H = \operatorname{proj}_M$. □

5.10. Corollary. *Let $C_1, C_2, C_1^\nu, C_2^\nu$ be subsets of $\mathbb{R}^d$ such that $C_1^\nu \xrightarrow{p} C_1$ and $C_2^\nu \xrightarrow{p} C_2$. Then $C_1^\nu + C_2^\nu \xrightarrow{p} C_1 + C_2$ provided that $C_1^\infty \times C_2^\infty = (C_1 \times C_2)^\infty$ and $C_1^\infty \cap -C_2^\infty = \{0\}$. In particular, if the sets C_1^ν, C_2^ν are convex, and $C_1^\nu \to C_1$, $C_2^\nu \to C_2$, then $C_1^\nu + C_2^\nu \xrightarrow{p} C_1 + C_2$ if $C_1^\infty \cap -C_2^\infty = \{0\}$.*

Proof. Apply 5.8 to $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $L(x,y) := x + y$, and use the results about the convergence of products 5.1. □

5.11. Corollary. *Suppose the sequence $C^\nu \subset \mathbb{R}^n$ is ultimately bounded and $C^\nu \to C$. Then $\operatorname{con} C^\nu \xrightarrow{p} \operatorname{con} C$.*

Proof. With $\Sigma = \{\,\alpha \in \mathbb{R}^{n+1} \mid \sum_{j=0}^{n} \alpha_j = 1, \alpha \geq 0\,\}$, the unit simplex, $S : \mathbb{R}^{(n+1)n} \times \mathbb{R}^{n+1} \to \mathbb{R}^n$ defined by

$$S(u^0, u^1, \ldots, u^n, \alpha) := \sum_{j=0}^{n} \alpha_j u^j, \quad x^j \in \mathbb{R}^n,$$

it follows from Caratheodory's theorem that for any $D \subset \mathbb{R}^n$,

$$\operatorname{con} D = S(D \times D \times \ldots \times D \times \Sigma).$$

The mapping S is continuous. Also C is bounded since the sequence C^ν is ultimately bounded, and not only does this imply that actually the C^ν converge cosmically to C (4.3(c)), but also that $(C \times C \times \ldots \times C \times \Sigma)^\infty = \{(0, 0, \ldots, 0, 0)\}$. In view of theorem (5.8), this allows us to conclude that $\operatorname{con} C^\nu \to \operatorname{con} C$. But these are convex sets, and then (ordinary) convergence implies paracosmic convergence (4.3(a)). ☐

The preceding result is remarkable in part for its limitations! In terms of our general result about the convergence of images (5.8), this may be attributable to the fact that for the horizon mapping S^∞ associated with the creation of convex hulls, one has $(S^\infty)^{-1}(0) = \mathbb{R}^{(n+1)n} \times \{0\}$ ($\operatorname{dom} S^\infty = \mathbb{R}^{(n+1)n} \times \{0\}$). This reflects the fact that some points in the convex hull could be generated by points that are arbitrarily close to the horizon in opposite (or very different) directions.

5.12. Corollary. *Let $C^\nu, C \subset \mathbb{R}^n$ be such that $C^\nu \to C$ and $0 \notin C$. Then $\operatorname{pos} C^\nu \xrightarrow{p} \operatorname{pos} C$.*

Proof. Let $S(u) := \operatorname{pos} u = \{\,\lambda u \,|\, \lambda \geq 0\,\}$, then S is a continuous mapping except at 0. Moreover S is positively homogeneous and thus $S = S^\infty$ and $(S^\infty)^{-1}(0) = \{0\}$. We apply the theorem and use the fact that the sets $\operatorname{pos} C^\nu$ and $\operatorname{pos} C$ are cones to pass from convergence to paracosmic convergence (4.3(b)). ☐

One of the implications of this last corollary is that if C^ν is a sequence of sets converging to a set C, then $\operatorname{pos}(C^\nu \times \{-1\}) \xrightarrow{p} \operatorname{pos}(C \times \{-1\})$, i.e., convergence of sets in $\mathbb{R}^n$ implies the convergence of the corresponding cones in the ray space.