Sensitivity Analysis of Aggregated Variational Inequality Problems, with Application to Traffic Equilibria

Michael Patriksson • R. Tyrrell Rockafellar
Department of Mathematics, Chalmers University of Technology, SE-412 96 Gothenburg, Sweden
Department of Mathematics, University of Washington, Seattle, Washington 98195-4350
mipat@math.chalmers.se • rtr@math.washington.edu

Some instances of variational inequality models over polyhedral sets can be stated in a disaggregated or aggregated formulation related by an affine variable transformation. For such problems, we establish that sensitivity analysis results under parameterizations rely neither on the strict monotonicity properties of the problem in terms of the disaggregated variables, nor on any particular choice of their values at the solution. We show how to utilize the affine transformation to devise computational tools for calculating sensitivity results and apply them to the sensitivity analysis of elastic demand traffic equilibrium problems. The results reached show that sensitivity results do not rely on the choice of any particular route or commodity flow solution. Further, the sensitivity analysis, including the calculation of the gradient of the equilibrium link flow if it exists, can be performed by means of solving linearized traffic equilibrium problems.

Introduction

The paper concerns the sensitivity analysis of variational problems of the form

$$-f(\rho, x) \in N_C(x), \tag{1}$$

where $\rho \in \Re^d$ is the parameter, $x \in \Re^n$ is the solution, $f\colon \Re^d \times \Re^n \mapsto \Re^n$ is a smooth function, $C \subseteq \Re^n$ is a nonempty *polyhedral* set, and N_C is the normal cone mapping to C; that is,

$$N_C(x) = \begin{cases} \{z \in \Re^n \mid z^{\mathrm{T}}(x-y) \le 0, \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

We focus our attention on the generalized differentiation of the solution mapping

$$S\colon \rho \mapsto S(\rho) = \big\{ x \mid -f(\rho, x) \in N_C(x) \big\}, \tag{2}$$

at a pair (ρ^*, x^*) with $x^* \in S(\rho^*)$. We further take a special interest in cases where problem (1) is generated by problems in a higher-dimensional setting, related by a linear transformation, as follows.

Let $D \in \Re^m$ be a nonempty polyhedron, $u \in \Re^m$ the solution, $\varphi\colon \Re^d \times \Re^m \mapsto \Re^m$ be smooth, and $A\colon \Re^m \mapsto \Re^n$ be a linear mapping. Consider the problem

$$-\varphi(\rho, u) \in N_D(u), \qquad u \in \Re^m. \tag{3}$$

If the two problems (1) and (3) are related by

$$C = A(D), \qquad x = Au, \quad \text{and}$$
$$A^{\mathrm{T}} f(\rho, A(\cdot)) = \varphi(\rho, \cdot), \tag{4}$$

we then say that (1) (respectively, (3)) is an *aggregated* (respectively, *disaggregated*) representation of the same problem. Note that the linearity of the mapping A implies that C is polyhedral if D is, and vice versa (Rockafellar and Wets 1998, Proposition 3.55). Further, in cases where $\varphi(\rho, \cdot)$ is monotone, this property is inherited by $A^{\mathrm{T}} f(\rho, A(\cdot))$ (Rockafellar and Wets 1998, Exercise 12.4), and, vice versa, if $f(\rho, \cdot)$ is monotone on C, then $\varphi(\rho, \cdot)$ is monotone on D.

Transportation Science © 2003 INFORMS
Vol. 37, No. 1, February 2003 pp. 56–68

0041-1655/03/3701/0056$05.00
1526-5447 electronic ISSN

The remainder of the paper is organized thus: In the next section we briefly outline the parameterization made, and state a basic sensitivity result for the case of problem (1). In §2, we then utilize the affine transformation in setting up rules for the central components in the sensitivity analysis calculations. A key factor in enabling these calculus rules is that an entity in the aggregated space (such as an element in the normal or tangent cones to C) does not depend on the selection of the element in the corresponding entity in the disaggregated space (such as in the corresponding cones to D).

The possibility to choose either a disaggregated or an aggregated representation is particularly pronounced in applications of variational problems associated with games and transportation, where the vector u may represent a disaggregated representation of the decision makers, or network flows in terms of individual commodities. The aggregated version of the problem then represents the decisions, or commodities, in some aggregated fashion which allows problem (3) to be solved through the solution of (1). Section 3 presents one such important application, the (elastic demand) *traffic equilibrium* problem, in which the mappings φ and f measure the cost of travel at different levels of aggregation, such as on routes or individual links, and their parameterizations may be associated with decisions on the part of society to change the level of service of the traffic infrastructure in order to influence the travellers' decisions on when, by which mode, and on which route, trips will be made. For these problems, the linear transformation described in (4) is a sum, and the corresponding special cases of the sensitivity results of §1 are given in the form of corollaries.

In §4, we then provide the corresponding results for the sensitivity analysis of traffic equilibria and establish that the calculations correspond to strictly monotone and affine variational inequality problems that can be solved as special network equilibrium problems. We also give an illustrative example, and compare the results obtained with some previous sensitivity analysis techniques for traffic equilibrium problems. The paper concludes by mentioning some further research work that is currently being conducted.

1. Parameterization and Sensitivity Analysis

Throughout this paper we will assume, without further reference, that the parameterization is done in such a way that at any (ρ^*, x^*) with $x^* \in S(\rho^*)$, the partial Jacobian matrix $\nabla_\rho f(\rho^*, x^*)$ for f with respect to ρ at (ρ^*, x^*) has full rank; in other words,

$$\operatorname{rank} \nabla_\rho f(\rho^*, x^*) = n, \qquad \nabla_\rho f(\rho^*, x^*) \in \Re^{n\times d}. \tag{5}$$

(In Dontchev and Rockafellar 2002, Definition 1.1, Property (5) is termed *ample*.) We note that this property can always be enforced, if necessary, through the addition of an additive vector of canonical (dummy) parameters to f. The paper by Dontchev and Rockafellar (2002) contains a full theory of Lipschitzian properties of the solution mapping $\rho \mapsto S(\rho)$ defined in (2), as well as of the graphical geometry associated with its generalized differentiation. For its present use, a small subset of the results obtained therein will suffice. We first recall the concept of protoderivative: the mapping S is said to be *protodifferentiable at ρ^* for x^** when $x^* \in S(\rho^*)$ and the difference quotient mappings

$$\Delta_\tau S(\rho^* \mid x^*)\colon \rho' \mapsto \tau^{-1}[S(\rho^* + \tau\rho') - x^*], \qquad \tau > 0,$$

converge graphically as $\tau \downarrow 0$; in other words, there is a mapping $D\colon \Re^d \mapsto \Re^n$ such that graph $\Delta_\tau S(\rho^* \mid x^*)$ converges to graph D as $\tau \downarrow 0$. (For more details on protoderivatives, consult Rockafellar and Wets 1998, Chapter 8.) We also say that a mapping is *piecewise polyhedral* if its graph is the union of a collection of finitely many polyhedral sets. Further, for a vector $z \in \Re^n$, $z^\perp = \{y \in \Re^n \mid z^\mathrm{T} y = 0\}$.

The following two results were established in Dontchev and Rockafellar (2002), as Theorems 7.1 and 7.4.

Theorem 1 (Protoderivatives). *Assume that Condition* (5) *holds. Then S is both graphically Lipschitzian of dimension d and protodifferentiable at ρ^* for x^*, with its protoderivatives given by an auxiliary variational inequality, namely*

$$DS(\rho^* \mid x^*)(\rho') = \{x' \mid r(\rho', x') + N_K(x') \ni 0\},$$

where

$$r(\rho', x') = \nabla_\rho f(\rho^*, x^*)\rho' + \nabla_x f(\rho^*, x^*)x' \quad and$$
$$K = T_C(x^*) \cap f(\rho^*, x^*)^\perp.$$

Furthermore, the mapping $DS(\rho^* \mid x^*)$ is itself graphically Lipschitzian of dimension d everywhere, and is piecewise polyhedral.

The set K in the definition of the derivative $DS(\rho^* \mid x^*)(\rho')$ is known as the *critical cone* associated with the variational inequality (1) for $\rho = \rho^*$ and $x = x^*$.

The main interest in the above result is in the special case where S is locally single-valued and Lipschitz continuous, when protodifferentiability turns into something stronger, extending the strong regularity results of Robinson (1980). If S happens to be single-valued and Lipschitz continuous around ρ^* with x^* being the unique element of $S(\rho^*)$, protodifferentiability implies that the mapping $DS(\rho^* \mid x^*)$ is likewise single-valued and Lipschitz continuous and that

$$S(\rho) = S(\rho^*) + DS(\rho^* \mid x^*)(\rho - \rho^*) + o(|\rho - \rho^*|),$$

where $o(\cdot)$ is such that $o(t)/t$ converges to zero as $t \to 0$. This property is known as *semidifferentiability* (see further Rockafellar and Wets 1998, §7.D). (This is further ordinary differentiability precisely when $DS(\rho^* \mid x^*)$, in addition, is linear.)

Theorem 2 (Semidifferentiability). *Assume that Condition* (5) *holds. Suppose further that S is convex valued* (*as it is when $f(\rho^*, \cdot)$ happens to be monotone*) *around ρ^*, in the sense that $S(\rho)$ is a convex set for all ρ in some neighborhood of ρ^*. Then the following properties are equivalent*:

(a) *S is single-valued and Lipschitz continuous on some neighborhood of ρ^*;*

(b) *$DS(\rho^* \mid x^*)$ is single-valued on some neighborhood of 0^d (hence everywhere).*

Moreover, then S is semidifferentiable at ρ^* for x^*, and $DS(\rho^* \mid x^*)$ is not only Lipschitz continuous and positively homogeneous, but also piecewise linear.

Suppose now that the situation is as described in the Introduction, that is, that problem (1) is generated through a linear transformation from the disaggregated problem (3). We next establish decomposition formulas providing the sensitivity calculations for (1) to be performed with the aid of data from a solution to problem (3).

2. Sensitivity Analysis for Aggregated Variational Problems

The existence of a sensitivity analysis amounts, by the above theorem, to the local existence of a single-valued and Lipschitz continuous mapping S. The fact that problem (1) also has a disaggregated representation and could also be described in terms of the variables u does not alter the analysis in any way; the only point that needs to be analyzed is whether the calculation of $DS(\rho^* \mid x^*)$ can be performed in some disaggregated fashion in terms in the variables u, thereby perhaps facilitating a more efficient sensitivity analysis. This question will be answered through the development of a disaggregated formulation of $DS(\rho^* \mid x^*)$. To this end, we first derive a disaggregated representation of the critical cone K. The following result is a special case of Theorem 6.43 in Rockafellar and Wets (1998), but deserves a special proof for polyhedral sets. (A related result was also previously given by Outrata 1997, Lemma 1.1.)

Theorem 3 (Cone Decomposition). *Let $\bar{x} \in C$, and consider* any *$\bar{u} \in D$ with $A\bar{u} = \bar{x}$.*

(a) $N_C(\bar{x}) = \{v \in \Re^n \mid A^{\mathrm{T}} v \in N_D(\bar{u})\}$;

(b) $T_C(\bar{x}) = \{w \in \Re^n \mid \exists z \in T_D(\bar{u}) \text{ with } Az = w\} = A(T_D(\bar{u}))$.

Proof. The result in (a) follows from the relations

$$\begin{aligned} v \in N_C(\bar{x}) &\iff \bar{x} \in \arg\max_{x \in C} v^{\mathrm{T}} x \\ &\iff \bar{u} \in \arg\max_{u \in D} v^{\mathrm{T}} A u = \arg\max_{u \in D} (A^{\mathrm{T}} v)^{\mathrm{T}} u \\ &\iff A^{\mathrm{T}} v \in N_D(\bar{u}) = (A^{\mathrm{T}})^{-1}(N_C(\bar{x})), \end{aligned}$$

where the first is by definition of $N_C(\bar{x})$, the second by the definition of C through the linear mapping A, and the fourth by definition of $N_D(\bar{u})$.

For the result in (b), we first note that $T_C(\bar{x}) = N_C(\bar{x})^*$, by polarity. But, on the other hand,

$$\begin{aligned} \big(A(T_D(\bar{u}))\big)^* &= \big\{v \in \Re^n \mid v^{\mathrm{T}} w \le 0, w \in A(T_D(\bar{u}))\big\} \\ &= \big\{v \in \Re^n \mid v^{\mathrm{T}} A h \le 0, h \in (T_D(\bar{u}))\big\} \end{aligned}$$

$$= \{v \in \Re^n \mid (A^{\mathrm{T}} v)^{\mathrm{T}} h \le 0,\, h \in (T_D(\bar{u}))\}$$
$$= \{v \in \Re^n \mid A^{\mathrm{T}} v \in (T_D(\bar{u})^*)\}$$
$$= \{v \in \Re^n \mid A^{\mathrm{T}} v \in (N_D(\bar{u}))\}$$
$$= N_C(\bar{x}),$$

by the result in (a). Hence,

$$T_C(\bar{x}) = N_(\bar{x})^* = A(T_D(\bar{u}))^{**} = \mathrm{cl}\, A(T_D(\bar{u})) = A(T_D(\bar{u})),$$

where the last equality follows because $T_D(\bar{u})$ is polyhedral, hence also $A(T_D(\bar{u}))$, which then in particular is closed. □

The following result is given in Rockafellar and Wets (1998, Exercise 6.44). The result will be used in the context of traffic equilibrium models in the next section.

COROLLARY 4 (AGGREGATION BY SUMS). *Assume further that $D = C_1 \times C_2 \times \cdots \times C_r$, where each set C_i is a polyhedron in $\Re^n$, and that $A(u_1, u_2, \ldots, u_r) = u_1 + u_2 + \cdots + u_r$ with $u_i \in \Re^n$ for all i, that is, $A = (I^n, I^n, \ldots, I^n)$ and $C = \sum_{i=1}^r C_i$. Let $\bar{x} \in C$ and consider* any *collection of vectors $\bar{u}_i \in C_i$ with $\bar{x} = \sum_{i=1}^r \bar{u}_i$.*

(a) $N_C(\bar{x}) = \{v \in \Re^n \mid v \in N_{C_i}(\bar{u}_i) \text{ for all } i\} = \bigcap_{i=1}^r N_{C_i}(\bar{u}_i)$;

(b) $T_C(\bar{x}) = \sum_{i=1}^r T_{C_i}(\bar{u}_i)$.

In the same context, consider now any $\bar{v} \in N_C(\bar{x})$ and the critical cone $T_C(\bar{x}) \cap \bar{v}^\perp$. Note that for the perturbed problem (1), the corresponding choice is $\bar{v} = -f(\rho^*, x^*)$. (Obviously, $[-f(\rho^*, x^*)]^\perp = f(\rho^*, x^*)^\perp$.)

THEOREM 5 (CRITICAL CONE DECOMPOSITION).

$$T_C(\bar{x}) \cap \bar{v}^\perp = A\left(T_D(\bar{u}) \cap (A^{\mathrm{T}} \bar{v})^\perp\right). \tag{6}$$

PROOF. Consider any $\bar{w} \in T_C(\bar{x}) \cap \bar{v}^\perp$, and $\bar{h} \in T_D(\bar{u})$ with $A\bar{h} = \bar{w}$, as guaranteed by Theorem 3(b). Then,

$$\begin{aligned} &\bar{w} \in T_C(\bar{x}) \cap \bar{v}^\perp \\ &\quad \Longleftrightarrow 0 = \bar{v}^{\mathrm{T}} \bar{w} \ge \bar{v}^{\mathrm{T}} w, \qquad w \in T_C(\bar{x}), \\ &\quad \Longleftrightarrow 0 = \bar{v}^{\mathrm{T}} A\bar{h} \ge \bar{v}^{\mathrm{T}} A h, \qquad h \in T_D(\bar{u}), \\ &\quad \Longleftrightarrow 0 = (A^{\mathrm{T}} \bar{v})^{\mathrm{T}} \bar{h} \ge (A^{\mathrm{T}} \bar{v})^{\mathrm{T}} h, \qquad h \in T_D(\bar{u}), \\ &\quad \Longleftrightarrow \bar{h} \in T_D(\bar{u}) \cap (A^{\mathrm{T}} \bar{v})^\perp, \end{aligned}$$

where the first equivalence follows by the definition of $T_C(\bar{x})$ and the relation $\bar{v} \in N_C(\bar{x})$ (cf. also Rockafellar and Wets 1998, Proposition 6.5), and the second by the definition of the mapping A. (We note that $A^{\mathrm{T}} \bar{v} \in N_D(\bar{u}) = T_D(\bar{u})^*$, as guaranteed by Theorem 3(b).) □

COROLLARY 6 (AGGREGATION BY SUMS). *In the special case given in Corollary* 4,

$$T_C(\bar{x}) \cap \bar{v}^\perp = \sum_{i=1}^r \left(T_{C_i}(\bar{u}_i) \cap \bar{v}^\perp\right).$$

PROOF. We note that $\bar{v} \in N_C(\bar{x})$ amounts to $\bar{v} \in N_{C_i}(\bar{u}_i)$ for all i; cf. Corollary 4(a). The result then follows from Theorem 5 and Corollary 4. □

We shall utilize the above cone decompositions to derive rules of calculus for the sensitivity of traffic equilibria. We first provide an outline of the modelling of such problems.

3. Traffic (Wardrop) Equilibrium

Let $\mathscr{G} = (\mathscr{N}, \mathscr{A})$ be a transportation network, where $\mathscr{N}$ and $\mathscr{A}$ are the sets of nodes and directed links (arcs), respectively. For certain ordered pairs of nodes, $(p, q) \in \mathscr{C}$, where node p is an origin, node q is a destination, and $\mathscr{C}$ is a subset of $\mathscr{N} \times \mathscr{N}$, there are travel demands d_{pq}, given by functions $g_{pq}(\rho, \cdot)$: $\Re^{|\mathscr{C}|} \mapsto \Re_+$ of the least costs of travel between the nodes p and q given the parameter vector $\rho \in \Re^d$; these functions are nonnegative, upper bounded, and continuous on $\Re^{|\mathscr{C}|}$ for each $(p, q) \in \mathscr{C}$. We assume that the network is strongly connected; that is, at least one route joins each origin-destination (OD) pair.

Wardrop's user equilibrium principle states that for every OD pair $(p, q) \in \mathscr{C}$, the travel costs of the routes utilized are equal and minimal for each individual user, and in the elastic context, that the total demand in the OD pair equals the demand function at this least cost, if positive. We denote by $\mathscr{R}_{pq}$ the set of simple (loop-free) routes for OD pair (p, q), by h_{pqr} the flow on route $r \in \mathscr{R}_{pq}$, and by $c_{pqr} = c_{pqr}(\rho, h)$ the travel cost on the route as experienced by an individual user given the vector $h \in \Re^{|\mathscr{R}|}$ of route flows, where $|\mathscr{R}|$ denotes the total number of routes in the network. With this notation, an equilibrium flow is defined by the conditions

$$h_{pqr} > 0 \Longrightarrow c_{pqr} = \pi_{pq}, \qquad r \in \mathscr{R}_{pq},\, (p, q) \in \mathscr{C}, \tag{7a}$$
$$h_{pqr} = 0 \Longrightarrow c_{pqr} \ge \pi_{pq}, \qquad r \in \mathscr{R}_{pq},\, (p, q) \in \mathscr{C}, \tag{7b}$$

where the value of $\pi_{pq} = \pi_{pq}(\rho, h)$ is the minimal (i.e., equilibrium) route cost in OD pair (p, q). By the nonnegativity of the route flows, the system (7) can be more compactly written as the complementarity system

$$0^{|\mathcal{R}|} \leq h \perp \left(c(\rho, h) - \Gamma\pi\right) \geq 0^{|\mathcal{R}|}, \tag{8}$$

where $c(\rho, \cdot)$: $\Re_+^{|\mathcal{R}|} \mapsto \Re_{++}^{|\mathcal{R}|}$ is the vector of route travel cost functions given the parameter vector $\rho \in \Re^d$, $\Gamma \in \Re^{|\mathcal{R}|\times|\mathcal{C}|}$ is the route-OD pair incidence matrix (i.e., the element γ_{rk} is 1 if route r joins OD pair $k = (p, q) \in \mathcal{C}$, and 0 otherwise), and $a \perp b$, for two arbitrary vectors $a, b \in \Re^n$, means that $a^{\mathrm{T}}b = 0$. The Wardrop conditions state that an equilibrium state is reached precisely when no traveller can decrease his/her travel cost by unilaterally shifting to another route.

The condition that the total demand d_{pq} in the OD pair $(p, q) \in \mathcal{C}$ equals the demand function at the least cost π_{pq} is stated as follows:

$$d_{pq} > 0 \Longrightarrow d_{pq} = g_{pq}(\rho, \pi), \qquad (p, q) \in \mathcal{C},$$
$$d_{pq} = 0 \Longrightarrow d_{pq} \geq g_{pq}(\rho, \pi), \qquad (p, q) \in \mathcal{C},$$

which, similar to conditions (7), can be written as a complementarity system as

$$0^{|\mathcal{C}|} \leq d \perp \left(d - g(\rho, \pi)\right) \geq 0^{|\mathcal{C}|}. \tag{9}$$

To cast the Wardrop conditions as a variational inequality problem, we need to decide in which space we wish to represent the flows and the flow feasibility requirements. A general form is obtained by describing the set of feasible, aggregate link flows as the solution in $v \in \Re^{|\mathcal{A}|}$ to the linear system

$$v = Vw, \tag{10a}$$
$$Ww = d, \tag{10b}$$
$$w \geq 0^l, \tag{10c}$$

where $w \in \Re^l$ is the (disaggregated) vector of the commodity flows, $V \in \Re^{|\mathcal{A}|\times l}$ is an incidence matrix which describes the aggregation of these flows into a corresponding link flow $v \in \Re^{|\mathcal{A}|}$, and W is an incidence matrix which describes the feasibility requirements with respect to the demand, d, in the commodity flow space.

The most common representation of the Wardrop conditions as a variational inequality problem is in terms of the route flow variables h_{pqr}. We obtain this formulation by identifying $w = h$, $d \in \Re^{|\mathcal{C}|}$ as the vector of each OD pair's demand, and $W = \Gamma^{\mathrm{T}}$. In a *disaggregated* version of the Wardrop conditions, we consider only utilizing the parts (10b)–(10c) of system (10) above, thus describing the (polyhedral) set

$$H_d = \left\{ (h, d) \in \Re_+^{|\mathcal{R}|} \times \Re^{|\mathcal{C}|} \mid \Gamma^{\mathrm{T}} h = d \right\}$$

of demand-feasible flows. The complementarity conditions (8) and (9) clearly are equivalent to h and $d = \Gamma^{\mathrm{T}} h$, satisfying

$$-\left[c(\rho, h) - \Gamma\pi, d - g(\rho, \pi)\right] \in N_{\Re_+^{|\mathcal{R}|}}(h) \times N_{\Re_+^{|\mathcal{C}|}}(d). \tag{11}$$

To further simplify the problem formulation, we now stipulate further that the function $-g(\rho, \cdot)$ is strictly monotone on $\Re^{|\mathcal{C}|}$, whence $-g(\rho, \cdot)$ is then also maximal (Rockafellar and Wets 1998, Example 12.7). Problem (11) can then be written as

$$\left[-c(\rho, h), g^{-1}(\rho, d)\right] \in N_{H_d}(h, d), \tag{12}$$

where $g^{-1}(\rho, \cdot)$ denotes the single-valued inverse of the demand function $g(\rho, \cdot)$. To see this directly, note that, together with the feasibility requirement that $\Gamma^{\mathrm{T}} h = d$ must hold, systems (8) and (9) describe the optimality conditions for (h, d) solving the linear program to minimize $c(\rho, h)^{\mathrm{T}} y_h - g^{-1}(\rho, d)^{\mathrm{T}} y_d$ over $y = (y_h, y_d) \in H_d$; utilizing the strict monotonicity of $-g(\rho, \cdot)$, this is precisely (12).

In the case where the travel cost of a route is the sum of the travel costs on the links defining it (i.e., the route costs are *additive*), then the above Wardrop conditions can be described in terms of link flows. We then further identify $V = \Lambda$ in (10a), where $\Lambda \in \{0, 1\}^{|\mathcal{A}|\times|\mathcal{R}|}$ is the link-route incidence matrix (i.e., the element λ_{ar} equals 1 if route r utilizes link a, and 0 otherwise), and thus the (polyhedral) set of demand-feasible link flows

$$\widehat{F}_d = \left\{(v, d) \in \Re^{|\mathcal{A}|} \times \Re^{|\mathcal{C}|} \mid \exists (h, d) \in H_d \text{ with } v = \Lambda h\right\}.$$

Then, problem (11) can be equivalently written as

$$\left[-t(\rho, v), g^{-1}(\rho, d)\right] \in N_{\widehat{F}_d}(v, d), \tag{13}$$

where $t(\rho, \cdot)\colon \Re^{|\mathcal{A}|}_{+} \mapsto \Re^{|\mathcal{A}|}_{++}$ is the vector of link travel cost functions given the parameter ρ. (The link and route costs are related by $c(\rho, h) = \Lambda^{\mathrm{T}} t(\rho, v) = \Lambda^{\mathrm{T}} t(\rho, \Lambda h)$, for any feasible pair (h, v).)

The set of feasible link flows can also be described by the OD-specific link flows that satisfy the demand for transportation and flow conservation constraints for all the nodes of the network; this is the other popular representation of feasible flows. In system (10), we then identify W as a block-diagonal matrix with $|\mathcal{C}|$ blocks W_k, with $W_k = E$, $E \in \{-1, 0, 1\}^{|\mathcal{N}| \times |\mathcal{A}|}$ being the node-link incidence matrix of the network. Further, d is a $(|\mathcal{C}| \cdot |\mathcal{N}|)$-vector, with $|\mathcal{C}|$ vectors d_k each being a vector of OD-specific demands, stacked on top of each other. (The elements of d_k sum to zero.) We also identify w as the $(|\mathcal{C}| \cdot |\mathcal{A}|)$-vector of commodity link flows w_{ak}. Hence, (10b)–(10c) correspond to the commodity-specific flow conservation constraints, giving rise to the set

$$W_d = \left\{ (w, d) \in \Re^{|\mathcal{C}| \cdot |\mathcal{A}|}_{+} \times \Re^{|\mathcal{C}| \cdot |\mathcal{N}|}_{+} \mid E w_k = d_k,\, k \in \mathcal{C} \right\}.$$

Finally, V is the block-diagonal $(|\mathcal{A}| \times |\mathcal{C}|)$-matrix $(I^{|\mathcal{A}|}, I^{|\mathcal{A}|}, \cdots, I^{|\mathcal{A}|})$, which describes the aggregation of the commodity link flows w_k into v: $v = \sum_{k \in \mathcal{C}} w_k$. Summarizing, then, system (10) describes the (polyhedral) set of demand-feasible link flows

$$F_d = \left\{ (v, d) \in \Re^{|\mathcal{A}|} \times \Re^{|\mathcal{C}| \cdot |\mathcal{N}|} \,\middle|\, \exists (w, d) \in \Re^{|\mathcal{C}| \cdot |\mathcal{A}|}_{+} \times \Re^{|\mathcal{C}| \cdot |\mathcal{N}|} \text{ with } v = \sum_{k \in \mathcal{C}} w_k \text{ and } E w_k = d_k \right\},$$

giving the problem

$$[-t(\rho, v), g^{-1}(\rho, d)] \in N_{F_d}(v, d). \tag{14}$$

In the present setting, of course, k is identified with an OD pair $(p, q) \in \mathcal{C}$, and, further, each vector d_k has precisely two nonzeros (so, $g^{-1}(\rho, \cdot)$ still operates in $\Re^{|\mathcal{C}|}$). We may, however, let k denote a less disaggregated flow such as flows from different origins, different vehicle types, etc. The two representations that we have chosen here are in that sense at the two extremes in terms of level of aggregation. We also note that in more generality, we may consider different networks, that is, different matrices E_k, for each commodity k, or type k of traffic. This will necessarily also lead to a proper modification of the matrix V above.

Note that the set of link flows in $\widehat{F}_d$ is included in that of F_d because the latter contains cyclic flows, but due to the positivity assumption on $t(\rho, \cdot)$ no equilibrium flow will utilize any cyclic flow, so this alternative representation is, in that sense, equivalent.

For further reading on these traffic equilibrium models, consult Patriksson (1994) and Nagurney (2000).

The connections between the above models and the perturbed systems (1) and (3) are given as follows. For the route-link representation, we can show that (12) corresponds to letting

$$D = H_d; \qquad u = \begin{pmatrix} h \\ d \end{pmatrix}; \qquad \varphi(\rho, u) = \begin{pmatrix} c(\rho, h) \\ -g^{-1}(\rho, d) \end{pmatrix}$$

in the disaggregated system (3), while the aggregated formulation (1) follows from letting

$$A = \begin{pmatrix} \Lambda & 0^{|\mathcal{A}| \times |\mathcal{C}|} \\ 0^{|\mathcal{C}| \times |\mathcal{R}|} & I^{|\mathcal{C}|} \end{pmatrix}; \qquad x = \begin{pmatrix} v \\ d \end{pmatrix};$$

$$f(\rho, x) = \begin{pmatrix} t(\rho, v) \\ -g^{-1}(\rho, d) \end{pmatrix} = f(\rho, Au). \tag{15}$$

As has already been shown, $\varphi(\rho, u) = A^{\mathrm{T}} f(\rho, Au)$ also holds.

For the node-link formulation, its disaggregated formulation corresponds to letting

$$D = W_d; \qquad u = \begin{pmatrix} w \\ d \end{pmatrix};$$

$$\varphi(\rho, u) = \begin{pmatrix} t(\rho, (I^{|\mathcal{A}|}, I^{|\mathcal{A}|}, \cdots, I^{|\mathcal{A}|}) w) \\ t(\rho, (I^{|\mathcal{A}|}, I^{|\mathcal{A}|}, \cdots, I^{|\mathcal{A}|}) w) \\ \vdots \\ t(\rho, (I^{|\mathcal{A}|}, I^{|\mathcal{A}|}, \cdots, I^{|\mathcal{A}|}) w) \\ -g^{-1}(\rho, d) \end{pmatrix},$$

while the aggregated formulation (14) follows from letting

$$A = \begin{pmatrix} I^{|\mathcal{A}|}, I^{|\mathcal{A}|}, \cdots, I^{|\mathcal{A}|} & 0^{|\mathcal{A}|\times|\mathcal{C}|} \\ 0^{|\mathcal{C}|\times|\mathcal{C}|\cdot|\mathcal{A}|} & I^{|\mathcal{C}|} \end{pmatrix}; \qquad x = \begin{pmatrix} v \\ d \end{pmatrix};$$

$$f(\rho, x) = \begin{pmatrix} t(\rho, v) \\ -g^{-1}(\rho, d) \end{pmatrix} = f(\rho, Au). \tag{16}$$

As above, it is also possible here to establish that $\varphi(\rho, u) = A^{\mathrm{T}} f(\rho, Au)$ holds.

When the travel cost function $t(\rho, \cdot)$ is *strictly* monotone on $\Re_+^{|\mathcal{A}|}$, the upper boundedness of the demand function implies the existence of unique demand and link flow solutions (d, v) (e.g., Patriksson 1994, and Nagurney 2000). However, it does not follow in general that the equilibrium route flow h is unique because the matrix Λ in general does not have full rank (that is, many route flows aggregate to the same equilibrium link flow). Seemingly, this fact might preclude the use of sensitivity analysis results, since they, in general, rely on some form of local single-valuedness of the solution mapping $\rho \mapsto S(\rho)$. The main conclusion of this paper is that a sensitivity analysis may indeed be performed, assuming a local uniqueness in the link flow space. The main reason for the concern about the nonuniqueness of route flows can be attributed to the formulation of the problem in the route flow space, while the appropriate space is that of link flows. In fact, the sensitivity analysis results can be stated, and established to be correct, without even providing any explicit representation of these link flows.

4. Sensitivity Analysis of Traffic Equilibria

4.1. The Critical Cone

The critical cone $K = T_Q(v^*, d^*) \cap [t(\rho^*, v^*), -g^{-1}(\rho^*, d^*)]^\perp$ (the set Q being either $\widehat{F}_d$ or F_d) is, roughly speaking, the set of feasible flow adjustments (circulations) that, on the aggregate, retain the equilibrium travel costs and demands, and is therefore an equilibrium. Indeed, the problem defined in calculating $DS(\rho^* \mid (v^*, d^*))(\rho')$ is a multicommodity flow problem, which we now turn to establish.

Theorem 7 (Critical Cone Decomposition for Traffic Equilibria).

(a) *Let* (v^*, d^*) *be a solution to* (13) *for a given* ρ^*, *and* h^* *an arbitrary vector of consistent equilibrium route flows; that is,* $v^* = \Lambda h^*$. *The set* $K = T_{\widehat{F}_d}(v^*, d^*) \cap [t(\rho^*, v^*), -g^{-1}(\rho^*, d^*)]^\perp$ *is given by*

$$A\left\{ (h', d') \in \Re^{|\mathcal{R}|} \times \Re^{|\mathcal{C}|} \;\middle|\; \Gamma^{\mathrm{T}} h' = d'; \quad \begin{array}{l} h'_{pqr} \text{ free if } h^*_{pqr} > 0 \\ h'_{pqr} \geq 0 \text{ if } h^*_{pqr} = 0 \text{ and } c_{pqr}(h^*) = \pi_{pq} \\ h'_{pqr} = 0 \text{ if } h^*_{pqr} = 0 \text{ and } c_{pqr}(h^*) > \pi_{pq} \\ {[r \in \mathcal{R}_{pq},\, (p,q) \in \mathcal{C}]} \end{array} \right\},$$

where A *is given in* (15).

(b) *Let* (v^*, d^*) *be a solution to* (14) *for a given* ρ^*, *and* w^* *an arbitrary vector of consistent equilibrium commodity link flows; that is,* $v^* = \sum_{k\in\mathcal{C}} w_k^*$. *The set* $K = T_{F_d}(v^*, d^*) \cap [t(\rho^*, v^*),\ -g^{-1}(\rho^*, d^*)]^\perp$ *is given by*

$$A\left\{ (w', d') \in \Re^{|\mathcal{C}|\times|\mathcal{A}|} \times \Re^{|\mathcal{C}|\cdot|\mathcal{N}|} \;\middle|\; \begin{array}{l} Ew'_k = d'_k \\ {[k \in \mathcal{C}]} \end{array}; \quad \begin{array}{l} w'_{ijk} \text{ free if } w^*_{ijk} > 0 \\ w'_{ijk} \geq 0 \text{ if } w^*_{ijk} = 0 \text{ and } t_{ij}(v^*) = \pi_{jk} - \pi_{ik} \\ w'_{ijk} = 0 \text{ if } w^*_{ijk} = 0 \text{ and } t_{ij}(v^*) > \pi_{jk} - \pi_{ik} \\ {[(i,j) \in \mathcal{A},\, k \in \mathcal{C}]} \end{array} \right\},$$

where A *is given in* (16), *and where now a link* $a \in \mathcal{A}$ *is identified by its origin node* i *and terminal node* j, *and where* π_{ik} *denotes the equilibrium* (*least*) *travel cost from the origin node of commodity* $k \in \mathcal{C}$ *to the node* $i \in \mathcal{N}$.

Proof. For the result in (a), we first note that by Corollary 4(b),

$$T_{\widehat{F}_d}(v^*, d^*) = \left\{ (v', d') \in \Re^{|\mathcal{A}|} \times \Re^{|\mathcal{C}|} \;\middle|\; v' = \Lambda h'; \Gamma^{\mathrm{T}} h' = d'; \quad \begin{array}{l} h'_{pqr} \text{ free if } h^*_{pqr} > 0 \\ h'_{pqr} \geq 0 \text{ if } h^*_{pqr} = 0 \\ {[r \in \mathcal{R}_{pq},\, (p,q) \in \mathcal{C}]} \end{array} \right\}.$$

For any $(v', d') \in T_{\widehat{F}_d}(v^*, d^*)$ it holds that $t(\rho^*, v^*)^{\mathrm{T}} v' = c(\rho^*, h^*)^{\mathrm{T}} h'$, so

$$\begin{aligned} &t(\rho^*, v^*)^{\mathrm{T}} v' - g^{-1}(\rho^*, d^*)^{\mathrm{T}} d' \\ &\quad = \sum_{(p,q)\in\mathscr{C}} \left(\sum_{r\in\mathscr{R}_{pq}} c_{pqr}(\rho^*, h^*) h'_{pqr} - g^{-1}_{pq}(\rho^*, d^*)\, d'_{pq} \right) \\ &\quad = \sum_{(p,q)\in\mathscr{C}} \left(\pi_{pq} \left(\sum_{r\in\mathscr{R}_{pq}} h'_{pqr} - d'_{pq} \right) + \sum_{r\in\mathscr{R}_{pq}} s_{pqr} h'_{pqr} \right) \\ &\quad = \sum_{(p,q)\in\mathscr{C}} \sum_{r\in\mathscr{R}_{pq}} s_{pqr} h'_{pqr}, \end{aligned}$$

where we have defined $s_{pqr} = c_{pqr}(\rho^*, h^*) - \pi_{pq}$, $r \in \mathscr{R}_{pq}$, $(p, q) \in \mathscr{C}$, to be the slack variables in the Wardrop conditions (7). (Also, $g^{-1}_{pq}(\rho^*, d^*)$ denotes the component (p, q) of the inverse demand function $g^{-1}(\rho^*, \cdot)$ evaluated at d^*.) Obviously, $s_{pqr} \geq 0$ always holds, and for the last sum to be zero, it is both necessary and sufficient that $h'_{pqr} = 0$ holds for routes where $h^*_{pqr} = 0$ and $s_{pqr} > 0$. □

Proof. The proof for (b) is analogous. First, note that, again by Corollary 4(b),

$$T_{F_d}(v^*, d^*) = \left\{ (v', d') \in \Re^{|\mathscr{A}|} \times \Re^{|\mathscr{C}|\cdot|\mathscr{N}|} \;\middle|\; v' = \sum_{k\in\mathscr{C}} w'_k;\ \begin{matrix} Ew'_k = d'_k; \\ [k \in \mathscr{C}] \end{matrix}\ \begin{matrix} w'_{ijk} \text{ free if } w^*_{ijk} > 0 \\ w'_{ijk} \geq 0 \text{ if } w^*_{ijk} = 0 \\ [(i, j) \in \mathscr{A},\ k \in \mathscr{C}] \end{matrix} \right\}.$$

The Wardrop Conditions (7) take, for the node-link representation, the following form:

$$0 \leq w_{ijk} \perp t_{ij}(\rho, v) + \pi_{ik} - \pi_{jk} \geq 0, \qquad (i, j) \in \mathscr{A}, k \in \mathscr{C}, \tag{17}$$

or,

$$0^{|\mathscr{A}|} \leq w_k \perp t(\rho, v) - E^{\mathrm{T}} \pi_k \geq 0^{|\mathscr{A}|}, \qquad k \in \mathscr{C}.$$

Now,

$$\begin{aligned} &t(\rho^*, v^*)^{\mathrm{T}} v' - g^{-1}(\rho^*, d^*)^{\mathrm{T}} d' \\ &\quad = \sum_{(i,j)\in\mathscr{A}} \sum_{k\in\mathscr{C}} \left(t_{ij}(\rho^*, v^*) w'_{ijk} - g^{-1}_k(\rho^*, d^*)^{\mathrm{T}} d'_k \right) \\ &\quad = \sum_{(i,j)\in\mathscr{A}} \sum_{k\in\mathscr{C}} \left((\pi_{jk} - \pi_{ik} + s_{ijk}) w'_{ijk} - \pi^{\mathrm{T}}_k d'_k \right) \\ &\quad = \sum_{(i,j)\in\mathscr{A}} \sum_{k\in\mathscr{C}} s_{ijk} w'_{ijk} + \sum_{k\in C} \pi_k \left(E w_k - d'_k \right) \\ &\quad = \sum_{(i,j)\in\mathscr{A}} \sum_{k\in\mathscr{C}} s_{ijk} w'_{ijk}, \end{aligned}$$

where, as above, $s_{ijk} = t_{ij}(\rho^*, v^*) - \pi_{jk} + \pi_{ik} \geq 0$, $(i, j) \in \mathscr{A}$, $k \in \mathscr{C}$, are the slack variables in the Wardrop conditions (17). The above sum is zero if and only if $w'_{ijk} = 0$ holds for the links $(i, j) \in \mathscr{A}$ and commodities $k \in \mathscr{C}$, where $w^*_{ijk} = 0$ and $s_{ijk} > 0$. □

The result (a) above was originally established (although in a different presentation) in Qiu and Magnanti (1989) as a special case of a general result for polyhedral sets having a special representation. The proof made here, however, is much more direct and intuitive.

A closer look at these critical cones reveals that they define multicommodity flow polyhedra where the flows are further restricted in sign for those commodity flow variables whose value in *some* equilibrium solution is zero. Those variables that happen to define a shortest route in this solution are restricted to be nonnegative; whereas those that are more expensive are restricted to be zero.

Next, the cost mapping r defined for the problem $DS(\rho^* \mid (v^*, d^*))(\rho')$ in Theorem 1 here becomes

$$r(\rho', (v', d')) = \begin{pmatrix} \nabla_v t(\rho^*, v^*) v' + \nabla_\rho t(\rho^*, v^*) \rho' \\ -\nabla_d g^{-1}(\rho^*, d^*) d' - \nabla_\rho g^{-1}(\rho^*, d^*) \rho' \end{pmatrix}, \tag{18}$$

that is, a linearization of the original mapping $(t, -g^{-1})$ at (ρ^*, v^*, d^*) in the direction of (ρ', v', d'). This mapping is affine in (v', d'), so the problem $DS(\rho^* \mid (v^*, d^*))(\rho')$ amounts to solving an affine cost multicommodity flow problem. We next provide sufficient conditions on the original mapping so that the solution to this problem provides the sensitivity analysis sought.

4.2. Sensitivity Analysis

Theorem 8 (Sensitivity Analysis of Traffic Equilibria). *Assume that condition* (5) *holds for problem* (13) (*or*, (14)). *Suppose that*, *for* ρ^* *given*, *problem* (13) (*or,* (14)) *has a solution* (v^*, d^*). *Suppose further that* $\rho \mapsto t(\rho, \cdot)$ (*respectively,* $\rho \mapsto -g^{-1}(\rho, \cdot)$) *is differentiable at* (ρ, v^*) $((\rho, d^*))$, *and monotone* (*respectively, strictly monotone*) *on the critical cone* K, *for every* ρ *in a neighborhood of* ρ^*. *Finally, assume that*

$$s_v^{\mathrm{T}} \nabla_v t(\rho^*, v^*) s_v - s_d^{\mathrm{T}} \nabla_d g^{-1}(\rho^*, d^*) s_d > 0,$$

$$s = \begin{pmatrix} s_v \\ s_d \end{pmatrix} \in (K - K) \setminus \{0^{|\mathscr{A}|+|\mathscr{C}|}\}$$

that is, that the Jacobian of $t(\rho^*, \cdot)$ (*and* $-g^{-1}(\rho^*, \cdot)$) *at* (ρ^*, v^*) $((\rho^*, d^*))$ *is positive definite on the critical subspace* $(K - K)$. *Then, the solution mapping* S *to the problem* (13) (*and* (14)) *is single-valued and Lipschitz continuous on some neighborhood of* ρ^* *and hence semidifferentiable at* ρ^* *for* (v^*, d^*), *while* $DS(\rho^* \mid (v^*, d^*))$ *is single valued everywhere, and hence Lipschitz continuous, positively homogeneous, and piecewise linear.*

If further, in the Wardrop conditions (7) (*respectively,* (17)), *the slack variables* s_{pqr} *are strictly positive for every* $r \in \mathscr{R}_{pq}$ *and* $(p, q) \in \mathscr{C}$ *with* $h^*_{pqr} = 0$ (*respectively,* $s_{ijk} > 0$ *for every* $(i, j) \in \mathscr{A}$ *and* $k \in \mathscr{C}$ *with* $w^*_{ijk} = 0$), *that is, strict complementarity holds for some disaggregated solution to the equilibrium problem, then the solution mapping* S *is differentiable at* ρ^*, *and* $DS(\rho^* \mid (v^*, d^*))$ *is linear.*

Proof. We establish that condition (b) in Theorem 2 is satisfied. That the solution set to (13) and (14) is convex valued follows from the monotonicity of the mappings $t(\rho, \cdot)$ and $-g^{-1}(\rho, \cdot)$ on the feasible set K of $DS(\rho^* \mid (v^*, d^*))(\rho')$. It is furthermore single valued by the assumption on the Jacobian of $[t(\rho, \cdot), -g^{-1}(\rho, \cdot)]$ being positive definite on the subspace spanned by K. The results, except the last one, then follow from Theorem 2. The last result follows from the observation that under the strict complementarity assumption, K locally changes linearly with (v^*, d^*). □

Quite obviously, the first result holds under a continuous differentiability and strong monotonicity assumption on the mapping $(v, d) \mapsto [t(\rho, v), -g^{-1}(\rho, d)]$ on the whole space, but this assumption is far from necessary. Whether all the common travel time and demand functions will actually fulfill the assumptions of this theorem is an interesting subject for further investigation.

We briefly mention two related results from the literature. In the context of problem (13), Qiu and Magnanti (1989, Theorem 4.1.1) establish that the mapping S is *strongly regular* in the sense of Robinson (1980) (that is, single valued and *locally Lipschitz* continuous, hence *directionally differentiable* since $\widehat{F}_d$ is polyhedral), at a reference parameter value ρ^*, if, in our setting, (a) in a neighborhood of (ρ^*, v^*, d^*) (for $(v^*, d^*) \in S(\rho^*)$), $(\rho, v, d) \mapsto (t(\rho, v), -g^{-1}(\rho, d))$ is continuous and further Lipschitz continuous with respect to ρ at (v^*, d^*), and differentiable with respect to (v, d) at (v^*, d^*); (b) the condition (18) holds. To reach their conclusion, they select a (uniquely determined) representative equilibrium route flow solution by means of solving a strictly convex quadratic projection problem; a process which is unnecessary, as seen above. Their result is stated under slightly milder assumptions, but on the other hand our results are more far-reaching. In particular, our results immediately imply the semismoothness of S, which will enable the use of bundle-type descent approaches in applications to bilevel optimization (see below). (See Patriksson and Rockafellar 2002, and Patriksson 2001 for detailed discussions on this topic, and Mifflin 1977 for a definition of semismoothness.) Outrata (1997, Proposition 1.2) establishes strong regularity for a wider class of problems, under conditions which here translate to strong monotonicity of the mapping $(v, d) \mapsto [t(\rho, v), -g^{-1}(\rho, d)]$ on $\widehat{F}_d$, and a condition like (18) but for a larger subspace. A similar result to Outrata's is also reached independently by Yen (1995, Theorem 4.1), where, however, the parameterized demand is inelastic.

A few final words are now stated on the use of the results of this paper in the solution to bilevel problems in transportation analysis and their relation to previous analyses in that context. If we consider a minimization in the parameter values ρ over some set $P \subset \Re^d$ and for some objective function ψ: $P \times \Re^{|\mathscr{A}|} \times \Re^{|\mathscr{C}|} \mapsto \Re \cup \{+\infty\}$ of the form $\psi(\rho^*, v^*, d^*)$, where (v^*, d^*) is a pair of equilibrium flows and demands given the parameter vector values ρ^*, then we speak of a *mathematical program with equilibrium constraints* (MPEC) (e.g., Luo et al. 1996, Outrata et al. 1998). For such problems, having sufficiently strong differentiability properties of the solution mapping S is essential for its efficient solution. Attempts to define descent algorithms for the minimization of ψ in the context of transportation analysis have most often been based on either ignoring the possible nondifferentiability of ψ due to the nondifferentiability of S, or on heuristic attempts to circumvent the underlying problem of the nonstrict complementarity of the equilibrium solutions. One such heuristic is given in Friesz et al. (1990), wherein a route flow is heuristically adjusted from an equilibrium to reach a strictly complementary flow. Another strategy stems from Tobin and Friesz (1988) (see also, for example, Yang and Bell 1997 and Cho et al. 2000), and is based on the calculation of a "derivative" of S through sensitivity information, also involving the choice of a *particular* representative commodity flow solution consistent with (v^*, d^*). The formulas include quite involved matrix calculations which, moreover, depend on relations between the number of OD pairs and links to be satisfied in order to even be applicable. Moreover, even if the formula is applicable, it may not yield a value which can be interpreted as a gradient (cf. the example in Patriksson 2001). In principle, the gradient, if it exists, can instead be obtained from d calculations of the directional derivative along coordinate directions; more efficient means of calculation are interesting to study further. Needless to say, such a strategy need not yield a true derivative unless strict complementarity holds, and further, as we have pointed out already, the sensitivity analysis of traffic equilibrium problems cannot be influenced by making any such particular choice. We mention finally that a descent algorithm which is valid even for nonmonotone travel cost mappings is found in Patriksson and Rockafellar (2002).

4.3. An Illustrative Example

An example problem for the *fixed* demand version of the problem (13) is found in Qiu and Magnanti (1989). In the following, we consider the elastic demand problem (14).

Consider the small-scale traffic network depicted in Figure 1.

For this network, we have the data in Table 1.

We consider the case where $\rho^* = (\rho_1^*, \rho_2^*)^{\mathrm{T}} = (0, 0)^{\mathrm{T}}$, and $\rho' = (2, 1)^{\mathrm{T}}$, which we can interpret as a case where tolls are to be introduced on the Links $(1, 2)$ and $(2, 4)$, the first toll being double the size of the second. At ρ^*, then, we have the following solution to the elastic demand traffic equilibrium problem:

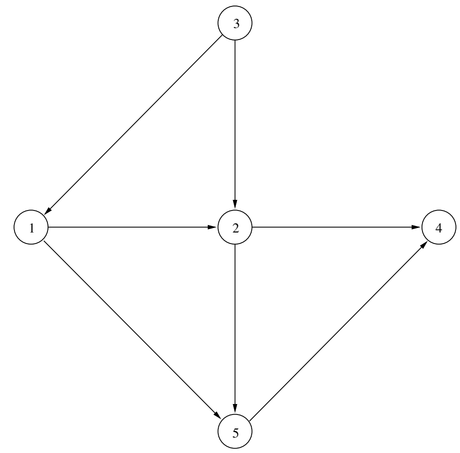

Figure 1 **A Traffic Network**

$$v^* = \begin{pmatrix} 5 \\ 2 \\ 2 \\ 5 \\ 3 \\ 2 \\ 2 \end{pmatrix}, \qquad t(\rho^*, v^*) = \begin{pmatrix} 5 \\ 10 \\ 15 \\ 5 \\ 15 \\ 20 \\ 10 \end{pmatrix},$$

$$d^* = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \qquad \pi^* = \begin{pmatrix} 20 \\ 25 \end{pmatrix}.$$

We remark that the commodity link flow solution is not unique; two possible combinations of commodity flows are:

$$\widetilde{w}_1^* = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \qquad \widetilde{w}_2^* = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 5 \\ 3 \\ 2 \\ 0 \end{pmatrix}, \quad \text{and}$$

$$\widehat{w}_1^* = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \qquad \widehat{w}_2^* = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 4 \\ 3 \\ 2 \\ 0 \end{pmatrix}.$$

Table 1 **Network Data**

Link	$t_{ij}(\rho, v_{ij})$	OD pair	$g_{pq}(\pi_{pq})$
1: (1, 2)	$\frac{5}{2} + \frac{5}{2}v_{12} + \rho_1$	1: (1,4)	$44 - 2\pi_{14}$
2: (1, 5)	$2 + 4v_{15}$	2: (3,5)	$55 - 2\pi_{35}$
3: (2, 4)	$11 + 2v_{24} + \rho_2$		
4: (2, 5)	$\frac{5}{2} + \frac{1}{2}v_{25}$		
5: (3, 1)	$3 + 4v_{31}$		
6: (3, 2)	$4 + 8v_{32}$		
7: (5, 4)	$2 + 4v_{54}$		

We note in particular that in the commodity flow solution $\widetilde{w}^*$, Link $(2, 5)$ has no flow in the first commodity and Link $(1, 5)$ has no flow in the second commodity. However, these links lie on a shortest route from Node 1 to Node 4, and from Node 3 to Node 5, respectively. So, $\widetilde{w}^*$ is an example of a nonstrictly complementary Wardrop solution. The solution $\widehat{w}^*$, however, is strictly complementary.

The travel cost function $t(\rho^*, \cdot)$ is affine and separable, and in particular strongly monotone. The same conclusion can be drawn for $-g(\rho^*, \cdot)$, whence the conditions of the first part of Theorem 8 are fulfilled. We also note that because $\widehat{w}^*$ is strictly complementary, the equilibrium link flow and demand solution are differentiable at ρ^*. (This is true for almost all values of ρ^*. Unfortunately, however, it *cannot* be expected to be true at optimal solutions to an MPEC problem involving an optimization in ρ.) Whether, for a given ρ^*, there exists a strictly complementary solution can be investigated through the solution of an entropy problem, whereby the aggregated solution is distributed over the commodities by means of maximizing the entropy function. If a feasible commodity solution exists, wherein all links lying on a shortest route have a positive flow, then such a flow will be generated. (See Larsson et al. 2001, for further details on this entropy problem in the context of problem (13).)

When setting up the problem $DS(\rho^* \mid (v^*, d^*))(\rho')$ we note again that whether we choose $\widetilde{w}^*$ or $\widehat{w}^*$, or any other disaggregated flow consistent with v^*, makes no difference whatsoever to the result in terms of (v', d'). (Of course, the perturbations will in general not be unique in the commodity space, however.) We here choose to work with $\widehat{w}^*$. Given this representation of the equilibrium link flow, we obtain the networks (in Figure 2) representing the critical cone K given in Theorem 7(b).

For these networks, we have the differentiated link costs and demand functions given in Table 2. (Note that $\nabla(g^{-1}) = (\nabla g)^{-1}$.) Due to the separability of the costs and demands, the affine variational problem is equivalent to an optimization problem over K, where the objective is the minimization of

$$2v'_{12} + \frac{5}{4}(v'_{12})^2 + 2(v'_{15})^2 + v'_{24} + (v'_{24})^2 + \frac{1}{4}(v'_{25})^2 + 2(v'_{31})^2 + 4(v'_{32})^2 + 2(v'_{54})^2 + \frac{1}{4}(d'_{14})^2 + \frac{1}{4}(d'_{35})^2.$$

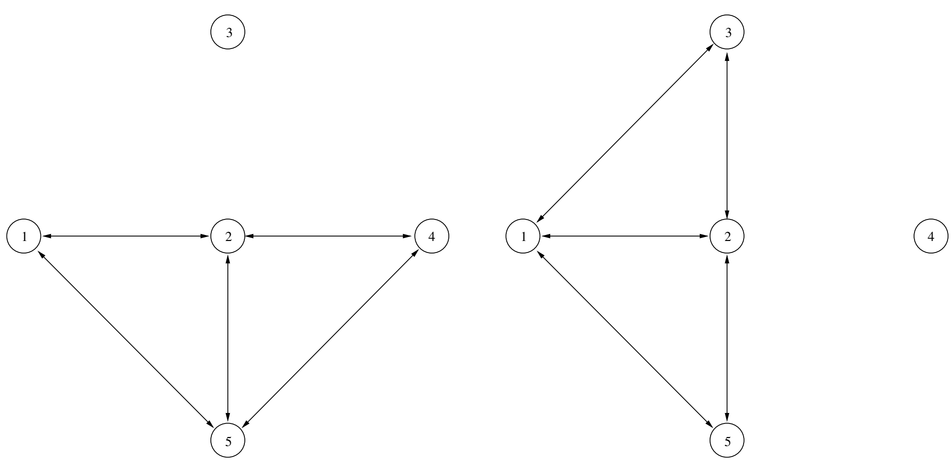

Figure 2 **Feasible Circulation Networks for the Two Commodities**

Table 2 Network Data

Link	$t'_{ij}(v'_{ij})$	OD pair	$g'_{pq}(\pi'_{pq})$
1: (1, 2)	$2+\frac{5}{2}v'_{12}$	1: (1,4)	$-2\pi'_{14}$
2: (1, 5)	$4v'_{15}$	2: (3,5)	$-2\pi'_{35}$
3: (2, 4)	$1+2v'_{24}$		
4: (2, 5)	$\frac{1}{2}v'_{25}$		
5: (3, 1)	$4v'_{31}$		
6: (3, 2)	$8v'_{32}$		
7: (5, 4)	$4v'_{54}$		

The solution to this problem is as follows:

$$v' = \begin{pmatrix} -0.6625 \\ 0.0700 \\ -0.5304 \\ -0.1271 \\ -0.0628 \\ 0.0050 \\ 0.0007 \end{pmatrix}, \qquad t'(v') = \begin{pmatrix} 0.3438 \\ 0.2800 \\ -0.0607 \\ -0.0636 \\ -0.2513 \\ 0.0400 \\ 0.0028 \end{pmatrix},$$

$$d' = \begin{pmatrix} -0.5297 \\ -0.0578 \end{pmatrix}, \qquad \pi' = \begin{pmatrix} 0.2648 \\ 0.0289 \end{pmatrix},$$

and

$$w'_1 = \begin{pmatrix} -0.4622 \\ -0.0674 \\ -0.5304 \\ 0.0681 \\ 0 \\ 0 \\ 0.0007 \end{pmatrix}, \qquad w'_2 = \begin{pmatrix} -0.2003 \\ 0.1374 \\ 0 \\ -0.1953 \\ -0.0628 \\ 0.0050 \\ 0 \end{pmatrix}$$

is one representative commodity flow adjustment. The result is the expected: d' is negative in both commodities, meaning that the demand is decreasing, and we see from v' that flow is not only reduced as a consequence, but also redistributed from the two tolled Links $(1, 2)$ and $(2, 4)$—that is, Links 1 and 3—to the other links.

5. Final Remarks

The results obtained in this paper have immediate application to solution algorithms for OD estimation or adjustment problems, where sensitivity analysis is used to direct the update of a prior OD matrix. Algorithms previously proposed for this problem (e.g., Spiess 1990, Drissi-Kaïtouni and Lundgren 1992, Yang 1995, and Codina and Barceló 2000) can be viewed as heuristic algorithms which use simplifications of the correct formula for the directional derivative of the demand matrix. In a forthcoming paper (Patriksson 2001), we provide further results on the sensitivity of traffic equilibria, including the existence and calculation of *gradients* or, in their absence, of *subgradients* of the equilibrium solution, and relationships between the sensitivity analyses of deterministic and stochastic user equilibria. These results are useful in devising bundle algorithms for MPEC problems arising in transportation science.

References

Cho, H.-J., T. E. Smith, T. L. Friesz. 2001. A reduction method for local sensitivity analyses of network equilibrium arc flows. *Transportation Res.* **34B** 31–51.

Codina, E., J. Barceló. 2000. Adjustment of O-D trip matrices from observed volumes: An algorithmic approach based on conjugate directions. Report, Statistics and Operations Research Department, Polytechnic University of Catalonia, Barcelona, Spain.

Dontchev, A. L., R. T. Rockafellar. 2002. Ample parameterization of variational inclusions. *SIAM J. Optim.* **12** 170–187.

Drissi-Kaïtouni, O., J. T. Lundgren. 1992. Bilevel origin-destination matrix estimation using a descent approach. Report LiTH-MAT-R-1992-49, Department of Mathematics, Linköping Institute of Technology, Linköping, Sweden.

Friesz, T. L., R. L. Tobin, H. J. Cho, N. J. Mehta. 1990. Sensitivity analysis-based heuristic algorithms for mathematical programs with variational inequality constraints. *Math. Programming* **48** 265–284.

Larsson, T., J. Lundgren, M. Patriksson, C. Rydergren. 2001. Most likely traffic equilibrium route flows: Analysis and computation. F. Giannessi, A. Maugeri, P. M. Pardalos, eds. *Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models.* Kluwer Academic Publishers, Dordrecht, The Netherlands, 129–159.

Luo, Z.-Q., J.-S. Pang, D. Ralph. 1996. *Mathematical Programs with Equilibrium Constraints*. Cambridge University Press, Cambridge, U.K.

Mifflin, R. 1977. Semismooth and semiconvex functions in constrained optimization. *SIAM J. Control and Optim.* **15** 959–972.

Nagurney, A. 2000. *Network Economics: A Variational Inequality Approach*, 2nd ed. Kluwer Academic Publishers, Dordrecht, The Netherlands.

Outrata, J. V. 1997. On a special class of mathematical programs with equilibrium constraints. P. Gritzmann, R. Horst, E. Sachs, R. Tichatschke, eds. *Recent Advances in Optimization*. *Lecture Notes in Economics and Mathematical Systems*, vol. 452. Springer-Verlag, Berlin, Germany, 246–260.

——, M. Kočvara, J. Zowe. 1998. *Nonsmooth Approach to Optimization Problems with Equilibrium Constraints*. Kluwer Academic Publishers, Dordrecht, The Netherlands.

Patriksson, M. 1994. *The Traffic Assignment Problem—Models and Methods. Topics in Transportation*. VSP BV, Utrecht, The Netherlands.

——. 2001. Sensitivity analysis of traffic equilibria. Report, Department of Mathematics, Chalmers University of Technology, Gothenburg, Sweden. Revised 2002 for *Transportation Sci.*

——, R. T. Rockafellar. 2002. A mathematical model and descent algorithm for bilevel traffic management. *Transportation Sci.* **36** 271–291.

Qiu, Y., T. L. Magnanti. 1989. Sensitivity analysis for variational inequalities defined on polyhedral sets. *Math. Oper. Res.* **14** 410–432.

Robinson, S. M. 1980. Strongly regular generalized equations. *Math. Oper. Res.* **5** 43–62.

Received: July 2001; accepted: October 2001.

Rockafellar, R. T., R.J.-B. Wets. 1998. *Variational Analysis—Grundlehren der mathematischen Wissenschaften*, vol. 317. Springer-Verlag, Berlin, Germany.

Spiess, H. 1990. A gradient approach for the O-D matrix adjustment problem. Publication CRT-693, Centre de recherche sur les transports, Université de Montréal, Montréal, Quebec, Canada.

Tobin, R. L., T. L. Friesz. 1988. Sensitivity analysis for equilibrium network flow. *Transportation Sci.* **22** 242–250.

Yang, H. 1995. Heuristic algorithms for the bilevel origin-destination matrix estimation problems. *Transportation Res.* **29B** 231–242.

——, M. G. H. Bell. 1997. Traffic restraint, road pricing and network equilibrium. *Transportation Res.* **31B** 303–314.

Yen, N. D. 1995. Lipschitz continuity of solutions of variational inequalities with a parametric polyhedral constraint. *Math. Oper. Res.* **20** 695–708.