MONOTONE RELATIONS AND NETWORK EQUILIBRIUM

R. T. Rockafellar

Departments of Mathematics and Applied Mathematics
University of Washington GN-50
Seattle, WA 98195, USA
Email: rtr@math.washington.edu

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Abstract: Conditions for network equilibrium are developed in terms of vector-valued flows and potentials and generalized resistance relations. The extent to which the equilibrium can be expressed by a variational inequality or characterized by optimization is analyzed. Emphasis is placed on maximal monotone relations, especially subgradient relations associated with convex optimization.

Key words: Variational inequalities, network equilibrium, vector flows, vector potentials, maximal monotone relations, dual problems of optimization.

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1. INTRODUCTION

Among the many important applications of variational inequalities is the expression of equilibrium conditions for flows in networks, in particular equilibrium involving different kinds of traffic. Variational inequalities are a relative newcomer to the theory of networks, however. Other approaches to equilibrium have grown out of the classical study of electrical networks and their generalization to hydraulic networks as well as the framework of transportation problems in operations research. A very useful idea has been the duality between flows and potentials as expressed by systems of possibly nonlinear or even multivalued relations imposed in the different elements of a network.

The aim of this article is to illuminate the connections between variational inequalities and these other approaches, with special attention paid to the extent to which equilibrium may correspond to some sort of optimization. It is hoped that the range of modeling possibilities thereby revealed will aid further in the formulation of traffic problems and also in their solution by a wider class of computational techniques.

Variational inequalities generalize conditions for optimality such as may be associated with a variational principle, so we begin by reviewing how this comes about. The need and desirability of working with multivalued mappings receives motivation in this way, and a remarkable degree of flexibility in the application of numerical methods is achieved. For simplicity the context here will be finite-dimensional.

Next we develop a general formulation of network equilibrium for vector-valued flows and potentials, paralleling the well known one for scalar-valued flows. We establish the circumstances in which this kind of equilibrium can be expressed by a variational inequality. We show that, even when this is not the case, an expression is available in which the many numerical approaches to calculating a zero of a possibly set-valued mapping can be applied. In addition we study situations where network equilibrium corresponds to solving a primal optimization problem for flows, a dual optimization problem for potentials, or a saddle point problems for flows and potentials together.

2. VARIATIONAL INEQUALITIES AND OPTIMIZATION

The variational inequality problem with respect to a nonempty, closed, convex set $Z \subset I\!R^N$ and a continuous, single-valued mapping $F : Z \to I\!R^N$ is usually stated in the form:

$$\text{(VI)} \qquad \text{find} \quad \bar{z} \in Z \quad \text{such that} \quad \langle F(\bar{z}),\, z - \bar{z} \rangle \geq 0 \quad \text{for all} \quad z \in Z.$$

An alternative form, which is preferable for many reasons and will be especially fruitful in what follows, is obtained by utilizing the notion of the *normal cone* $N_Z(\bar{z})$ to Z at $\bar{z}$, which in convex analysis consists of all vectors w such that $\langle w,\, z - \bar{z} \rangle \leq 0$ for all $z \in Z$, cf. [1]. The problem is then:

$$\text{(VI}'\text{)} \qquad \text{find} \quad \bar{z} \in Z \quad \text{such that} \quad -F(\bar{z}) \in N_Z(\bar{z}).$$

In either form the inspiration comes from the case where F is the gradient mapping ∇f associated with a continuously differentiable function f defined on a neighborhood of Z. The variational inequality then expresses the first-order necessary condition for optimality in the minimization of f over Z, this being not just necessary but sufficient when f happens to be convex.

Variational inequalities in which $F = \nabla f$ are usually called *symmetric*, whereas all others are *asymmetric*. Really, this terminology is appropriate only under the additional assumption that F is continuously differentiable, since that allows the existence of a function f with $F = \nabla f$ to be identified with the property of F that the Jacobian matrix $\nabla F(z)$ is symmetric everywhere. It is possible, of course, to have $F = \nabla f$ without F being differentiable at all, so that the Jacobian does not even exist. On the other hand, in circumstances where F is differentiable but not continuously differentiable the Jacobian might exist but not be symmetric, as indicated by classical examples of twice differentiable functions f for which the matrix of second partial derivatives is not symmetric.

The distinction between the symmetric and asymmetric cases is sometimes interpreted as marking the division between the variational inequality problems directly reducible to optimization and the ones not so reducible. But this view is inaccurate and potentially misleading. Variational inequalities can correspond to optimization despite asymmetry, and whenever that is true they can be solved by optimization techniques just as well as if they were symmetric, and without resorting to the introduction of an artificial "gap" function.

For example, the Kuhn-Tucker conditions for a minimization problem with functional constraints express first-order optimality in terms of an asymmetric variational inequality in the primal and dual variables jointly. Suppose the problem consists of minimizing $f_0(x)$ over all $x \in X$ satisfying $(f_1(x), \ldots, f_m(x)) \in K$, where K is a closed, convex cone in $I\!R^m$, X is a nonempty, closed, convex set in $I\!R^n$, and the functions f_i are continuously differentiable. Let $l(x, y)$ stand for the Lagrangian expression $f_0(x)+\sum_{i=1}^m y_i f_i(x)$. The generalized Kuhn-Tucker conditions that apply to this setting, as established in [2] (Thms. 4.2, 10.6) under a basic constraint qualification, say that for $\bar{x} \in X$ to give a local minimum there must be a vector $\bar{y} \in Y$, where Y is the cone polar to K, such that

$$-\nabla_x l(\bar{x}, \bar{y}) \in N_X(\bar{x}), \qquad \nabla_y l(\bar{x}, \bar{y}) \in N_Y(\bar{y}). \tag{2.1}$$

The search for a pair $(\bar{x}, \bar{y})$ satisfying this double relation is the variational inequality problem (VI$'$) in the case of

$$z = (x, y), \qquad Z = X \times Y, \qquad F(z) = (\nabla_x l(x, y), -\nabla_y l(x, y)). \tag{2.2}$$

The variational inequality is asymmetric because F is not actually the gradient mapping of any function. Indeed, when F is continuously differentiable (through the functions f_i being twice continuously differentiable) its Jacobian matrix has an obvious lack of symmetry:

$$\nabla F(z) = \begin{bmatrix} \nabla^2_{xx} l(x, y) & \nabla^2_{xy} l(x, y) \\ -\nabla^2_{yx} l(x, y) & -\nabla^2_{yy} l(x, y) \end{bmatrix}.$$

Beyond Kuhn-Tucker conditions, a similar pattern of asymmetry holds in the characterization of optimality for broader problem models of composite type, cf. [2], Sec. 10. These likewise concern dual vectors y along with primal vectors x as in (2.1) and (2.2), but Y not necessarily a cone, and $l(x,y)$ not necessarily the classical Lagrangian. Also in this class are general problems of finding a saddle point $(\bar{x},\bar{y})$ of a differentiable function $l(x,y)$ relative to closed, convex sets X and Y, since (2.1) is necessary for $l(x,\bar{y})$ to have its minimum over $x\in X$ at $\bar{x}$ while $l(\bar{x},y)$ has its maximum over $y\in Y$ at $\bar{y}$.

In all these examples an *asymmetric* variational inequality is seen to be reducible directly to optimization and therefore open to solution by methods of numerical optimization. Such methods, whether they are posed in a primal, dual, or primal-dual context, inevitably aim at producing not only a primal vector $\bar{x}$ but an associated dual vector $\bar{y}$ such that (2.1) holds.

The concept of monotonicity plays the key role in the theory of variational inequalities that convexity plays in optimization. A variational inequality is *monotone* if its mapping $F: Z\to I\!R^N$ is monotone in the sense that

$$\langle F(z')-F(z),\, z'-z\rangle \geq 0 \quad \text{for all} \quad z,\, z'\in Z. \tag{2.3}$$

(We denote by $\langle\cdot,\cdot\rangle$ the canonical inner product.) When F is continuously differentiable this property is equivalent to the positive semidefiniteness of the (possibly asymmetric) Jacobian matrix $\nabla F(z)$ at every point $z\in Z$. In the symmetric case with $F=\nabla f$ it corresponds to f being convex. The variational inequality then describes the solution(s) to a problem of minimizing a convex function over a convex set.

An important asymmetric example of monotonicity is encountered when F has the form (2.2) relative to a product of convex sets X and Y, and the function $l(x,y)$ is convex in $x\in X$ and concave in $y\in Y$. Such a variational inequality corresponds to convex optimization as well. It characterizes solutions $\bar{x}$ to a certain primal problem of minimization by means of a saddle point $(\bar{x},\bar{y})$, where $\bar{y}$ solves a certain dual problem of maximization.

As valuable as the notion of a variational inequality has turned out to be, it has definite limitations which need to be appreciated if connections with optimization are fully to be understood. One limitation, which fortunately is easy to get around, is the single-valuedness of the mapping F. On the surface, this excludes applications to areas like nonsmooth optimization. A more serious limitation, however, is the requirement that the set Z be convex. When a variational inequality problem is stated in the form (VI), the convexity of Z is essential for it to make good sense, but in form (VI$'$) the way is open to assigning to the normal cone $N_Z(\bar{z})$ a definition appropriate not only for convex sets Z but nonconvex sets as well. For instance, $N_Z(\bar{z})$ can be taken to be the Clarke normal cone or the smaller cone that has received special emphasis in the work of Mordukhovich; cf. [2], Sec. 10.

The point is that although variational inequalities in which Z is convex do cover some problems of nonconvex optimization through extended Kuhn-Tucker conditions, as already discussed, there is something rather strained about the formulation. The case where Z is convex and F is monotone is natural in providing a platform for a theory of variational inequalities that mirrors convex optimization. The case where Z is potentially not convex and F not monotone is well motivated too, if interpreted in the manner just described. But the hybrid case where Z is convex, yet F is not monotone, draws boundaries rather artificially.

Still another way of stating the basic variational inequality problem, which will serve as a guide in our discussion of network equilibrium, is:

$$\text{(VI}''\text{)} \qquad \text{find} \quad \bar{z} \in Z \quad \text{such that} \quad 0 \in T(\bar{z}), \qquad \text{where} \quad T(z) = \begin{cases} F(z) + N_Z(z) & \text{if } z \in Z, \\ \emptyset & \text{if } z \notin Z. \end{cases}$$

At first this format may seem unappealing because it requires working with a set-valued mapping T. As support for an alternative point of view, however, it is rich in theoretical implications.

A general mapping T that assigns to each $z \in I\!R^N$ a subset $T(z) \subset I\!R^N$ can be regarded as an ordinary single-valued mapping from $I\!R^N$ to the space $2^{I\!R^N}$. For most purposes, though, there is much more to be gained by identifying T with the set

$$\operatorname{gph} T := \{(z,w) \in I\!R^N \times I\!R^N \mid w \in T(z)\,\} \tag{2.4}$$

as its "graph" and thinking of it thus as expressing a relation between vectors z and w. The "effective domain" $\operatorname{dom} T$ and "effective range" $\operatorname{rge} T$ of T are defined then by

$$\operatorname{dom} T := \{z \mid T(z) \neq \emptyset\}, \qquad \operatorname{rge} T := \{w \mid \exists\, z,\ w \in T(z)\}. \tag{2.5}$$

In this framework, which we signal this framework by writing $T : I\!R^N \rightrightarrows I\!R^N$ in place of $T : I\!R^N \to I\!R^N$, T is regarded as single-valued, empty-valued or multivalued at z according to whether $T(z)$ is a singleton, the empty set, or a set with more than one element. The "inverse" of T is the mapping $T^{-1} : I\!R^N \rightrightarrows I\!R^N$ defined by

$$T^{-1}(w) = \{z \mid w \in T(z)\,\}. \tag{2.6}$$

Clearly $\operatorname{dom} T^{-1} = \operatorname{rge} T$ and $\operatorname{rge} T^{-1} = \operatorname{dom} T$.

The monotonicity property introduced in (2.3) for a mapping $F : Z \to I\!R^N$ has the following generalization. A mapping $T : I\!R^N \rightrightarrows I\!R^N$ is called *monotone* if

$$\langle z' - z,\, w' - w \rangle \geq 0 \quad \text{whenever} \quad w \in T(z),\ w' \in T(z). \tag{2.7}$$

It is *maximal monotone* if it is monotone but its graph cannot be enlarged without losing monotonicity, i.e., if for every choice of vectors $\hat{z}$ and $\hat{w}$ with $\hat{w} \notin T(\hat{z})$, there exist $\tilde{z}$ and $\tilde{w}$ with $\tilde{w} \in T(\tilde{z})$ such that $\langle \hat{z} - \tilde{z},\, \hat{w} - \tilde{w} \rangle < 0$.

Theorem 1 (Rockafellar [3], Thm. 3)

For the variational inequality problem that corresponds to a nonempty, closed, convex set $Z \subset I\!R^N$ and a continuous mapping $F : Z \to I\!R^N$ as expressed in (VI″), if F is monotone, then the associated mapping $T : I\!R^N \rightrightarrows I\!R^N$ is maximal monotone.

This result, which characterizes monotone variational inequality problems as problems of solving $0 \in T(\bar{z})$ for certain kinds of maximal monotone mappings T, will enable us to identify different ways in which conditions for network equilibrium can be cast in terms of a variational inequality as long as monotonicity is present. In appealing to it we will essentially be limiting our attention to problems with overtones of convexity, as already explained. An extension beyond monotonicity would no doubt be possible, but we will not undertake it in this article.

3. NETWORK EQUILIBRIUM

For purposes here, a *network* consists of a finite set of *nodes* indexed by $i \in I = \{1, \ldots, m\}$ and a finite set of arcs indexed by $j \in J = \{1, \ldots, n\}$. Each arc j has an *initial node* and a *terminal node*, which are different. The information about these nodes is embodied in the $m \times n$ *incidence matrix* $E = (e_{ij})$ for the network, where

$$e_{ij} = \begin{cases} 1 & \text{if node } i \text{ is the initial node of arc } j, \\ -1 & \text{if node } i \text{ is the terminal node of arc } j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Classical network theory is concerned with scalar-valued flows, but here we will be occupied with vector-valued flows. A d-dimensional *flow* x is a "supervector" $(x_1, \ldots, x_n)$ where each component x_j is a vector $(x_{j1}, \ldots, x_{jd}) \in I\!R^d$. In applications, x_{jk} will represent the amount of scalar flow of type k in arc j. Constraints on the magnitude and direction of such flow amounts may be imposed later, but for now we note merely that a quantity $x_{jk} > 0$ is to be interpreted as flowing from the initial node of arc j to its terminal node, whereas a quantity $x_{jk} < 0$ refers to physical flow in the opposite direction.

The *divergence* of the flow x at node i is the vector $y_i = (y_{i1}, \ldots, y_{id}) \in I\!R^d$ in which y_{ik} gives the net amount of flow type k that originates at node i. This is expressed by

$$y_i = \sum_{j \in J} e_{ij} x_j \quad \text{for each} \quad i \in I, \quad \text{or in summary,} \quad y = Ex, \tag{3.2}$$

where y is the supervector $(y_1, \ldots, y_m)$. Node i is a source for flow type k under x if $y_{ik} > 0$ and a sink if $y_{ik} < 0$. Flow type k is conserved at node i if $y_{ki} = 0$.

Dual to the concept of flow is that of potential. A d-dimensional *potential* u is a supervector $(u_1, \ldots, u_m)$, each component of which designates a vector $u_i = (u_{i1}, \ldots, u_{id})$. The quantity u_{ik} refers to the potential of type k at node i, and abstract quantity which in economic applications may have a price interpretation. Relative to such a vector-valued potential u, the *tension* v_j in arc j is the difference $u_{i'} - u_i$, where i is the initial node of arc j and i' is the terminal node. In terms of the incidence matrix E this comes out as

$$v_j = -\sum_{i \in I} u_i e_{ij} \quad \text{for each} \quad j \in J, \quad \text{or in summary,} \quad v = -E^T u, \tag{3.3}$$

where $v = (v_1, \ldots, v_n)$. Each tension vector $v_j = (v_{j1}, \ldots, v_{jd}) \in I\!R^d$ has components v_{jk} giving the difference in potential type k in arc j.

Equilibrium problems in this context can usefully be set up on several levels. To begin with, we consider the case of fixed supplies and demands. By a *supply* $b = (b_1, \ldots, b_m)$ in the network we will mean the assignment to each node i of a vector $b_i = (b_{i1}, \ldots, b_{id})$, where b_{ik} designates the supply of flow type k at node i, this being the value that the divergence y_{ki} will be required to have. Negative supply values b_{ik} correspond of course to demand. A value $b_{ik} = 0$ indicates that flow type k is required to be conserved at node i.

By a *flow-tension* relation in arc j we will mean a subset of $I\!\!R^d \times I\!\!R^d$ specifying the flow-tension pairs (x_j, v_j) permitted to coexist in arc j. We interpret this subset as the graph $\operatorname{gph} R_j$ of a mapping $R_j : I\!\!R^d \rightrightarrows I\!\!R^d$; thus, x_j and v_j are related in the required manner if and only if $v_j \in R_j(x_j)$, or equivalently $x_j \in R_j^{-1}(v_j)$.

The classical analogy for scalar-valued flows in an electrical network lies with resistance and conductance. In such a network each arc j represents an electrical component with a certain "characteristic curve" which describes how the flow (electrical current) through j corresponds to the tension (voltage difference) across j. This characteristic curve is the graph of R_j, and the R_j is "resistance mapping" for the arc j; the inverse R_j^{-1} is the "conductance mapping" for the arc.

If arc j represents an ideal resistor, behaving in accordance with Ohm's Law with resistance value $r_j > 0$, its characteristic curve is a line in $I\!\!R \times I\!\!R$ with slope r_j. Then both R_j and R_j^{-1} are single-valued and linear. Nonlinear resistors correspond to more complicated curves in $I\!\!R \times I\!\!R$. Sometimes R_j or R_j^{-1}, or both, can fail to be single-valued in such a context. For instance, in the case of an ideal diode, the graph of R_j is the subset of $I\!\!R \times I\!\!R$ formed by the union of the nonnegative x_j-axis and the nonpositive v_j-axis.

Equilibrium Problem 1 *Given for each arc j a mapping $R_j : I\!\!R^d \rightrightarrows I\!\!R^d$ and for each node i a supply vector $b_i \in I\!\!R^d$, find a flow x and a potential u for which the corresponding divergence y and tension v satisfy*

$$\begin{cases} v_j \in R_j(x_j) & \text{for all } j \in J, \\ y_i = b_i & \text{for all } i \in I. \end{cases}$$

A more general formulation of equilibrium dispenses with fixed supplies and demands and instead allows a divergence-potential relation to be assigned to each node. Again, we think of such a relation in terms of a subset of $I\!\!R^d \times I\!\!R^d$ viewed as the graph $G(S_i)$ of a mapping $S_i : I\!\!R^d \rightrightarrows I\!\!R^d$. The divergence y_i and potential u_i at node i are related in the required manner when $u_i \in S_i(y_i)$, or equivalently, $y_i \in S_i^{-1}(u_i)$. In Equilibrium Problem 1, the graph of S_i is the set $\{b_i\} \times I\!\!R^d$ for every node i; in other words, we have

$$S_i^{-1} : u_i \mapsto b_i \quad \text{(constant mapping)}. \tag{3.4}$$

But instead now, S_i^{-1} might for instance be a nonconstant, single-valued mapping. An economic interpretation in some models where the kinds of flow represent different kinds of commodities is that u_i is a vector of prices at i for these commodities, and $S_i^{-1}(u_i)$ gives the amounts supplied (produced), or with negative signs, demanded (consumed) at i in response to these prices.

Equilibrium Problem 2 *Given for each arc j a mapping $R_j : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ and for each node i a mapping $S_i : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$, find a flow x and a potential u for which the corresponding divergence y and tension v satisfy*

$$\begin{cases} v_j \in R_j(x_j) & \textit{for all } j \in J, \\ u_i \in S_i(y_i) & \textit{for all } i \in I. \end{cases}$$

By the *maximal monotone* version of Equilibrium Problem 1, we will mean the version where every mapping R_j is maximal monotone. Likewise, by the maximal monotone version of Equilibrium Problem 2, we will mean the version where every mapping R_j and every mapping S_i is maximal monotone. (Note that S_i is maximal monotone in particular when S_i^{-1} is a constant mapping, as seen when Equilibrium Problem 1 is imbedded within Equilibrium Problem 2.) For scalar-valued flows and potentials ($d = 1$), the role of maximal monotonicity was first explored by Minty [4], who concentrated on Equilibrium Problem 1 with $b_i = 0$ for all i. The theory of this case is fully presented in the book [5].

One of the many nice features of maximal monotonicity with $d = 1$ is that the graphs of the relations are indeed "curves," i.e., sets nicely parameterized by a real variable. For $d > 1$ there is the following generalization.

Theorem 2 (Minty [6])
The graph $\operatorname{gph} T$ *of any maximal monotone mapping $T : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is homeomorphic to $\mathbb{R}^d$. Moreover, the homeomorphism can be set up to be Lipschitz continuous in both directions.*

In applications to traffic equilibrium, the following model is basic. For each node i consider a supply vector $b_i = (b_{i1}, \ldots, b_{id})$. For each arc j let $X_j = [0, \xi_{j1}] \times \cdots \times [0, \xi_{jd}]$, where ξ_{jk} is the upper bound for flow of type k in the arc in question, $\xi_{jk} \geq 0$. (When $\xi_{jk} = 0$, flow of type k is forbidden in this arc.) For $x_j \in X_j$ let

$$F_j(x_j) = c_j(w_{j1}x_{j1} + \cdots + w_{jd}x_{jd})$$

for a continuous, nondecreasing function $c_j : [0, \infty) \to [0, \infty)$ and fixed weights $w_{jk} \geq 0$. The conditions for traffic equilibrium are taken then to be those of Equilibrium Problem 1 with

$$v_j \in R_j(x_j) \quad \Longleftrightarrow \quad x_j \in X_j, \quad v_j - F_j(x_j) \in N_{X_j}(x_j).$$

In models of this kind it is common to have only one source and one sink for each type of traffic. Then for each k there is exactly one node i with $b_{ik} > 0$ and exactly one other node i' with $b_{i'k} < 0$. Often the models are set up in terms of flows along particular paths instead of just flow amounts in each arc. Such models are much more complicated to work with, yet they seem not to offer any serious advantages, because the flow of traffic of type k can readily be represented, at any stage of computation or analysis where desired, as a sum of flows along paths from source to sink. See Rockafellar [5], Secs. 4A and 4B, for the elementary algorithm that is involved.

Generalized traffic models in the format of Equilibrium Problem 2 instead of Equilibrium Problem 1 might arise from situations in which the supply and demand for the different kinds of flow could be affected by the state of congestion. Models attempting to treat the difficulties of passing through various nodes would not necessarily require passage to Equilibrium Problem 2. Instead one might use the device of introducing "internal arcs" in such nodes; the needed equilibrium conditions could then be centered on such arcs, see [5], Sec. 3L. In either approach, *dynamical* networks could be formulated in a space-time framework to handle traffic equilibrium in the sense of a day-to-day cycle; see [5], Secs. 1H and 3L.

4. EQUILIBRIUM AS A VARIATIONAL INEQUALITY

To what extent are the equilibrium conditions in Equilibrium Problems 1 and 2 expressible in terms of a variational inequality? In general they go beyond the limited format served by variational inequalities, but there are important cases where they fit with it. It is instructive to see that this can occur in several different ways.

Let us say that a flow-tension relation for arc j is of *primal VI-type* if its mapping $R_j : I\!R^d \rightrightarrows I\!R^d$ has the form

$$R_j(x_j) = \begin{cases} F_j(x_j) + N_{X_j}(x_j) & \text{if } x_j \in X_j, \\ \emptyset & \text{if } x_j \notin X_j, \end{cases} \tag{4.1}$$

where X_j is a nonempty, closed, convex subset of $I\!R^d$ and $F_j : X_j \to I\!R^d$ is continuous (then $X_j = \operatorname{dom} R_j$). On the other hand, let us say that this relation is of *dual VI-type* if the inverse mapping R_j^{-1} has such form:

$$R_j^{-1}(v_j) = \begin{cases} \Phi_j(v_j) + N_{V_j}(v_j) & \text{if } v_j \in V_j, \\ \emptyset & \text{if } v_j \notin V_j, \end{cases} \tag{4.2}$$

where V_j is a nonempty, closed, convex subset of $I\!R^d$ and $\Phi_j : V_j \to I\!R^d$ is continuous (then $V_j = \operatorname{rge} R_j$). In a similar vein, let us say that a divergence-potential relation for node i is of primal VI-type if its mapping $S_i : I\!R^d \to I\!R^d$ has the form

$$S_i(y_i) = \begin{cases} G_j(y_j) + N_{Y_j}(y_j) & \text{if } y_i \in Y_i, \\ \emptyset & \text{if } y_i \notin Y_i, \end{cases} \tag{4.3}$$

where Y_i is a nonempty, closed, convex subset of $I\!R^d$ and $G_i : Y_i \to I\!R^d$ is continuous (then $Y_i = \operatorname{dom} S_i$), while it is of dual VI-type if the inverse mapping S_i^{-1} has the form

$$S_i^{-1}(u_i) = \begin{cases} \Psi_j(u_j) + N_{U_j}(u_j) & \text{if } u_i \in U_i, \\ \emptyset & \text{if } u_i \notin U_i, \end{cases} \tag{4.4}$$

where U_i is a nonempty, closed, convex subset of $I\!R^d$ and $\Psi_i : U_i \to I\!R^d$ is continuous (then $U_i = \operatorname{rge} S_i$).

The divergence-potential relations in the special case (3.4) used in imbedding Equilibrium Problem 1 within Equilibrium Problem 2 are obviously of dual VI-type with

$U_i = I\!R^d$ and $\Psi(u_i) \equiv b_i$, but they are also of primal VI-type with $Y_i = \{b_i\}$ and $G_i(y_i) = 0$.

Our results will utilize the theory of relative interiors. Recall that the *relative interior* $\mathrm{ri}\, C$ of a convex set C is the interior of C relative to its affine hull (see [1], Sec. 6). An affine set is its own relative interior; in particular, if $C = \{a\}$ (a singleton set) then $\mathrm{ri}\, C = \{a\}$. Recall further the C is *polyhedral* when it is representable as the intersection of a finite collection of closed half-spaces, or equivalently as the set of solutions to a system of finitely many (weak) linear inequalities.

Theorem 3 (Rockafellar [1], Cor. 23.8.1)

Suppose that $C = C_1 \cap \cdots \cap C_r$, where each C_l is a convex subset of $I\!R^N$, and suppose there exists $\tilde{z} \in C$ such that actually $\tilde{z} \in \mathrm{ri}\, C_l$ for each l such that C_l is not polyhedral. Then at all points $z \in C$ one has $N_C(z) = N_{C_1}(z) + \cdots + N_{C_r}(z)$.

Because Equilibrium Problem 1 is covered by Equilibrium Problem 2, we develop results in terms of Equilibrium Problem 2 and then specialize.

Theorem 4 (Variational Inequalities from Equilibrium Problem 2).

(a) (primal case). *In Equilibrium Problem 2, suppose all the flow-tension relations and divergence-potential relations are of primal VI-type: (4.1) and (4.3) hold. Assume there is at least one flow $\tilde{x}$ which, with its divergence $\tilde{y}$, satisfies $\tilde{x}_j \in \mathrm{ri}\, X_j$ for all $j \in J$ and $\tilde{y}_i \in \mathrm{ri}\, Y_i$ for all $i \in I$; in this assumption, "ri" can be omitted for any X_j or Y_i that is polyhedral. The problem is equivalent then to solving the variational inequality for*

$$\begin{aligned} Z &= \{z = (x_1, \dots, x_n, y_1, \dots, y_m) \mid x_j \in X_j,\ y_i \in Y_i,\ y = Ex\}, \\ F(z) &= (F_1(x_1), \dots, F_n(x_n), G_1(y_1), \dots, G_m(y_m)). \end{aligned}$$

This variational inequality is monotone when every F_j and G_i is monotone; then one has a maximal monotone version of Equilibrium Problem 2.

(b) (dual case). *In Equilibrium Problem 2, suppose all the flow-tension relations and divergence-potential relations are of dual VI-type: (4.2) and (4.4) hold. Assume there is at least one potential $\tilde{u}$ which, with its tension $\tilde{v}$, satisfies $\tilde{v}_j \in \mathrm{ri}\, V_j$ for all $j \in J$ and $\tilde{u}_i \in \mathrm{ri}\, U_i$ for all $i \in I$; in this assumption, "ri" can be omitted for any V_j or U_i that is polyhedral. The problem is equivalent then to solving the variational inequality for*

$$\begin{aligned} Z &= \{z = (v_1, \dots, v_n, u_1, \dots, u_m) \mid v_j \in V_j,\ u_i \in U_i,\ v = -E^T u\}, \\ F(z) &= (\Phi_1(v_1), \dots, \Phi_n(v_n), \Psi_1(u_1), \dots, \Psi_m(u_m)). \end{aligned}$$

This variational inequality is monotone when every Φ_j and Ψ_i is monotone; then one has a maximal monotone version of Equilibrium Problem 2.

(c) (primal-dual case). *In Equilibrium Problem 2, suppose all the flow-tension relations are of primal VI-type and all the divergence-potential relations are of dual VI-type: (4.1) and (4.4) hold. The problem is equivalent then to solving the variational inequality for*

$$\begin{aligned} Z &= \{z = (x_1, \dots, x_n, u_1, \dots, u_m) \mid x_j \in X_j,\ u_i \in U_i\}, \\ F(z) &= (F_1(x_1), \dots, F_n(x_n), \Psi_1(u_1), \dots, \Psi_m(u_m)) + (E^T u, -Ex). \end{aligned}$$

This variational inequality is monotone when every F_j and Ψ_i is monotone; then one has a maximal monotone version of Equilibrium Problem 2.

Proof. The analysis of the normal cone $N_Z(z)$ at points $z \in Z$ is crucial in each case. In (a) and (b) the main tool for this purpose will be Theorem 3.

In (a) we have $Z = L \cap Z_0$ for the subspace $L = \{z = (x,y) \mid y = Ex\}$ and the product set $Z_0 = X_1 \times \cdots \times X_n \times \cdots \times Y_1 \times \cdots \times Y_m$. We can also express Z_0 as the intersection $X'_1 \cap \cdots \cap X'_n \cap Y'_1 \cap \cdots \cap Y'_m$ by taking X'_j to be the subset of $(\mathbb{R}^d)^{n+m}$ having the same formula as Z_0 but with all factors except X_j replaced by $\mathbb{R}^d$, and likewise for Y'_i. Then $\operatorname{ri} X'_j$ and $\operatorname{ri} Y'_i$ have this form as well, with $\operatorname{ri} X_j$ and $\operatorname{ri} Y_i$ replacing the factors X_j and Y_i. Also, X'_j and Y'_i are polyhedral when X_j and Y_i are polyhedral. Since $\operatorname{ri} L = L$ we see that the assumption in (a) about a certain flow $\tilde{x}$ corresponds to the hypothesis of Theorem 3 when applied to the intersection $Z = L \cap X'_1 \cap \cdots \cap X'_n \cap Y'_1 \cap \cdots \cap Y'_m$. We deduce thereby that

$$N_Z(z) = N_L(z) + N_{X'_1}(z) + \cdots + N_{X'_n}(z) + N_{Y'_1}(z) + \cdots + N_{Y'_m}(z).$$

Here $N_L(z) = L^\perp = \{(v,u) \mid v = -E^T u\}$, whereas

$$\begin{aligned} N_{X'_1}(z) + \cdots + {} & N_{X'_n}(z) + N_{Y'_1}(z) + \cdots + N_{Y'_m}(z) \\ = {} & N_{X_1}(x_1) \times \cdots \times N_{X_n}(x_n) \times N_{Y_1}(y_1) \times \cdots \times N_{Y_n}(y_n) = N_{Z_0}(z). \end{aligned}$$

Thus, $N_Z(z) = L^\perp + N_{Z_0}(z)$.

The variational inequality for Z and F, expressed in form (VI′), refers therefore to the existence of $\bar{z} \in L \cap Z_0$ such that there exists $\bar{w} \in L^\perp$ with $\bar{w} - F(\bar{z}) \in N_{Z_0}(\bar{z})$. To say that $\bar{z} \in L \cap Z_0$ is to say that $\bar{z} = (\bar{x}, \bar{y})$ with $\bar{y} = E\bar{x}$, $\bar{x}_j \in X_j$ and $\bar{y} \in Y_i$. To say that $\bar{w} \in L^\perp$ with $\bar{w} - F(\bar{z}) \in N_{Z_0}(\bar{z})$ is to say that $\bar{w} = (\bar{v}, \bar{u})$ with $\bar{v} = -E^T \bar{u}$, $\bar{v}_j - F_j(\bar{x}_j) \in N_{X_j}(\bar{x}_j)$ and $\bar{u}_i - G_i(\bar{y}_i) \in N_{Y_i}(\bar{y}_i)$ for all arcs j and nodes i. From (4.1) and (4.3) we conclude that the variational inequality comes down to the equilibrium conditions in Equilibrium Problem 2.

When all the mappings F_j and G_i in (a) are monotone, F is obviously monotone as well. Then too, every relation R_j and S_i is maximal monotone by Theorem 1, so we have a maximal monotone version of Equilibrium Problem 2.

In case (b) the argument is closely parallel. We have $Z = L \cap Z_0$ for the subspace $L = \{\, (v,u) \mid v = -E^T u\}$ and set $Z_0 = V_1 \times \cdots \times V_n \times U_1 \times \cdots \times U_m$. Again through Theorem 3, the assumption about a potential $\tilde{u}$ guarantees that $N_Z(z) = N_L(z) + N_{Z_0}(z)$ with $N_L(z) = L^\perp = \{(x,y) \mid y = Ex\}$ and $N_{Z_0}(z) = N_{V_1}(v_1) \times \cdots \times N_{V_n}(v_n) \times N_{U_1}(u_1) \times \cdots \times N_{U_m}(u_m)$. The specified variational inequality in form (VI′) reduces then to the existence of $\bar{z} = (\bar{v}, \bar{u}) \in L$ and $\bar{w} = (\bar{x}, \bar{y}) \in L^\perp$ such that $\bar{v}_j \in V_j$ and $\bar{x}_j - \Phi_j(\bar{v}_j) \in N_{V_j}(\bar{v}_j)$ for all arcs j, while $\bar{u}_i \in U_i$ and $\bar{y}_i - \Psi_i(\bar{u}_i) \in N_{U_i}(\bar{u}_i)$ for all nodes i. Because of (4.2) and (4.4), these conditions are identical to $\bar{x}_j \in R_j^{-1}(\bar{v}_j)$ and $\bar{y}_i \in S_i^{-1}(\bar{u}_i)$, which are just another way of writing the ones in Equilibrium Problem 2.

When all the mappings Φ_j and Ψ_i in (b) are monotone, F is monotone too, and through Theorem 1 the relations R_j^{-1} and S_i^{-1} are maximal monotone. Then R_j and

S_i are maximal monotone and we have a maximal monotone version of Equilibrium Problem 2.

Case (c) is simpler and does not require Theorem 3. Without having to invoke any constraint qualification we know that

$$N_Z(z) = N_{X_1}(x_1) \times \cdots \times N_{X_n}(x_n) \times N_{U_1}(u_1) \times \cdots \times N_{U_m}(u_m).$$

Consider $\bar{z} = (\bar{x}, \bar{u})$ and let $\bar{y} = E\bar{x}$ and $\bar{v} = -E^T\bar{u}$. To say that $\bar{z} \in Z$ is to say that $\bar{x}_j \in X_j$ for all j and $\bar{u}_i \in U_i$ for all i. The condition $-F(\bar{z}) \in N_Z(\bar{z})$ takes the form then that $-F_j(\bar{x}_j) + \bar{v}_j \in N_{X_j}(\bar{x}_j)$ and $-\Psi_i(\bar{u}_i) + \bar{y}_i \in N_{U_i}(\bar{u}_i)$ for all i and j. By (4.1) and (4.4) these properties are equivalent to having $\bar{v}_j \in R_j(\bar{x}_j)$ and $\bar{y}_i \in S^{-1}(\bar{u}_i)$, which are the same as the equilibrium conditions in Equilibrium Problem 2.

When F_j and Ψ_i are monotone, the mapping

$$(x, u) \mapsto (F_1(x_1), \dots, F_n(x_n), \Psi_1(u_1), \dots, \Psi_m(u_m))$$

in (c) is monotone. The linear mapping $(x, u) \mapsto (E^T u, -Ex)$ is always monotone (because its matrix is antisymmetric). Then F, as the sum of two monotone mappings, is itself monotone. In this case the variational inequality in (c) is monotone. At the same time the mappings R_j in (4.1) and S_i^{-1} in (4.4) are maximal monotone by virtue of Theorem 1, so we have a maximal monotone version of Equilibrium Problem 2. □

Theorem 5 (Variational Inequalities from Equilibrium Problem 1)

(a) (primal case). *In Equilibrium Problem 1, suppose all the flow-tension relations are of primal VI-type: (4.1) holds. Assume there is at least one flow* $\tilde{x}$ *with* $E\tilde{x} = b$ *such that* $\tilde{x}_j \in \operatorname{ri} X_j$ *for all* j*; in this assumption, "*ri*" can be omitted for any* X_j *that is polyhedral. The problem is equivalent then to solving the variational inequality for*

$$\begin{aligned} Z &= \{\, z = (x_1, \dots, x_n) \mid x_j \in X_j,\ Ex = b \,\}, \\ F(z) &= (F_1(x_1), \dots, F_n(x_n)). \end{aligned}$$

This variational inequality is monotone when every F_j *is monotone; then one has a maximal monotone version of Equilibrium Problem 1.*

(b) (dual case). *In Equilibrium Problem 1, suppose all the flow-tension relations are of dual VI-type: (4.2) holds. Suppose there is at least one potential* $\tilde{u}$ *whose tension* $\tilde{v}$ *satisfies* $\tilde{v}_j \in \operatorname{ri} V_j$ *for all* j*; in this assumption, "*ri*" can be omitted for any* V_j *that is polyhedral. The problem is equivalent then to solving the variational inequality for*

$$\begin{aligned} Z &= \{z = (v_1, \dots, v_n, u_1, \dots, u_m) \mid v_j \in V_j,\ u_i \in \mathbb{R}^d,\ v = -E^T u\}, \\ F(z) &= (\Phi_1(v_1), \dots, \Phi_n(v_n), b_1, \dots, b_m). \end{aligned}$$

This variational inequality is monotone when every Φ_j *is monotone; then one has a maximal monotone version of Equilibrium Problem 1.*

(c) (primal-dual case). *In Equilibrium Problem 1, suppose all the flow-tension relations are of primal VI-type: (4.1) holds. The problem is equivalent then to solving the variational inequality for*

$$\begin{aligned} Z &= \{z = (x_1, \ldots, x_n, u_1, \ldots, u_m) \mid x_j \in X_j,\ u_i \in I\!R^d\}, \\ F(z) &= (F_1(x_1), \ldots, F_n(x_n), b_1, \ldots, b_m) + (E^T u, -Ex). \end{aligned}$$

This variational inequality is monotone when every F_j is monotone; then one has a maximal monotone version of Equilibrium Problem 1.

Proof. Here we take the divergence-potential relations in Theorem 4 all to have form (3.4). As observed, these relations are simultaneously of primal VI-type and of dual VI-type, and they are maximal monotone. In (a) the form is so special that the y_i arguments trivialize. □

The standard traffic equilibrium model described at the end of Section 3 fits case (a) of Theorem 5. Because c_j is nondecreasing in this example, F_j is monotone and we have a monotone variational inequality corresponding to a maximal monotone version of Equilibrium Problem 1.

5. OPTIMAL FLOWS AND POTENTIALS

The network equilibrium problems in Section 3 correspond to in some important cases to problems of optimization of flows and potentials, and vice versa. This correspondence is more general than that associated with the variational inequality format in its multiple modes in Section 4. We concentrate here on optimization of convex type and describe the connections in terms of subgradients of convex functions. The results obtained generalize the ones presented for scalar-valued flows and potentials in Rockafellar [5], Sec. 8H.

Let $\overline{I\!R}$ denote the extended real line $I\!R \cup \{\pm\infty\}$. Recall that a function $f : I\!R^N \to \overline{I\!R}$ is called *convex* if its "epigraph" $\operatorname{epi} f := \{(z, \alpha) \in I\!R^N \times I\!R \mid \alpha \geq f(z)\}$ is a convex set. The "effective domain" $\operatorname{dom} f := \{z \mid f(z) < \infty\}$ of such a function is then a convex set in particular.

A convex function f on $I\!R^N$ is *proper* if $f(z) > -\infty$ for all z, and $f(z) < \infty$ for at least one z. It is *lower semicontinuous* (lsc) if the set $\operatorname{epi} f$ is closed in $I\!R^N \times I\!R$. The function f^* *conjugate* to f is defined by

$$f^*(w) = \sup_{z \in R^N} \{\langle w, z\rangle - f(z)\}.$$

When f is convex, proper and lsc, the conjugate function f^* likewise is convex, proper and lsc, and the function f^{**} conjugate to f^* is in turn f:

$$f(z) = \sup_{w \in R^N} \{\langle w, z\rangle - f^*(w)\}.$$

These matters are developed in detail in [1].

For a convex, proper, lsc function f on $I\!R^N$, the *subgradients* of f at a point $\bar{z}$ are the vectors $\bar{w}$ (if any) such that

$$f(z) \geq f(\bar{z}) + \langle \bar{w},\, z - \bar{z} \rangle \quad \text{for all} \quad z.$$

The set of these is denoted by $\partial f(\bar{z})$. Thus ∂f denotes a mapping $I\!R^N \rightrightarrows \overline{I\!R}^N$ in the general sense adopted in Section 2. Furthermore, the inverse mapping $(\partial f)^{-1}$ is the subgradient mapping associated with the conjugate function f^*:

$$\bar{w} \in \partial f(\bar{z}) \quad \Longleftrightarrow \quad \bar{z} \in \partial f^*(\bar{w}). \tag{5.1}$$

Theorem 6 (Moreau [6], Rockafellar [1], Cor. 31.5.2)
When $f : I\!R^N \to \overline{I\!R}$ is convex, proper and lsc, the mapping $T = \partial f : I\!R^N \rightrightarrows I\!R^N$ is maximal monotone.

When $N > 1$, not every maximal monotone mapping $T : I\!R^N \rightrightarrows I\!R^N$ is of the form ∂f. The ones that are have the property of *maximal cyclic monotonicity*; see [1], Thm. 24.9. Maximal cyclic monotonicity is the same as maximal monotonicity in the one-dimensional case, however.

As with network equilibrium problems, it will be useful to consider network optimization problems on two levels.

Primal Problem 1 *Given for each arc j a convex, proper, lsc function f_j on $I\!R^d$ and for each node i a supply vector $b_i \in I\!R^d$, minimize $\sum_{j \in J} f_j(x_j)$ over all flows x with divergence y satisfying $y_i = b_i$ for all $i \in I$.*

Primal Problem 2 *Given for each arc j a convex, proper, lsc function f_j on $I\!R^d$ and for each node i a convex, proper, lsc function g_i on $I\!R^d$, minimize $\sum_{j \in J} f_j(x_j) + \sum_{i \in I} g_i(y_i)$ over all flows x, where y is the divergence of x.*

It should be kept in mind that these problems have implicit constraints represented through ∞. In Primal Problem 1, a flow x with divergence $y = b$ is not regarded as feasible unless $f_j(x_j) < \infty$ for all j, i.e., x_j belongs to the convex set $\operatorname{dom} f_j \subset I\!R^d$ for all j. In Primal Problem 2, there is the further requirement that y_i should belong to the convex set $\operatorname{dom} g_i$ for all i. Primal Problem 2 reduces to Primal Problem 1 when

$$g_i(y_i) = \begin{cases} 0 & \text{if } y_i = b_i, \\ \infty & \text{if } y_i \neq b_i. \end{cases} \tag{5.2}$$

With these primal problems we associate the following dual problems in terms of the conjugate convex functions.

Dual Problem 1 *Given for each arc j a convex, proper, lsc function f_j on $I\!R^d$ and for each node i a supply vector $b_i \in I\!R^d$, maximize $-\sum_{j \in J} f_j^*(v_j) - \sum_{i \in I} -\langle b_i, u_i \rangle$ over all potentials u and their tensions v.*

Dual Problem 2 *Given for each arc j a convex, proper, lsc function f_j on $\mathbb{R}^d$ and for each node i a convex, proper, lsc function g_i on $\mathbb{R}^d$, maximize $-\sum_{j\in J} f_j^*(v_j) - \sum_{i\in I} g_i^*(u_i)$ over all potentials u and their tensions v.*

In Dual Problem 1, the constraint $v_j \in \operatorname{dom} f_j^*$ is implicit. In Dual Problem 2, one also needs $u_i \in \operatorname{dom} g_i^*$, since otherwise the expression being maximized has the value $-\infty$. Note that Dual Problem 2 reduces to Dual Problem 1 under the choice (5.2) for g_i, because the conjugate convex function is then

$$g_i^*(u_i) \equiv \langle b_i, u_i \rangle. \tag{5.3}$$

By the *optimal values* in these primal and dual problems we will mean the values (in $\overline{\mathbb{R}}$) giving the infimum or supremum in each case. The *optimal solutions* are the elements (flows or potentials) for which these values are achieved, if any.

Finally, we formulate saddle point problems of Lagrangian type corresponding to the optimization problems on both levels.

Saddle Problem 1 *Given for each arc j a convex, proper, lsc function f_j on $\mathbb{R}^d$ and for each node i a supply vector $b_i \in \mathbb{R}^d$, find a saddle point of the Lagrangian function*

$$L(x,u) = \sum_{j\in J} f_j(x_j) - \sum_{i\in I} \langle b_i, u_i \rangle + \sum_{j\in J,\, i\in I} e_{ij} \langle u_i, x_j \rangle$$

with respect to minimizing over flows x but maximizing over potentials u.

Saddle Problem 2 *Given for each arc j a convex, proper, lsc function f_j on $\mathbb{R}^d$ and for each node i a convex, proper, lsc function g_i on $\mathbb{R}^d$, find a saddle point of the Lagrangian function*

$$L(x,u) = \sum_{j\in J} f_j(x_j) - \sum_{i\in I} g_i^*(u_i) + \sum_{j\in J,\, i\in I} e_{ij} \langle u_i, x_j \rangle$$

with respect to minimizing over flows x but maximizing over potentials u.

In Saddle Problem 2, the expression $L(x,u)$ is interpreted as $-\infty$ unless $u_i \in \operatorname{dom} g_i^*$ for every $i \in I$. A pair $(\bar{x}, \bar{u})$ furnishes a saddle point relative to all flows and potentials if and only if it furnishes a saddle point relative to $X \times U$, where X is the product of the sets $\operatorname{dom} f_j$, and U is the product of the sets $\operatorname{dom} g_i^*$; see [1], Sec. 36.

Theorem 7 *In Equilibrium Problem 2, suppose the flow-tension and divergence-potential relations have the form $R_j = \partial f_j$ for all $j \in J$ and $S_i = \partial g_i$ for all $i \in I$, where f_j and g_i are convex, proper, lsc functions on $\mathbb{R}^d$. Then one has a maximal monotone version of the problem in which the following conditions on a flow $\bar{x}$ with divergence $\bar{y}$ and a potential $\bar{u}$ with tension $\bar{v}$ are equivalent to each other:*

(a) *$\bar{x}$ and $\bar{u}$ solve Equilibrium Problem 2.*

(b) *$\bar{x}$ and $\bar{u}$ solve Saddle Problem 2.*

(c) *$\bar{x}$ is an optimal solution to Primal Problem 2, $\bar{u}$ is an optimal solution to Dual Problem 2, and the optimal values in these two problems are equal.*

Proof. The condition for $(\bar{x}, \bar{u})$ to be a saddle point in (b) is that the expression $L(x, \bar{u})$ should achieve its minimum over all flows x at $\bar{x}$, whereas $L(\bar{x}, u)$ should achieve its maximum over all potentials u at $\bar{u}$. The minimization part is equivalent to having $f_j(x_j) - \langle \bar{v}_j, x_j \rangle$ achieve its minimum over $x_j \in \mathbb{R}^d$ at $\bar{x}_j$ for each $j \in J$, where $\bar{v}_j = -\sum_{i \in I} e_{ij} \bar{u}_i$. This means that $\bar{v}_j \in \partial f_j(\bar{x}_j)$ for all j. The maximization part is equivalent to having $\langle \bar{y}_i, u_i \rangle - g_i^*(u_i)$ achieve its maximum over $u_i \in \mathbb{R}^d$ at $\bar{u}_i$ for each $i \in I$, where $\bar{y}_i = \sum_{j \in J} e_{ij} \bar{x}_j$. This means that $\bar{y}_i \in \partial g_i^*(\bar{u}_i)$ for all i, or since $\partial g_i^* = (\partial g_i)^{-1}$, that $\bar{u}_i \in \partial g_i(\bar{y}_i)$. These subgradient conditions are the same as $\bar{v}_j \in R_j(\bar{x}_j)$ and $\bar{u}_i \in S_i(\bar{y}_i)$ under our hypothesis. Thus, (a) and (b) are equivalent.

To establish the equivalence of (b) with (c), let $r(x) = \sup_u L(x, u)$ and $s(u) = \inf_x L(x, u)$. It is elementary and well known in general minimax theory (cf. [1], Sec. 36) that $(\bar{x}, \bar{u})$ furnishes a saddle point of L if and only if $\bar{x}$ minimizes $r(x)$, $\bar{u}$ maximizes $s(u)$, and the minimum value $r(\bar{x})$ agrees with the maximum value $s(\bar{u})$. We merely need to observe now that $r(x) = \sum_{j \in J} f_j(x_j) + \sum_{i \in I} g_i(y_i)$ (with y_i standing for $\sum_{j \in J} e_{ij} x_j$), whereas $s(u) = -\sum_{j \in J} f_j^*(v_j) - \sum_{i \in I} g_i^*(u_i)$ (with v_j standing for $-\sum_{i \in I} e_{ij} u_i$). □

Theorem 8 *In Equilibrium Problem 1, suppose that the flow-tension relations are of the form $R_j = \partial f_j$ for all $j \in J$, where each f_j is a convex, proper, lsc function on $\mathbb{R}^d$. Then one has a maximal monotone version of the problem in which the following conditions on a flow $\bar{x}$ with divergence $\bar{y}$ and a potential $\bar{u}$ with tension $\bar{v}$ are equivalent to each other:*

(a) *$\bar{x}$ and $\bar{u}$ solve Equilibrium Problem 1.*

(b) *$\bar{x}$ and $\bar{u}$ solve Saddle Problem 1.*

(c) *$\bar{x}$ is an optimal solution to Primal Problem 1, $\bar{u}$ is an optimal solution to Dual Problem 1, and the optimal values in these two problems are equal.*

Proof. This specializes Theorem 7 to the case of (5.1)–(5.2). □

To give these results their full force, supplementary conditions need to be provided under which the optimal values in the primal and dual problems are equal. Then the pairs $(\bar{x}, \bar{u})$ satisfying the equilibrium problem in question are precisely those such that $\bar{x}$ solves the primal problem and $\bar{u}$ solves the dual problem. In other words, equilibrium is fully reducible to optimization.

Theorem 9 *Either one of the following assumptions suffices to guarantee that the optimal values in Primal Problem 2 and Dual Problem 2 are equal:*

(a) *There is a flow $\tilde{x}$ with divergence $\tilde{y}$ such that $\tilde{x}_j \in \operatorname{ri}(\operatorname{dom} f_j)$ for all arcs $j \in J$, and $\tilde{y}_i \in \operatorname{ri}(\operatorname{dom} g_i)$ for all nodes $i \in I$.*

(b) *There is a potential $\tilde{u}$ with tension $\tilde{v}$ such that $\tilde{v}_j \in \operatorname{ri}(\operatorname{dom} f_j^*)$ for all arcs $j \in J$, and $\tilde{u}_i \in \operatorname{ri}(\operatorname{dom} g_i^*)$ for all nodes $i \in I$.*

Proof. This follows from the Fenchel duality theorem in convex optimization, specifically as a case of [1], Cor. 31.2.1. □

Theorem 10 *Either one of the following assumptions suffices to guarantee that the optimal values in Primal Problem 1 and Dual Problem 1 are equal:*

(a) *There is a flow* $\tilde{x}$ *with divergence* $\tilde{y}$ *such that* $\tilde{x}_j \in \mathrm{ri}\,(\mathrm{dom}\, f_j)$ *for all arcs* $j \in J$*, and* $\tilde{y}_i = b_i$ *for all nodes* $i \in I$*.*

(b) *There is a potential* $\tilde{u}$ *with tension* $\tilde{v}$ *such that* $\tilde{v}_j \in \mathrm{ri}\,(\mathrm{dom}\, f_j^*)$ *for all arcs* $j \in J$*.*

Proof. Again we merely choose g_i as in (5.1), so that g_i^* is given by (5.2). □

The constraint qualifications in Theorem 9 could be refined along the lines of the one in Theorem 3. Recall that a convex function is *polyhedral* when its epigraph is a polyhedral set; then the conjugate function is polyhedral as well; see [1], Secs. 19 and 20. Theorem 10 remains valid if in (a) the condition $\bar{x}_j \in \mathrm{ri}\,(\mathrm{dom}\, f_j)$ is weakened to $\bar{x}_j \in \mathrm{dom}\, f_j$ when f_j is polyhedral, while the condition $\bar{y}_i \in \mathrm{ri}\,(\mathrm{dom}\, g_i)$ is weakened to $\bar{y}_i \in \mathrm{dom}\, g_i$ when g_i is polyhedral; similarly in (b) and Theorem 10. This refinement can be verified on the basis of the polyhedral results in [1], Theorem 31.1, by splitting up the polyhedral and nonpolyhedral parts of the problem in a suitable way.

For scalar-valued flows and potentials, "ri" can be dropped entirely from Theorems 9 and 10—see [5], Sec. 8H.

An example of particular importance is the one where the functions f_j have the form

$$f_j(x_j) = \begin{cases} \phi_j(x_j) & \text{if } x_j \in X_j, \\ \infty & \text{if } x_j \notin X_j, \end{cases} \tag{5.4}$$

with X_j a nonempty, closed, convex subset of $\mathbb{R}^d$ and ϕ_j a differentiable convex function defined on a open set containing X_j. Then

$$\partial f_j(x_j) = \begin{cases} \nabla\phi_j(x_j) + N_{X_j}(x_j) & \text{if } x_j \in X_j, \\ \emptyset & \text{if } x_j \notin X_j. \end{cases} \tag{5.5}$$

In other words, the relation $R_j = \partial f_j$ is of primal VI-type (4.1) with $F_j = \nabla\phi_j$ (this mapping being continuous, because any differentiable convex function is continuously differentiable). Then Primal Problem 1 corresponds to minimizing $\sum_{j\in J}\phi_j(x_j)$ subject to $x_j \in X_j$ and $y_i = b_i$.

Other cases, where R_j is of dual VI-type, or where S_i is of primal or dual VI-type, can be identified in like manner. This reveals the extent to which the variational inequalities developed for Equilibrium Problems 1 and 2 in Theorems 4 and 5 can be viewed as coming from optimization with convexity. It should be noted that the primal-dual variational inequalities correspond to the saddle point problems.

6. EQUILIBRIUM AS FINDING A ZERO OF A MAPPING

It was seen in formulation (VI″) of the standard variational inequality problem that such a problem in $\mathbb{R}^N$ can be stated as the "zero problem"

$$\text{(ZP)} \qquad \text{find} \quad \bar{z} \quad \text{satisfying} \quad 0 \in T(\bar{z})$$

for a certain mapping $T : I\!R^N \rightrightarrows I\!R^n$. If the variational inequality is monotone, this mapping T is maximal monotone by Theorem 1. When Equilibrium Problems 1 and 2 reduce to variational inequalities in the manner of Theorems 4 and 5, they fit this pattern in particular. We will demonstrate now, though, that these problems always have an expression as (ZP), even when they lie beyond the scope of variational inequalities, and that in their maximal monotone versions the associated mappings T are maximal monotone—under mild assumptions.

Our motivation for this effort, beyond the nicety of tying several ideas together, is that many computational methods have been developed for solving (ZP), even with T not single-valued. The results open the way to applying those methods to Equilibrium Problems 1 and 2.

As background we will need the following fact about the effective domain and range of a maximal monotone mapping.

Theorem 11 (Minty [7])
The effective domain $\operatorname{dom} T$ *and effective range* $\operatorname{rge} T$ *of any maximal monotone mapping* $T : I\!R^n \rightrightarrows I\!R^N$ *are almost convex, in the sense that the sets* $C = \operatorname{cl}(\operatorname{dom} T)$ *and* $D = \operatorname{cl}(\operatorname{rge} T)$ *are convex and such that* $\operatorname{dom} T \supset \operatorname{ri} C$ *and* $\operatorname{rge} T \supset \operatorname{ri} D$.

Because of this property of $\operatorname{dom} T$ and $\operatorname{rge} T$ we can speak of the relative interiors $\operatorname{ri}(\operatorname{dom} T)$ and $\operatorname{ri}(\operatorname{rge} T)$, these being the same as the relative interiors of the convex sets C and D in Theorem 11. When $T = \partial f$ for a convex, proper, lsc function $f : I\!R^N \to \overline{I\!R}$ (cf. Theorem 6), one has $\operatorname{ri}(\operatorname{dom} T) = \operatorname{ri}(\operatorname{dom} f)$ and $\operatorname{ri}(\operatorname{rge} T) = \operatorname{ri}(\operatorname{dom} f^*)$, see [1], Thm. 23.4.

Theorem 12 (Rockafellar [3], Thm. 2)
If $T_1 : I\!R^N \rightrightarrows I\!R^N$ *and* $T_2 : I\!R^N \rightrightarrows I\!R^N$ *are maximal monotone and such that* $\operatorname{ri}(\operatorname{dom} T_1) \cup \operatorname{ri}(\operatorname{dom} T_2) \neq \emptyset$, *then the mapping* $T = T_1 + T_2$ *likewise is maximal monotone.*

This result holds not only for the sum of two mappings, but any number. A proof by induction is immediate from the fact that for convex sets $C_1, \ldots, C_r$ one has

$$\operatorname{ri}(C_1 \cap \cdots \cap C_r) = \operatorname{ri} C_1 \cap \cdots \cap \operatorname{ri} C_r \quad \text{when} \quad \operatorname{ri} C_1 \cap \cdots \cap \operatorname{ri} C_r \neq \emptyset,$$

cf. [1], Thm. 6.5.

Theorem 13 (Equilibrium Problem 2 as a Zero Problem)
(a) (primal case). *Equilibrium Problem 2 can be identified with (ZP) in the case of* $z = (x, y)$ *and the mapping*

$$T(z) = \begin{cases} (R_1(x_1), \ldots, R_n(x_n), S_1(y_1), \ldots, S_m(y_m)) + L^\perp & \text{if } z \in L, \\ \emptyset & \text{if } z \notin L, \end{cases}$$

where $L = \{(x, y) \mid y = Ex\}$, *so that* $L^\perp = \{(v, u) \mid v = -E^T u\}$. *In the maximal monotone version of Equilibrium Problem 2,* T *is maximal monotone as long as there* *exists a flow* $\tilde{x}$ *with divergence* $\tilde{y}$ *such that* $\tilde{x}_j \in \operatorname{ri}(\operatorname{dom} R_j)$ *for all* $j \in J$ *and* $\tilde{y}_i \in \operatorname{ri}(\operatorname{dom} S_i)$ *for all* $i \in I$.

(b) (dual case). *Equilibrium Problem 2 can be identified with (ZP) in the case of* $z = (v, u)$ *and the mapping*

$$T(z) = \begin{cases} (R_1^{-1}(v_1), \dots, R_n^{-1}(v_n), S_1^{-1}(u_1), \dots, S_m^{-1}(u_m)) + L^{\perp} & \text{if } z \in L, \\ \emptyset & \text{if } z \notin L, \end{cases}$$

where $L = \{(v, u) \mid v = -E^T u\}$, *so that* $L^{\perp} = \{(x, y) \mid y = Ex\}$. *In the maximal monotone version of Equilibrium Problem 2,* T *is maximal monotone as long as there exists a potential* $\tilde{u}$ *with tension* $\tilde{v}$ *such that* $\tilde{v}_j \in \operatorname{ri}(\operatorname{rge} R_j)$ *for all* $j \in J$ *and* $\tilde{u}_i \in \operatorname{ri}(\operatorname{rge} S_i)$ *for all* $i \in I$.

(c) (primal-dual case). *Equilibrium Problem 2 can be identified with (ZP) in the case of* $z = (x, u)$ *and the mapping*

$$T(z) = (R_1(x_1), \dots, R_n(x_n), S_1^{-1}(u_1), \dots, S_m^{-1}(u_m)) + (E^T u, -Ex).$$

In the maximal monotone version of Equilibrium Problem 2, T *is maximal monotone.*

Proof. In all cases the equivalence of (ZP) with Equilibrium Problem 2 is elementary. What we have to verify are the maximal monotonicity assertions. For these we will rely on Theorem 12. In (a), let

$$\begin{aligned} T_1(z) &= (R_1(x_1), \dots, R_n(x_n), S_1(y_1), \dots, S_m(y_m)), \\ T_2(z) &= \begin{cases} L^{\perp} & \text{if } z \in L, \\ \emptyset & \text{if } z \notin L. \end{cases} \end{aligned}$$

Clearly $T = T_1 + T_2$. The maximal monotonicity of T_1 follows at once from that of every R_j and S_i. That of T_2 is seen from Theorem 1 with $F(z) \equiv 0$ through the fact that $T_2(z) = N_L(z)$. Alternatively, T_2 is maximal monotone by Theorem 6 because $T_2 = \partial \delta_L$ for the indicator function δ_L (which has the value 0 on L but ∞ everywhere else). We have $\operatorname{dom} T_2 = L$, a set which is its own relative interior, whereas $\operatorname{dom} T_1$ is the product of the sets $\operatorname{dom} R_j$ and $\operatorname{dom} S_i$. The assumption in (a) about $\tilde{x}$ is equivalent therefore to the assumption that $\operatorname{ri}(\operatorname{dom} T_1) \cap \operatorname{ri}(\operatorname{dom} T_2) \neq \emptyset$, and Theorem 12 then gives us the maximal monotonicity of T.

Similarly in (b), let

$$\begin{aligned} T_1(z) &= (R_1^{-1}(x_1), \dots, R_n^{-1}(x_n), S_1^{-1}(y_1), \dots, S_m^{-1}(y_m)), \\ T_2(z) &= \begin{cases} L^{\perp} & \text{if } z \in L, \\ \emptyset & \text{if } z \notin L. \end{cases} \end{aligned}$$

Again, $T = T_1 + T_2$. The maximal monotonicity of T_1 is implied by that of every R_j and S_i. The maximal monotonicity of T_2 is justified by the same arguments used in case (a). We have $\operatorname{dom} T_2 = L = \operatorname{ri} \operatorname{dom} T_2$, while $\operatorname{dom} T_1$ is the product of the sets $\operatorname{rge} R_j$ and $\operatorname{rge} S_i$. The assumption in (b) about $\tilde{u}$ means that $\operatorname{ri}(\operatorname{dom} T_1) \cap \operatorname{ri}(\operatorname{dom} T_2) \neq \emptyset$, and it thus guarantees through Theorem 12 the maximal monotonicity of T.

In case (c) we get $T = T_1 + T_2$ by taking

$$\begin{aligned} T_1(z) &= (R_1(x_1), \dots, R_n(x_n), S_1^{-1}(u_1), \dots, S_m^{-1}(u_m)), \\ T_2(z) &= (E^T u, -Ex). \end{aligned}$$

Here T_1 is maximal monotone when R_j and S_i are maximal monotone for all j and i. On the other hand, T_2 is maximal monotone by Theorem 1 as invoked for the linear mapping $F(z) = (E^T u, -Ex)$ with Z taken to be the whole space (so that $N_Z(z) = \{0\}$ for all z). Then $\operatorname{dom} T_2$ is the whole space, so that the condition $\operatorname{ri}(\operatorname{dom} T_1) \cap \operatorname{ri}(\operatorname{dom} T_2) \neq \emptyset$ is satisfied trivially. Hence T is maximal monotone by Theorem 12. □

Theorem 14 (Equilibrium Problem 1 as a Zero Problem)

(a) (primal case). *Equilibrium Problem 1 can be identified with (ZP) in the case of* $z = x$ *and the mapping*

$$T(z) = \begin{cases} (R_1(x_1), \dots, R_n(x_n)) + \{v \mid \exists u,\ v = -E^T u\} & \text{if } Ex = b, \\ \emptyset & \text{if } Ex \neq b. \end{cases}$$

In the maximal monotone version of Equilibrium Problem 1, T *is maximal monotone as long as there exists a flow* $\tilde{x}$ *with* $E\tilde{x} = b$ *and* $\tilde{x}_j \in \operatorname{ri}(\operatorname{dom} R_j)$ *for all* $j \in J$.

(b) (dual case). *Equilibrium Problem 1 can be identified with (ZP) in the case of* $z = (v, u)$ *and the mapping*

$$T(z) = \begin{cases} (R_1^{-1}(v_1), \dots, R_n^{-1}(v_n), b_1, \dots, b_m) + L^\perp & \text{if } z \in L, \\ \emptyset & \text{if } z \notin L, \end{cases}$$

where $L = \{(v, u) \mid v = -E^T u\}$, *so that* $L^\perp = \{(x, y) \mid y = Ex\}$. *In the maximal monotone version of Equilibrium Problem 1,* T *is maximal monotone as long as there exists a potential* $\tilde{u}$ *with tension* $\tilde{v}$ *such that* $\tilde{v}_j \in \operatorname{ri}(\operatorname{rge} R_j)$ *for all* $j \in J$.

(c) (primal-dual case). *Equilibrium Problem 1 can be identified with (ZP) in the case of* $z = (x, u)$ *and the mapping*

$$T(z) = (R_1(x_1), \dots, R_n(x_n), b_1, \dots, b_m) + (E^T u, -Ex).$$

In the maximal monotone version of Equilibrium Problem 1, T *is maximal monotone.*

Proof. We specialize Theorem 13 here to (3.4). □

References

[1] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.

[2] R. T. Rockafellar, "Lagrange multipliers and optimality," *SIAM Review* **35** (1993), 183–238.

[3] R. T. Rockafellar, "On the maximality of sums of nonlinear monotone operators," *Transactions Amer. Math. Soc.* **149** (1970), 75–88.

[4] G. J. Minty, "Monotone networks," *Proc. Roy. Soc. London A* **257** (1960), 194–212.

[5] R. T. Rockafellar, *Network Flows and Monotropic Optimization*, Wiley, New York, 1984.

[6] J. J. Moreau, "Proximité et dualité dans un espace hilbertien," *Bull. Soc. Math. France* **93** (1965), 273–299.

[7] G. J. Minty, "On the maximal domain of a 'monotone' function," *Michigan Math. J.* **8** (1961), 135–137.