CHARACTERIZATIONS OF LIPSCHITZIAN STABILITY IN NONLINEAR PROGRAMMING[1]

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Abstract

Nonlinear programming problems are analyzed for Lipschitz and upper-Lipschitz behavior of their solutions and stationary points under general perturbations. Facts from a diversity of sources are put together to obtain new characterizations of several local stability properties.

1. INTRODUCTION

In this paper we consider the following nonlinear programming problem with canonical perturbations:

$$\text{minimize } g_0(w,x) + \langle v, x\rangle \text{ over all } x \in C(u,w), \tag{1}$$

where $C(u,w)$ is given by the constraints

$$g_i(w,x) - u_i \begin{cases} = 0 & \text{for } i \in [1,r], \\ \leq 0 & \text{for } i \in [r+1,m], \end{cases} \tag{2}$$

for $\mathcal{C}^2$ functions $g_i : I\!R^d \times I\!R^n \to I\!R$, $i = 0, 1, \ldots, m$. The vectors $w \in I\!R^d$, $v \in I\!R^n$ and $u = (u_1, \ldots, u_m) \in I\!R^m$ are parameter elements. Consolidating them as $p = (v,u,w)$, we denote by $X(p)$ the set of local minimizers of (1) and refer to the map $p \mapsto X(p)$ as the *solution map.* An element $x \in X(p)$ is *isolated* if $X(p) \cap U = \{x\}$ for some neighborhood U of x. To fit with this notational picture, we write $C(p)$ for the set of feasible solutions, even though this only depends on the (u,w) part of p; the map $p \mapsto C(p)$ is the *constraint map.*

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In terms of the basic Lagrangian function

$$L(w,x,y) = g_0(w,x) + y_1 g_1(w,x) + \cdots + y_m g_m(w,x),$$

the Karush-Kuhn-Tucker (KKT) system associated with problem (1) has the form:

$$\begin{cases} \quad v + \nabla_x L(w,x,y) = 0, \\ -u + \nabla_y L(w,x,y) \in N_Y(y) \quad \text{for } \; Y = I\!\!R^r \times I\!\!R^{m-r}_+, \end{cases} \tag{3}$$

where $N_Y(y)$ is the normal cone to the set Y at the point y. For a given $p = (v,u,w)$ the set of solutions (x,y) of the KKT system (the set of the KKT pairs) is denoted by $S_{\mathrm{KKT}}(p)$; the map $p \mapsto S_{\mathrm{KKT}}(p)$ is called the *KKT map*. We write $X_{\mathrm{KKT}}(p)$ for the set of stationary points; that is, $X_{\mathrm{KKT}}(p) = \{x \,|\; \text{there exists } y \text{ such that } (x,y) \in S_{\mathrm{KKT}}(p)\}$; the map $p \mapsto X_{\mathrm{KKT}}(p)$ is the *stationary point map*. The set of Lagrange multiplier vectors associated with x and p is $Y_{\mathrm{KKT}}(x,p) = \{y \,|(x,y) \in S_{\mathrm{KKT}}(p)\}$.

Recall that the *Mangasarian-Fromovitz condition* holds at (p,x) if $y = 0$ is the only vector satisfying

$$\begin{aligned} &y = (y_1, y_2, \ldots, y_m) \in N_{\mathcal{K}}(g_1(w,x) - u_1, \ldots, g_m(w,x) - u_m) \\ &\text{and } \; y_1 \nabla_x g_1(w,x) + \cdots + y_m \nabla_x g_m(w,x) = 0, \end{aligned}$$

where $\mathcal{K}$ is the convex and closed cone in $I\!\!R^m$ with elements whose first r components are zeros and the remaining $m-r$ components are nonpositive numbers. Under the Mangasarian-Fromovitz condition, the KKT system (3) represents a necessary condition for a feasible point x for (1) to be locally optimal, see e.g. [37].

In this paper we study Lipschitz-type properties of the maps S_{KKT} and X_{KKT}. We complement and unify a number of results scattered in the literature by putting together facts from a diversity of sources and exploiting the canonical form of the perturbations in our model (1). In Section 2 we discuss the robustness of the local upper-Lipschitz property with respect to higher-order perturbations and give a characterization of a stronger version of this property for the KKT map S_{KKT}. Section 3 is devoted to the stationary point map X_{KKT}. In Theorem 3.1 we present a characterization of the local upper-Lipschitz property of this map by utilizing a condition for its proto-derivative. Theorem 3.3 complements a known result of Kojima; we prove that if the map X_{KKT} is locally single-valued and Lipschitz continuous with its values locally optimal solutions, then both the Mangasarian-Fromovitz condition and the strong second-order condition hold. The converse is true under the constant rank condition for the constraints. Further, in the line of our previous paper [8], we show in Theorem 3.6 that the combination of the Mangasarian-Fromovitz condition and the Aubin continuity of the map X_{KKT} is equivalent to the local single-valuedness and Lipschitz continuity of this map, provided that the values of $X_{\rm KKT}$ are locally optimal solutions. In Section 4 we present a sharper version of the characterization of the local Lipschitz continuity of the solution-multiplier pair obtained in [8].

The literature on stability of nonlinear programming problems is enormous, and even a short survey would be beyond the scope of the present paper. We refer here to papers that are explicitly related to the results presented. For recent surveys also on other aspects of the subject, see [4] and [15].

Throughout we denote by $I\!B_a(x)$ the closed ball with center x and radius a. The ball $I\!B_1(0)$ is denoted simply by $I\!B$. For a (potentially set-valued) map Γ from $I\!R^m$ to $I\!R^n$ we denote by $\mathrm{gph}\,\Gamma$ the set $\{(u,x)\,|\,u \in I\!R^m,\ x \in \Gamma(u)\}$. We associate with any point $(u_0, v_0, w_0, x_0, y_0) \in \mathrm{gph}\, S_{\rm KKT}$ the index sets I_1, I_2, I_3 in $\{1, 2, \ldots, m\}$ defined by

$$
\begin{aligned}
I_1 &= \{i \in [r+1, m]\,|\, g_i(w_0, x_0) - u_{0i} = 0,\ y_{0i} > 0\} \cup \{1, \ldots, r\},\\
I_2 &= \{i \in [r+1, m]\,|\, g_i(w_0, x_0) - u_{0i} = 0,\ y_{0i} = 0\},\\
I_3 &= \{i \in [r+1, m]\,|\, g_i(w_0, x_0) - u_{0i} < 0,\ y_{0i} = 0\}.
\end{aligned}
$$

Recall that the *strict Mangasarian-Fromovitz condition* holds at a point (p_0, x_0) if there is a Lagrange multiplier vector $y_0 \in Y_{\rm KKT}(x_0, p_0)$ such that:
(a) the vectors $\nabla_x g_i(w_0, x_0)$ for $i \in I_1$ are linearly independent;
(b) there is a vector $z \in I\!R^n$ such that

$$
\begin{aligned}
\nabla_x g_i(w_0, x_0)^\top z = 0 &\quad \text{for all } i \in I_1\\
\nabla_x g_i(w_0, x_0)^\top z < 0 &\quad \text{for all } i \in I_2.
\end{aligned}
$$

It is known that the strict Mangasarian-Fromovitz condition holds at (p_0, x_0) if and only if there is a unique multiplier vector y_0 associated with (p_0, x_0); that is, $Y_{\rm KKT}(x_0, p_0) = \{y_0\}$ (cf. Kyparisis [21]).

Let (x_0, y_0) satisfy the KKT conditions (3) for a given $p_0 = (v_0, u_0, w_0)$. In the notation $A = \nabla^2_{xx} L(w_0, x_0, y_0)$, $B = \nabla^2_{yx} L(w_0, x_0, y_0)$, the *linearization* of (3) at $(u_0, v_0, w_0, x_0, y_0)$ is the linear variational inequality:

$$
\begin{cases} v + \nabla_x L(w_0, x_0, y_0) + A(x - x_0) + B^\top (y - y_0) = 0, \\ -u + g(w_0, x_0) + B(x - x_0) \in N_Y(y). \end{cases} \tag{4}
$$

We denote by $L_{\rm KKT}$ the map assigning to each (u, v) the set of all pairs (x, y) that solve (4).

2. THE LOCAL UPPER-LIPSCHITZ PROPERTY

Robinson [31] introduced the following definition. The set-valued map $\Gamma : I\!R^n \to I\!R^m$ is *locally upper-Lipschitz* at y_0 with modulus M if there is a neighborhood V of y_0 such that

$$
\Gamma(y) \subset \Gamma(y_0) + M\|y - y_0\| I\!B \quad \text{for all } y \in V. \tag{5}
$$

In [31] he proved that if $F : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a set-valued map whose graph is a (possibly nonconvex) polyhedron, then F is locally upper-Lipschitz at every point y in $\mathbb{R}^n$, moreover with a modulus M that is independent of the choice of y.

The upper-Lipschitz property is not a completely local property of the graph of a map, so for the sake of investigating stability under local perturbations we work with the following variant.

Definition 2.1. *The map* $\Gamma : \mathbb{R}^n \to \mathbb{R}^m$ *is* locally upper-Lipschitz *with modulus* M *at a point* (y_0, x_0) *in its graph if there exist neighborhoods* U *of* x_0 *and* V *of* y_0 *such that*

$$\Gamma(y) \cap U \subset \{x_0\} + M\|y - y_0\|\mathbb{B} \text{ for all } y \in V. \tag{6}$$

The local upper-Lipschitz property at a point $(y_0, x_0) \in \operatorname{gph}\Gamma$ implies that $\Gamma(y_0) \cap U = \{x_0\}$ for some neighborhood U of x_0. Conversely, if $\Gamma(y_0) \cap U = \{x_0\}$ for some neighborhood U of x_0, then the local upper-Lipschitz property at y_0 in the Robinson's sense implies the local upper-Lipschitz property at the point $(y_0, x_0) \in \operatorname{gph}\Gamma$; and the latter is in turn equivalent to the local upper-Lipschitz property of $\Gamma \cap U$ holding at y_0 with respect to some neighborhood U of x_0. Note that $\Gamma(y) \cap U$ might be empty for some y near y_0. Of course, if Γ is single-valued and locally upper-Lipschitz at $(y_0, \Gamma(y_0))$, it need not be Lipschitz continuous in a neighborhood of y_0.

Bonnans [2] studied a version of the local upper-Lipschitz property in Definition 2.1 for solution mappings of variational inequalities under the name "semistability". Levy [22] called it the "local upper-Lipschitz property at y_0 for x_0." Pang [27], in the context of the linear complementarity problem, introduced a stronger property of a map Γ; in addition to the condition that Γ is locally upper-Lipschitz at the point (y_0, x_0) in its graph (Definition 2.1) he also requires that there exist neighborhoods U of x_0 and V of y_0 such that

$$\Gamma(y) \cap U \neq \emptyset \text{ for all } y \in V.$$

The latter is equivalent to the openness of the inverse Γ^{-1} at (x_0, y_0). Throughout we call such maps locally nonempty-valued and upper-Lipschitz at the point in the graph.

We show first that, for maps defined by solutions to generalized equations, the local upper-Lipschitz property at a point in the graph is "robust under higher-order perturbations." Note that the local openness at a point, and hence the property introduced by Pang, are not robust in this sense.

Let $P = \mathbb{R}^d \times \mathbb{R}^m$ and consider the map Σ from P to the subsets of $\mathbb{R}^n$ defined by

$$\Sigma(p) = \{x \in \mathbb{R}^n \mid y \in f(w, x) + F(w, x)\} \text{ for } p = (w, y), \tag{7}$$

where $f : I\!R^d \times I\!R^n \to I\!R^m$ is a function and $F : I\!R^d \times I\!R^n \to I\!R^m$ is a possibly set-valued map. Assume that $x_0 \in \Sigma(p_0)$ for some $p_0 = (w_0, y_0) \in P$ and that the function $f(w_0, \cdot)$ is differentiable at x_0 with Jacobian matrix $\nabla_x f(w_0, x_0)$. As in (7), consider the map obtained by the linearization of f:

$$\Lambda(p) = \{x \in I\!R^n \,|\, y \in f(w_0, x_0) + \nabla_x f(w_0, x_0)(x - x_0) + F(w, x)\}. \quad (8)$$

The following result was established by Dontchev [6] in a more abstract setting.

Theorem 2.2. *Suppose there exist neighborhoods U of x_0 and W of w_0 along with a constant l such that, for every $x \in U$ and $w \in W$,*

$$\|f(w, x) - f(w_0, x)\| \le l\|w - w_0\|. \quad (9)$$

Then the following are equivalent:
(i) *Λ is locally upper-Lipschitz at the point (p_0, x_0) in its graph;*
(ii) *Σ is locally upper-Lipschitz at the point (p_0, x_0) in its graph.*

We note that a result closely related to the implication (i) $\Rightarrow$ (ii), but in a different setting, is proved in Robinson [32], Theorem 4.1. If the map F is polyhedral and independent of w, we obtain the following fact by combining Theorem 2.2 with Robinson's result in [31] mentioned at the beginning of this section.

Corollary 2.3. *Let the assumptions of Theorem 2.2 be fulfilled, and let $F : I\!R^n \to I\!R^m$ be a polyhedral map. Then the following are equivalent:*
(i) *there exists a neighborhood U of x_0 such that*

$$[f(w_0, x_0) + \nabla f(w_0, x_0)(\cdot - x_0) + F(\cdot)]^{-1}(y_0) \cap U = \{x_0\};$$

(ii) *The map Σ is locally upper-Lipschitz at the point (p_0, x_0) in its graph.*

Proof. The map $\Lambda = [f(w_0, x_0) + \nabla f(w_0, x_0)(\cdot - x_0) + F(\cdot)]^{-1}$ is polyhedral, hence from [31] it is locally upper-Lipschitz in $I\!R^m$. Then (i) implies that Λ is locally upper-Lipschitz at the point (y_0, x_0) in the graph. Applying Theorem 2.2, Σ is locally upper-Lipschitz at the point (p_0, x_0) in its graph. The converse implication follows again from Theorem 2.2. $\square$

Bonnans in [2], Theorem 3.1(a), showed that the local upper-Lipschitz property at a point in the graph of the solution map of a variational inequality over a polyhedral set is equivalent to the requirement that the reference point be an isolated solution of the linearized variational inequality. This conclusion follows immediately from Corollary 2.3. The precise result, specialized for the KKT system (3) (where N_Y is a polyhedral set) is as follows.

Corollary 2.4. *The following are equivalent:*
(i) (x_0, y_0) *is an isolated point of the set* $L_{\mathrm{KKT}}(p_0)$*;*
(ii) *The map* S_{KKT} *is locally upper-Lipschitz at* $(p_0, x_0, y_0) \in \operatorname{gph} S_{\mathrm{KKT}}$.

We note that Corollary 2.4 can be also deduced by the characterization of the local upper-Lipschitz property at a point in the graph of a map in terms of its graphical derivative, see Section 3 for an application of this result to the stationary point map.

Recall that a set-valued map Γ from $I\!R^m$ to the subsets of $I\!R^n$ has the *Aubin property*[2] at $(y_0, x_0) \in \operatorname{gph}\Gamma$ with constant M if there exist neighborhoods U of x_0 and V of y_0 such that

$$\Gamma(y_1) \cap U \ \subset\ \Gamma(y_2) + M\|y_1 - y_2\| I\!B \text{ for all } y_1, y_2 \in V.$$

The following lemma is a particular case of Theorem 1 of Klatte [14], see also Theorem 4.3 in Robinson [33]; for completeness we present a short proof.

Lemma 2.5. *Suppose* x_0 *is an isolated local minimizer of (1) for* $p = p_0$*, and let the Mangasarian-Fromovitz condition hold at* (p_0, x_0)*. Then the map* X *is lower semicontinuous at* (p_0, x_0)*; that is, for every neighborhood* U *of* x_0 *there exists a neighborhood* V *of* p_0 *such that for every* $p \in V$ *the set* $X(p) \cap U$ *is nonempty.*

Proof. The constraint map C defined by (2) has the Aubin property at (w_0, u_0, x_0) if and only if the Mangasarian-Fromovitz condition holds at (w_0, u_0, x_0), see e.g., [26], Corollary 4.5. Let a, b and γ be the constants in the definition of the Aubin property of the map C; that is, for $p_1, p_2 \in I\!B_b(p_0)$,

$$C(p_1) \cap I\!B_a(x_0) \subset C(p_2) + \gamma(\|p_1 - p_2\|) I\!B.$$

Let U be an arbitrary neighborhood of x_0. Choose $\alpha \in (0, a)$ in such a way that x_0 is the unique minimizer in $I\!B_\alpha(x_0)$ of (1) with $p = p_0$ and $I\!B_\alpha(x_0) \subset U$.

For this fixed α and for $p \in I\!B_b(p_0)$ consider the map

$$p \mapsto C_\alpha(p) = \{x \in C(p) \mid \quad \|x - x_0\| \le \alpha + \gamma\|p - p_0\|\}.$$

It is clear that the map C_α is upper semicontinuous at $p = p_0$. Let us show that it is lower semicontinuous at $p = p_0$ as well. Take $x \in C_\alpha(p_0) = C(p_0) \cap I\!B_\alpha(x_0)$. From the Aubin property of the map C, for any p near p_0 there exists $x_p \in C(p)$ such that $\|x_p - x\| \le \gamma\|p - p_0\|$. Then $\|x_p - x_0\| \le \|x_p - x\| + \|x - x_0\| \le \alpha + \gamma\|p - p_0\|$. Thus $x_p \in C_\alpha(p)$ and $x_p \to x$ as $p \to p_0$. Hence C_α is lower semicontinuous at $p = p_0$.

[2] In [1], J.-P. Aubin used the name “pseudo-Lipschitz continuity”. Following [8], we prefer to call this concept the Aubin property.

The problem

$$\text{minimize } g_0(w,x) + \langle x, v\rangle \text{ in } x \text{ subject to } x \in C_\alpha(p) \tag{10}$$

has a solution for every p near p_0, because $C_\alpha(p)$ is nonempty and compact. Moreover, because of the choice of α, x_0 is the unique minimizer of this problem for $p = p_0$. From the Berge theorem (see e.g. Chapter 9, Theorem 3, in [9]), the solution map X_α giving the argmin in (10) is upper semicontinuous at $p = p_0$; in other words, for any $\delta > 0$ there exists $\eta \in (0, b)$ such that for any $p \in \mathbb{B}_\eta(p_0)$ the set of (global) minimizers of (10) is nonempty and included within $\mathbb{B}_\delta(x_0)$. Since $X_\alpha(p_0) = \{x_0\}$, the map X_α is actually continuous at p_0. Let δ' be such that $0 < \delta' < \alpha$. Then there exists $\eta' > 0$ such that for every $p \in \mathbb{B}_{\eta'}(p_0)$ any solution $x \in X_\alpha(p)$ satisfies $\|x - x_0\| \le \delta' < \alpha + \gamma\|p - p_0\|$. Hence for $p \in \mathbb{B}_{\eta'}(p_0)$ the constraint $\|x - x_0\| \le \alpha + \gamma\|p - p_0\|$ is inactive in the problem (10). Then for every $p \in \mathbb{B}_{\eta'}(p_0)$ we have $X_\alpha(p) \subset X(p) \cap \mathbb{B}_{\delta'}(x_0)$. The proof is now complete. □

Recall that the *second-order sufficient condition* holds at $(p_0, x_0, y_0) \in \operatorname{gph} S_{\mathrm{KKT}}$ if

$$\langle x', \nabla^2_{xx} L(w_0, x_0, y_0)x'\rangle > 0 \text{ for all } x' \neq 0 \text{ in the cone}$$

$$D = \{x' \mid \nabla_x g_i(w_0, x_0)x' = 0 \text{ for } i \in I_1,\ \nabla_x g_i(w_0, x_0)x' \le 0 \text{ for } i \in I_2\}.$$

Theorem 2.6. *The following are equivalent:*

(i) *The map S_{KKT} is upper-Lipschitz at the point (p_0, x_0, y_0) in its graph and is locally nonempty-valued there, and x_0 is a locally optimal solution to problem (1) for p_0;*

(ii) *The strict Mangasarian-Fromovitz condition and the second-order sufficient condition for local optimality hold for (p_0, x_0, y_0).*

Proof. Suppose that (i) holds. Then y_0 is an isolated point in $Y_{\mathrm{KKT}}(x_0, p_0)$. Noting that $Y_{\mathrm{KKT}}(x_0, p_0)$ is convex, we get $Y_{\mathrm{KKT}}(x_0, p_0) = \{y_0\}$. Hence the strict Mangasarian-Fromovitz condition holds, see [21] for instance. Further, from Corollary 2.4, there is no (x, y) close to (x_0, y_0) such that $(x, y) \in L_{\mathrm{KKT}}(p_0)$. Without loss of generality, suppose that $I_1 = \{1, 2, \ldots, m_1\}$ and $I_2 = \{m_1 + 1, \ldots, m_2\}$ and denote by B_1, B_2 the submatrices of B corresponding to the indices I_1, I_2, respectively. Then the vector $(x, y) = (0, 0)$ is an isolated solution of the variational system

$$\begin{aligned} &Ax + B^\top y = 0,\\ &B_1 x = 0,\\ &B_2 x \le 0, \quad y_i \ge 0, \quad y_i (Bx)_i = 0 \quad \text{for } i \in [m_1 + 1, m_2]. \end{aligned} \tag{11}$$

Note that there is no restriction here on the sign of y_i for $i \in I_1$, since $y_{0i} > 0$ for $i \in I_1$. As a matter of fact, $(0, 0)$ is the unique solution to (11), because the set of solutions to (11) is a cone. Applying the second-order necessary condition for local optimality of x_0 we get

$$\langle x', Ax' \rangle \geq 0 \quad \text{for all} \quad x' \neq 0 \quad \text{in} \quad D.$$

All we need is to show that this inequality is $>$. Suppose to the contrary that there exists a nonzero vector $x' \in D$ with $Ax' = 0$. Then the nonzero vector $(x', 0)$ is a solution to (11), a contradiction.

If (ii) holds, then it is known that x_0 is an isolated local solution of (1) for $p = p_0$ and y_0 is the corresponding unique multiplier vector. Suppose that the index set I_1 associated with (p_0, x_0) is nonempty and $\mathcal{U}$ and $\mathcal{W}$ are neighborhoods of x_0 and w_0 respectively such that the vectors $\nabla_x g_i(w, x)$ for $i \in I_1$ are linearly independent for all $x \in \mathcal{U}$ and $w \in \mathcal{W}$. From Lemma 2.5, $X(p) \cap \mathcal{U} \neq \emptyset$ for p near p_0. Then for all p near p_0 and $x(p) \in X(p)$ near x_0 there exist $y_i(p)$ for $i \in I_1$ which are close to the values y_{0i} for $i \in I_1$ and such that

$$v + \nabla_x g_0(w, x(p)) + \sum_{i \in I_1} y_i(p) \nabla_x g_i(w, x(p)) = 0.$$

Note that $y_i(p) > 0$ for all $i \in I_1$. Taking $y_i(p) = 0$ for $i \in I_2 \cup I_3$, we obtain that the vector $y(p) = (y_1(p), \ldots, y_m(p))$ is a Lagrange multiplier for the perturbed problem which is close to y_0. Hence, if U is a neighborhood of (x_0, y_0) and p is sufficiently close to p_0, then $S_{\mathrm{KKT}}(p) \cap U \neq \emptyset$.

If $I_1 = \emptyset$, then $y_0 = 0 = Y_{\mathrm{KKT}}(x_0, p_0)$. From Lemma 2.5, $X(p) \cap \mathcal{U} \neq \emptyset$ for any neighborhood $\mathcal{U}$ of x_0 provided that p is sufficiently close to p_0. Further, the Mangasarian-Fromovitz condition yields that for p near p_0 and x near x_0 the set of Lagrange multipliers $Y_{\mathrm{KKT}}(x, p)$ is nonempty and contained in a bounded set, see e.g. Theorem 2.3 in [32]. Suppose that there exist $\alpha > 0$, a sequence $p_k \to p_0$ and a sequence $x_k \to x_0$ such that $\|y\| \geq \alpha$ for all $y \in Y_{\mathrm{KKT}}(x_k, p_k)$, $k = 1, 2, \ldots$. Take a sequence $y_k \in Y_{\mathrm{KKT}}(x_k, p_k)$; this sequence is bounded, hence it has an accumulation point, say $\bar{y}$, and then $\bar{y} \neq 0$. Passing to the limit with k in the KKT system we obtain that $\bar{y} \in Y_{\mathrm{KKT}}(x_0, p_0)$ which means that $Y_{\mathrm{KKT}}(x_0, p_0)$ is not a singleton. This contradicts the strict Mangasarian-Fromovitz condition. Hence, for any neighborhood $\mathcal{Y}$ of $y_0 = 0$, $Y_{\mathrm{KKT}}(x, p) \cap \mathcal{Y} \neq \emptyset$ when p is sufficiently close to p_0 and $x \in X(p)$ is sufficiently close to x_0. Thus, for a neighborhood U of (x_0, y_0) and for p close to p_0, $S_{\mathrm{KKT}}(p) \cap U \neq \emptyset$ also in this case.

Assume that the map S_{KKT} is not locally upper-Lipschitz at the point (p_0, x_0, y_0) in its graph. Then, from Corollary 2.4, the system (11) has a nonzero solution (x', y') and this solution can be taken as close to $(0, 0)$ as desired. With a slight abuse of the notation, suppose that $y' \in \mathbb{R}^m$ with $y'_i = 0$ for $i \in I_3$. If $x' = 0$, then $y' \neq 0$. Note that if $y'_i \neq 0$ for some $i \in I_2$, then $y'_i > 0$. Since $y_{0i} > 0$ for $i \in I_1$, and y' is close to zero, the vector $y_0 + y'$ is a Lagrange multiplier for x_0 and p_0. This contradicts the strict Mangasarian-Fromovitz condition. Hence $x' \neq 0$. But $x' \in D$. Multiplying the first equality in (11) by x', we obtain $\langle x', Ax' \rangle = 0$, a contradiction. This proves the theorem. □

Theorem 2.6 can be also derived by combining the equivalence between the strict Mangasarian-Fromovitz condition and the uniqueness of the Lagrange multiplier with Proposition 6.2 of Bonnans [2], where it is assumed that (x_0, y_0) is an isolated solution of (3) and x_0 is a local solution to (1) for $p = p_0$, and then it is shown that the local upper-Lipschitz property of S_{KKT} at (p_0, x_0, y_0) is equivalent to the second-order sufficient condition at (p_0, y_0, x_0). In a different setting, Pang [28] considered the KKT system for a variational inequality and proved (roughly speaking, see Theorem 5 in [28]) that under the strict Mangasarian-Fromovitz condition and a second-order necessary optimality condition, the map S_{KKT} is locally nonempty-valued and upper-Lipschitz at the point in its graph if and only if the second-order sufficient optimality condition holds. For results relating upper-Lipschitz properties of local minimizers to growth conditions for the objective function, see Klatte [15].

3. THE STATIONARY POINT MAP

One can get a characterization of the local upper-Lipschitz property at a point of the stationary point map X_{KKT} in terms of the proto-derivative of this map by combining results from Levy [22] and Levy and Rockafellar [24]. Namely, it was shown in King and Rockafellar [13], Proposition 2.1, and Levy [22], Proposition 4.1, that a map has the local upper-Lipschitz property at a point in its graph if and only if its graphical (contingent) derivative at that point has image $\{0\}$ at 0. In our case the graphical derivative is actually the proto-derivative of X_{KKT}, a formula for which is given in Theorems 3.1 and 3.2 of Levy and Rockafellar [24] (cf. also Theorem 5.1 in Levy [22]).

Theorem 3.1. *Let $x_0 \in X_{\mathrm{KKT}}(p_0)$, $p_0 = (v_0, u_0, w_0)$, be such that the Mangasarian-Fromovitz condition is fulfilled. Then the following condition is necessary and sufficient for the map X_{KKT} to be locally upper-Lipschitz at (p_0, x_0): there exists no vector $x' \neq 0$ which for some choice of*

$$y_0 \in \operatorname{argmax} \left\{ \langle x', \nabla^2_{xx} L(w_0, x_0, y) x' \rangle \;\middle|\; y \text{ with } (x_0, y) \in S_{\mathrm{KKT}}(p_0) \right\}$$

satisfies the KKT conditions for the subproblem having objective function $h_0(x') = \langle x', \nabla^2_{xx} L(w_0, x_0, y_0) x' \rangle$ and constraint system

$$\begin{cases} \langle \nabla_x g_0(w_0, x_0) - v_0, x' \rangle = 0, \\ \langle \nabla_x g_i(w_0, x_0), x' \rangle = 0 \text{ for } i \in [1, r], \\ \langle \nabla_x g_i(w_0, x_0), x' \rangle \leq 0 \text{ for } i \in [r+1, m] \text{ with } g_i(w_0, x_0) - u_{0i} = 0. \end{cases}$$

Proof. According to the cited Theorems 3.1 and 3.2 of Levy and Rockafellar [24] as specialized to this situation, the vectors x' that satisfy the KKT conditions for one of the subproblems in question form the image of 0 under the proto-derivative mapping associated with $X_{\rm KKT}$ at (p_0, x_0). (The first of the cited theorems establishes the proto-differentiability.) Applying Proposition 2.1 of King and Rockafellar [13], we see that the nonexistence of a vector $x' \neq 0$ in this set is equivalent to the property we wish to characterize. □

On the basis of our Lemma 2.5 and Theorem 3.1 we now are able to get the following.

Corollary 3.2. *Let x_0 be an isolated local minimizer of (1) for $p_0 = (v_0, u_0, w_0)$. Suppose that the Mangasarian-Fromovitz condition holds for (p_0, x_0), and let the condition in Theorem 3.1 hold as well. Then the solution map X of (1) is locally nonempty-valued and upper-Lipschitz at the point (p_0, x_0).*

Note that both the local optimality of x_0 and the condition in Theorem 3.1 are satisfied if the second-order sufficient condition holds at (p_0, x_0, y_0) for *every* choice of a Lagrange multiplier vector y_0. Under this stronger form of the second-order sufficient condition, the property of the solution map X obtained in Corollary 3.2 can be derived by combining Theorem 3.2 and Corollary 4.3 of Robinson [32]; for more recent results in this direction see [3], [16], [36] and [38].

Recall that a map Σ from $I\!R^m$ to the subsets of $I\!R^n$ with $(y_0, x_0) \in \operatorname{gph} \Sigma$ is *locally single-valued and (Lipschitz) continuous around* (y_0, x_0) if there exist neighborhoods U of x_0 and V of y_0 such that the map $y \mapsto \Sigma(y) \cap U$ is single-valued and (Lipschitz) continuous on V. Our next result is related to Theorem 7.2 in Kojima [18]. Kojima showed that, under the Mangasarian-Fromovitz condition and for C^2 perturbations of the functions in the problem, as long as the reference point is a local minimizer, the stationary point map is locally single-valued and continuous if and only if the strong second-order sufficient condition holds. Note that for the case when the perturbations are represented by parameters, the continuity of the stationary point map does not imply the strong second-order condition (consider the example of $\min_x \{x^4 + vx \mid x \in I\!R\}$, $v \in I\!R$).

The theorem below complements the "only if" part of Kojima's theorem in the following way: we use a narrower class of perturbations represented by parameters in canonical form and show that a stronger condition, namely of the stationary point map being locally single-valued and *Lipschitz* continuous with its values locally optimal solutions, implies both the Mangasarian-Fromovitz condition and the strong second-order condition for local optimality.

Theorem 3.3. *Suppose that $x_0 \in X_{\rm KKT}(p_0)$, $p_0 = (v_0, u_0, w_0)$ and assume* *that the stationary point map $X_{\rm KKT}$ is locally single-valued and Lipschitz continuous around (p_0, x_0), moreover with the property that for all $(p, x) \in$ $\operatorname{gph} X_{\rm KKT}, p = (v, u, w)$ in some neighborhood of (p_0, x_0), x is a locally optimal solution to the nonlinear programming problem (1) for p. Then the following conditions hold:*

(i) *The Mangasarian-Fromovitz condition holds at (p_0, x_0);*

(ii) *The strong second-order sufficient condition for local optimality holds at (p_0, x_0); that is, for every $y_0 \in Y_{\rm KKT}(x_0, p_0)$, if I_1 is the set of indices with positive y_{0i}, then*

$$\langle x', \nabla^2_{xx} L(w_0, x_0, y_0)x' \rangle > 0 \text{ for all } x' \neq 0 \tag{12}$$

in the subspace $M = \{x' \in I\!R^n \,|\, x' \perp \nabla_x g_i(w_0, x_0) \text{ for all } i \in I_1\}$.

In the proof of the theorem we use the following general result from [6]. Let Γ be a set-valued map from $I\!R^n$ to the subsets of $I\!R^m$ and let $x_0 \in \Gamma(y_0)$. The function $s : Y \to X$ is said to be a local selection of Γ around (y_0, x_0) if $x_0 = s(y_0)$ and there exists a neighborhood V of y_0 such that $s(y) \in \Gamma(y)$ for all $y \in V$.

Let us consider the maps Σ and Λ defined in (7) and (8) under the following two conditions:

(A) *f is differentiable with respect to x with Jacobian matrix $\nabla_x f(w, x)$ depending continuously on (w, x) in a neighborhood of (w_0, x_0);*

(B) *f is Lipschitz continuous in w uniformly in x around (w_0, x_0); that is, there exist neighborhoods U of x_0 and V of w_0 and a number $l > 0$ such that $\|f(w_1, x) - f(w_2, x)\| \leq l\|w_1 - w_2\|$ for all $x \in U$ and $w_1, w_2 \in V$.*

The following result, proved in Dontchev [6], Theorem 4.1 (see also [7], Theorem 2.4) shows that, similarly to the local upper-Lipschitz property (and also to the Aubin property, see [6], Theorem 2.4) the existence of a Lipschitz continuous local selection is robust under (non)linearization:

Theorem 3.4. *Consider the maps Σ and Λ defined in (7) and (8) respectively, let $x_0 \in \Sigma(p_0)$ for some $p_0 = (w_0, y_0) \in P$ and let the conditions (A) and (B) hold. Then the following are equivalent:*

(i) *Λ has a Lipschitz continuous local selection around (p_0, x_0);*

(ii) *Σ has a Lipschitz continuous local selection around (p_0, x_0).*

Proof of Theorem 3.3. The assumption that the stationary point map $X_{\rm KKT}$ is locally single-valued and Lipschitz continuous in x around (p_0, x_0) implies that the feasible map $p \mapsto C(p)$ has a Lipschitz continuous local selection around (p_0, x_0). Then, from Theorem 3.4 it follows that for every u near u_0 there exists a solution of the linearized system of constraints,

$$-u_i + g_i(w_0, x_0) + \nabla_x g_i(w_0, x_0)(x - x_0) \in \mathcal{K} \tag{13}$$

where $\mathcal{K}$, as in the introduction, denotes the convex and closed cone in $I\!R^m$ of vectors whose first r components are zeros and the remaining $m-r$ components are nonpositive numbers. The map $\mathcal{T} : I\!R^n \mapsto I\!R^m$, defined as

$$x \mapsto \mathcal{T}(x) := u_0 - g(w_0, x_0) - \nabla_x g(w_0, x_0)(x - x_0) + \mathcal{K},$$

has convex and closed graph and the condition (13) means that $0 \in I\!R^m$ is in the interior of the range of $\mathcal{T}$. Then we can apply the Robinson-Ursescu theorem, see e.g. [6], Theorem 2.2, obtaining that the constraint map C is Aubin continuous at (w_0, u_0, x_0); the latter is in turn equivalent to the Mangasarian-Fromovitz condition. Thus (i) is established. Note that in obtaining (i) the only condition we use is that $X_{\rm KKT}$ has a Lipschitz continuous local selection. Actually, it is sufficient to assume that $X_{\rm KKT}$ has a set-valued local selection which is Aubin continuous.

Let us prove (ii). It is known that, under the Mangasarian-Fromovitz condition, the set $Y_{\rm KKT}(x_0, p_0)$ is a nonempty polyhedron, and moreover, if y_0 is any extreme point of $Y_{\rm KKT}(x_0, p_0)$ and if I_i for $i = 1, 2, 3$, are the sets of indices associated with (p_0, x_0, y_0), then the gradients $\nabla_x g(x_0, w_0)$, $i \in I_1$, must be linearly independent. Consider the variational system (3) with the following value of the parameter vector $p = (v, u, w)$ denoted by p_ε: $v = v_0$, $u_i = u_{0i}$ for $i \in I_1 \cup I_3$, $u_i = u_{0i} + \varepsilon$ for $i \in I_2$, and $w = w_0$, where ε is a positive number. For a sufficiently small $\varepsilon > 0$ there exist neighborhoods V of p_ε and U of x_0 such that for every $p \in V$ the set $X_{\rm KKT}(p) \cap U$ is a singleton and the map $p \mapsto X_{\rm KKT}(p) \cap U$ is Lipschitz continuous in V. Choose U and V smaller, if necessary, so if $p \in V$ and $x \in U$, one has $-u_i + g_i(w, x) < 0$ for $i \in I_2 \cup I_3$; moreover the vectors $\nabla_x g_i(w, x)$, $i \in I_1$, are linearly independent, and if y_i, $i \in I_1$, satisfy

$$v + \nabla_x g_0(w, x) + \sum_{i \in I_1} y_i \nabla_x g_i(w, x) = 0, \tag{14}$$

then $y_i > 0$ for $i \in I_1$ (the latter being true by the linear independence of the vectors $\nabla_x g_i(w, x)$ for $i \in I_1$ and the fact that $y_{0i} > 0$ for $i \in I_1$).

If $p \in V$ and $x \in X_{\rm KKT}(p) \cap U$, then every associated multiplier vector y for (3) must satisfy $y_i = 0$ for $i \in I_2 \cup I_3$; furthermore, y_i for $i \in I_1$ must satisfy (14), so that $y_i > 0$ for $i \in I_1$. Denoting by W a neighborhood of y_0 such that $y_i > 0$ for $i \in I_1$ whenever $y \in W$, we obtain that $(x, y) \in U \times W$ is a solution of the variational system:

$$\begin{aligned} &v + \nabla_x g_0(w, x) + \textstyle\sum_{i \in I_1} y_i \nabla_x g_i(w, x) = 0, \\ &-u_i + g_i(w, x) = 0, \ \text{for } i \in I_1, \\ &-u_i + g_i(w, x) \le 0, \ y_i = 0 \ \text{for } i \in I_2 \cup I_3. \end{aligned} \tag{15}$$

Further, if $(x, y) \in U \times W$ is a solution of (15) for some $p \in V$, then $x \in X_{\rm KKT}(p) \cap U$ and, because of the linear independence of $\nabla_x g_i(w, x)$

for $i \in I_1$, y is uniquely defined and the function $p \mapsto y(p)$ is Lipschitz continuous in V. Thus, the solution map of (15) is locally single-valued and Lipschitz continuous around $(p_\varepsilon, x_0, y_0)$. Observe that (x_0, y_0) is a locally unique solution of (15) and x_0 is a locally optimal solution of (1), both for $p = p_\varepsilon$. In particular, the map $p \mapsto (x(p), y(p))$ is locally nonempty-valued and upper-Lipschitz at the point $(p_\varepsilon, x_0, y_0)$. From Theorem 2.6 it follows that the second-order sufficient condition holds at $(p_\varepsilon, x_0, y_0)$. Noting that the set I_2 associated with $(p_\varepsilon, x_0, y_0)$ is empty, and $\nabla^2_{xx} L$ does not depend on ε and is affine in y, we see that (12) holds for every $y_0 \in Y_{\text{KKT}}(x_0, p_0)$. □

Observe that, from the above proof, Theorem 3.3 remains valid for a smaller class of perturbations $p = (v, u, w)$ where $w = w_0$ is kept constant.

As indicated by a counterexample of Robinson [32], the statement converse to Theorem 3.3 is false, in general. It has been noted recently, see Liu [25] and Ralph and Dempe [29], that, under the constant rank condition for the set of constraints at the reference point, the converse statement holds; that is the combination of the Mangasarian-Fromovitz condition and the strong second-order sufficient condition implies that both the solution map and the stationary point map are locally single-valued and Lipschitz continuous (and B-differentiable). Recall that the set of constraints (2) satisfies the *constant rank condition* if there exists a neighborhood $\mathcal{W}$ of (w_0, x_0) such that for every $I \subset I_1 \cup I_2$, $\text{rank}\{\nabla_x g_i(w, x) \mid i \in I\}$ is constant for every $(w, x) \in \mathcal{W}$. By combining the above mentioned result with Theorem 3.3 we obtain the following characterization of the Lipschitzian stability of the stationary point map:

Corollary 3.5. *Suppose that $x_0 \in X_{\text{KKT}}(p_0)$ and the constant rank condition holds. Then the following are equivalent:*

(i) *The stationary point map X_{KKT} is locally single-valued and Lipschitz continuous around (p_0, x_0), moreover with the property that for all $(p, x) \in \text{gph}\, X_{\text{KKT}}$ in some neighborhood of (p_0, x_0), x is a locally optimal solution to the nonlinear programming problem (1) for p;*

(ii) *The Mangasarian-Fromovitz condition and the strong second-order sufficient condition for local optimality (12) hold for (p_0, x_0).*

In our previous paper [8] we proved that if the solution map, say Σ, of a variational inequality over a convex polyhedral set has the Aubin property at $(y_0, x_0) \in \text{gph}\, \Sigma$, it must be locally single-valued and Lipschitz continuous around (y_0, x_0) (for more recent related results see [11] and [23]). In particular, if the KKT map S_{KKT} has the Aubin property, it is locally single-valued and Lipschitz continuous. Here we present an analogue of this result for the stationary point map X_{KKT}, but under the additional assumptions that the values of this map are locally optimal solutions.

Theorem 3.6. *Let $(p_0, x_0) \in \operatorname{gph} X_{\mathrm{KKT}}$, $p_0 = (v_0, u_0, w_0)$, and suppose that if $(p, x) \in \operatorname{gph} X_{\mathrm{KKT}}$ in some neighborhood of (p_0, x_0) then x is a locally optimal solution to (1) for p, that is $(p, x) \in \operatorname{gph} X$. Then the following are equivalent:*

(i) *The Mangasarian-Fromovitz condition holds for (p_0, x_0) and the map X_{KKT} is Aubin continuous at (p_0, x_0);*

(ii) *The map X_{KKT} is locally single-valued and Lipschitz continuous around (p_0, x_0).*

Proof. The implication (ii) $\Rightarrow$ (i) follows from Theorem 3.3 (note that the local optimality is not needed in this direction). If we prove that (i) implies the strong second-order condition for local optimality (12), then from Theorem 7.2 in Kojima [18] it follows that the map X_{KKT} is locally single-valued and continuous, hence Lipschitz continuous, and we obtain (ii).

Invoking the argument given in the proof of Theorem 3.3, we choose an extreme point y_0 of $Y_{\mathrm{KKT}}(x_0, p_0)$ and consider the KKT system (3) for $p = (v, u, w)$ in a neighborhood of the value p_ε whose components are: $v = v_0$, $u_i = u_{0i}$ for $i \in I_1 \cup I_3$, $u_i = u_{0i} + \varepsilon$ for $i \in I_2$, and $w = w_0$. For a sufficiently small $\varepsilon > 0$ the map X_{KKT} is Aubin continuous at (p_ε, x_0), by the very definition of the Aubin continuity at (p_0, x_0). Choose neighborhoods V of p_ε and U of x_0 such that for every $p \in V$ and $x \in U$, one has $-u_i + g_i(w, x) < 0$ for $i \in I_2 \cup I_3$; moreover the vectors $\nabla_x g_i(w, x)$, $i \in I_1$, are linearly independent, and if y_i, $i \in I_1$, satisfy (14), then $y_i > 0$ for $i \in I_1$. From the definition of the Aubin continuity one can find neighborhoods $U' \subset U$ of x_0 and $V' \subset V$ of p_ε such that if $p', p'' \in V'$ and $x' \in X_{\mathrm{KKT}}(p') \cap U'$, then there exists $x'' \in X_{\mathrm{KKT}}(p'')$ with $\|x'' - x'\| \le M\|p' - p''\|$; moreover, U' and V' can be chosen so small that $x'' \in U$ for every choice of $p', p'' \in V'$ and $x' \in U'$. Denoting by W a neighborhood of y_0 such that $y_{0i} > 0$ for $i \in I_1$, let $(x', y') \in U' \times W$ be a KKT point for (1) for p', that is, (x', y') solves (3) for p'. Then (x', y') must satisfy the system (15) for p' and the Lagrange multiplier y' is unique, from the linear independence of $\nabla_x g_i(w', x'), i \in I_1$. Let $x'' \in X_{\mathrm{KKT}}(p'')$ be such that $\|x'' - x'\| \le M\|p' - p''\|$, and let y'' be an associate Lagrange multiplier. Then, from the choice of the neighborhoods U' and V', y'' must satisfy (15) and then $y''_i > 0$ for $i \in I_1$ and $y''_i = 0$ for $i \in I_2 \cup I_3$. Further, since y', y'' satisfy (14) for (p', x') and $(p,'' x'')$, respectively, and because of the linear independence of $\nabla_x g_i(w, x), i \in I_1$ for $p \in V', x \in U'$, there exists a constant $c > 0$ such that

$$\|y'' - y'\| \le c(\|x'' - x'\| + \|p'' - p'\|) \le c(M + 1)\|p'' - p'\|.$$

This means that the KKT map S_{KKT} of the problem (1) is Aubin continuous at $(p_\varepsilon, x_0, y_0)$.

Applying the characterization of the Aubin property from Theorem 5 in [8] with $I'_1 = I_1, I'_3 = I_2 \cup I_3$, see Theorem 4.1 in the following section, we obtain that

$$\langle x', \nabla^2_{xx} L(w_0, x_0, y_0) x' \rangle \neq 0 \text{ for all } x' \neq 0, x \in M. \tag{16}$$

Since every $y_0 \in Y_{\rm KKT}(x_0, p_0)$ can be represented as a convex combination of the extreme points of $Y_{\rm KKT}(x_0, p_0)$ and $\nabla^2_{xx} L$ is affine in y, we get (16) for every choice of the Lagrange multiplier y_0. Taking into account the second-order necessary conditions for local optimality of x_0 for (1) with p_ε, we complete the proof. □

4. STRONG REGULARITY

In Robinson's terminology [30], the KKT system (3) is *strongly regular* if the map $L_{\rm KKT}$ defined by the linearization (4) of the KKT system (3) is locally single-valued and Lipschitz continuous around (q_0, x_0), $q_0 = (v_0, u_0)$. In [30] it is proved that if the KKT system (3) is strongly regular, then the KKT map $S_{\rm KKT}$ is locally single-valued and Lipschitz continuous; moreover, for the problem (1), the strong second-order sufficient condition for local optimality together with the linear independence of the gradients of the active constraints implies the strong regularity.

From the result in [8] mentioned in the preceding section it follows that the map $S_{\rm KKT}$ has the Aubin property if and only if this map is locally single-valued and Lipschitz continuous. In [8] we obtained a characterization of the strong regularity in a form of a "critical face condition." The precise result is as follows.

Theorem 4.1 ([8], Theorem 5)**.** *The KKT system (3) is strongly regular for* (p_0, x_0, y_0), $p_0 = (v_0, u_0, w_0)$ *if and only if the following two requirements are fulfilled:*

(a) *The vectors* $\nabla_x g_i(w_0, x_0)$ *for* $i \in I_1 \cup I_2$ *are linearly independent;*

(b) *For each partition of* $\{1, 2, \ldots, m\}$ *into index sets* I_1', I_2', I_3' *with* $I_1 \subset I_1' \subset I_1 \cup I_2$ *and* $I_3 \subset I_3' \subset I_3 \cup I_2$, *the cone* $K(I_1', I_2') \subset I\!R^n$ *consisting of all the vectors* x' *satisfying*

$$\langle \nabla_x g_i(w_0, x_0), x' \rangle \begin{cases} = 0 & \text{for } i \in I_1', \\ \leq 0 & \text{for } i \in I_2', \end{cases}$$

should be such that

$$x' \in K(I_1', I_2'),\ \nabla^2_{xx} L(w_0, x_0, y_0) x' \in K(I_1', I_2')^* \quad \Longrightarrow \quad x' = 0.$$

Here K^* denotes the polar to K. For other characterizations of strong regularity and a detailed discussion of related results, see Klatte and Tammer [17] and Kummer [19]. In particular, Kummer's condition is based on a general implicit-function theorem for nonsmooth functions, while we use the equivalence of the Aubin property and the strong regularity established in [8] and then apply Mordukhovich's characterization of the Aubin property. It is not clear to us how one could derive the equivalence between these various conditions directly.

Relying on the above result, in [8], Theorem 6 we showed that, under canonical perturbations, the combination of the strong second-order sufficient condition for local optimality with the linear independence of the gradients of the active constraints is actually equivalent to the requirement the map S_{KKT} be locally single-valued and Lipschitz continuous with the x-component being a locally optimal solution. Below, we present a refinement of this theorem with a short proof.

Theorem 4.2. *The following are equivalent:*

(i) *The map* $p \mapsto S(p) = \{(x, y) \in S_{\text{KKT}}(p) \mid x \in X(p)\}$ *is locally single-valued and Lipschitz continuous around* $(p_0, x_0, y_0), p_0 = (u_0, v_0, w_0)$;

(ii) *The map* S_{KKT} *is locally single-valued and Lipschitz continuous around* (p_0, x_0, y_0), *moreover with the property that for all* $(p, x, y) \in \operatorname{gph} S_{KKT}$ *in some neighborhood of* $(u_0, v_0, w_0, x_0, y_0)$, x *is a locally optimal solution to the nonlinear programming problem (1) for* $p = (u, v, w)$;

(iii) x_0 *is an isolated local minimizer of (1) and the associated KKT system (3) is strongly regular around* (p_0, x_0, y_0);

(iv) *The constraint gradients* $\nabla_x g_i(w_0, x_0)$ *for* $i \in I_1 \cup I_2$ *are linearly independent and the strong second-order sufficient condition for local optimality holds for* (p_0, x_0, y_0): *one has*

$$\langle x', \nabla^2_{xx} L(w_0, x_0, y_0)x' \rangle > 0 \text{ for all } x' \neq 0 \text{ in the subspace}$$

$$M = \{x' \mid x' \perp \nabla_x g_i(w_0, x_0) \text{ for all } i \in I_1\}.$$

Proof. Let (i) hold. Since $S(p) \subset S_{\text{KKT}}(p)$, the KKT map S_{KKT} has a Lipschitz continuous local selection around (p_0, x_0, y_0). Consider the Kojima map associated with the KKT system (3):

$$F(x, y) = \begin{pmatrix} \nabla_x g_0(w_0, x) + \sum_{i=1}^r y_i \nabla_x g_i(w_0, x) + \sum_{i=r+1}^m y_i^+ \nabla_x g_i(w_0, x) \\ -g_1(w_0, x) \\ . \\ . \\ -g_r(w_0, x) \\ -g_{r+1}(w_0, x) + y_{r+1}^- \\ . \\ . \\ -g_m(w_0, x) + y_m^- \end{pmatrix},$$

where $y^+ = \max\{0, y\}$ and $y^- = \min\{0, y\}$. Let $G(x, y) = F(x, y+g(w_0, x))$, where $g = (g_1, \cdots, g_m)$. Then every $(x, y) \in S_{\text{KKT}}(p)$ for $p = (u, v, w_0)$ is a solution of the equation

$$G(x, y) = \left(\begin{array}{c} -v \\ -u \end{array} \right).$$

The continuous map $G : I\!\!R^{n+m} \mapsto I\!\!R^{n+m}$ has the property that its inverse G^{-1} has a Lipschitz continuous local selection around (p_0, x_0, y_0). Then the map G^{-1} must be locally single-valued, see Lemma 1 in Kummer [20]. Hence S_{KKT} is locally single-valued and Lipschitz continuous and thus (ii) is established.

The equivalence (ii) $\Leftrightarrow$ (iii) follows from Proposition 2 in [8]. The implication (iii) $\Rightarrow$ (iv) is a consequence of Theorems 3.3 and 4.1. Finally, (iv) $\Rightarrow$ (i) is proved in Robinson [30]; for other proofs see [9], p. 370 and [17]. □

Under the linear independence of the gradients of the active constraints, the equivalence of (iii) and (iv) follows from a combination of earlier results of Kojima [18] and Jongen et al. [12]; alternative proofs of this equivalence have been furnished recently by Bonnans and Sulem [5] and Pang [28].

At the end, as a consequence of Theorem 4.2 we present a characterization of the continuous differentiability of the pair solution-Lagrange multiplier which complements a basic result due to Fiacco [10]. Using the implicit function theorem, Fiacco showed that the combination of the linear independence of the gradient of the active constraints, the second-order sufficient condition and the strict complementary slackness implies that the map S is locally single-valued and continuously differentiable around the reference point (p_0, x_0, y_0). The following result shows that, for canonical perturbations, these conditions are also necessary for the latter property. Recall that strict complementary slackness condition holds if there are no zero Lagrange multipliers associated with active constraints at the reference point; that is, $I_2 = \emptyset$.

Corollary 4.3. *The following are equivalent:*

(i) *The map* $p \mapsto S(p) = \{(x, y) \in S_{\text{KKT}}(p) \mid x \in X(p)\}$ *is locally single-valued and continuously differentiable around* (p_0, x_0, y_0), $p_0 = (u_0, v_0, w_0)$*;*

(ii) *The strict complementary slackness (i.e.,* $I_2 = \emptyset$*), the linear independence of the gradients of the active constraint, and the second-order sufficient condition for local optimality hold for* (p_0, x_0, y_0).

Proof. All we need to prove is that (i) implies the strict complementarity; the rest follows from Theorem 4.2 and Fiacco's theorem. On the contrary, assume that $I_2 \neq \emptyset$ and (i) holds. Let $i \in I_2$ and consider the problem (1) with the following values of the parameter p denoted p_ε: $w = w_0$, $v = v_0$, $u_i = u_{0i} + \varepsilon$ and $u_j = u_{0j}$ for $j \neq i$, where ε is a real parameter from a neighborhood of zero. As already noted in the previous proof, since $y_{0i} = 0$, for every nonnegative and sufficiently small ε and for some neighborhood W of (x_0, y_0) the only element in $S(p_\varepsilon) \cap W$ is (x_0, y_0). Then the derivative of x must satisfy

$$\frac{d}{d\varepsilon} x(p_\varepsilon)|_{\varepsilon=0} = 0. \tag{17}$$

On the other hand, for $\varepsilon < 0$ the corresponding solution $x(p_\varepsilon)$ is feasible, that is,

$$-u_{0i} - \varepsilon + g_i(w_0, x(p_\varepsilon)) \leq 0.$$

Combining this inequality with the assumed equality $-u_{0i} + g_i(w_0, x_0) = 0$, we obtain

$$-1 + \frac{1}{\varepsilon}[g(w_0, x(p_\varepsilon)) - g(w_0, x_0)] \geq 0.$$

Passing to zero with ε and using (17) results in $-1 \geq 0$, a contradiction. This completes the proof. □

References

[1] J.-P. Aubin, Lipschitz behavior of solutions to convex minimization problems, *Math. Oper. Res.* **9**, 1984, 87–111.

[2] J. F. Bonnans, Local analysis of Newton-type methods for variational inequalities and nonlinear programming, *Appl. Math. and Optim.* **29** (1994), 161–186.

[3] J. F. Bonnans, A. D. Ioffe, Quadratic growth and stability in convex programming problems with multiple solutions, *J. Convex Analysis* **2** (1995) 41-57.

[4] J. F. Bonnans, A. Shapiro, Optimization problems with perturbations, a guided tour, INRIA preprint 2872, Théme 4, INRIA Rocquencourt, April 1996.

[5] J. F. Bonnans, A. Sulem, Pseudopower expansion of solutions of generalized equations and constrained optimization problems, *Math. Programming* **70** (1995), 123–148.

[6] A. L. Dontchev, Characterizations of Lipschitz stability in optimization; in *Recent Developments in Well-Posed Variational Problems*, R. Lucchetti and J. Revalski (eds.), Kluwer, 1995, 95–116.

[7] A. L. Dontchev, Implicit function theorems for generalized equations, *Math. Programming* **70** (1996) 91-106.

[8] A. L. Dontchev, R. T. Rockafellar, Characterizations of strong regularity for variational inequalities over polyhedral convex sets, *SIAM J. Optim.*, accepted 1995.

[9] A. L. Dontchev, T. Zolezzi, Well-Posed Optimization Problems, Lecture Notes in Mathematics, vol. **1543**, Springer-Verlag, 1993.

[10] A. V. Fiacco, Sensitivity analysis for nonlinear programming using penalty methods, *Math. Programming* **10** (1976), 287–311

[11] M. Seetharama Gowda, R. Sznajder, On the pseudo-Lipschitzian behavior of the inverse of a piecewise affine function, In *Proc. International Conference on Complementarity Problems*, Baltimore, Nov. 1995, to appear.

[12] H. Th. Jongen, T. Möbert, J. Rückmann, K. Tammer, Implicit functions and sensitivity of stationary points, *Linear Algebra and Appl.* **95** (1987), 97–109.

[13] A. J. King, R. T. Rockafellar, Sensitivity analysis for nonsmooth generalized equations, *Math. Programming* **55** (1992), 341–364.

[14] D. Klatte, On the stability of local and global optimal solutions in parametric problems of nonlinear programming, Part I: Basic results, *Seminarbericht 75 der Sektion Mathematik der Humboldt-Universität zu Berlin,* Berlin, 1985, 1–21.

[15] D. Klatte, On quantitative stability for non-isolated minima, *Control Cybernet.* **23** (1994), 183–200.

[16] D. Klatte, On quantitative stability for $C^{1,1}$ programs; in *Proceeding of the French-German Conference on Optimization*, R. Durier and C. Michelot (eds.), Lecture Notes in Economics and Mathematical Systems, vol. **429**, Springer-Verlag, 1995, 214–230.

[17] D. Klatte, K. Tammer, Strong stability of stationary solutions and Karush-Kuhn-Tucker points in nonlinear optimization, *Annals of Oper. Research* **27** (1990), 285–308.

[18] M. Kojima, Strongly stable stationary solutions in nonlinear programming; in *Analysis and computation of fixed points*, S. M. Robinson (ed.), Academic Press, New York, 1980, 93–138.

[19] B. Kummer, Lipschitzian inverse functions, directional derivatives and applications in $C^{1,1}$ optimization, *J. Optim. Theory Appl.* **70** (1991), 561–582.

[20] B. Kummer, Lipschitzian and pseudo-Lipschitzian inverse functions and applications to nonlinear optimization, preprint, Dept. of Mathematics, Humboldt University, Berlin 1996.

[21] J. Kyparisis, On uniqueness of Kuhn-Tucker multipliers in nonlinear programming, *Math. Programming* **32** (1985), 242–246.

[22] A. B. Levy, Implicit multifunction theorems for the sensitivity analysis of variational conditions, *Math. Programming*, **74** (1996) 333–350.

[23] A. B. Levy, R. A. Poliquin, Characterizations of the local single-valuedness of multifunctions, preprint, Dept. of Mathematics, Bowdoin College, 1996.

[24] A. B. Levy, R. T. Rockafellar, Sensitivity of solutions in nonlinear programming problems with nonunique multipliers, in *Recent Advances in Nonsmooth Optimization*, D. Z. Du, L. Qi and R. S. Womersley (eds.), World Scientific Press, 1995, 215–223.

[25] Jiming Liu, Sensitivity analysis in nonlinear programs and variational inequalities via continuous selections, *SIAM J. Control Optim.* **33** (1995) 1040-1060.

[26] B. Mordukhovich, Lipschitzian stability of constraint systems and generalized equations, *Nonlinear analysis*, **22** (1994) 173–206.

[27] J.-S. Pang, Convergence of splitting and Newton methods for complementarity problems: an application of some sensitivity results, *Math. Programming* **58** (1993), 149–160.

[28] J.-S. Pang, Necessary and sufficient conditions for solution stability of parametric nonsmooth equations, in *Recent Advances in Nonsmooth Optimization*, D. Z. Du, L. Qi and R. S. Womersley (eds.), World Scientific Press, 1995, 261–288.

[29] D. Ralph, S. Dempe, Directional derivatives of the solution of a parametric nonlinear program, *Math. Programming* **70** (1995) 159–172.

[30] S. M. Robinson, Strongly regular generalized equations, *Math. Oper. Res.* **5** (1980), 43–62.

[31] S. M. Robinson, Some continuity properties of polyhedral multifunctions, *Math. Programming Study* **14** (1981), 206–214.

[32] S. M. Robinson, Generalized equations and their solutions, Part II: Applications to nonlinear programming, *Math. Programming Study* **19** (1982), 200–221.

[33] S. M. Robinson, Local epi-continuity and local optimization, *Mathematical Programming* **37** (1987), 208–223.

[34] R. T. Rockafellar, Proto-differentiability of set-valued mappings and its applications in optimization, *Ann. Inst. H. Poincaré: Analyse Non Linéaire* **6** (1989), suppl., 449–482.

[35] S. Scholtes, Introduction to piecewise differentiable equations, preprint 53/1994, Institut für Statistik und Mathematische Wirtschaftstheorie, Universität Karlsruhe, May 1994.

[36] A. Shapiro, Sensitivity analysis of nonlinear programs and differentiability properties of metric projections, *SIAM J. Control Optim.* **26** (1988), 628–645.

[37] G. Still, M. Streng, Optimality conditions in smooth nonlinear programming, *J. Optim. Theory Appl.* **90** (1996) 483–515.

[38] D. E. Ward, Characterizations of strict local minima and necessary conditions for weak sharp minima, *J. Optim. Theory Appl.* **80** (1994) 551-571.