Chapter 1

A VARIATIONAL INEQUALITY SCHEME FOR DETERMINING AN ECONOMIC EQUILIBRIUM OF CLASSICAL OR EXTENDED TYPE

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Abstract The existence of an equilibrium in an extended Walrasian economic model of exchange is confirmed constructively by an iterative scheme. In this scheme, truncated variational inequality problems are solved in which the agents' budget constraints are relaxed by a penalty representation. Epi-convergence arguments are employed to show that, in the limit, a virtual equilibrium is obtained, if not actually a classical equilibrium. A number of technical hurdles are, in this way, surmounted.

Keywords: variational inequalities, Walras exchange equilibrium, virtual equilibrium, epi-convergence, penalization, equilibrium computations

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1. INTRODUCTION

Mathematical models of equilibrium in economics attempt to capture the effects of competing interests among different "agents" in face of the limited availability of goods and other resources. They typically revolve around the existence of prices for the goods under which the optimization carried out by these agents, individually, leads collectively to a balance between supply and demand.

Although the fundamental ideas go back to Walras and others, the work of Arrow and Debreu [1], [3], initiated the solidly mathematical form of the subject, still continuing in its development. Notions from game theory, such as Nash equilibrium and its counterpart for generalized games (where each agent's strategy set can depend on the other agents' actions), have entered strongly too. Nowadays, influences are also coming from applications beyond the academic, for instance to traffic equilibrium and the practical consequences of deregulation of markets in electrical power.

In the economics literature, fixed-point theory has long provided the environment for establishing whether an equilibrium exists. Fixed-point approaches to calculation were promoted by Scarf [15], [16]. The emphasis on the theory side, though, has largely been on broadening the models so as to encompass preference relations expressed by set-valued mappings that satisfy weakened semicontinuity assumptions and the like. The question of how agents might discover an equilibrium through a Walras-type procedure of tatonnement has been of interest as well, but economists have not devoted much effort to achieving a structured format conducive to large-scale numerical computation. General fixed-point algorithms are notoriously slow and unpromising in anything but simple, low-dimensional situations.

Alternative approaches have been opening up, however, in the optimization literature in connection with variational inequality formulations, including "complementarity" models; see [2], the 1990 survey of Harker and Pang [9], and the 2003 book of Facchinei and Pang [6] for background. Such approaches offer ways of tying the computation of equilibrium into the major advances that have been made in numerical optimization, although this kind of computation is nevertheless much more difficult than mere minimization or maximization.

The task of setting up a variational inequality model for equilibrium involves not only challenges but compromises for the sake of tractibility. Some levels of generality have to be abandoned, at least within present capabilities. For example, the expression of preferences by abstract relations has to be dropped in favor of expression by utility functions, which moreover may need to satisfy assumptions like differentiability. Certain constraints need to be handled with Lagrange multipliers. Such maneuvers run into some serious technical issues, however, two of the main ones being the existence of an equilibrium and the existence of a solution to the proposed variational inequality.

The question of whether an equilibrium exists can be very subtle, even in a purely economic framework. The Arrow-Debreu model [1], as applied to pure exchange, for instance, effectively requires that each agent start out with a tradable quantity of every possible good. Much effort has successfully gone into weakening that sort of provision, but the techniques appear, at least on the surface, to conflict with the features desired for a readily computible representation. The constraint qualifications ordinarily invoked to ensure access to Lagrange multipliers can fail, in particular. On the other side, the variational inequality models achieved by introducing Lagrange multipliers have the drawback of leading to problems in which the underlying convex sets are unbounded and adequate coercivity is absent. They tend then to fall outside the domain of the standard criteria for confirming that a solution exists.

Our aim in this paper is to demonstrate how these difficulties can be overcome in the fundamental case of a Walras equilibrium, which we take for simplicity (rather than technical necessity) to be a pure exchange equilibrium among consumers, with no producers. We carefully introduce assumptions that enable us to prove the existence, at least, of a "virtual" exchange equilibrium, which might have some agents just barely surviving without optimizing, but can be approximated arbitrarily closely by an exchange equilibrium in the classical sense. Moreover, we show that a *virtual equilibrium* can be computed in principle by solving a sequence of variational inequality problems in which the underlying convex sets are actually compact.

A key contribution lies in showing how the iterative truncations needed technically in order to achieve compactness in the variational inequality, for existence of solutions, can be interpreted as corresponding to penalty representations of the agents' budget constraints, which surprisingly, however, furnish classical equilibrium relative to nearby endowments in place of the original ones. In verifying that the equilibrium sequence from the truncated problems yields, in the limit, a virtual equilibrium, we develop detailed progress estimates and break new ground in utilizing arguments about epi-convergence.

We do not try to answer, here, the question of how the truncated variational inequalities can, themselves, be solved. Some guidance toward the future prospects is available, though, in the recent papers [10], [11], which deal with generalized games, and of course in the book [6], which addresses variational inequalities more generally.

Beyond computation, it should be noted that variational inequality representations of equilibrium are able also to take advantage of the extensive theory on how solutions to variational inequality problems respond to data perturbations, as for instance in [14], [5]. Our work can be viewed as contributing also in that direction.

2. EQUILIBRIUM MODEL

The space of goods is $I\!R^l_+$; the goods are indexed by $j = 1, \ldots, l$. Each agent $a \in \mathcal{A}$ has an endowment $e_a \in I\!R^l_+$ and a utility function u_a to be applied to consumption choices. The consumption vector x_a must belong to a certain subset $X_a \subset I\!R^l_+$. The condition $x_a \in X_a$ is the *survival constraint*, and X_a is the *survival set*. In elementary models, $X_a = I\!R^l_+$.

Subject to survival and the feasibility of exchanging the goods j at appropriate prices p_j, which are not given but have to be determined from the data elements e_a, X_a and u_a, the agents seek individually to arrange their consumption so as to maximize their utility. The focus is on *relative* price vectors, i.e., vectors p that belong to the price simplex

$$P = \{\, p = (p_1, \ldots, p_l) \in I\!R^l \,|\, p_j \geq 0,\ p_1 + \cdots p_l = 1\}. \tag{1}$$

Definition 1 (exchange equilibrium)**.** *A* classical *exchange equilibrium consists of a price vector $\bar{p}$ and consumption vectors, $\bar{x}_a$, such that*

(a) $\Sigma_{a\in\mathcal{A}}\, \bar{x}_{aj} \leq \Sigma_{a\in\mathcal{A}}\, e_{aj}$ *for all goods j, with equality holding if $\bar{p}_j > 0$,*

(b) $\bar{x}_a \in \operatorname{argmax}\{\, u_a(x_a) \,|\, x_a \in X_a,\ \bar{p}\cdot x_a \leq \bar{p}\cdot e_a)\}$*, and* $\bar{p}\cdot\bar{x}_a = \bar{p}\cdot e_a$.

A two-tier *exchange equilibrium is the same, except that some of the agents a may satisfy as a substitute for* (b) *the condition*

(b$^-$) $\bar{x}_a \in \operatorname{argmin}\{\, \bar{p}\cdot x_a \,|\, x_a \in X_a\}$*, and* $\bar{p}\cdot\bar{x}_a = \bar{p}\cdot e_a$.

An agent satisfying (b) *will be called an* optimizing *agent , whereas an agent satisfying* (b$^-$) *will be called a* barely surviving *agent.*

The requirement that $\bar{p}\cdot x_a \leq \bar{p}\cdot e_a$ is the *budget* constraint for agent a. In a two-tier equilibrium, the barely surviving agents have their budgets so tight that they can only choose cheapest possible consumption vectors from their survival sets, and that uses up all their wealth.

If the argmin in (b$^-$) consists of a unique vector, that is what must be chosen. In that case, (b$^-$) trivially entails (b), so the situation special interest in (b$^-$) is mainly the one where the argmin isn't just a singleton. It's conceivable then that a small amount of freedom may be left for utility optimization while keeping to lowest cost. No such secondary optimization is claimed in the definition, but we don't exclude the possibility that an optimizing agent might also be a barely surviving agent. However, we will really be concerned with a sharpened form of two-tier equilibrium, defined next, in which the barely surviving agents, if any, are "arbitrarily close" to being optimizing agents and fall short only because of a slightest lack of resources.

Definition 2 (virtual exchange equilibrium)**.** *A two-tier exchange equilibrium, with price vector $\bar{p}$ and consumption vectors $\bar{x}_a$, is a* virtual *exchange equilibrium if (when not itself actually a classical equilibrium) it includes at least one optimizing agent and can be approximated arbitrarily closely by a classical* *equilibrium in the following sense. There are price vectors p^ν and consumption vectors x_a^ν, $\nu = 1, 2, \ldots$, with*

$$\lim_{\nu\to\infty} p^\nu = \bar{p}, \qquad \lim_{\nu\to\infty} x_a^\nu = \bar{x}_a,$$

which for each ν furnish a classical exchange equilibrium with respect to the same sets X_a and functions u_a but possibly different endowments e_a^ν satisfying

$$e_a^\nu \geq e_a, \qquad \lim_{\nu\to\infty} e_a^\nu = e_a.$$

Although any classical equilibrium is a virtual equilibrium in particular (and fits the sequence prescription with $p^\nu = \bar{p}$, $x_a^\nu = \bar{x}_a$, $e_a^\nu = e_a$), the converse is false. Likewise, not every two-tier equilibrium is a virtual equilibrium. Examples of these differences will be provided in the final section of this paper.

In the economic literature, what we are calling a classical exchange equilibrium in Definition 1 is a special case of a Walras equilibrium, namely one in which preferences are expressed by utilities, free disposal is assumed, and "production" has not been introduced. Production is omitted here mainly for the sake of simplicity. The results that will be described can be extended in that way, but we wish to avoid the notational complications in order to focus here on the newer features more clearly.

What we call a two-tier exchange equilibrium in Definition 1 corresponds, under the same specializations, to a model first developed by Debreu [4] as a *quasi*-equilibrium. We prefer to speak of a two-tier equilibrium because the term quasi-equilibrium has shifted over the years to mean something different from what Debreu originally indicated. It regularly refers now, in a utility context like ours, to substituting for (b) the condition that $\bar{x}_a \in X_a$ with $\bar{p}\cdot\bar{x}_a = \bar{p}\cdot e_a$, but there is no $x_a \in X_a$ satisfying both $\bar{p}\cdot x_a < \bar{p}\cdot e_a$ and $u_a(x_a) > u_a(\bar{x}_a)$. This property is not as sharp as (b$^-$); it is implied by (b$^-$) but is insufficient to yield (b$^-$) in return.

The notion of a virtual exchange equilibrium in Definition 2 does not seem to have been introduced or explored previously in economics. Beyond its potential in the theoretical understanding of equilibrium, it has natural significance for numerical work, where limits of computed sequences of approximations to a desired equilibrium may inevitably need to be contemplated anyway.

In our variational approach to equilibrium, each agent's utility maximization problem will be translated into optimality conditions involving a Lagrange multiplier. It is partly for the extra benefit accruing from such conditions, but also for enhancing the computational possibilities when given specific data, that we concentrate on utility functions (instead of abstract preference relations) and furthermore make the following restrictions. Although these restrictions could be relaxed in several ways, they will assist us here in getting some basic ideas across without too many technical complications.

Ongoing Assumptions (utility and constraint structure)**.**

(A1) X_a *is convex and closed, with nonempty interior.*

(A2) u_a *is concave and continuously differentiable on* X_a*.*

(A3) u_a *does not attain a maximum on* X_a*.*

Because we are operating in an environment of free disposal, there is no real loss of generality in stipulating in (A1) that $\operatorname{int} X_a \neq \emptyset$; we could harmlessly replace X_a by $\hat{X}_a = X_a + \mathbb{R}^l_+$ while extending u_a to the nondecreasing utility $\hat{u}_a$ defined by $\hat{u}_a(x_a) = \sup\{\, u_a(\hat{x}_a) \,|\, \hat{x}_a \leq x_a\}$. The continuous differentiability in (A2) can be interpreted merely as continuous differentiability on $\operatorname{int} X_a$ with the mapping ∇u_a having a continuous extension from $\operatorname{int} X_a$ to the boundary of X_a.

Definition 3 (utility scaling)**.** *By an equilibrium with* utility scaling *will be meant an equilibrium in the sense of Definition 1 or Definition 2 in which condition* (b) *is replaced by the existence of a coefficient* $\bar{\lambda}_a$*, called a* utility scale factor *for agent* a*, such that*

(b$^+$) $\bar{x}_a \in \operatorname{argmax}\{\, u_a(x_a) - \bar{\lambda}_a \bar{p}\cdot(x_a - e_a) \,|\, x_a \in X_a\}$ *with*

$$\bar{\lambda}_a \in [0,\infty) \text{ and } \bar{p}\cdot(\bar{x}_a - e_a) \begin{cases} \leq 0 & \text{if } \bar{\lambda}_a = 0, \\ = 0 & \text{if } \bar{\lambda}_a > 0, \end{cases}$$

and, in Definition 2, this also to the sequence of approximate equilibria.

Proposition 1 (status of utility scale factors)**.** *Condition* (b$^+$) *implies condition* (b) *always. Thus, an exchange equilibrium with utility scaling (whether classical or two-tier) in the sense of Definition 3 always entails the corresponding equilibrium in Definition 1 or Definition 2. Conversely,* (b) *implies* (b$^+$) *in particular when there exists* $x_a \in X_a$ *such that* $\bar{p}\cdot x_a > \bar{p}\cdot e_a$*.*

Proof. In fact, (b$^+$) gives the Kuhn-Tucker conditions for the maximization problem in (b), inasmuch as X_a is convex by (A1) and u_a is concave by (A2). These conditions are always sufficient for optimality, and they are necessary under a Slater assumption, which by virtue of (A3) comes out here as the existence of an $x_a \in X_a$ satisfying the budget constraint strictly. □

The point is that (b$^+$) is, in general, an enhancement of (b), so that in establishing the existence of an equilibrium with utility scaling, we will be accomplishing more than just proving the existence of a equilibrium by itself.

Proposition 2 (positivity of utility scale factors)**.** *Because of* (A3)*, condition* (b$^+$) *can only hold with* $\bar{\lambda}_a > 0$ *and* $\bar{p}\cdot(\bar{x}_a - e_a) = 0$*.*

Proof. If we had $\lambda_a = 0$ in (b$^+$), the maximum of u_a over X_a would be attained at $\bar{x}_a$, in contradiction to (A3). □

The reason for calling $\bar{\lambda}_a$ a utility scale factor is that it acts as a coefficient for converting the price $\bar{p}_j$ for a good j into to a price $\bar{\lambda}_a \bar{p}_j$ measured in the utility units of agent a. According to (b$^+$), once such utility prices are available they can be brought into play by maximizing $u_a(x_a) - \bar{\lambda}_a \bar{p}_a \cdot (x_a - e_a)$ instead of $u_a(x_a)$, with the original budget constraint pushed into the background. This alternative maximization converts the cost $\bar{p} \cdot (x_a - e_a)$ of passing from e_a to x_a into an adjustment of the utility associated with x_a, as compared to e_a.

If u_a were strictly concave, the maximization in (b$^+$) would by itself determine $\bar{x}_a$ uniquely, and the budget constraint would therefore turn out to be satisfied automatically. Even when the maximization in (b$^+$) doesn't determine $\bar{x}_a$ uniquely, however, the budget constraint is not invoked directly in this maximization and is only needed, if at all, in the aftermath, for the purpose of eliminating some of the vectors in the argmin set.

Theorem 1 (existence of virtual equilibrium)**.** *A two-tier exchange equilibrium that is a virtual exchange equilibrium with utility scaling is sure to exist under the following assumptions on the initial endowments:*

(S1) *for every agent a there is a vector $x_a \in X_a$ such that $x_a \leq e_a$,*
(S2) *there are vectors $x_a \in X_a$ such that $\sum_{a \in \mathcal{A}} x_a < \sum_{a \in \mathcal{A}} e_a$.*

The proof of Theorem 1 will come later and, in a major respect, it will be "constructive" (as elaborated in Theorem 3). In contrast to Theorem 1, the existence result of Debreu [4] for this sort of model, although posed in a somewhat broader setting, was not constructive and didn't provide utility scaling. It didn't confirm the presence of at least one optimizing agent or yield the approximation property that distinguishes a virtual equilibrium.

Of course, any agent a for which there exists $x_a \in X_a$ such that $x_a < e_a$ must in particular be an optimizing agent, since this strict vector inequality precludes (b$^-$). Other, more subtle criteria for an agent to be optimizing are known as well; cf. [7], [8], and their references. In combination with Theorem 1, such criteria immediately lead to conclusions about the existence of a *classical* equilibrium in our setting. We omit the details, because our interest centers on the proof of Theorem 1 by way of a variational inequality formulation having computational potential.

Nonetheless, it's worth noting that both of our survival assumptions (S1) and (S2) automatically do hold when every agent a has some $x_a \in X_a$ with $x_a < e_a$ (which amounts to the main case treated in [1] by Arrow and Debreu).

3. VARIATIONAL REPRESENTATION

The variational inequality representation of an equilibrium with utility scaling will now be set up. In general in a space $\mathbb{R}^L$ of vectors v, the variational inequality problem $\mathrm{VI}(C, F)$ associated with a nonempty, convex set $C \subset \mathbb{R}^L$ and a mapping $F : C \to \mathbb{R}^L$ consists of finding

$$\bar{v} \in C \text{ such that } -F(\bar{v}) \in N_C(\bar{v}),$$

where $N_C(\bar{v})$ is the normal cone to C at $\bar{v}$:

$$w \in N_C(\bar{v}) \iff w \cdot (v - \bar{v}) \leq 0 \text{ for all } v \in C.$$

It's well known that if C is compact and F is continuous, a solution $\bar{v}$ to problem $\mathrm{VI}(C, F)$ exists.

In our formulation of equilibrium, the variational inequality we set up will have C closed and F continuous, but C unbounded, so this criterion for the existence of a solution to $\mathrm{VI}(C, F)$ will not be applicable directly. That will oblige us to introduce truncations to create compactness. Such truncations will be construed as corresponding to penalty formulations of the budget constraints in the agent's maximization problems. In obtaining an equilibrium through an iterative process of truncation, we will employ an argument crucially based on epi-convergence, which is a concept of variational analysis associated with the convergence of solutions when optimization problems are approximated.

Theorem 2 (variational inequality format for classical equilibrium)**.** *A classical exchange equilibrium with utility scaling is furnished by* $\bar{p}$, $\{\bar{x}_a\}_{a\in\mathcal{A}}$ *and* $\{\bar{\lambda}_a\}_{a\in\mathcal{A}}$, *if and only if the variational inequality* $\mathrm{VI}(C, F)$ *in the form*

$$-F(\bar{p}; \ldots, \bar{x}_a, \ldots; \ldots, \bar{\lambda}_a, \ldots) \in N_C(\bar{p}; \ldots, \bar{x}_a, \ldots; \ldots, \bar{\lambda}_a \ldots) \qquad (2)$$

holds for the nonempty, closed, convex set $C \subset I\!R^l \times [\Pi_{a\in\mathcal{A}} I\!R^l] \times [\Pi_{a\in\mathcal{A}} I\!R]$ *defined by*

$$C = P \times [\Pi_{a\in\mathcal{A}} X_a] \times [\Pi_{a\in\mathcal{A}} [0, \infty)] \qquad (3)$$

and the continuous mapping $F : C \to I\!R^l \times [\Pi_{a\in\mathcal{A}} I\!R^l] \times [\Pi_{a\in\mathcal{A}} I\!R]$ *defined by*

$$\begin{aligned} &F(p; \ldots, x_a, \ldots; \ldots, \lambda_a, \ldots) \\ &\quad = (\Sigma_{a\in\mathcal{A}}[e_a - x_a]; \ldots, \lambda_a p - \nabla u_a(x_a), \ldots; \ldots, p \cdot [e_a - x_a], \ldots) . \end{aligned} \qquad (4)$$

Proof. The closedness and convexity claimed for C and the continuity claimed for F are evident from (A1) and (A2). The variational inequality in question decomposes into the conditions

$$\begin{aligned} \Sigma_{a\in\mathcal{A}}[\bar{x}_a - e_a] &\in N_P(\bar{p}), \\ \nabla u_a(\bar{x}_a) - \bar{\lambda}_a \bar{p} &\in N_{X_a}(\bar{x}_a) \text{ for all } a \in \mathcal{A}, \\ \bar{p} \cdot (\bar{x}_a - e_a) &\in N_{[0,\infty)}(\bar{\lambda}_a) \text{ for all } a \in \mathcal{A}. \end{aligned} \qquad (5)$$

The second condition means that the function $x_a \mapsto u_a(x_a) - \bar{\lambda}_a \bar{p} \cdot (x_a - e_a)$, which is concave, has its maximum over X_a at $\bar{x}_a$, whereas the third condition refers to the complementarity relations

$$\bar{p} \cdot (\bar{x}_a - e_a) \leq 0, \qquad \bar{\lambda}_a \geq 0, \qquad \bar{\lambda}_a \bar{p} \cdot (\bar{x}_a - e_a) = 0. \qquad (6)$$

Those two conditions together, therefore, are equivalent to (b$^+$) holding for every $a \in \mathcal{A}$.

The first condition in (5) is not, on the surface, the same as the market condition (a) in Definition 1, which in principle would be stronger. We will see, however, that in the presence of the other conditions, the first condition in (5) implies (a). In terms of

$$\zeta = \max_{j=1,\dots,l} \{\Sigma_{a\in\mathcal{A}}[\bar{x}_{aj} - e_{aj}]\}, \tag{7}$$

the first condition in (5) says that

$$\bar{p}_j = 0 \text{ unless } \Sigma_{a\in\mathcal{A}}[\bar{x}_{aj} - e_{aj}] = \zeta, \tag{8}$$

so that in particular

$$\bar{p} \cdot \Sigma_{a\in\mathcal{A}}[\bar{x}_a - e_a] = \zeta. \tag{9}$$

Now we bring in Proposition 2: we actually must have $\bar{p} \cdot [\bar{x}_a - e_a] = 0$ for all $a \in \mathcal{A}$. Then (9) implies $\zeta = 0$, hence $\Sigma_{a\in\mathcal{A}}[\bar{x}_{aj} - e_{aj}] \le 0$ for every j in (7), and we are able to conclude through (8) that (a) holds. □

Although the unboundedness of the set C in our variational inequality representation of classical equilibrium is an unavoidable consequence of the multiplier conditions we have introduced, a partial kind of boundedness, at least, can be achieved under our assumptions by a trouble-free truncation of the survival sets X_a.

Proposition 3 (underlying boundedness of consumption)**.** *Under* (S1) *and* (S2), *there exist bounded subsets $X_a^b \subset X_a$ still satisfying these assumptions and such that a classical exchange equilibrium with utility scaling is furnished for $\{X_a^b\}_{a\in\mathcal{A}}$ by $\bar{p}$, $\{\bar{x}_a\}_{a\in\mathcal{A}}$ and $\{\lambda_a\}_{a\in\mathcal{A}}$, if and only if these elements give such an equilibrium for $\{X_a\}_{a\in\mathcal{A}}$. Specifically, this is true when*

$$X_a^b = \{\, x_a \in X_a \,|\, x_a \le b\} \text{ for any } b > \sum\nolimits_{a\in\mathcal{A}} e_a, \tag{10}$$

in which case there definitely exist elements $x_a \in X_a^b$ satisfying $x_a < b$, whereas any elements $x_a \in X_a^b$ satisfying $\sum_{a\in\mathcal{A}} x_a \le \sum_{a\in\mathcal{A}} e_a$ must satisfy $x_a < b$.

Proof. Take b and X_a^b as in (10). Clearly X_a^b is still convex and closed, but also bounded, and (S2) is preserved. Since $e_a \ge 0$ for every $a \in \mathcal{A}$, the strict inequality in (10) implies that $e_a < b$ for every $a \in \mathcal{A}$. The condition that $e_a \in X_a$, from (S1), thus carries over to having $e_a \in X_a^b$ and in particular informs us, by taking $x_a = e_a$, that there exists $x_a \in X_a^b$ satisfying $x_a < b$. Indeed, in the background of e_a belonging to $\{\, x_a \,|\, x_a < b\} \cap X_a$, we get from (A1) that $\operatorname{int} X_a^b = \{\, x_a \,|\, x_a < b\} \cap \operatorname{int} X_a \ne \emptyset$ (cf. [12, Theorem 6.5]) and can conclude that (A1) holds for X_a^b. Trivially, (A2) persists when X_a is replaced by the truncation X_a^b.

Because $X_a \subset I\!R^l_+$, the conditions $x_a \in X_a$ and $\sum_{a\in\mathcal{A}} x_a \leq \sum_{a\in\mathcal{A}} e_a$ in the definition of an equilibrium imply $x_a < b$. Hence any equilibrium with respect to the sets X_a is an equilibrium with respect to the sets X^b_a, and conversely as well, the constraints $x_a \leq b$ necessarily being inactive in either case. □

According to this observation, we can replace the sets X_a by bounded sets X^b_a in the formulation of the variational inequality in Theorem 2 without undermining the equivalence with the desired equilibrium. This still leaves the unboundedness caused by the multiplier conditions, however. To handle that, our approach is to truncate the interval $[0, \infty)$ to $[0, r]$ for a value $r > 0$, which will turn out to act as a penalty parameter.

Proposition 4 (truncated variational inequality)**.** *Consider the variational inequality* $\mathrm{VI}(C^b_r, F)$ *for the same mapping* F *as in Theorem 2 but with the set* C *there replaced for* $r > 0$ *by*

$$C^b_r = P \times [\Pi_{a\in\mathcal{A}} X^b_a] \times [\Pi_{a\in\mathcal{A}}[0, r]],$$

the sets X^b_a *being defined as in Proposition 3. Then* C^b_r *is nonempty, closed and convex, but also bounded, and a solution to* $\mathrm{VI}(C^b_r, F)$ *therefore exists. A solution to* $\mathrm{VI}(C^b_r, F)$ *is comprised of a relative price vector* $\bar{p}$ *along with* $\{\bar{x}_a\}_{a\in\mathcal{A}}$ *and* $\{\bar{\lambda}_a\}_{a\in\mathcal{A}}$ *for which there is a value* $\zeta \in I\!R$ *such that*

(a$_r$) $\Sigma_{a\in\mathcal{A}} \bar{x}_{aj} \leq \Sigma_{a\in\mathcal{A}} e_{aj} + \zeta$ *for all goods* j*, with equality when* $\bar{p}_j > 0$,

(b^+_r) $\bar{x}_a \in \mathrm{argmax}\{\, u_a(x_a) - \bar{\lambda}_a \bar{p}\cdot(x_a - e_a) \,|\, x_a \in X^b_a\}$*, with*

$$\bar{\lambda}_a \in [0, r] \;\; \textit{and} \;\; \bar{p}\cdot(\bar{x}_a - e_a) \begin{cases} \leq 0 & \textit{if } \bar{\lambda}_a = 0, \\ = 0 & \textit{if } 0 < \bar{\lambda}_a < r, \\ \geq 0 & \textit{if } \bar{\lambda}_a = r. \end{cases}$$

Proof. The standard existence criterion for variational inequalities, invoked for the compact set C^b_r, produces $\bar{p}$, $\{\bar{x}_a\}_{a\in\mathcal{A}}$ and $\{\bar{\lambda}_a\}_{a\in\mathcal{A}}$ for which the corresponding $\bar{v} = (\bar{p}; \ldots, \bar{x}_a, \ldots; \ldots, \bar{\lambda}_a, \ldots)$ solves $\mathrm{VI}(C^b_r, F)$, i.e., has $-F(\bar{v}) \in N_{C^b_r}(\bar{v})$. Adopting the pattern in the proof of Theorem 2, we decompose this variational inequality into the conditions

$$\begin{aligned} \Sigma_{a\in\mathcal{A}}[\bar{x}_a - e_a] &\in N_P(\bar{p}), \\ \nabla u_a(\bar{x}_a) - \bar{\lambda}_a \bar{p} &\in N_{X^b_a}(\bar{x}_a) \;\text{ for all }\; a \in \mathcal{A}, \\ \bar{p}\cdot[\bar{x}_a - e_a] &\in N_{[0,r]}(\bar{\lambda}_a) \;\text{ for all }\; a \in \mathcal{A}. \end{aligned}$$

The fact that the first of these conditions is equivalent to (a$_r$) was effectively argued already in the proof of Theorem 2. The second and third of these conditions is (b^+_r). □

In working with the truncated variational inequality and understanding its meaning, it will be helpful to have the notation

$$[t]_+ = \max\{0, t\} \;\text{ for }\; t \in I\!R.$$

We use it to set up a linear penalty approximation to the budget constraint $p\cdot(x_a - e_a) \le 0$ in terms of the expression

$$r[p\cdot(x_a-e_a)]_+ = \begin{cases} 0 & \text{when } p\cdot(x_a-e_a)\le 0,\\ rp\cdot(x_a-e_a) & \text{when } p\cdot(x_a-e_a) > 0.\end{cases}$$

Proposition 5 (penalty interpretation)**.** *Condition (b$_r^+$) of Proposition 4 holds with respect to $\bar{p}$ for $\bar{x}_a$ and some $\bar{\lambda}_a$ if and only if $\bar{x}_a$ satisfies*

(b$_r$) $\bar{x}_a \in \operatorname{argmax}\{\, u_a(x_a) - r[\bar{p}\cdot(x_a-e_a)]_+ \mid x_a \in X_a^b\}$.

Proof. The equivalence can be seen by thinking of (b$_r$) as referring to the minimization of $\varphi_a + \psi_a$ over $I\!R^l$, where

$$\varphi_a(x_a) = \begin{cases} -u_a(x_a) & \text{when } x_a \in X_a^b,\\ \infty & \text{when } x_a \notin X_a^b,\end{cases} \qquad \psi_a(x_a) = r[\bar{p}\cdot(x_a-e_a)]_+.$$

Here φ_a is a lower semicontinuous, proper, convex function, while ψ_a is a finite convex function on $I\!R^l$. The subgradient condition both necessary and sufficient for the minimum of $\varphi_a + \psi_a$ to occur at $\bar{x}_a$, namely $0 \in \partial(\varphi_a+\psi_a)(x_a)$, comes out therefore as the existence of a subgradient $z_a \in \partial\psi_a(\bar{x}_a)$ such that $-z_a \in \partial\varphi_a(\bar{x}_a)$, where moreover $\partial\varphi_a(\bar{x}_a) = -\nabla u_a(\bar{a}) + N_{X_a^b}(\bar{x}_a)$ (cf. [12, Theorem 23.8]). The necessary and sufficient condition thus refers to the existence of $\nabla u_a(\bar{x}_a) - z_a \in N_{X_a^b}(\bar{x}_a)$.

By a basic chain rule in convex analysis (cf. [12, Theorem 23.9]), we have $z_a \in \partial\psi_a(\bar{x}_a)$ if and only if $z_a = \bar{\lambda}_a\bar{p}$ for some $\bar{\lambda}_a$ satisfying the conditions in (b$_r^+$). In this manner, we have $\nabla u_a(\bar{x}_a) - z_a \in N_{X_a^b}(\bar{x}_a)$ if and only if $\nabla u_a(\bar{x}_a) - \bar{\lambda}_a\bar{p} \in N_{X_a^b}(\bar{x}_a)$ for some such $\bar{\lambda}_a$, and this can be recognized as the necessary and sufficient condition for optimality in the maximization in condition (b$_r$). □

4. ITERATIVE SCHEME

The existence result in Theorem 1 will be derived by an iterative scheme based on the variational inequality representations of equilibrium we have been developed above. In this scheme, we replace the survival sets X_a to the bounded sets X_a^b specified in 10 and consider for $\nu = 1, 2, \ldots$, a sequence of penalty parameter values $r^\nu \nearrow \infty$, denoting by C^ν the set C_b^r of Proposition 4 in the case of $r = r^\nu$. For each ν we solve the variational inequality $\mathrm{VI}(C^\nu, F)$, which is possible by Proposition 4 in principle (and moreover should be approachable numerically by methods developed along the lines of those in [6], [10], [11], as mentioned in the introduction).

This way, we generate a sequence of price vectors $p^\nu \in P$ together with sequences of consumption vectors $x_a^\nu \in X_a^b$, multipliers λ_a^ν and values ζ^ν satisfying

(a$_{r^\nu}$) $\Sigma_{a\in\mathcal{A}} x^\nu_{aj} \le \Sigma_{a\in\mathcal{A}} e_{aj} + \zeta^\nu$ for all goods j, with equality when $p^\nu_j > 0$,
(b$^+_{r^\nu}$) $x^\nu_a \in \operatorname{argmax}\{\, u_a(x_a) - \lambda^\nu_a p^\nu \cdot (x_a - e_a) \,|\, x_a \in X^b_a\}$ with

$$\lambda^\nu_a \in [0, r^\nu] \;\text{ and }\; p^\nu \cdot (x^\nu_a - e_a) \begin{cases} \le 0 & \text{if } \lambda^\nu_a = 0, \\ = 0 & \text{if } 0 < \lambda^\nu_a < r^\nu, \\ \ge 0 & \text{if } \lambda^\nu_a = r^\nu. \end{cases}$$

Note that because the components p^ν_j of p^ν are nonnegative, but not all zero, condition (a$_{r^\nu}$) means that

$$\zeta^\nu = \max_{j=1,\dots,l} \sum_{a\in\mathcal{A}} [x^\nu_{aj} - e_{aj}]. \tag{11}$$

Condition (b$^+_{r^\nu}$), on the other hand, can be interpreted through Proposition 5 as the condition

(b$_{r^\nu}$) $x^\nu_a \in \operatorname{argmax}\{\, u_a(x_a) - r^\nu [p^\nu \cdot (x_a - e_a)]_+ \,|\, x_a \in X^b_a\}$,

which relaxes the budget constraint $p^\nu \cdot (x_a - e_a) \le 0$ by allowing it to be exceeded at a penalty rate which is increased in each iteration.

Theorem 3 (limits in the iterative scheme)**.** *Once r^ν is higher than a certain threshold value, p^ν and x^ν_a furnish a classical equilibrium, with utility scaling, respect to the same sets X_a and functions u_a but possibily different endowment vectors $e^\nu_a \ge e_a$ with $e^\nu_a \to e_a$. The sequence of these nearby classical equilibria $(p^\nu, \{x^\nu_a\}_{a\in\mathcal{A}})$ is bounded, and every cluster point $(\bar p, \{\bar x_a\}_{a\in\mathcal{A}})$ furnishes a virtual equilibrium for the original data. Hence if only one virtual equilibrium exists, the entire sequence must converge to it.*

Obviously, in proving Theorem 3 we will have proved Theorem 1, so we can concentrate on Theorem 3. Since the sets P and X^b_a containing p^ν and x^ν_a are closed and bounded, the sequences of vectors p^ν and x^ν_a are bounded, as claimed in Theorem 3, and cluster points do exist. Note, however, that no such claim is made about the sequences of multipliers λ^ν_a. The possible unboundedness of such a sequence is exactly what can lead to an agent a being only a barely surviving agent. The following facts will be crucial, in view of the definition of a virtual equilibrium,

Proposition 6 (convergence estimates)**.** *For each $a \in \mathcal{A}$, choose any $\hat x_a \in X_a$ with $\hat x_a \le e_a$, as exists by (S1), and let $\mu^b_a = \max\{\, u_a(x_a) \,|\, x_a \in X^b_a\}$. Then $\mu^b_a \ge u_a(\hat x_a)$, and one has*

$$p^\nu \cdot (x^\nu_a - e_a) \;\le\; \frac{\mu^b_a - u_a(\hat x_a)}{r^\nu} \;\textit{ for all }\; a \in \mathcal{A}, \tag{12}$$

This implies that

$$x^\nu_a < b \;\textit{ for all }\; a \in \mathcal{A} \;\textit{ when } r^\nu \textit{ is sufficiently large,} \tag{13}$$

as is true specifically when

$$r^\nu \ \geq\ \hat{r} \ \ \textit{for} \ \ \hat{r} = \frac{N}{\beta} \max_{a\in\mathcal{A}} \{\, \mu_a^b - u_a(\hat{x}_a)\}, \tag{14}$$

where N is the number of agents $a \in \mathcal{A}$ and β is any positive number small enough that $\sum_{a\in\mathcal{A}} e_{aj} \leq b_j - \beta$ for every good j. Thereafter, one will have

$$p^\nu \cdot (x_a^\nu - e_a) \geq 0 \ \textit{for all} \ a \in \mathcal{A}, \tag{15}$$

and the vectors p^ν and x_a^ν will furnish a classical equilibrium with respect to the sets X_a, functions u_a, and the endowment vectors e_a^ν defined by

$$e_{aj}^\nu = e_{aj} + \zeta_a^\nu \ \textit{with} \ \zeta_a^\nu = p^\nu \cdot (x_a^\nu - e_a), \tag{16}$$

in which the multipliers λ_a^ν are positive and serve as utility scale factors. Thus, one will have

(a$^\nu$) $\Sigma_{a\in\mathcal{A}} x_{aj}^\nu \leq \Sigma_{a\in\mathcal{A}} e_{aj}^\nu$ *for all j, with equality holding if $p_j^\nu > 0$,*
(b$^{\nu+}$) $x_a^\nu \in \operatorname{argmax}\{\, u_a(x_a) - \lambda_a^\nu p^\nu \cdot (x_a - e_a^\nu) \,|\, x_a \in X_a\}$ *with*

$$\lambda_a^\nu > 0, \qquad p^\nu \cdot (x_a^\nu - e_a^\nu) = 0.$$

Proof. Because $\hat{x}_a \leq e_a$, we have $\hat{x}_a \in X_a^b$ with $p^\nu \cdot (\hat{x}_a - e_a) \leq 0$. Hence $\mu_a^b \geq 0$ and

$$\begin{aligned} u_a(\hat{x}_a) &= u_a(\hat{x}_a) - r^\nu [p^\nu \cdot (\hat{x}_a - e_a]_+ \\ &\leq u_a(x_a^\nu) - r^\nu [p^\nu \cdot (x_a^\nu - e_a]_+ \leq \mu_a^b - r^\nu [p^\nu \cdot (x_a^\nu - e_a]_+, \end{aligned}$$

so that $r^\nu [p^\nu \cdot (x_a^\nu - e_a]_+ \leq \mu_a^b - u_a(\hat{x}_a)$. This inequality guarantees (12). From (a$_{r^\nu}$) we have $\sum_{a\in\mathcal{A}} p^\nu \cdot (x_a^\nu - e_a) = p^\nu \cdot \sum_{a\in\mathcal{A}} (x_a^\nu - e_a) = \zeta^\nu$ with ζ^ν expressed by (11), and therefore $\sum_{a\in\mathcal{A}} [x_{aj}^\nu - e_{aj}] \leq \zeta^\nu$. It follows that $\sum_{a\in\mathcal{A}} x_{aj}^\nu \leq \sum_{a\in\mathcal{A}} e_{aj} + \beta$ when $\zeta^\nu \leq \beta$, and in particular

$$x_{aj}^\nu \leq b_j - \beta \ \text{ when } \ \zeta^\nu \leq \beta \ \text{ with } \ \beta \leq b_j - \sum\nolimits_{a\in\mathcal{A}} e_{aj}. \tag{17}$$

Thus, $x_a^\nu < b$ as claimed in (13) when r^ν is beyond the value $\hat{r}$ in (14).

Once we have $x_a^\nu < b$, the maximum over X_a^b in condition (b$_{r^\nu}^+$) is the same as the maximum over X_a, due to convexity. Then necessarily $\lambda_a^\nu > 0$, since otherwise our nonsatiation assumption (A3) would be violated. In (b$_{r^\nu}^+$) we then have $p^\nu \cdot (x_a^\nu - e_a) \geq 0$ for all $a \in \mathcal{A}$. In that case, by taking $\zeta^\nu = p^\nu \cdot (x_a^\nu - e_a)$ and defining e_a^ν as indicated, we get $e_a^\nu \geq e_a$ and $p^\nu \cdot (x_a^\nu - e_a^\nu) = 0$, so that conditions (a$_{r^\nu}$) and (b$_{r^\nu}^+$) have been converted to (a$^\nu$) and (b$^{\nu+}$). That implies by Proposition 1 that the elements p^ν, x_a^ν and λ_a^ν furnish a classical equilbrium with respect to the endowments e_a^ν. □

These estimates immediately reveal key properties of our iterative scheme. As $r^\nu \to \infty$, we eventually have (14) and, in the augmentation rule in (16),

$$0 \le \zeta_a^\nu \le \frac{\mu_a^b - u_a(\hat{x}_a)}{r^\nu} \to 0, \text{ so that } e_a^\nu \to e_a.$$

By taking limits of in (a^ν) and ($b^{\nu+}$), we see then that cluster points $\bar{p}$ and $\bar{x}_a$ must satisfy the market clearing condition (a) and the budget condition $\bar{p}\cdot(\bar{x}_a - e_a) = 0$. The extent to which they satisfy (b^+) or (b^-), however, remains to be established.

The key to further analysis lies in the utility scale factors λ_a^ν. For simplicity of notation in this analysis, we can suppose we have passed to subsequences so that actually $p^\nu \to \bar{p}$ and $x_a^\nu \to \bar{x}_a$ for every agent $a \in \mathcal{A}$, and that ($b^{\nu+}$) and (15) hold for all ν, furthermore with $\bar{x}_a < b$, as comes out of the uniformity of the bound derived in (17). We look at

$$\begin{aligned} \mathcal{A}_+ &= \{ \text{ agents } a \in \mathcal{A} \text{ such that } \{\lambda_a^\nu\}_{\nu=1}^\infty \text{ is bounded} \}, \\ \mathcal{A}_- &= \{ \text{ agents } a \in \mathcal{A} \text{ such that } \{\lambda_a^\nu\}_{\nu=1}^\infty \text{ is unbounded} \}. \end{aligned}$$

By a further reduction to subsequences if necessary, we can arrange that

$$\begin{cases} \text{for each } a \in \mathcal{A}_+, \text{ actually } \lambda_a^\nu \to \bar{\lambda}_a \ge 0, \\ \text{for each } a \in \mathcal{A}_-, \text{ actually } \lambda_a^\nu \to \infty. \end{cases}$$

Consider now an agent $a \in \mathcal{A}_+$. Define the functions φ_a^ν and φ_a on the entire space $I\!R^l$ by

$$\begin{aligned} \varphi_a^\nu(x_a) &= \begin{cases} -u_a(x_a) + \lambda_a^\nu p^\nu \cdot (x_a - e_a^\nu) & \text{if } x_a \in X_a^b, \\ \infty & \text{if } x_a \notin X_a^b, \end{cases} \\ \varphi_a(x_a) &= \begin{cases} -u_a(x_a) + \bar{\lambda}_a \bar{p} \cdot (x_a - e_a) & \text{if } x_a \in X_a^b, \\ \infty & \text{if } x_a \notin X_a^b, \end{cases} \end{aligned}$$

these functions being convex and lower semicontinuous by virtue of (A1) and (A2). Conditions ($b^{\nu+}$) and (b^+) correspond respectively to

$$x_a^\nu \in \operatorname*{argmin}_{x_a \in \boldsymbol{R}^l} \varphi_a^\nu(x_a), \qquad \bar{x}_a \in \operatorname*{argmin}_{x_a \in \boldsymbol{R}^l} \varphi_a(x_a), \tag{18}$$

inasmuch as $x_a^\nu < b$ and $\bar{x}_a < b$, along with $\lambda_a^\nu > 0$ and $p^\nu \cdot (x_a^\nu - e_a^\nu) = 0$, as well as $\bar{\lambda} \ge 0$ and $\bar{p}\cdot(\bar{x}_a - e_a) = 0$. Therefore, if we can show that the second condition in (18) follows in the limit from the first condition as $\nu \to \infty$, we will be able to conclude that $\bar{p}$, $\bar{x}_a$ and $\bar{\lambda}_a$ satisfy (b^+) and thus that agent a is an optimizing agent.

This is an issue addressed, in general, by the theory of “epi-convergence” of sequences of functions and its role in minimization, as expounded for instance in [14, Chapter 7]. Here, the circumstances are especially simple because the functions are convex and all have the same effective domain, namely X_a^b, which moreover has nonempty interior. As $\nu \to \infty$, we have $\varphi_a^\nu(x_a) \to \varphi_a(x_a)$ for each $x_a \in X_a^b$, and that guarantees the epi-convergence of φ_a^ν to φ_a by [14, Theorem 7.17]. Then by [14, Theorem 7.33], because these functions are lower semicontinuous with their effective domains uniformly bounded, the first condition in (18) yields the second, as required.

Next, consider instead an agent $x \in \mathcal{A}_-$. Define the functions ψ_a^ν and ψ_a on the entire space $I\!R^l$ by

$$\psi_a^\nu(x_a) = \begin{cases} -(1/\lambda_a^\nu)u_a(x_a) + p^\nu \cdot (x_a - e_a^\nu) & \text{if } x_a \in X_a^b, \\ \infty & \text{if } x_a \notin X_a^b, \end{cases}$$
$$\psi_a(x_a) = \begin{cases} \bar{p} \cdot (x_a - e_a) & \text{if } x_a \in X_a^b, \\ \infty & \text{if } x_a \notin X_a^b. \end{cases}$$

Again, these functions are convex and lower semicontinuous by virtue of (A1) and (A2), so ψ_a^ν epi-converges to ψ_a for the reasons already mentioned, coming from [14, Theorem 7.17]. On the basis of ($b^{\nu+}$), we have $x_a^\nu \in \operatorname{argmin} \psi_a^\nu$, and can conclude through [14, Theorem 7.33] that $\bar{x}_a \in \operatorname{argmin} \psi_a$. That tells us that $\bar{x}_a$ minimizes $\bar{p} \cdot x_a$ subject to $x_a \in X_a^b$, and since $\bar{x}_a < b$, it establishes that (b^-) holds. Thus, agent a is a barely surviving agent.

Finally, we confirm that the agents can't all be just barely surviving. If indeed $\mathcal{A}_- = \mathcal{A}$, we would have

$$\inf\Big\{ \bar{p} \cdot \textstyle\sum_{a\in\mathcal{A}} x_a \,\Big|\, x_a \in X_a \Big\} = \bar{p} \cdot \textstyle\sum_{a\in\mathcal{A}} e_a.$$

But that's incompatible with our assumption (S2), inasmuch as $\bar{p} \neq 0$.

In summary, we have demonstrated that $\bar{p}$ and $\{\bar{x}_a\}_{a\in\mathcal{A}}$ provide a virtual equilibrium as in Definition 2, in which moreover the agents $a \in \mathcal{A}_-$ are barely surviving, whereas the agents $a \in \mathcal{A}_+$ are optimizing and have the limits $\bar{\lambda}_a$ as utility scale factors. □

5. EXAMPLES

Illustrations will now be provided of the distinctions between the various equilibrium concepts in Definitions 1 and 2 and how they relate to the existence result in Theorem 1 and the iterative scheme addressed in Theorem 3.

In these examples, we have just two goods and two agents: here $l = 2$ and $\mathcal{A} = \{1, 2\}$. Price vectors have the form $p = (p_1, p_2)$ with $p_1 \geq 0$, $p_2 \geq 0$ and $p_1 + p_2 = 1$. Agent $a = 1$ has an endowment vector $e_1 = (e_{11}, e_{12})$ and chooses a consumption vector $x_1 = (x_{11}, x_{12})$ with utility $u_1(x_{11}, x_{12})$ from a survival set $X_1 \subset I\!R^2_+$, whereas agent $a = 2$ has an endowment vector $e_2 = (e_{21}, e_{22})$ and chooses a consumption vector $x_2 = (x_{21}, x_{22})$ with utility $u_2(x_{21}, x_{22})$ from a survival set $X_2 \subset I\!R^2_+$.

Example 1 (a classical equilibrium without strict feasibility). *Let* $X_1 = I\!R^2_+$ *and* $X_2 = I\!R^2_+$*, and take*

$$\begin{cases} e_1 = (1,1), & u_1(x_{11},x_{12}) = x_{11}, \\ e_2 = (1,0), & u_2(x_{21},x_{22}) = x_{21} + x_{22}. \end{cases}$$

In this case there is an $x_1 \in X_1$ *with* $x_1 < e_1$*, but no* $x_2 \in X_2$ *with* $x_2 < e_2$*. Nonetheless, a classical equilibrium exists, given by*

$$\bar{p} = (1/2, 1/2), \quad \bar{x}_1 = (2,0), \quad \bar{x}_2 = (0,1).$$

There is no other equilibrium, even two-tier. The iterative scheme, applied to this data, would necessarily converge to the unique classical equilibrium.

Detail. Here $p_2 = 1 - p_1$, so $p = (p_1, 1 - p_1)$ with $0 \le p_1 \le 1$. For agent $a = 1$ the utility maximizing set is

$$\begin{aligned} M_1 &= \operatorname{argmax}\{\, u_1(x_1) \,|\, x_1 \in X_1,\ p\cdot x_1 \le p\cdot e_1\} \\ &= \operatorname{argmax}\{\, x_{11} \,|\, x_{11} \ge 0,\ x_{12} \ge 0, \\ &\qquad\qquad\qquad\qquad p_1 x_{11} + (1-p_1)x_{12} \le 1\} \\ &= \begin{cases} \emptyset & \text{if } p_1 = 0, \\ \{\,(p_1^{-1}, 0)\} & \text{if } p_1 > 0, \end{cases} \end{aligned}$$

whereas for agent $a = 2$ the utility maximizing set is

$$\begin{aligned} M_2 &= \operatorname{argmax}\{\, u_2(x_2) \,|\, x_2 \in X_2,\ p\cdot x_2 \le p\cdot e_2\} \\ &= \operatorname{argmax}\{\, x_{21} + x_{22} \,|\, x_{21} \ge 0,\ x_{22} \ge 0, \\ &\qquad\qquad\qquad\qquad p_1 x_{21} + (1-p_1)x_{22} \le p_1\} \\ &= \begin{cases} \emptyset & \text{if } p_1 = 0, \\ \{(1,0)\} & \text{if } 0 < p_1 < 1/2, \\ \{\,(\tau, 1-\tau) \,|\, 0 \le \tau \le 1\} & \text{if } p = 1/2, \\ \{(0, (1-p_1)^{-1})\} & \text{if } 1/2 < p_1 < 1, \\ \emptyset & \text{if } p_1 = 1. \end{cases} \end{aligned}$$

The total endowment $e_1 + e_2$ is $(2,1)$, so the condition for market clearing is

$$\begin{cases} x_{11} + x_{21} \le 2, & \text{with equality if } p_1 > 0, \\ x_{12} + x_{22} \le 1, & \text{with equality if } p_1 < 1. \end{cases} \tag{19}$$

Having $p_1 = 0$ or $p_1 = 1$ in a classical equilibrium is excluded by the emptiness then of M_2, so any candidates would have to have $0 < p_1 < 1$ and obey both of the inequalities in (19) as equations. In choosing (x_{11}, x_{12}) from M_1 and (x_{21}, x_{22}) from M_2, it's impossible to get the second of these equations satisfied when $0 < p_1 < 1/2$, or to get the first satisfied when $1/2 < p_1 < 1$. Hence the only available candidate is $p_1 = 1/2$. And indeed, for $\bar{p} = (1/2, 1/2)$ we can take $\bar{x}_1 = (2,0)$ from M_1 and and $\bar{x}_2 = (0,1)$ from M_2 and have $\bar{x}_1 + \bar{x}_2 = (2,1)$, as required for a classical equilibrium.

This is the only possibility for a classical equilibrium, but what about a two-tier equilibrium more generally? The investigation of that requires us to look at the set of cheapest consumption vectors, which here happens to be the same for both agents:

$$\begin{aligned} M_- &= \mathrm{argmin}\{\, p \cdot x_1 \,|\, x_1 \in X_1\} = \mathrm{argmin}\{\, p \cdot x_2 \,|\, x_2 \in X_2\} \\ &= \begin{cases} \{\,(\tau, 0) \,|\, \tau \geq 0\} & \text{if } p_1 = 0, \\ \{\,(0,0)\} & \text{if } 0 < p_1 < 1, \\ \{\,(0, \tau) \,|\, \tau \geq 0\} & \text{if } p_1 = 1. \end{cases} \end{aligned}$$

In a two-tier equilibrium with both agents barely surviving, both (x_{11}, x_{12}) and (x_{21}, x_{22}) would be selected from M_-. Thus, both would have 0 in the first component if $p_1 > 0$, or both would have 0 in the second component if $p_1 < 1$, which would be inconsistent with (19), no matter how p_1 is selected.

For a two-tier equilibrium with agent $a = 1$ barely surviving and agent $a = 2$ optimizing, we would need to satisfy (19) with a choice of $(x_{11}, x_{12}) \in M_-$ and $(x_{21}, x_{22}) \in M_2$. Again, the cases $p_1 = 0$ and $p_1 = 1$ are excluded by the emptiness of M_2 for those values, but on the other hand, when $0 < p_1 < 1$ we are forced to take $(x_{11}, x_{12}) = (0, 0)$, and yet both of the conditions in (19) are required to be fulfilled as equations. But there is no way to choose p_1 to get $(x_{21}, x_{22}) \in M_2$ with $(x_{21}, x_{22}) = (2, 1)$.

For a two-tier equilibrium with agent $a = 1$ optimizing and agent $a = 2$ barely surviving, we would need (19) to hold for some $(x_{11}, x_{12}) \in M_1$ and $(x_{21}, x_{22}) \in M_-$. Because $M_1 = \emptyset$ when $p_1 = 0$, we are limited to $0 < p_1 \leq 1$ and $(x_{11}, x_{12}) = (p_1^{-1}, 0)$, with at least the first condition in (19) holding as an equation. Since x_{21} has to be 0 when $p_1 > 0$, we can only get this equation with $p_1 = 1/2$, but then the second condition in (19) must hold as an equation too, even though x_{22} has to be 0. Thus, this mode of equilibrium is impossible as well. □

Example 2 (a nonclassical virtual equilibrium along with other equilibria)**.** *Let $X_1 = \mathbb{R}^2_+$ and $X_2 = \mathbb{R}^2_+$ and take*

$$\begin{cases} e_1 = (1,1), & u_1(x_{11}, x_{12}) = x_{11}, \\ e_2 = (0,1), & u_2(x_{21}, x_{22}) = x_{21} + x_{22}. \end{cases}$$

In this case there is no classical equilibrium, but two-tier equilibria in which agent $a = 1$ is optimizing and agent $a = 2$ is barely surviving are furnished by

$$\bar{p} = (1, 0), \quad \bar{x}_1 = (1, 0), \quad \bar{x}_2 = (0, \theta), \qquad \textit{for any } \theta \in [0, 2]. \tag{20}$$

These are the only two-tier equilibria, and among them, only the one for $\theta = 2$ is a virtual equilibrium. That unique virtual equilibrium, with utility scaling, must be the limit of any sequence of vectors p^ν and x_a^ν generated by the iterative scheme.

Detail. This is close in many respects to Example 1, having the same sets M_1 and M_- and only a coordinate-switched version of M_2, namely

$$\begin{aligned} M_2' &= \operatorname{argmax}\{\, u_2(x_2) \,|\, x_2 \in X_2,\ p \cdot x_2 \le p \cdot e_2 \} \\ &= \operatorname{argmax}\{\, x_{21} + x_{22} \,|\, x_{21} \ge 0,\ x_{22} \ge 0, \\ &\qquad\qquad p_1 x_{21} + (1-p_1) x_{22} \le 1 - p_1 \} \\ &= \begin{cases} \emptyset & \text{if } p_1 = 0, \\ \{((p_1^{-1} - 1, 0)\} & \text{if } 0 < p_1 < 1/2, \\ \{\, (\tau, 1-\tau) \,|\, 0 \le \tau \le 1 \} & \text{if } p = 1/2, \\ \{(0,1)\} & \text{if } 1/2 < p_1 < 1, \\ \emptyset & \text{if } p_1 = 1. \end{cases} \end{aligned}$$

The total endowment $e_1 + e_2$ is $(1, 2)$, so now the condition for market clearing takes the form

$$\begin{cases} x_{11} + x_{21} \le 1, & \text{with equality if } p_1 > 0, \\ x_{12} + x_{22} \le 2, & \text{with equality if } p_1 < 1. \end{cases} \tag{21}$$

A classical equilibrium requires $0 < p_1 < 1$ because of the emptiness otherwise of M_2', and therefore two equations in (21). That can't be met; no choice of $(x_{11}, x_{12}) \in M_1$ and $(x_{21}, x_{22}) \in M_2'$ can yield $x_{12} + x_{22} \ge 1$.

A two-tier equilibrium with both agents barely surviving is impossible for the reasons already explained in Example 1. A two-tier equilibrium with agent $a = 1$ barely surviving and agent $a = 2$ optimizing is likewise impossible for the reasons seen earlier.

A two-tier equilibrium with agent $a = 1$ optimizing and agent $a = 2$ barely surviving does turn out to be possible, however. For this, we need $0 < p_1 \le 1$ in order to avoid M_1 being empty. But $0 < p_1 < 1$ would make the choice of $(x_{21}, x_{22}) \in M_-$ reduce to $(0, 0)$ while requiring two equations in (21), which doesn't work. In taking $p_1 = 1$, we merely have to satisfy the first condition in (21) with equality. The only vector in M_1 is $(1, 0)$, whereas M_- consists of the vectors $(0, \tau)$ with $\tau \ge 0$. We have (21) fulfilled when $0 \le \tau \le 2$.

In view of Theorem 2 (and Theorem 3), at least one of these two-tier equilibria must be a virtual equilibrium, but which? To sort that out, we have to inspect the possibilities for having a classical equilibrium when the endowment vectors e_1 and e_2 are perturbed to

$$e_1^\varepsilon = (1 + \varepsilon_{11}, 1 + \varepsilon_{12}), \qquad e_2^\varepsilon = (\varepsilon_{21}, 1 + \varepsilon_{22}),$$

where the increments are all ≥ 0. The calculations focus then on the set

$$\begin{aligned} M_1^\varepsilon &= \operatorname{argmax}\{\, u_1(x_1) \,|\, x_1 \in X_1,\ p \cdot x_1 \le p \cdot e_1^\varepsilon \} \\ &= \operatorname{argmax}\{\, x_{11} \,|\, x_{11} \ge 0,\ x_{12} \ge 0, \\ &\qquad\qquad p_1 x_{11} + (1-p_1) x_{12} \le 1 + p_1 \varepsilon_{11} + (1-p_1)\varepsilon_{21} \} \\ &= \begin{cases} \emptyset & \text{if } p_1 = 0, \\ \{\, (p_1^{-1}(1 + p_1 \varepsilon_{11} + (1-p_1)\varepsilon_{21}), 0)) \} & \text{if } p_1 > 0, \end{cases} \end{aligned}$$

for agent $a = 1$ and the set

$$\begin{aligned} M_2^\varepsilon &= \operatorname{argmax}\{\, u_2(x_2) \,|\, x_2 \in X_2,\ p \cdot x_2 \le p \cdot e_2^\varepsilon \} \\ &= \operatorname{argmax}\{\, x_{21} + x_{22} \,|\, x_{21} \ge 0,\ x_{22} \ge 0, \\ &\qquad\qquad p_1 x_{21} + (1-p_1) x_{22} \le p_1 \varepsilon_{21} + (1-p_1)(1+\varepsilon_{22}) \} \\ &= \begin{cases} \emptyset & \text{if } p_1 = 0, \\ \{\, (\varepsilon_{21} + p_1^{-1}(1-p_1)(1+\varepsilon_{22}), 0) \} & \\ & \text{if } 0 < p_1 < 1/2, \\ \{\, (\tau, 1-\tau) \,|\, 0 \le \tau \le 1 \} & \text{if } p = 1/2, \\ \{\, (0, 1 + p_1(1-p_1)^{-1} \varepsilon_{21} + \varepsilon_{22})) \} & \text{if } 1/2 < p_1 < 1, \\ \emptyset & \text{if } p_1 = 1, \end{cases} \end{aligned}$$

for agent $a = 2$. The perturbed total endowment is

$$e_1^\varepsilon + e_2^\varepsilon = (1 + \varepsilon_{11} + \varepsilon_{21}, 2 + \varepsilon_{12} + \varepsilon_{22}),$$

and the market clearing conditions come out therefore as

$$\begin{cases} x_{11} + x_{21} \le 1 + \varepsilon_{11} + \varepsilon_{21}, & \text{with equality if } p_1 > 0, \\ x_{12} + x_{22} \le 2 + \varepsilon_{12} + \varepsilon_{22}, & \text{with equality if } p_1 < 1. \end{cases} \tag{22}$$

Once more the cases where $p_1 = 0$ or $p_1 = 1$ can be eliminated because of emptiness in M_2^ε, so we must have $0 < p_1 < 1$ along with equality in both of the conditions in (22). This can be achieved with

$$\begin{aligned} p^\varepsilon &= (1 + \varepsilon_{12} + \varepsilon_{21})^{-1}(1 + \varepsilon_{12}, \varepsilon_{21}), \\ x_1^\varepsilon &= (1 + \varepsilon_{11} + \varepsilon_{12}, 0), \quad x_2^\varepsilon = (0, 2 + \varepsilon_{21} + \varepsilon_{22}), \end{aligned}$$

when $\varepsilon_{21} > 0$, but not otherwise. As the increments tend to 0, the only possible limit of such classical equilibria is the two-tier equilibrium in (20) for $\theta = 2$. Hence that is the unique virtual equilibrium, and the iterative scheme must converge to it. □

Other insights can be gleaned from Example 2 as well. The utility scale factors associated with agent $a = 2$ in the perturbed equilibria must tend to ∞ as the increments go to 0, specifically as $\varepsilon_{21} \to 0$. If that were not the case, the virtual equilibrium obtained in the limit would actually be a classical equilibrium. This feature of the iterative scheme came out in the proof of Theorem 3. The interpretation is that, as the amount of good 1 available to agent 2 shrinks to nothing, the interest of agent 2 in acquiring some of good 1 increases without bound. The scaling between utility for agent 2 and the relative prices at equilibrium blows up. This emerges as the essential reason why agent 2 ends up barely surviving without optimizing, even though, with an infinitesimal amount of good 1, optimization would be possible.

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