Sensitivity of Solutions in Nonlinear Programming Problems with Nonunique Multipliers

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Abstract We analyze the perturbations of quasi-solutions to a parameterized nonlinear programming problem, these being feasible solutions accompanied by a Lagrange multiplier vector such that the Karush-Kuhn-Tucker optimality conditions are satisfied. We show under a standard constraint qualification, not requiring uniqueness of the multipliers, that the quasi-solution mapping is differentiable in a generalized sense, and we present a formula for its derivative. The results are distinguished from previous ones in the subject, in that they do not entail having to impose conditions to ensure that dual as well as primal elements behave well with respect to sensitivity.

1 Introduction

A standard parameterized nonlinear programming problem can be formulated in terms of smooth functions f_i on $\mathbb{R}^n \times \mathbb{R}^d$ as follows:

$$\text{minimize } f_0(x,w) - \langle v, x\rangle \text{ over all } x \in C(w), \tag{1}$$

where the set $C(w) \subset \mathbb{R}^n$ is defined by

$$C(w) := \{x : f_1(x,w) \le 0, ..., f_s(x,w) \le 0, f_{s+1}(x,w) = 0, ..., f_m(x,w) = 0\}.$$

Here $w \in \mathbb{R}^d$ and $v \in \mathbb{R}^n$ both serve as parameter elements. In principle, the "tilt" perturbations represented through v could be subsumed into w, but they have an essential role in theory, and we therefore keep them explicit.

We concentrate our attention on points x that are *quasi-solutions* to the minimization problem (1) in the sense of satisfying, in association with some multiplier vector y, the Karush-Kuhn-Tucker (K-K-T) optimality conditions:

$$\begin{gathered}\exists y=(y_1,\ldots,y_m)\in N_K(f_1(x,w),\ldots,f_m(x,w)) \text{ with} \\ v=\nabla_x f_0(x,w)+y_1\nabla_x f_1(x,w)+\cdots+y_m\nabla_x f_m(x,w),\end{gathered} \tag{2}$$

written here for convenience in terms of $N_K(u)$ denoting the set of normal vectors at u to the polyhedral cone

$$K:=\{u\in\mathbb{R}^m : u_1\le 0,\ldots,u_s\le 0, u_{s+1}=0,\ldots,u_m=0\}. \tag{3}$$

Thus, for any $u\in K$ the vectors $y\in N_K(u)$ are the ones with $y_i\ge 0$ for indices $i\in\{1,\ldots,m\}$ having $u_i=0$ but $y_i=0$ for indices $i\in\{1,\ldots,m\}$ with $u_i<0$, whereas y_i is unrestricted for indices $i\in\{s+1,\ldots,m\}$. (By convention, $N_K(u)$ is the empty set when $u\notin K$.) The notation $y\in N_K(u)$ saves us from repeatedly having to write down such complicated details, and it has the further advantage of adapting in the framework of variational analysis to a broad range of circumstances beyond those that come into play here. The K-K-T conditions are of course necessary for a feasible solution x to be locally optimal under the Mangasarian-Fromovitz constraint qualification, which in turn takes the form

$$\begin{gathered}\nexists y=(y_1,\ldots,y_m)\in N_K(f_1(x,w),\ldots,f_m(x,w)) \text{ with} \\ y_1\nabla_x f_1(x,w)+\cdots+y_m\nabla_x f_m(x,w)=0, \text{ except } y=0.\end{gathered} \tag{4}$$

Quasi-solutions are sure to be optimal solutions when the minimization problem exhibits convexity with respect to x, but this is not an issue of concern to us here.

The *quasi-solution mapping* S in this framework associates with each parameter element $(w,v)\in\mathbb{R}^{d+n}$ the set

$$S(w,v):=\{x\in\mathbb{R}^n : \text{ the K-K-T conditions (2) hold}\}.$$

Since, in general, $S(w,v)$ is not a singleton, this equation defines a *multifunction* (set-valued mapping) $S:\mathbb{R}^{d+n}\rightrightarrows\mathbb{R}^n$. In the main result of this paper, we calculate a kind of generalized derivative of S with respect to (w,v).

Until now, differentiability properties of S have been studied by distinctly different means than will be used here. Most researchers (cf. [2], [6], and [1] for a survey) have looked at the sensitivity of solution multifunctions defined by K-K-T *pairs* (x,y),

$$T(w,v):=\{(x,y) : x \text{ solves the K-K-T conditions (2) with } y \text{ as multiplier}\},$$

being forced by this strategy to make strong assumptions about the multipliers y (e.g. uniqueness) in order to draw conclusions about the x-components of these pairs. Some exceptions to this approach are described in [6] where, however, single-valuedness of the solution mapping S is essential. Our approach is new and has the advantage of enabling us to study the "primal" solution multifunction S directly, without any restrictions on the multipliers. In such a setting, much broader than previously has been accessible, our approach leads to formulas describing the magnitude and direction of perturbations of quasi-solutions in terms of approximations based on set convergence. It does not in itself, though, provide tests for whether S is single-valued in a localized sense.

Our methodology derives from our recent work in [3], where we studied the sensitivity of parameterized "variational conditions" over a set which itself can depend on the parameters. Associated with any closed set $C \subset \mathbb{R}^n$ and mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is the *variational condition*

$$F(x) + N_C(x) \ni 0, \quad x \in C. \tag{5}$$

When C is convex, this expresses the *variational inequality* for C and F. When C is not convex, $N_C(x)$ is interpreted as the cone of "limiting proximal normals" in nonsmooth analysis, rather than the cone of normal vectors in the sense of convex analysis. The parameterized variational conditions studied in [3] are of the form

$$F(x, w) + N_{C(w)}(x) \ni v, \quad x \in C(w), \tag{6}$$

with parameter element $(w, v) \in \mathbb{R}^d \times \mathbb{R}^n$. As long as the Mangasarian-Fromovitz constraint qualification (4) holds, the K-K-T optimality conditions (2) can be reformulated in terms of this type of parameterized variational condition by taking $F(x, w) = \nabla_x f_0(x, w)$ (details in Section 3). The quasi-solution multifunction S is given then by the solutions to the parameterized variational condition (6), the perturbations of which were analyzed in [4]. Here we show that by applying the results of [3] to this formulation of the K-K-T optimality conditions, it can be established that S is "proto-differentiable," moreover with a specific formula for the proto-derivatives.

2 Proto-derivatives

Proto-differentiability, a concept of generalized differentiability which was introduced in [9], is distinguished from other differentiability concepts through its utilization of set convergence of graphs. Consider any multifunction $\Gamma : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ and any pair $(\bar{w}, \bar{z})$ in the graph of Γ, i.e., with $\bar{z} \in \Gamma(\bar{w})$. For each $t > 0$ one can form the difference quotient multifunction

$$(\Delta_t \Gamma)_{\bar{w},\bar{z}} : \omega \mapsto [\Gamma(\bar{w} + t\omega) - \bar{z}]/t.$$

Instead of asking the difference quotient multifunctions $(\Delta_t \Gamma)_{\bar{w},\bar{z}}$ to converge in some kind of pointwise sense as $t \downarrow 0$, proto-differentiability asks that they converge graphically, i.e., that their graphs converge as subsets of $\mathbb{R}^m \times \mathbb{R}^n$ to the graph of some multifunction Δ. Then Δ is the *proto-derivative* multifunction at $\bar{w}$ for $\bar{z}$ and is denoted by $\Gamma'_{\bar{w},\bar{z}}$; for each $\omega \in \mathbb{R}^d$, $\Gamma'_{\bar{w},\bar{z}}(\omega)$ is a certain (possibly empty) subset of $\mathbb{R}^n$.

The concept of Painlevé-Kuratowski set convergence underlies the formation of these graphical limits. It refers to a kind of approximation described from two sides as follows. The *inner set limit* of a parameterized family of sets $\{G_t\}_{t>0}$ in $\mathbb{R}^N$ is the set of points η such that for *every* sequence $t_k \downarrow 0$ there is a sequence of points $\eta_k \in G_{t_k}$ with $\eta_k \to \eta$. The *outer set limit* of the family is the set of points η such that for *some* sequence $t_k \downarrow 0$ there is a sequence of points $\eta_k \in G_{t_k}$ with $\eta_k \to \eta$. When the inner and outer set limits coincide, the common set G is the *limit* as $t \downarrow 0$.

In our framework, this is applied to sets that are the graphs of multifunctions. For a multifunction $\Gamma : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ and any pair $(\bar{w}, \bar{z})$ in $\operatorname{gph}\Gamma$, i.e., with $\bar{z} \in \Gamma(\bar{w})$, the graph of the difference quotient mapping $(\Delta_t\Gamma)_{\bar{w},\bar{z}}$ is $t^{-1}[\operatorname{gph}\Gamma - (\bar{w}, \bar{z})]$. The multifunction $\Gamma'^{+}_{\bar{w},\bar{z}} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ having as its graph the outer limit of the sets $\operatorname{gph}(\Delta_t\Gamma)_{\bar{w},\bar{z}}$ as $t \downarrow 0$ is called the *outer graphical derivative* of Γ at $\bar{w}$ for $\bar{z}$. Similarly, the multifunction $\Gamma'^{-}_{\bar{w},\bar{z}} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ having as its graph the inner limit of these sets is the *inner graphical derivative*. Proto-differentiability of Γ at $\bar{w}$ for $\bar{z}$ is the case where the outer and inner derivatives agree, the common mapping being then the *proto-derivative*: $\Gamma'_{\bar{w},\bar{z}} = \Gamma'^{+}_{\bar{w},\bar{z}} = \Gamma'^{-}_{\bar{w},\bar{z}}$, cf. Rockafellar [8].

For the sake of better understanding of the approximation inherent in proto-differentiability, we furnish a description of the kind of uniformity that the concept involves.

Proposition 2.1 *Under the assumption that $\Gamma : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ and $\Delta : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ are multifunctions having closed graph, the following is necessary and sufficient for Γ to be proto-differentiable at $\bar{w}$ for $\bar{z}$ (where $\bar{z} \in \Gamma(\bar{w})$) with proto-derivative $\Gamma'_{\bar{w},\bar{z}} = \Delta$: for every $\epsilon > 0$ there exists $\tau > 0$ such that, for all $t \in (0, \tau)$,*

(a) whenever $\bar{z} + t\zeta \in \Gamma(\bar{w} + t\omega)$ with $|\zeta| < \epsilon^{-1}$ and $|\omega| < \epsilon^{-1}$, there exist ζ' and ω' with $|\zeta' - \zeta| < \epsilon$, $|\omega' - \omega| < \epsilon$, $\zeta' \in \Delta(\omega')$.

(b) whenever $\zeta \in \Delta(\omega)$ with $|\zeta| < \epsilon^{-1}$ and $|\omega| < \epsilon^{-1}$, there exist ζ' and ω' with $|\zeta' - \zeta| < \epsilon$, $|\omega' - \omega| < \epsilon$, $\bar{z} + t\zeta' \in \Gamma(\bar{w} + t\omega')$.

Proof. Let G_t be the graph of $(\Delta_t\Gamma)_{\bar{w},\bar{z}}$ and G the graph of Δ. Proto-diffentiability corresponds to having G_t converge to G as $t \downarrow 0$. Such set convergence is known to mean that for every neighborhood V of the origin of $\mathbb{R}^{m+n}$ and every bounded set B in $\mathbb{R}^{m+n}$ there exists $\tau > 0$ such that, for all $t \in (0, \tau)$, one has

$$G_t \cap B \subset G + V \text{ and } G \cap B \subset G_t + V.$$

It suffices in this to consider neighborhoods V of $(0, 0) \in \mathbb{R}^{m+n}$ formed by the product of an ϵ ball around the origin of $\mathbb{R}^m$ and such a ball in $\mathbb{R}^n$, and on the other hand to consider bounded sets B formed by the product of an ϵ^{-1} ball around the origin of $\mathbb{R}^m$ and such a ball in $\mathbb{R}^n$. The two inclusions reduce then to (a) and (b). □

The proto-derivative notation simplifies when Γ happens to be single-valued at $\bar{w}$, i.e., such that the set $\Gamma(\bar{w})$ is just a singleton $\{\bar{z}\}$, then it suffices to write $\Gamma'_{\bar{w}}$. The next result clarifies the relationship between proto-differentiability in this case and B-differentiability as defined by Robinson [7].

Proposition 2.2 *Suppose that* Γ *is single-valued on a neighborhood of* $\bar{w}$*. Then* Γ *is B-differentiable at* $\bar{w}$ *if and only if* Γ *is continuous at* $\bar{w}$ *and proto-differentiable at* $\bar{w}$ *with* $\Gamma'_{\bar{w}}$ *single-valued, in which event one has the local expansion*

$$\Gamma(\bar{w} + t\omega) \;=\; \Gamma(\bar{w}) + t\Gamma'_{\bar{w}}(\omega) + o(t|\omega|) \textit{ for } t > 0. \tag{7}$$

Proof. B-differentiability corresponds to having an expansion of the form described, but in which the middle term on the right is $t\Delta(\omega)$ for a continuous (single-valued) mapping Δ. When this holds it is clear that Γ is continuous at $\bar{w}$ and proto-differentiable there with $\Gamma'_{\bar{w}} = \Delta$. Conversely, if the latter properties hold with respect to a single-valued mapping Δ, then by [9, Theorem 4.1] there exists $\kappa > 0$ such that $|\Gamma(w) - \Gamma(\bar{w})| \leq \kappa|w - \bar{w}|$ for all w in some neighborhood of $\bar{w}$. In particular, this yields $\Delta(\omega) \leq \kappa|\omega|$ for all ω. Because the graph of Δ, being a limit under graph convergence, is closed, it follows that Δ must be continuous. The characterization of proto-differentiability in 2.1 specializes then to show that the mappings $(\Delta_t\Gamma)_{\bar{w}}$ (which are single-valued on ever larger neighborhoods of 0 and bounded there by κ) converge uniformly on bounded sets to Δ. That is the meaning of the expansion expressing B-differentiability. □

This result means that proto-differentiability extends to set-valued mappings, just in the manner that might be wished, the notion of one-sided directional differentiability deemed most appropriate in the sensitivity analysis of single-valued mappings, smooth or nonsmooth. The question of whether a certain mapping is single-valued or not can be dealt with as a separate issue, which need not be resolved before progress can be made on quantitative stability of solutions.

3 Sensitivity Theorem

Our main result rests on the reformulation of the K-K-T optimality conditions (2) as a parameterized variational condition (6), so we give the details of this reformulation next. In terms of the mapping $G : \mathbb{R}^{n+d} \to \mathbb{R}^m$ defined by

$$G(x, w) := (f_1(x, w), \ldots, f_m(x, w)),$$

the K-K-T conditions simply require that

$$\nabla_x f_0(x, w) + \nabla_x G(x, w)^* N_K(G(x, w)) \;\ni\; v, \tag{8}$$

where $\nabla_x G(x, w)^*$ is the transpose of the partial Jacobian matrix for G with respect to x. In [3, Theorem 5.1], we have shown that if the Mangasarian-Fromovitz constraint qualification holds at $(\bar{x}, \bar{w})$, then for all pairs $(x, w) \in \mathbb{R}^n \times \mathbb{R}^d$ that are sufficiently close to $(\bar{x}, \bar{w})$, the set $\nabla_x G(x,w)^* N_K(G(x,w))$ is equal to the normal cone mapping associated with the set $C(w)$ at x. Under these circumstances then, the K-K-T optimality conditions (2) come out as the variational condition

$$\nabla_x f_0(x,w) + N_{C(w)}(x) \ni v. \tag{9}$$

Theorem 3.1 *Let $S : \mathbb{R}^{n+d} \to \mathbb{R}^m$ be the K-K-T solution mapping defined by*

$$S(w,v) := \{x : \text{ the K-K-T conditions (2) hold}\},$$

and let $\bar{x} \in S(\bar{w}, \bar{v})$ be such that the Mangasarian-Fromovitz constraint qualification (4) is satisfied. Then for all (w,v) sufficiently close to $(\bar{w}, \bar{v})$ and for all $x \in S(w,v)$, S is proto-differentiable at (w,v) for x with proto-derivative given by the formula:

$$S'_{(w,v),x}(w',v') := \{x' : \nabla^2_{xx} f_0(x,w)x' + \nabla^2_{xw} f_0(x,w)w' + M'_{(x,w),z}(x',w') \ni v'\}$$
$$\text{with } z = v - \nabla_x f_0(x,w) \text{ and } M(x,w) := N_{C(w)}(x),$$

the multifunction M being proto-differentiable at (x,w) for z.

Proof. From the equivalence of the K-K-T optimality conditions (2) to the variational condition (9), we get the K-K-T solution mapping S to reduce to the solution mapping associated with this variational condition, namely

$$S(w,v) = \{x : \nabla_x f_0(x,w) + N_{C(w)}(x) \ni v\}.$$

This is exactly the kind of solution mapping whose proto-differentiability was studied in [3]. The proto-differentiability of $M(w,x) = N_{C(w)}(x)$ immediately follows along with that of S from [3, Theorem 5.2]. □

To carry this further, a formula for the proto-derivatives of M is required. We can obtain such a formula from viewing $C(w)$ for each w as the x-section at w of the set

$$E = \{(x,w) \in \mathbb{R}^n \times \mathbb{R}^d : G(x,w) \in K\}.$$

Poliquin and Rockafellar [5] show that when the Mangasarian-Fromovitz constraint qualification holds at $(x,w) \in E$, the multifunction $N_E : (x,w) \mapsto N_E(x,w)$ is proto-differentiable at (x,w) for any $(z,q) \in N_E(x,w)$. Then from [3, Theorem 5.2] we have

$$M'_{(x,w),z}(x',w') = \bigcup_q \left\{ z' : \ \exists q' \text{ with } (z',q') \in (N_E)'_{(x,w),(z,q)}(x',w') \right\}. \tag{10}$$

Here the formula for the proto-derivatives of N_E is a key ingredient. To develop it and employ it put our various pieces together, we need the following notation.

We let $I_s(x,w)$ and I_m denote the sets of active indices at (x,w) in the specification of E, namely

$$I_s(x,w) = \{i \in \{1, ..., s\} : f_i(x,w) = 0\} \quad \text{and} \quad I_m = \{s+1, ..., m\},$$

and we define the polyhedral cone $Q(x,w) \subset \mathbb{R}^{n+d}$ by

$$Q(x,w) := \left\{ (x',w') : \begin{array}{l} \nabla f_i(x,w) \cdot (x',w') \leq 0 \text{ for } i \in I_s(x,w), \\ \nabla f_i(x,w) \cdot (x',w') = 0 \text{ for } i \in I_m \end{array} \right\}. \tag{11}$$

Next we introduce certain sets of multiplier vectors, first the bounded, polyhedral set

$$Y(x,w,z,q) := \left\{ \begin{array}{r} y = (y_1,\ldots,y_m) \in N_K(f_1(x,w),\ldots,f_m(x,w)) : \\ \sum_{i=1}^m y_i \nabla_x f_i(x,w) = z, \ \sum_{i=1}^m y_i \nabla_w f_i(x,w) = q \end{array} \right\},$$

and its face

$$Y_{\max}(x,w,z,q;x',w') = \operatorname*{argmax}_{y \in Y(x,w,z,q)} \sum_{i=1}^m y_i \Big\langle (x',w'), \nabla^2 f_i(x,w)(x',w') \Big\rangle,$$

and then the polyhedral cone

$$Y'(x,w;x',w') = \left\{ \begin{array}{r} y' = (y'_1,\ldots,y'_m) \in N_K(f_1(x,w),\ldots,f_m(x,w)) : \\ y'_i = 0 \text{ for } i \text{ with } \Big\langle \nabla f_i(x,w), (x',w') \Big\rangle \neq 0 \end{array} \right\}.$$

Theorem 3.2 *Under the assumptions of 3.1, the proto-derivatives of the multifunction M are given as follows. For $(x',w') \notin Q(x,w)$, the set $M'_{(x,w),z}(x',w')$ is empty. But for $(x',w') \in Q(x,w)$, the set $M'_{(x,w),z}(x',w')$ consists of all vectors z' having the form*

$$z' = \sum_{i=1}^m y_i \Big[\nabla^2_{xx} f_i(x,w)x' + \nabla^2_{xw} f(x,w)w' \Big] + \sum_{i=1}^m y'_i \nabla_x f_i(x,w) - y'_0 z$$

generated by arbitrary choices of $y' \in Y'(x,w;x',w')$ and $y'_0 \in \mathbb{R}$ along with choices of y for which there exists q with $\langle (z,q),(x',w') \rangle = 0$ such that $y \in Y_{\max}(x,w,z,q)$. (If no such choice of y is possible, then again $M'_{(x,w),z}(x',w')$ is empty.)

Proof. Our strategy is to apply (10) in conjunction with the formula for proto-derivatives that can be gleaned from [5, Theorem 3]. That formula utilizes the cone

$$\widetilde{Q}(x,w;z,q) = \{(x',w') \in Q(x,w) : \langle (z,q),(x',w') \rangle = 0\}.$$

We have $(z,q) \in N_E(x,w)$ if and only if there exists $y \in Y(x,w,z,q)$. Suppose that holds. The formula in question says that the set $(N_E)'_{(x,w),(z,q)}(x',w')$ is empty if $(x',w') \notin \widetilde{Q}(x,w;z,q)$ whereas if $(x',w') \in \widetilde{Q}(x,w;z,q)$ this set consists of all pairs (z',q') of the form

$$(z',q') = \sum_{i=1}^m y_i \nabla^2 f_i(x,w)(x',w') + \sum_{i=1}^m y'_i \nabla f_i(x,w) - y'_0 z$$

generated by arbitrary choices of $y \in Y_{\max}(x, w, z, q)$, $y' \in Y'(x, w; x', w')$ and $y'_0 \in \mathbb{R}$. When this is plugged into (10) we get the formula claimed here. □

Theorems 3.1 and 3.2 can be extended to cover other solution mappings associated with much more general optimization problems, but we will not take this up here. As seen, these results rely heavily on the those in of [3], in particular on [3, Theorem 5.2]. The theory developed in [3] allows a direct sensitivity analysis of parameterized optimization problems to a degree that has not been possible before. We are able on this foundation to obtain in Theorems 1 and 2 proto-derivatives in the sensitivity analysis of the "primal" solution mapping S without making any restrictions on the multiplier vectors y in the K-K-T optimality conditions.

When the solution mapping S happens to be single-valued, Theorem 3.1 gives results about the B-differentiability of S.

Theorem 3.3 *Let $S : \mathbb{R}^{n+d} \to \mathbb{R}^m$ be the K-K-T solution mapping defined by*

$$S(w, v) := \{x : \text{ the K-K-T conditions (2) hold}\},$$

and let $\bar{x} \in S(\bar{w}, \bar{v})$ be such that the Mangasarian-Fromovitz constraint qualification (4) is satisfied. If S is single-valued on some neighborhood of $(\bar{w}, \bar{v})$ and continuous at $(\bar{w}, \bar{v})$, and $S'_{(\bar{w},\bar{v})}$ is single-valued as well, then S is B-differentiable at $(\bar{w}, \bar{v})$ with the expansion

$$S(\bar{w} + tw', \bar{v} + tv') \;=\; S(\bar{w}, \bar{v}) + tS'_{(\bar{w},\bar{v})}(w', v') + o(t|(w', v')|),$$

the B-derivative $S'_{(\bar{w},\bar{v})}(w', v')$ being given by the formula in Theorem 3.1 in combination with the one in Theorem 3.2.

Proof. This combines the preceding results with Proposition 2. □

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