THE ROBUST STABILITY OF EVERY EQUILIBRIUM IN ECONOMIC MODELS OF EXCHANGE EVEN UNDER RELAXED STANDARD CONDITIONS

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Abstract. In an economic model of exchange of goods, the preference structure can be specified by utility functions. Under utility conditions identified here more broadly than usual, except for concavity in place of quasi-concavity, every equilibrium will be stable in a doubly local sense with respect to shifts in the agent's holdings and Walrasian tâtonnement. This result, fully allowing the boundary of the goods orthant to come into play, is obtained by paying attention not only to prices but also to the closeness of initial holdings to equilibrium holdings.

The utility conditions are classically standard for stability investigations, in that they invoke properties coming from second derivatives, but are significantly relaxed in not forcing all goods to be held in positive amounts. Agents can be more than consumers, and the goods are viewed very generally, not just as commodities and not only for immediate disposal. For a given agent some goods are allowed to have no effect at all on utility, while others, although insatiably interesting, may anyway end up at zero in equilibrium. Recent advances in variational analysis provide the support needed for working in that context, which requires in particular a convenient "ample" survivability condition for existence to replace the usual assumption that agents start with at least a little bit of every good.

The stability results also point the way toward further developments in which an equilibrium might evolve in response, say, to incremental consumption or inputs in the agents' holdings as stockpiles or other sources of benefit.

Key Words. General equilibrium, exchange equilibrium, comparative statics, Walrasian tâtonnement, robust stability, shift stability, tâtonnement stability, evolution of equilibrium, monotonicity, variational analysis

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1 Introduction

The general theory of equilibrium is fundamental in mathematical economics, but its biggest success has been in confirming existence. Questions about how an equilibrium might be identified by some market mechanism such as Walrasian tâtonnement, or how readily it might adjust to shifts in the initial holdings of the agents, have been harder to answer. Yet such questions are crucial to assessing whether the concept of equilibrium, as defined and developed so far, is truly satisfactory.[1] Examples of bizarre configurations of equilibrium prices and instability in the face of changes in resources have led researchers to fall back on merely generic forms of answers. Technical limitations have prevented realistic assessment of the effects of survival constraints on the adjustment behavior of agents, even generically.

Here we provide affirmative answers which go beyond such limitations. We show that equilibrium in models of exchange enjoys properties of stability which are much more robust and universal than the existing literature might lead one to expect, if viewed from the proper perspective of local behavior. The results are obtained under assumptions on preference structure that are significantly relaxed from the usual ones in this subject. The key is utilization of methodology from variational analysis, [40], [16], which makes it feasible to treat stability issues effectively when the amounts of some goods in an agent's holdings might be zero in equilibrium or out of it, or might repeatedly hit the boundary of the goods space during a process of adjustment. The tools of smooth differential analysis could not cope with that and anyway have promoted conclusions about uniqueness and stability that are only guaranteed to hold generically; cf. the books of Mas-Colell [35] (1985), Balasko [7] (1988) and [9] (2009). The one restrictive aspect of our assumptions is that we take utility functions to be concave instead of quasi-concave.[2] This enables us to work more effectively with Lagrange multipliers for budget constraints and to rely on the duality theory of convex optimization in developing a simple alternative to the common requirement that initial holdings be positive in all goods.

A conceptual distinction accompanying our approach is that goods can be more than just "commodities," and agents can be more than just consumers. What might happen to the holdings amassed by the agents is left unsaid. That opens the way to new ideas of how such holdings might evolve in time, as will be explained below.

In models of exchange where agents maximize utility under budget constraints dictated by prices and initial holdings, there are demand mappings that go from prices and initial holdings to desired holdings. An equilibrium is achieved when the desired holdings, in aggregate, are in balance with the initial holdings, and the issue is whether a price vector can bring this about. A prime object of study is the mapping from the initial holdings to such a price vector, or vectors, if any—there might be multivaluedness. However, it is essential for our efforts to include the desired holdings along with the prices, thereby getting an equilibrium mapping from initial

[1] The importance of stability issues in economic equilibrium was strongly emphasized early-on by Samuelson [42] (1941) and recently all the more by Kirman [31] (2011).

[2] Preferences coming from a quasi-concave utility can be approximated arbitrarily closely by those coming from a concave utility; see Kannai [28] (1977). On the other hand, finite demand data cannot test the difference; see Brown and Shannon [12] (2008).

holdings to prices and terminal holdings combined. In that picture, a combination of prices and holdings is an *equilibrium* if those holdings, as inputs, lead to themselves as outputs.[3]

The important thing to keep in mind from this perspective is that a given equilibrium combination of prices and holdings, although induced by those holdings themselves, can also arise from many other instances of initial holdings, in fact from any that offer the same total quantities of goods without changing the agents' budgets with respect to the equilibrium prices. The behavior of the equilibrium mapping can be very different for some of those instances than for others. This has perhaps clouded some of the perceptions about whether "stability" is a property of equilibrium that can more or less be counted upon. Poor behavior is well documented and has provided a discouraging undercurrent to research.[4]

An instructive example of Dontchev and Rockafellar [17] (2010) has two agents, two goods, and strongly concave utilities. An instance of initial holdings is given which leads to two different equilibrium configurations. One of them behaves very nicely with respect to perturbations of the initial holdings, but the other either bifurcates or abruptly vanishes, depending on the direction of perturbation. Does this mean that the second equilibrium is unstable? Not necessarily, because, in our view, "stability" ought to tied to an equilibrium in itself instead of being something "accidental" in the singling out a particular choice of initial holdings that gives rise to it. Indeed, as demonstrated further for this example in [17], if the initial holdings are taken closer to those in the second equilibrium, the trouble goes away. The second equilibrium becomes the unique equilibrium for those alternative initial holdings and behaves nicely with respect to their perturbation. The potential of this proximity phenomenon seems largely to have gone unrecognized despite some results about it published by Balasko [6] and Sattinger [43] in 1975.

These considerations are all the more crucial in appreciating difficulties associated with Walrasian tâtonnement, which tries to identify an equilibrium by gradually adjusting prices down or up in reaction to excess supply or demand as indicated by tentative trading responses of the agents. A classical model for this is an ordinary differential equation in a space of price vectors as formulated from the demand functions of the agents. If those functions are adequately differentiable, traditional stability analysis around an equilibrium price vector (this being a stationary point for the system) can be carried out in terms of the matrix in the linearization of the equation at that point. That approach, seen early in Hicks [20] (1939) and Metzler [37] (1945), led further to the investigation of various matrix conditions with added meaning for economics, as in Arrow and McManus [5] (1958). Other research, as in Arrow and Hahn [2] (1971), explored non-Walrasian adjustment processes which go beyond the development of supply-demand information and enter into iterative trading; cf. Keisler [30] (1996) and its references.[5]

Scarf [44] (1960) gave the counterexamples that became a major turning point for hopes of

[3]We prefer "holdings" to "endowments" as better reflecting this possibly double environment.

[4]Whether such behavior is possible with concave utility is unsettled. Examples like those of Debreu [14] and Mas-Colell [34] with strange infinite sets of equilibrium price vectors have only quasi-concave utility. Concave utility imposes more regularity; cf. Mas-Colell [35] and Kannai [28]. It is interesting to speculate that an updated approach to the axiomatization of preferences which addresses comparative utility as in Kahneman and Tversky [27] might lead directly to concave utility and its advantages.

[5]The complicated history of interpretation of the original ideas of Walras [49] (1874) is discussed in this article as well. See also Walker [48] (1987).

convergence of Walrasian tâtonnement in the manner in which it had been envisioned.[6] The process might get nowhere, regardless of how close the initial prices are to equilibrium prices. Saari [41] (1985) underscored the challenges faced by almost any price adjustment mechanism in achieving successful performance globally, unless the economy had very special characteristics like gross substitutability. A separate blow to hopes for a reassuringly broad convergence result came with the realization that the differential equation behind tâtonnement suffered hardly any restriction in having to emerge from an economic model of supply and demand.[7]

Most of the convergence attention was focused however on the starting point for prices without consideration of the starting point for holdings. The initial holdings, which dictate the budgets on the basis of the prices, are kept fixed throughout the adjustment process, and the trouble with convergence may therefore come from them simply being too far out of kilter. While it would be good news if tâtonnement furnished a sort of globally effective algorithm for finding an equilibrium from arbitrary initial circumstances, this is evidently too much to ask of it.[8] The question of whether the properties of the tâtonnement differential equation might nonetheless be strongly influenced by localization with respect to *both goods and prices* stayed below the surface of discussion despite the observations of Balasko [6] and Sattinger [43].[9]

Here, we pose the question about tâtonnement in the framework of a stability property which may or may not be enjoyed by an equilibrium, independently of the different instances of initial holdings that would lead to it. Focusing on the classical continuous-time version of tâtonnement as an ordinary differential equation, we define an equilibrium to be tâtonnement-stable if the solution trajectory converges to the price vector in that equilibrium when initiated not only from prices not too far away, but also from initial holdings not too far away from those in the equilibrium. This property has a clear economic implication. If the holdings of the agents in the equilibrium were redistributed in some way, but not by too much, then tâtonnement, starting from the current prices and the altered holdings, would be able to re-identify the lost equilibrium. An equilibrium with tâtonnement stability would thus have a certain attractivity. In combination with shift stability, tâtonnement could similarly even locate a new equilibrium after a shift of goods that is more than just a redistribution of holdings.

Shift stability is reminiscent of, but distinct from "regularity" of an economy, e.g. as in Mas-Colell [35], which is featured in arguments on genericity. Such regularity relies on some degree of smoothness for its very definition and is inoperable as a concept in situations where nonnegativity

[6]The microeconomics textbooks of Hildenbrand and Kirman [21] and Mas-Colell et al. [36, Chapter 17.H] further explore many of the things that could go wrong.

[7]The recent volume of articles put together by Brown and Kubler [11](2008) is a remarkable eye-opener in that regard, although inklings were present much earlier in the results of Debreu [15] and others.

[8]Analogy can be made with the simpler situation where a function to be minimized is not convex and may have have many isolated points that are locally optimal. In the vast computational literature on this, most algorithms are only locally effective.

[9]Balasko's 1975 theorem was extended by Keenan [29] in 1982, but reference to it can hardly be found elsewhere in equilibrium literature. It is not cited by Hirota [22] (1981), where an example indicating such localization influence was offered. More recently, the paper of Brown and Shannon [12] (2008), containing insights into the extent that "rationalization" of an economy from finite data can be carried out so as to promote tâtonnement, presents the convergence issue in terms of prices only; no localization in goods comes up.

constraints on goods can be active, as in the our framework here. Shift stability builds instead on recent results of Dontchev and Rockafellar [17] which are not of a generic character. Those results will be key ingredients in our efforts here. A greatly weakened survivability assumption, shown in [17] to be enough for the existence of an equilibrium, will help crucially as well.

The conclusion we arrive at, that both stability properties prevail in surprisingly broad circumstances, suggests that classical equilibrium is a more satisfying notion than opinion has generally allowed, at least for exchange economies when localization in goods as well as prices is appreciated. A stronger point still is that we reach this conclusion without resorting to some of the unrealistic restrictions on goods that have dominated many studies. We fully encompass the prospect that the attitudes of agents towards particular goods can differ sharply, with some goods having no effect on utility at all and some being indispensable. Others can be attractive without being indispensable and can be present in zero amounts in equilibrium, or initially, or both.[10] These provisions are handled without disrupting the existence of equilibrium or the two types of stability. Equilibrium in this setting cannot conveniently be reduced to a system of smooth equations. A variational inequality model works instead, and that is what we build on in our proofs.

Our double form of local stability of equilibrium entails a local uniqueness with the potential for orderly change through exogenous or endogenous influences. We explore this by considering the possibility of a law of continuous-time evolution in which adjustments to equilibrium are induced incrementally. The increments could enter through subsidies or production effects, or in negative mode through taxes and rates of consumption or deterioration of goods. Rich possibilities are evident without the need, already here, to develop a full-blown model that prescribes them. Such a model might, for instance, address the dynamical control of equilibrium in a principal-agent formulation.

2 Statement of assumptions and the main results

Proceeding toward a precise formulation, we take the nonnegative orthant $I\!R^{n+1}_+$ as the space of goods[11] and suppose that the agents, indexed by $i = 1, \dots, r$, have preferences on it which are given by utility functions u_i. To fully appreciate the equilibrium context we are aiming at, it is important to keep in mind that the goods can be very general, not just commodities destined for consumption. They can be anything physical, or perhaps even "rights," that an agent might wish to acquire and are available for trading in fixed supply. The question of what an agent might do with them is separate and will be revisited later.

[10] After all, in an economy with a potentially huge number of goods, agents should be able to display total disinterest in some of them and, on other hand, forgo other goods of some interest because other goods are still more compelling. A theory of equilibrium that assumes this away in order to concentrate only on mathematically "regular" cases, even if they are somehow generic, risks a lack of plausibility.

[11] Having $n+1$ instead of n will shortly be seen to help in the presentation. We could just as well work with survival sets in the form of displaced orthants specifying various nonnegative lower bounds on the goods required by the agents. But that can be reduced to the basic orthant case by a change of variables, so for the sake of a simpler presentation we leave this as an obvious *implicit* enhancement.

Assumption A1 (utility fundamentals). *Each utility function u_i on $I\!R_+^{n+1}$ is nondecreasing, concave*[12] *and upper semicontinuous. It may take on $-\infty$, but if so, only at points on the boundary of $I\!R_+^{n+1}$. Relative to the set where it is finite, u_i is continuous.*[13]

An important provision will depend on classifying goods according to the interest that an agent has in them. A good will be called *attractive* for agent i if every increase in that good leads to a higher value of u_i. It will be called indispensable for agent i if it is attractive and, at any point in which the quantity of that good (but not every attractive good) is zero, either u_i takes on $-\infty$ or u_i is finite but the marginal utility of the good is $+\infty$.[14]

Assumption A2 (indispensability). *There is a good that is indispensable to all agents. Every good is indispensable to at least one agent.*

The presence of a good that is indispensable to all agents will have an central role in what follows. It implies insatiability of all the utility functions, but it will have other major consequences as well. Classically standard assumptions, requiring the level surfaces of utility to curve away from orthant boundaries, actually force *every* good to be indispensable to *every* agent. We need just one such good and will work with it as a numéraire.

In our picture, some goods, far from being indispensable, can fail to be attractive at all to agent i. Other goods can be attractive without being indispensable. Our next condition sharpens the distinction.

Assumption A3 (unattractiveness). *If a good is not attractive to agent i, then it has no effect on the utility function u_i.*

In this condition we forgo the possibility of goods that might have positive marginal utility up to some level but zero marginal utility thereafter.

Assumption A4 (partial strict concavity). *With respect to the suborthants of the goods space that are defined by*

$$O_i = \{ \text{ vectors in } I\!R_+^{n+1} \text{ having positive components for goods indispensable to agent } i \}, \quad (1)$$

the utility functions u_i are twice continuously differentiable.[15] *Furthermore, the Hessian matrices, formed by the second partial derivatives, are negative definite with respect to the goods that are attractive to agent i.*[16]

[12]The distinction between concave and quasi-concave is vital here. For results about approximating quasi-concave utility by concave utility while taking into account various properties of strictness and differentiability, such as enter below, see Kannai [28], Mas-Colell [35, Chapter 2], and recently Connell and Rasmusen [13].

[13]Continuity on the interior of the orthant, where u_i is surely finite, is automatic from concavity, so this technical provision refers only to boundary behavior.

[14]Marginal utility refers here to the one-sided directional derivative with respect to an increase in the good in question. That derivative exists from the concavity.

[15]By this we mean that first and second partial derivatives not only exist continuously on the interior of O_i, namely the positive orthant itself, but also that these derivatives can be extended continuously, through limits, to the boundary points belonging to O_i.

[16]This refers to the submatrix of the Hessian obtained by excluding the goods that are not attractive. Under A3, such goods only yield zero derivatives anyway.

In contrast to A4, it would classically be standard—with concave utility—to insist on the entire Hessian being negative definite. This would be combined with limiting attention to the interior of the goods space; the orthant boundary would not be allowed to complicate the analysis. But for economic theory to be more realistic, it ought to be permitted to do so.

With these assumptions at our disposal, we can move to more specific symbolism and formulations of stability. We concentrate on a particular good that is indispensable to all agents, as guaranteed by A2, calling it *money* for short.[17] [18] We designate quantities of money in the hands of agent i by m_i, and vectors giving quantities of the other goods by x_i, so that the elements of the goods space have the form (m_i, x_i) with $m_i \in \mathbb{R}_+$ and $x_i \in \mathbb{R}^n_+$. Initial holdings will have the notation (m_i^0, x_i^0).

Prices will always be denominated in money. Since money has price 1 with respect to itself, we will only need to be concerned with price vectors $p \in \mathbb{R}^n_+$ for the remaining goods.

Utility maximization problems. *The goal of agent i with respect to a price vector p and initial holdings $(m_i^0, x_i^0) \in O_i$ is to maximize the utility $u_i(m_i, x_i)$ over all goods vectors (m_i, x_i) satisfying the budget constraint*

$$m_i + p{\cdot}x_i = m_i^0 + p{\cdot}x_i^0. \tag{2}$$

In dealing often with agents collectively, it will be expedient to use the "supervector" notation

$$(m, x) \text{ for } m = (\ldots, m_i, \ldots),\; x = (\ldots, x_i, \ldots),$$

and similarly (m^0, x^0) in the case of initial holdings, and so forth.

Definition of equilibrium. *An equilibrium is a triple $(\bar{p}, \bar{m}, \bar{x})$ such that each $(\bar{m}_i, \bar{x}_i)$ component solves the utility maximization problem of agent i relative to $\bar{p}$ when $(m_i^0, x_i^0) = (\bar{m}_i, \bar{x}_i)$.*

More generally, $(\bar{p}, \bar{m}, \bar{x})$ is an equilibrium with respect to initial holdings (m^0, x^0), possibly differing from $(\bar{m}, \bar{x})$, written

$$(\bar{p}, \bar{m}, \bar{x}) \in E(m^0, x^0),$$

if each pair $(\bar{m}_i, \bar{x}_i)$ solves the utility maximization problem of agent i relative to $\bar{p}$ and (m_i^0, x_i^0), and moreover

$$\sum\nolimits_{i=1}^r \bar{x}_i = \sum\nolimits_{i=1}^r x_i^0. \tag{3}$$

The goods equation (3) requires supply to equal demand in all the goods other than money. Money can be left out because the corresponding condition

$$\sum\nolimits_{i=1}^r \bar{m}_i = \sum\nolimits_{i=1}^r m_i^0 \tag{4}$$

[17]Our assumptions will guarantee that this good can serve as a numéraire for prices. However, it is possible to go further and interpret it as money in the sense of the printed bills and coins of some currency, e.g., dollars. Such money can be treated as a good because it is available for "trading" in limited supply only. Moreover agents may well consider it indispensable in their desired holdings, as argued in our papers [25] and [26].

[18]In their 1988 textbook [21], Hildenbrand and Kirman likewise appeal to numéraire prices as an aid to the study of tâtonnement, but they do not speak of the numéraire as "money"; the same for MacKenzie [33] (2002). This is reflected also in the earlier work of Uzawa [47]. In the recent paper of Kitti [32] (2010), the reduction to numéraire prices is called price "normalization." For us, however, money does more than normalize. It enters significantly into survivability conditions for existence and other assumptions.

follows at once from (3) and the budget constraints (2). Apart from the money feature, this definition is fairly ordinary, but an equilibrium is often conceived in terms of the price vector alone. We think this is a shortcut which may have lulled researchers into neglecting the very influences of holdings that we find essential to the analysis of stability issues.

Some observations can now be made which will simplify the discussions to come. First,

$$\text{in any equilibrium, all prices must be positive.} \tag{5}$$

This follows from A2 through the fact that if the price of some good were zero, then the maximization problem for an agent considering that good to be indispensable, or even just attractive, could not have a solution. Next,[19]

$$\begin{array}{l}\text{for } p > 0 \text{ the problem of an agent } i \text{ has a unique solution,}\\ \text{and it has to belong to the suborthant } O_i \text{ defined by (1).}\end{array} \tag{6}$$

Indeed, the budget constraint defines a compact set of goods vectors which meets the interior of $\mathbb{R}^{n+1}$, where utility is surely finite. The upper semicontinuity of u_i in A1 guarantees then that the maximum is finitely attained. Quantities of goods that are not attractive are pushed to zero, since otherwise they would drag down the budget available for attractive goods, but goods with infinite marginal utility at zero are forced to be positive. The partial strict concavity guaranteed by A4 then provides the uniqueness.

On the platform of (5) and (6) we can introduce, with respect to price vectors $p > 0$, the demand mappings

$$X_i(p; m_i^0, x_i^0) = \text{ the corresponding unique optimal } x_i \text{ for agent } i, \tag{7}$$

observing that the associated optimal money amount will then be

$$m_i = M_i(p; m_i^0, x_i^0) = m_i^0 + p{\cdot}[x_i^0 - X_i(p; m_i^0, x_i^0)], \tag{8}$$

and the excess demand mapping will be

$$Z(p; m^0, x^0) = \sum\nolimits_{i=1}^{r} \Big[X_i(p; m_i^0, x_i^0) - x_i^0\Big]. \tag{9}$$

In this notation we can say for the mapping E in the definition of equilibrium that, with respect to $\bar{x}_i = X_i(p; m_i^0, x_i^0)$ and $\bar{m}_i = M_i(p; m_i^0, x_i^0)$,

$$(\bar{p}, \bar{m}, \bar{x}) \in E(m^0, x^0) \iff \bar{p} > 0,\ Z(\bar{p}; m^0, x^0) = 0. \tag{10}$$

In particular, $(\bar{p}, \bar{m}, \bar{x})$ is an equilibrium (unto itself) when this holds with $(m^0, x^0) = (\bar{m}, \bar{x})$.

The neighborhoods appearing in the following definitions can be regarded as closed balls with respect to the Euclidean norm $||\cdot||$.

[19] For us, a strict vector inequality refers to a strict inequality in each component.

Definition of shift stability. *An equilibrium* $(\bar{p}, \bar{m}, \bar{x})$ *is shift-stable if there are neighborhoods* N_0 *of* $(\bar{m}, \bar{x})$ *and* N_1 *of* $(\bar{p}, \bar{m}, \bar{x})$ *such that*

$$\text{each } (m^0, x^0) \in N_0 \text{ yields a unique equilibrium } (p, m, x) \in N_1,$$

and the corresponding localized equilibrium mapping

$$E : (m^0, x^0) \in N_0 \mapsto E(m^0, x^0) = (p, m, x) \in N_1, \text{ having } E(\bar{m}, \bar{x}) = (\bar{p}, \bar{m}, \bar{x}), \tag{11}$$

is Lipschitz continuous. The equilibrium is semidifferentiably shift stable if E is not only Lipschitz continuous but also possesses, with respect to all choices of $(m^{0\prime}, x^{0\prime})$*, the one-sided directional derivative*[20]

$$DE(m^0, x^0; m^{0\prime}, x^{0\prime}) = \lim_{h \to 0^+} \frac{1}{h}\Big[E(m^0 + hm^{0\prime}, x^0 + hx^{0\prime}) - E(m^0, x^0)\Big]. \tag{12}$$

A perturbation result that bears closely on this property has recently been obtained in [17, Theorem 3], but not quite in the same framework. That result, not couched as shift stability of the equilibrium $(\bar{p}, \bar{m}, \bar{x})$, is presented as a property of (m^0, x^0) when close enough to $(\bar{m}, \bar{x})$ as long as a something further is satisfied: the initial holdings (m^0, x^0) are *amply survivable*. This condition, shown in [17] to furnish the existence of an equilibrium, is *far weaker* than the customary assumption that initial goods belong to survival set interiors.[21] That perturbation result will be central to our development here, with the new feature being that, in our setting, *any initial holdings* (m^0, x^0) *with components* (m_i^0, x_i^0) *drawn from the suborthants* O_i *will automatically be amply survivable*, and the same then for any equilibrium holdings $(\bar{m}, \bar{x})$, in particular.

Note that the possibility of there being more than one equilibrium associated with (m^0, x^0) from the neighborhood N_0 of $(\bar{m}, \bar{x})$ is not ruled out by shift stability. The requirement is only that there cannot be a second equilibrium in the neighborhood N_1 of $(\bar{p}, \bar{m}, \bar{x})$. For $(m^0, x^0) \notin N_0$ there might be multiple equilibria in N_1, but none could be an equilibrium associated with $(\bar{m}, \bar{x})$.

The one-sided limit in (12) with $h \to 0^+$ refers to h tending to 0 only from above; the classical two-sided directional derivative would have $h \to 0$ with no such restriction. The companion derivative expression with the opposite one-sided limit is tacitly covered by this as well, because

$$\lim_{h \to 0^-} \frac{1}{h}\Big[E(m^0 + hm^{0\prime}, x^0 + hx^{0\prime}) - E(m^0, x^0)\Big] = -DE(m^0, x^0; -m^{0\prime}, -x^{0\prime}).$$

By virtue of the Lipschitz continuity in the definition of shift stability, we get a sort of Taylor expansion of the localized equilibrium mapping:

$$E(m^0 + hm^{0\prime}, x^0 + hx^{0\prime}) = E(m^0, x^0) + hDE(m^0, x^0; m^{0\prime}, x^{0\prime}) + o(h).$$

[20] The primes here do not, themselves, refer to derivatives.

[21] Apart from that strict positivity assumption there are other, more subtle equilibrium-supporting conditions in [1] and later in [18], [19], involving "irreducibility." However, these are all rather unwieldy in comparison with "ample survivability."

This corresponds to the differentiability of E at (m^0, x^0) if and only if $DE(m^0, x^0; m^{0\prime}, x^{0\prime})$ is *linearly* dependent on $(m^{0\prime}, x^{0\prime})$. Without that linearity it still signals *semidifferentiability,*[22] which serves as the tool enabling us to handle the one-sided effects on equilibrium distributions of goods that may be caused by the orthant boundary coming into play.

In turning next to tâtonnement, backed by shift stability, we follow Arrow and Hurwicz [3] in posing it in terms of an ordinary differential equation; see also [4]. Versions in discrete time could similarly be laid out along the lines in [21] and [47], but in our opinion the case of continuous time puts the ideas in sharper focus.[23]

Definition of tâtonnement stability. *An equilibrium $(\bar{p}, \bar{m}, \bar{x})$ is tâtonnement-stable if there is a neighborhood N of $\bar{p}$ and a neighborhood N_0 of $(\bar{m}, \bar{x})$ such that, for all $p^0 \in N$ and all $(m^0, x^0) \in N_0$ having $E(m^0, x^0) = (\bar{p}, \bar{m}, \bar{x})$,*[24] *the differential equation of tâtonnement, namely*[25]

$$\dot{p}(t) = Z(p(t); m^0, x^0) \text{ for } t \geq 0 \text{ with } p(0) = p^0, \tag{13}$$

has a unique solution $p(t)$ converging to $\bar{p}$, while the associated demands $x_i(t) = X_i(p(t); m^0, x^0)$ and $m_i(t) = M_i(p(t); m^0, x^0)$ converge then to $\bar{x}_i$ and $\bar{m}_i$. It is strongly tâtonnement stable if, for such neighborhoods, there is a constant $\mu > 0$ such that

$$(p' - p)\cdot(Z(p'; m^0, x^0) - Z(p; m^0, x^0)) \leq -\mu ||p' - p||^2 \text{ for all } p', p \in N, \ (m^0, x^0) \in N_0. \tag{14}$$

According to the equation (13), the price of a good rises at a rate equal to the current excess demand for that good (which amounts to falling if that is negative).[26] An equilibrium price vector $\bar{p}$, in having $Z(\bar{p}; m^0, x^0) = 0$, furnishes a stationary point for the differential equation (6): starting from $p^0 = \bar{p}$ one would get $p(t) \equiv \bar{p}$ as a solution.

The property in (14), which in the weaker form of having

$$(p' - p)\cdot(Z(p'; m^0, x^0) - Z(p, m^0, x^0)) < 0 \text{ when } p' \neq p, \tag{15}$$

is known as "monotonicity" among economists,[27] has already been recognized as guaranteeing the convergence of $p(t)$ to $\bar{p}$, which is easy to prove,[28] as for instance in the textbook of Hildenbrand and Kirman [21, page 237]. (The convergence of the demands $x_i(t)$ then follows from the

[22] See [40, Chapter 7] for more on semidifferentiability.

[23] Anyway, the issue here is a conceptual property of stability of an equilibrium. It deserves to be seen in simpler terms without getting into a myriad possible variants, which perhaps would not add much in overall understanding to stability theory. See Kitti [32] for a comprehensive discussion of the efforts that have been put into the discrete-time case and the latest accomplishments in that direction.

[24] In the sense of (11); this N_0 shrinks the one there, if necessary.

[25] We employ $\dot{p}(t)$ for the derivative of $p(t)$ in order to preserve primed symbols like p' for other uses.

[26] It is easy to make the rates depend instead on different proportionality coefficients for different goods. However, this requires no additional mathematics because it really amounts only to changing the units of measurement for the goods other than money. With the prices adjusted accordingly, the budgets then come out the same. Arrow and Hurwicz observed this already in [3]. Other formulations in which the right side of (13) depends in extra ways on $p(t)$ have been explored by MacKenzie [33].

[27] There is an unfortunate conflict with long-established terminology in mathematics, according to which (15) is the *strict* monotonicity of $-Z(\cdot; m^0, x^0)$, not $Z(\cdot; m^0, x^0)$. Plain monotonicity would have $\leq$ in place of < 0, whereas (14) is the *strong* monotonicity of $-Z(\cdot; m^0, x^0)$. For an introduction to the remarkable theory of such monotonicity, which comprises a valuable and much applied a branch of convex analysis, see [40, Chapter 12].

[28] A simple tactic is to show by differentiation that $||p(t) - \bar{p}||^2$ is a decreasing function of t that must go to 0.

continuity of the mappings X_i^0, which will be shown later.) From the stronger property in (14) an additional conclusion readily follows about the convergence rate:

$$||p(t) - \bar{p}|| \leq e^{-\mu t}||p^0 - \bar{p}||. \tag{16}$$

Indeed, (14) implies even that two price trajectories $p_1(t)$ and $p_2(t)$ starting from different initial states $\bar{p}_1^0$ and $\bar{p}_2^0$ have $||p_1(t) - p_2(t)|| \leq e^{-\mu t}||p_1^0 - p_2^0||$.

The question of what properties of utility might induce the "monotonicity" of excess demand has received attention from a number of researchers over the years; see the article of Quah [38] (2000) and its references. However, the results in that literature are not applicable in our context. They concern an excess demand mapping which differs from the one in (9) by being defined in terms of a wealth parameter that suppresses the role of initial holdings and their proximity to equilibrium holdings.

It deserves to be emphasized that tâtonnement, as formulated here, does not represent a process in which distributions of goods get adjusted in real time. Rather it is conceived as a scheme for exchange of information in "virtual time" by means of which a Walrasian broker or auctioneer determines prices that will bring supply and demand into balance.[29]

Robust Stability Theorem. *Under assumptions A1, A2, A3 and A4, there exists for any instance of initial holdings (m^0, x^0) having $(m_i^0, x_i^0) \in O_i$ for all agents i at least one equilibrium $(\bar{p}, \bar{m}, \bar{x}) \in E(m^0, x^0)$. Moreover, every equilibrium $(\bar{p}, \bar{m}, \bar{x}) \in E(m^0, x^0)$ is robustly stable in the sense of being both semidifferentiably shift-stable and strongly tâtonnement-stable.*

In this setting the demand mappings X_i and the excess demand mapping Z in tâtonnement are themselves Lipschitz continuous with one-sided directional derivatives, hence semidifferentiable.

This result may help to alleviate worries about the fragility of equilibrium and open new horizons for exploration. The idea that the goods acquired by the agents through trading yield general "holdings" with many conceivable attractions besides immediate "consumption" is crucial for contemplating a broad range of new possibilities. From this angle equilibrium does not need be viewed statically. It can interpreted as modeling an observable phenomenon over time in which supply and demand, with respect to maintaining the agents' holdings, stay close to being in balance, and the balance continuously shifts due to the influence of various factors, both internal and external (but not as yet incorporated into the model).[30]

Those factors could have many forms. A good that stands for a consumable commodity, for which the holding is a sort of stockpile, might be subject to a rate of consumption dictated by an agent's needs, or for that matter, deterioration due to environmental circumstances. Money, as a good, could be taken from an agent through exogenously instituted taxation, or on the other hand enhanced by an ongoing subsidy. And so forth.[31]

[29] This is made especially clear by Uzawa in [47]; see also the text of Mas-Colell [35, page 621].

[30] This vision can be found in the 1941 paper of Samuelson [42] in the era before equilibrium had achieved an adequate mathematical formulation.

[31] This extended view of holdings and their potential persistence also underlies our work with financial market modeling in [26].

The localized (or truncated) mapping $E : (m^0, x^0) \mapsto (p, m, x)$ described in the Robust Stability Theorem can help us clarify what should then happen. To make this specific, imagine we have a equilibrium $(p(t), m(t), x(t))$ at time t,

$$(p(t), m(t), x(t)) = E(m(t), x(t)),$$

but additions to $m(t)$ and $x(t)$ are coming in at rates $m_+(t)$ and $x_+(t)$,[32] which force the equilibrium to evolve. As an approximation over a time increment $h > 0$, the holdings would shift slightly to $m(t) + hm_+(t)$ and $x(t) + hx_+(t)$ and no longer be matched by $p(t)$. However, an adjusted equilibrium

$$(p(t+h), m(t+h), x(t+h)) = E(m(t) + hm_+(t), x(t) + hx_+(t)), \tag{17}$$

identifiable by tâtonnement, will exist uniquely nearby. Then also

$$(p(t+h), m(t+h), x(t+h)) = E(m(t+h), x(t+h)),$$

so the corresponding rate of change in the components of the equilibrium is

$$\begin{aligned} &\tfrac{1}{h}[(p(t+h), m(t+h), x(t+h)) - (p(t), m(t), x(t))] \\ &\qquad = \tfrac{1}{h}[E(m(t) + hm'(t), x(t) + hx'(t)) - E(m(t), x(t))]. \end{aligned}$$

In taking the limit of this as $h \to 0^+$ we obtain, on the right, the one-sided directional derivative $DE(m(t), x(t); m_+(t), x_+(t))$. This gives us the following insight.

Evolution of equilibrium. *Consider an equilibrium $(\bar{p}, \bar{m}, \bar{x})$ and let input rates $(m_+(t), x_+(t))$ depend continuously on t. Then, over a time interval $[0, \tau]$ (with $\tau > 0$ and sufficiently small), an equilibrium trajectory $(p(t), m(t), x(t))$ should exist which is Lipschitz continuous, one-sidedly differentiable, and characterized in terms of the directional derivatives (12) of the localized equilibrium mapping E by the one-sided differential equation*

$$(\dot{p}^+(t), \dot{m}^+(t), \dot{x}^+(t)) = DE(m(t), x(t); m_+(t), x_+(t)), \quad (p(0), m(0), x(0)) = (\bar{p}, \bar{m}, \bar{x}). \tag{18}$$

One-sided, instead of two-sided, differentiability of the trajectory is unavoidable because some goods components in the $x(t)$ trajectory could start at 0, or drop to zero at a later time, only to eventually rise up and perhaps again drop down. For a locally Lipschitz continuous function $y(t)$ (which necessarily has a derivative $\dot{y}(t)$ almost everywhere), one-sided differentiability refers to the existence of the right and left limits

$$\dot{y}^+(t) = \lim_{h \to 0^+} \frac{1}{h}[y(t+h) - y(h)], \qquad \dot{y}^-(t) = \lim_{h \to 0^-} \frac{1}{h}[y(t+h) - y(h)], \tag{19}$$

in which case $\dot{y}(t+h) \to \dot{y}^+(t)$ as $h \to 0^+$ and $\dot{y}(t+h) \to \dot{y}^-(t)$ as $h \to 0^-$; differentiability corresponds to having $\dot{y}^+(t) = \dot{y}^-(t)$.

[32]Some components in these input rate vectors could be negative, thereby acting as subtractions.

The one-sidedness in the right side of the differential equation (18) brings up a need for mathematical innovation and casts our assertion about evolution into more of a conjecture than a theorem. But there is the interesting prospect that, within the confines of the model, both the prices and holdings in an equilibrium will evolve in time according to a fixed rule, dictated only by the utility functions of the agents, in response to internally/externally driven inputs. For instance, what might be expected if agents had their money holdings "controlled" by a government through subsidies or taxation? Of course, the limitations of the idea are indeed many, and most important among them is the absence in this formulation of any modeling of uncertainty over the future.

We wish to reiterate that the complications in (18) with one-sidedness originate with the imperative of letting the quantities of attractive goods sometimes be 0. Around an equilibrium with everything positive, as compelled in particular by the classical inability to handle the orthant boundary in stability analysis, the differential equation (18) loses its one-sided aspects and takes the ordinary form

$$(\dot{p}(t), \dot{m}(t), \dot{x}(t)) = DE(m(t), x(t); m_+(t), x_+(t)), \qquad (p(0), m(0), x(0)) = (\bar{p}, \bar{m}, \bar{x}),$$

in which E is continuously differentiable instead of just semidifferentiable. Standard theory of differential equations is applicable then, and evolution passes from just being a conjecture. We do then have a solid theorem.

3 Proof of the robust stability theorem, first part

Everything is much easier in the setting of *full indispensability* of goods, where boundary effects are avoided. Direct arguments for that case are provided in the Appendix for readers who may wish to understand the simpler situation first, before getting into newer methodology.

For the general case, the assertions about shift-stability will be an application of the stability result of Dontchev and Rockafellar [17, Theorem 3]. First, we must deal with the claim that, for any choice of initial holdings (m^0, x^0) with $(m_i^0, x_i^0) \in O_i$ there exists at least one corresponding equilibrium $(\bar{p}, \bar{m}, \bar{x})$. That will rely on specializing the existence result of Dontchev and Rockafellar [17, Theorem 1]. The key to that result, besides the utility conditions in A1, is the following replacement for the usual assumption that all agents start with positive quantities of every good.

Ample survivability. *The initial holdings (m^0, x^0) give ample survivability if the agents i have choices $(\hat{m}_i, \hat{x}_i)$ with $u_i(\hat{m}_i, \hat{x}_i) > -\infty$ such that*
(a) $\hat{x}_i \leq x_i^0$ *but* $\hat{m}_i < m_i^0$, *and*
(b) $\sum_{i=1}^r \hat{x}_i < \sum_{i=1}^r x_i^0$.

The interpretation is that the agents could, if they wished, survive without any trading at all and do so with individual surpluses of money and collective surpluses in every other good.

To justify the basic existence claim made here, it will be enough to demonstrate that

$$\text{all choices of } (m^0, x^0) \text{ with } (m_i^0, x_i^0) \in O_i \text{ give ample survivability.} \tag{20}$$

The argument is elementary and merely depends on the observing the extent to which various components in (m^0, x^0) can be lowered slightly to get new holdings $(\hat{m}, \hat{x})$ such that $(\hat{m}_i, \hat{x}_i)$ still lies in O_i. This is evidently possible when the money holdings of all agents are positive and each other good is possessed in positive quantity by at least one agent. The definition of O_i in (1) ensures this through the indispensability in A2.

An immediate consequence of (20), in combination with the fact noted earlier in (6) about solutions to the agents' optimization problems, is that

$$\text{in any equilibrium } (\bar{p}, \bar{m}, \bar{x}), \text{ the holdings } (\bar{m}, \bar{x}) \text{ as } (m^0, x^0) \text{ give ample survivability.} \tag{21}$$

Through this, the parametric stability result of Dontchev and Rockafellar [17, Theorem 3] can applied to an equilibrium with respect to its own holdings. The result then asserts the shift stability of the equilibrium. (The cited result is applicable to any initial holdings (m^0, x^0) in some small-enough neighborhood of $(\bar{m}, \bar{x})$ as long as (m^0, x^0) gives ample survivability. Shift stability of an equilibrium, not defined or considered in [17], needs $(m^0, x^0) = (\bar{m}, \bar{x})$ itself to give ample survivability, and the guarantee of that is what is new here.)[33] The existence of one-sided derivatives of the localized equilibrium mapping is provided by [17, Theorem 3] as well.

That work relies, in particular, on conditions that characterize optimality in the agent's maximization problems. Those conditions will again have to come into play, so we record then next before going on with the remainder of the proof.

Because u_i is concave[34] and differentiable on the convex set O_i where any solution must lie, a condition both necessary and sufficient for (m_i, x_i) to be optimal for a given $p > 0$ can be given in terms of the gradient of u_i at (m_i, x_i) and a Lagrange multiplier λ_i for the budget constraint:

$$(m_i, x_i) \geq (0, 0), \quad \lambda_i(1, p) - \nabla u_i(m_i, x_i) \geq (0, 0), \quad (m_i, x_i) \cdot [\lambda_i(1, p) - \nabla u_i(m_i, x_i)] = 0, \tag{22}$$

where

$$m_i = m_i^0 + p \cdot (x_i^0 - x_i). \tag{23}$$

The so-called complementary slackness conditions (22), expressed in a manner typical in optimization, say that for each good the corresponding components of the nonnegative vectors (m_i, x_i) and $\lambda_i(1, p) - \nabla u_i(m_i, x_i)$ cannot both be positive; at least one or the other must be 0. Specializations can be gleaned from the categorization of goods in our model by attractiveness and indispensability. Let the goods other than money be indexed by $j = 1, \ldots, n$, so that

$$x_i = (\ldots, x_{ij}, \ldots) \text{ with } x_{ij} \geq 0, \qquad p = (\ldots, p_j, \ldots) \text{ with } p_j > 0.$$

Since indispensable goods, including money, occur only in positive amounts in O_i, we can reduce

[33] The format in [17] is that of survival sets U_i not necessarily of orthant type, but the cited result depends on having orthant-like structure locally around the equilibrium under investigation. That is true automatically here for the same reasons that have been laid out in deriving (20).

[34] Plain quasi-concavity of the utility function u_i would not suffice for this.

(22) through assumptions A2 and A3 to

$$
\begin{array}{ll}
\lambda_i = (\partial u_i/\partial m_i)(m_i, x_i) & \text{(hence } \lambda_i > 0\text{)},\\
\lambda_i p_j = (\partial u_i/\partial x_{ij})(m_i, x_i) & \text{for indispensable goods } j \text{ of agent } i,\\
\lambda_i p_j \geq (\partial u_i/\partial x_{ij})(m_i, x_i) & \text{for attractive but not indispensable goods } j,\\
 & \text{with equality holding when } x_{ij} > 0,\\
x_{ij} = 0 \text{ for goods } j \text{ that are not attractive for agent } i. &
\end{array}
\tag{24}
$$

The conditions for solutions to these problems for $i = 1, \ldots, r$ to constitute an equilibrium (p, m, x) associated with (m^0, x^0) are the combination of (23) and (24) with

$$\textstyle\sum_{i=1}^{r} x_{ij} = \sum_{i=1}^{r} x_{ij}^0 \ \text{ for } \ j = 1, \ldots, n. \tag{25}$$

4 Proof of the robust stability theorem, second part

The remainder of the proof, which is concerned with tâtonnement stability, must delve deeper into the variational analysis through which the results in [17] that we have been applying were themselves derived. Some background in [16], concerning solution mappings associated with variational inequality models for expressing optimality conditions and equilibrium, will be essential.[35] To make things easier for readers not familiar with that subject, we start with a brief overview.

Variational inequalities. *The variational inequality associated with a nonempty, closed, convex set $C \subset I\!R^N$ and a mapping $f : C \to I\!R^N$ with parameter $p \in I\!R^n$ takes the form finding $w \in C$ such that*

$$-f(p, w) \in N_C(w), \tag{26}$$

where $N_C(w)$ is the normal cone to C at w, defined by

$$v \in N_C(w) \iff w \in C \text{ and } v{\cdot}(w' - w) \leq 0 \text{ for all } w' \in C. \tag{27}$$

The normal cone $N_C(w)$ at any $w \in C$ is closed and convex. It always contains $v = 0$, and that is its only element when w is an interior point of C, which is true of course for every w in the special case when $C = I\!R^N$. The variational inequality reduces then to the vector equation $f(p, w) = 0$, and this is the sense in which variational inequality models expand on equation models. Vectors $v \neq 0$ necessarily exist in $N_C(w)$ when w is a boundary point of C. They can be of any length and are the outward normals to the (closed) supporting half-spaces to C at w.

The *solution mapping* associated with (26), which may be set-valued (i.e., a relation, or a correspondence in terminology common to economics literature), is

$$S : p \mapsto \{\, w \mid -f(p, w) \in N_C(w) \,\}. \tag{28}$$

[35]Variational analysis as laid out in [40] has also been the key to our other papers on economic equilibrium, namely [23], [24], [25] and [26].

Results in [16] generalize the classical implicit function theorem for equations by providing criteria under which, *in localization* around a pair $(\bar{p},\bar{w})$ with $\bar{w}\in S(\bar{p})$, the mapping S is single-valued and Lipschitz continuous, moreover with one-sided derivatives having a specific formula. The best case, which will be in play here, centers on C being polyhedral, i.e., expressible as the intersection of a finite collection of closed half-spaces. A useful object then is the *critical cone* to C at a point $w\in C$ with respect to a normal $v\in N_C(w)$, which is the polyhedral cone

$$K(w,v)=\{\,w'\in T_C(w)\,|\,v\cdot w'=0\,\},$$

where $T_C(w)$ is the *tangent cone* to C at w, equal to the polar of $N_C(w)$. All these cones are important in the study of optimality conditions, and to a large extent the passage from equations to variational inequalities is motivated by modeling circumstances that involve first-order optimality conditions associated with inequality constraints, such as the nonnegativity of goods in our economic setting.

The form of generalized implicit function theorem for (26) that was basic in Dontchev and Rockafellar [17], and will be basic here again, refers to the smallest linear subspace $K^+(w,v)$ containing the critical cone $K(w,v)$ as well as the largest linear subspace, $K^-(w,v)$ contained within $K(w,v)$. Under the assumption that the function f in (26) is continuously differentiable, it focuses on a particular solution $w\in S(p)$ and invokes for the normal vector $v=-f(p,w)$ the criterion that

$$w'\in K^+(w,-f(p,w)),\;\; \nabla_w f(p,w)w'\perp K^-(p,-f(p,w)),\;\; w'\cdot\nabla_w f(p,w)w'\le 0 \quad\Longrightarrow\quad w'=0,$$

where $\nabla_w f(p,w)$ denotes the $N\times N$ Jacobian of f with respect to the w argument. The conclusion then is that the solution mapping S does have a single-valued Lipschitz continuous localization around (p,w) for which the one-sided derivatives relative to vectors p' exist and are given by

$$\begin{aligned} DS(p;p') = \;&\text{the unique solution } w' \text{ to the auxiliary variational inequality}\\ &-[\nabla_p f(p,w)p'+\nabla_w f(p,w)w']\in N_{K(p,w)}(w'). \end{aligned}$$

This is from [16, Theorem 2E.8]. Note that in the equation case, with $C=I\!\!R^N$ and $f(p,w)=0$, the critical cone and its associated subspaces are all just $I\!\!R^N$ itself. The criterion to be invoked reverts then to having $\nabla_w f(p,w)w'=0$ imply $w'=0$, or in other words, the full rank condition on the Jacobian matrix $\nabla_w f(p,w)$, as in the classical implicit function theorem.

For utilizing this general perturbation theory here, the target is the excess demand mapping Z in (9) and specifically the monotonicity-type property we claim for it in (14). That property will be deduced from a formula for one-sided derivatives of Z. Clearly from (9), the key ingredient in that has to be formulas for one-sided derivatives of the agents' demand mappings X_i in (7). From now on the initial holdings (m^0,x^0) will be fixed, so in working with these mappings we can pass to simpler notation:

$$X_i^0(p)=X_i(p;m_i^0,x_i^0),\qquad Z^0(p)=\textstyle\sum_{i=1}^r[X_i^0(p)-x_i^0]=Z(p;m^0,x^0). \tag{29}$$

It has already been noted that (through ample survivability) $X_i^0(p)$ is a uniquely determined goods vector in O_i for every price vector $p > 0$, and indeed that it is the unique solution to the conditions in (24) with m_i given by (23) (and λ_i given by the first line in (24)). We are involved, in other words, with solving these conditions for x_i as a function of p. If it were not for the third line in (24), we could view this from the classical perspective of solving a system of equations and try to apply the implicit function theorem. The inequality complication would drop away, of course, if we could be sure that the demand vector x_i would be > 0 in all its components, but allowing goods that are attractive but not indispensable to have zero demand for some combinations of prices is an important goal of our efforts.

It will help to reconfigure our task as the analysis of the enlarged mapping

$$S_i^0 : p \mapsto \{\, (m_i, x_i, \lambda_i) \text{ satisfying (23)–(24) } \}. \tag{30}$$

From that analysis, the properties we require of X_i^0, as a component mapping, will be easy to extract. By interpreting S_i^0 as the solution mapping associated with a "variational inequality" problem, we will have available the above extension of the implicit function theorem, which can handle the inequality condition in (24).

In the case to which we want to apply this, the solution mapping will be S_i^0, already known to be single-valued. This case identifies (23)–(24) with the variational inequality[36]

$$-f_i(p, w_i) \in N_{C_i}(w_i) \text{ for } \begin{cases} w_i = (m_i, x_i, \lambda_i) \in C_i = I\!R_+^{n+1} \times I\!R, \\ f_i(p, w_i) = -(\nabla u_i(m_i, x_i) - \lambda_i(1, p), m_i - m_i^0 - p{\cdot}(x_i^0 - x_i)). \end{cases} \tag{31}$$

We will be analyzing this relative to an arbitrary $p > 0$ and $(m_i, x_i, \lambda_i) = w_i = S_i^0(p)$. Then $(m_i, x_i) \in O_i$, and since the analysis is local, the fact that ∇u_i is undefined at points of $I\!R_+^{n+1}$ outside O_i will not matter. The analysis will utilize the Jacobian expressions

$$\begin{aligned} &\nabla_p f_i(p, w_i)p' = [\, \lambda_i(0, p'), p'{\cdot}(x_i - x_i^0) \,], \\ &\nabla_{w_i} f_i(p, w_i)w_i' = [-\nabla^2 u_i(m_i, \bar{x}_i)(m_i', x_i') + \lambda_i'(1, p), -m_i' - p{\cdot}x_i' \,], \end{aligned} \tag{32}$$

where $\nabla^2 u_i$ is the matrix of second partial derivatives of u_i. It will involve us not only with the normal cone $N_{C_i}(w_i)$, but also its polar, the tangent cone $T_{C_i}(w_i)$, and the "critical cone"

$$K_i(p, w_i) = \{\, w_i' = (m_i', x_i', \lambda_i') \in T_{C_i}(w_i) \mid f_i(p, w_i){\cdot}w_i' = 0 \,\}. \tag{33}$$

Because C_i is a polyhedral convex set (actually a cone itself), the critical cone $K_i(p, w_i)$ is polyhedral convex as well. The theorem about solution mappings to variational inequalities over polyhedral sets that we are going to apply requires us also to look at

$$\begin{aligned} &K_i^+(p, w_i) = K_i(p, w_i) - K_i(p, w_i) = \text{smallest subspace} \supset K_i(p, w_i), \\ &K_i(p, w_i) \cap K_i(p, w_i) = \text{largest subspace} \subset K_i(p, w_i). \end{aligned} \tag{34}$$

Perturbation Result to be Applied (as specialized from [16, Theorem 2E.8])**.** *Under the criterion that*

$$w_i' \in K_i^+(p, w_i), \quad \nabla_{w_i} f_i(p, w_i)w_i' \perp K_i^-(p, w_i), \quad w_i'{\cdot}\nabla_{w_i} f_i(p, w_i)w_i' \le 0 \implies w_i' = 0, \tag{35}$$

[36]Again, this formulation could not be reached without utility being concave instead of just quasi-concave.

the solution mapping S_i^0 for the variational inequality (31) is Lipschitz continuous in a neighborhood of p and semidifferentiable there with one-sided directional derivatives given by

$$\begin{array}{rl} DS_i^0(p;p') = & \textit{the unique solution } w_i' \textit{ to the auxiliary variational inequality} \\ & -[\nabla_p f_i(p,w_i)p' + \nabla_{w_i} f_i(p,w_i)w_i'] \in N_{K_i(p,w_i)}(w_i'). \end{array} \tag{36}$$

The next step is to work out the details of this in our context of (31). The cone $K_i(p,w_i)$ and subspaces $K_i^+(p,w_i)$ and $K_i^-(p,w_i)$ come out, in expression with respect to the goods j, as

$$\begin{array}{l} K_i(p,w_i) = I\!\!R \times \Pi_{j=1}^n K_{ij}(p_j,m_i,x_{ij},\lambda_i) \times I\!\!R, \text{ where} \\ \quad K_{ij}(p_j,m_i,x_{ij},\lambda_i) = \begin{cases} I\!\!R & \text{if } x_{ij} > 0, \\ I\!\!R_+ & \text{if } x_{ij} = 0 \text{ and } (\partial u_i/\partial x_{ij}(m_i,x_i) = \lambda_i p_j, \\ \{0\} & \text{if } x_{ij} = 0 \text{ and } (\partial u_i/\partial x_{ij})(m_i,x_i) < \lambda_i p_j, \end{cases} \end{array} \tag{37}$$

$$\begin{array}{l} K_i^+(p,w_i) = I\!\!R \times \Pi_{j=1}^n K_{ij}^+(p_j,m_i,x_{ij},\lambda_i) \times I\!\!R, \text{ where} \\ \quad K_{ij}^+(p_j,m_i,x_{ij},\lambda_i) = \begin{cases} I\!\!R & \text{if } x_{ij} > 0, \\ I\!\!R & \text{if } x_{ij} = 0 \text{ and } (\partial u_i/\partial x_{ij})(m_i,x_i) = \lambda_i p_j, \\ \{0\} & \text{if } x_{ij} = 0 \text{ and } (\partial u_i/\partial x_{ij})(m_i,x_i) < \lambda_i p_j, \end{cases} \end{array} \tag{38}$$

$$\begin{array}{l} K_i^-(p,m_i,x_i,\lambda_i) = I\!\!R \times \Pi_{j=1}^n K_{ij}^-(p_j,m_i,x_{ij},\lambda_i) \times I\!\!R, \text{ where} \\ \quad K_{ij}^-(p_j,m_i,x_{ij},\lambda_i) = \begin{cases} I\!\!R & \text{if } x_{ij} > 0, \\ \{0\} & \text{if } x_{ij} = 0 \text{ and } (\partial u_i)/\partial x_{ij})(m_i,x_i) = \lambda_i p_j, \\ \{0\} & \text{if } x_{ij} = 0 \text{ and } (\partial u_i)/\partial x_{ij}(m_i,x_i)) < \lambda_i p_j. \end{cases} \end{array} \tag{39}$$

In (37), (38) and (39) the same three categories of indices j are involved, and it will be convenient to speak of them as categories 1, 2 and 3. Unattractive goods j are clearly always in category 3, which gives $\{0\}$ in every case. Let J^+ refer to all the indices j in categories 1 and 2, and let J^- refer to those only in category 1. In these terms we proceed toward verifying (35), which can written as

$$\begin{array}{l} \text{only } (m_i',x_i',\lambda_i') = (0,0,0) \text{ satisfies the conditions} \\ \quad (m_i',x_i',\lambda_i') = w_i' \in K_i^+(p,w_i), \\ \quad [-\nabla^2 u_i(m_i,x_i)(m_i',x_i') + \lambda_i'(1,p), -m_i' - p{\cdot}x_i'] \perp K_i^-(p,w_i), \\ \quad (m_i',x_i',\lambda_i'){\cdot}[-\nabla^2 u_i(m_i,x_i)(m_i',x_i') + \lambda_i'(1,p), -m_i' - p{\cdot}x_i'] \leq 0, \end{array} \tag{40}$$

The first of the three conditions in (40) narrows our attention to cases of (m_i',x_i',λ_i') having $x_{ij}' = 0$ for all $j \notin J^+$, while the second further narrows it to $\lambda_i' = 0$ and $m_i' + \bar{p}{\cdot}x_i' = 0$, along with having the components of the vector $\nabla^2 u_i(\bar{m}_i,\bar{x}_i)(m_i',x_i')$ be 0 except possibly for some of them that belong to indices $j \notin J^-$. The quadratic expression in the third condition reduces then to $-(m_i',x_i'){\cdot}\nabla^2 u_i(\bar{m}_i,\bar{x}_i)(m_i',x_i')$. Because of the negative definiteness coming from our assumption A4, this expression cannot be ≤ 0 unless $m_i' = 0$ and $x_{ij}' = 0$ for all attractive goods j. But $x_{ij}' = 0$ already for unattractive goods, so we conclude that (40) does hold, and with it the properties of S_i^0 listed in the Perturbation Result above.

It follows then that the demand mapping X_i^0 is likewise locally Lipschitz continuous and semidifferentiable. More specifically, we have from (36) through (32) that

$$\begin{array}{l} DS_i^0(p;p') \text{ is the unique solution } (m_i',x_i',\lambda_i') \text{ to the variational inequality} \\ \quad -[\lambda_i(0,p') + \lambda_i'(1,p) - \nabla^2 u_i(m_i,x_i)(m_i',x_i'), -p'{\cdot}(x_i - x_i^0) - m_i' - p{\cdot}x_i'] \\ \quad\quad \in N_{K_i(p,w_i)}(w_i') \text{ with } w_i = (m_i,x_i,\lambda_i),\ w_i' = (m_i',x_i',\lambda_i'), \end{array} \tag{41}$$

where the details of the cone $K_i(p,w_i)$ are in (37). The one-sided directional derivatives of X_i^0 are given then by

$$DX_i^0(p;p') = x_i' \text{ for the } (m_i', x_i', \lambda_i') \text{ in } (41). \tag{42}$$

It is evident now that the excess demand mapping Z^0 in (29) is Lipschitz continuous locally as well, and semidifferentiable with its one-sided derivatives given by

$$DZ^0(p;p') = \sum\nolimits_{i=1}^r x_i' \text{ where } x_i' = DX_i^0(p;p'). \tag{43}$$

This brings us to the stage where we have confirmed all of the claims of the Robust Stability Theorem except for the "monotonicity" property (14). Directional derivatives will help with that, as follows. The inequality in (14) can be equivalently be rewritten (with a change of variables that alters the meaning of p') as

$$-\mu||p'||^2 \;\geq\; ([p+p'] - p)\cdot(Z^0(p+p') - Z^0(p)) = \int_0^1 p'\cdot DZ^0(p+tp';p')dt,$$

inasmuch as the (Lipschitz continuous) function $z(t) = Z^0(p+tp')$ is differentiable for almost every t with $\dot{z}^+(t) = DZ^0(p+tp';p')$. Our task in these terms is reduced to demonstrating the existence of $\mu > 0$ for which

$$-\mu||p'||^2 \;\geq\; p'\cdot DZ^0(p;p') \;=\; p'\cdot\sum\nolimits_{i=1}^r DX_i^0(p;p') \text{ when } p\in N,\; p+p'\in N,\; p'\neq 0, \tag{44}$$

provided that the ball N around the equilibrium price vector $\bar{p}$ and the ball N_0 around the equilibrium holdings $(\bar{m},\bar{p})$ in (14) are chosen small enough. Here by (42) we have

$$p'\sum\nolimits_{i=1}^r DX_i^0(p;p') = \sum p'\cdot x_i' \text{ with } x_i' \text{ from } (m_i', x_i', \lambda_i') \text{ solving } (41) \tag{45}$$

and can make that the platform for our analysis.

It is important now to notice a sort of uniformity in the local behavior of the sets in (37), (38) and (39), namely that

$$\begin{gathered} K_i^-(\bar{p},\bar{w}_i) \subset K_i^-(p,w_i) \subset K_i(p,w_i) \subset K_i^+(p,w_i) \subset K_i^+(\bar{p},\bar{w}_i) \\ \text{for } (p,w_i) \text{ near enough to } (\bar{p},\bar{w}_i). \end{gathered} \tag{46}$$

This is evident from the formulas for these sets and the continuity of the partial derivatives of u_i. A follow-up to this observation, taking advantage of the fact that making p be close to $\bar{p}$ also makes $w_i = S^0(p)$ be close to $\bar{w}_i = S_i^0(\bar{p})$ through the continuity of S_i^0, is that

$$\begin{aligned} &\text{if } (m_i', x_i', \lambda_i') \text{ solves in (41) with } p \text{ near enough to } \bar{p}, \text{ then } (m_i', x_i', \lambda_i') \in K_i^+(\bar{p},\bar{w}_i), \\ &-[\lambda_i(0,p') + \bar{\lambda}_i'(1,p) - \nabla^2 u_i(m_i,x_i)(m_i',x_i'), -p'\cdot(x_i - x_i^0) - m_i' - p\cdot x_i'] \perp K_i^-(\bar{p},\bar{w}_i), \\ &(m_i', x_i', \lambda_i')\cdot\Big(- [\lambda_i(0,p') + \lambda_i'(1,p) - \nabla^2 u_i(m_i,x_i)(m_i',x_i'), -p'\cdot(x_i - x_i^0) - m_i' - p\cdot x_i']\Big) = 0. \end{aligned} \tag{47}$$

In view of (38) and (39), we must then have

$$\begin{gathered} p'\cdot(x_i - x_i^0) + m_i' + p\cdot x_i' = 0 \text{ for all agents } i, \text{ and} \\ x_{ij}' = 0 \text{ when a good } j \text{ is unattractive to agent } i. \end{gathered} \tag{48}$$

The equation in the third condition of (47) reduces in this case to

$$-\lambda_i{\cdot}x_i'{\cdot}p' + \lambda_i'{\cdot}(m_i' + x_i'{\cdot}p) - (m_i', x_i'){\cdot}\nabla^2 u_i(m_i, x_i)(m_i', x_i') = 0.$$

Since $\lambda_i > 0$, we can rewrite this, using the first line of (48), as

$$p'{\cdot}x_i' = \lambda_i^{-1}\Big[(m_i', x_i'){\cdot}\nabla^2 u_i(m_i, x_i)(m_i', x_i') + \lambda_i' p'{\cdot}[x_i - x_i^0]\Big], \tag{49}$$

where moreover $(m_i', x_i'){\cdot}\nabla^2 u_i(m_i, x_i)(m_i', x_i') < 0$ unless $(m_i', x_i') = (0,0)$ through the second line of (48) and the negative definiteness of the submatrix of $\nabla^2 u_i(m_i, x_i)$ with respect to the attractive goods for agent i in our assumption A4. Hence

$$p'{\cdot}\sum\nolimits_{i=1}^r x_i' \;=\; \sum\nolimits_{i=1}^r (m_i', x_i'){\cdot}\Big[\frac{1}{\lambda_i}\nabla^2 u_i(m_i, x_i)\Big](m_i', x_i') + \sum\nolimits_{i=1}^r \frac{\lambda_i'}{\lambda_i} p'{\cdot}[x_i - x_i^0], \tag{50}$$

where the first sum on the right is < 0 unless $(m_i', x_i') = (0,0)$ for all i.

The crux of the matter emerges as making sure that the negativity of the quadratic sum in (50) cannot be overpowered by the second sum. The quadratic sum has the form

$$(m', x'){\cdot}A(m, x, \lambda)(m', x') \;\text{ for }\; m' = (\ldots, m_i', \ldots),\; x' = (\ldots, x_i', \ldots),$$

where $A(m, x, \lambda)$ is a negative definite matrix depending continuously on (m, x, λ), which in turn is comprised of elements $(m_i, x_i, \lambda_i) = S_i^0(p) = S_i(p; m_i^0, x_i^0)$ that depend continuously on p and also on (m^0, x^0).[37] Its eigenvalues can therefore be bounded locally away from 0:

$$\begin{gathered}\text{there exist } \varepsilon > 0 \text{ and closed balls } N \text{ at } \bar{p} \text{ and } N_0 \text{ at } (\bar{m}, \bar{x}) \text{ such that}\\ \sum\nolimits_{i=1}^r (m_i', x_i'){\cdot}\Big[\frac{1}{\lambda_i}\nabla^2 u_i(m_i, x_i)\Big](m_i', x_i') \;\le\; -\varepsilon\|(m', x')\|^2 \;\text{ when }\; p \in N,\; (m^0, x^0) \in N_0.\end{gathered} \tag{51}$$

An upper estimate of the size of the second sum in (50) must next come into play. For this we return to the conditions in (47). With $(\ldots, \lambda_i', \ldots)$ denoted by λ', let

$$W(p; m^0, x^0) = \{\, (m', x', \lambda', p') \text{ satisfying the first two conditions in (47) for all } i \,\}.$$

Because $K_i^+(\bar{p}, \bar{w}_i)$ and $K_i^-(\bar{p}, \bar{w}_i)$ are linear subspaces, $W(p; m^0, x^0)$ is a linear subspace as well. We claim

$$\begin{gathered}\text{there exist } \rho > 0 \text{ and balls } N' \text{ at } \bar{p} \text{ and } N_0' \text{ at } (\bar{m}, \bar{x}) \text{ such that}\\ \|(p', \lambda')\| \;\le\; \rho\|(m', x')\|\\ \text{for all } (p', m', x', \lambda') \in W(p; m^0, x^0) \text{ when } p \in N',\; (m^0, x^0) \in N_0'.\end{gathered} \tag{52}$$

If this were not true, there would be sequences of elements

$$\begin{gathered}(p'^k, m'^k, x'^k, \lambda'^k) \in W(p^k; m^{0k}, x^{0k}) \text{ with } p^k \in N',\; (m^{0k}, x^{0k}) \in N_0',\\ \text{such that } \|(p'^k, \lambda'^k)\| = 1 \text{ for all } k \text{ and } |(m'^k, x'^k)\| \to 0.\end{gathered}$$

[37]The continuity of $S_i(p; m_i^0, x_i^0)$ with respect to (m_i^0, x_i^0) is seen from the optimality conditions (23)–(24) for the optimization problem of agent i, which involve (m_i^0, x_i^0) only through (23). A limit of solutions to these conditions coming from a convergent sequence of such initial holdings must be another solution.

Passing to convergent subsequences, we would arrive in the limit at elements

$$(p'^*, m'^*, x'^*, \lambda'^*) \in W(p^*; m^{0\,*}, x^{0\,*}) \text{ with } p^* \in N',\ (m^{0\,*}, x^{0\,*}) \in N_0', \\ \text{such that } (p'^*, \lambda'^*) \neq (0,0) \text{ but } (m'^*, x'^*) = (0,0).$$

This is an impossible situation for the following reason. It entails

$$-[\lambda_i^*(0, p'^*) + \bar{\lambda}_i'^*(1, p^*), -p'{\cdot}(x_i^* - x_i^{0*})\,] \perp K_i^-(\bar{p}, \bar{m}_i, \bar{x}_i, \bar{\lambda}_i),$$

which first implies $\bar{\lambda}_i'^* = 0$ and then that $p_j'^* = 0$ for all goods j having $x_{ij}^* > 0$. But then $p_j'^* = 0$ for all goods j, yielding a contradiction because our assumptions make it impossible for any j to have $x_{ij}^* = 0$ for every agent i. Thus, (52) is confirmed.

Putting (52) now to use, and noting that $||(p', \lambda')|| \le \rho||(m', x')||$ implies that $||p'|| \le \rho||(m', x')||$ and $|\lambda_i')| \le \rho||(m', x')||$ for all i, as well as $||x_i - x_i^0|| \le ||x - x^0||$, we get the upper bound

$$\sum\nolimits_{i=1}^r \frac{\lambda_i'}{\lambda_i} p'{\cdot}[x_i - x_i^0] \ \le\ \rho\Big[\sum\nolimits_{i=1}^r \frac{1}{\lambda_i}\Big]||x - x^0||\,||(m', x')||^2,$$

which holds when p is close enough to $\bar{p}$ and (m^0, x^0) is close enough to $(\bar{m}, \bar{x})$. Since $\lambda \to \bar{\lambda}$ and $x \to \bar{x}$ as $p \to \bar{p}$ because $(\bar{m}_i, \bar{x}_i, \bar{\lambda}_i) = S_i(\bar{p}; m^0, x^0)$ for all i, there is also a local upper bound

$$\Big[\sum\nolimits_{i=1}^r \frac{1}{\lambda_i}\Big]||x - x^0|| \le \nu||\bar{x} - x^0||.$$

Putting all this together with (51), we obtain from (50) and (52) that

$$p'{\cdot}\sum\nolimits_{i=1}^r x_i' \ \le\ -\Big(\varepsilon - \nu\rho||\bar{x} - x^0||\Big)||(m', x')||^2 \ \le\ -\rho^{-2}\Big(\varepsilon - \nu\rho||\bar{x} - x^0||\Big)||p'||^2$$

when p is close enough to $\bar{p}$. The coefficient $\mu = \rho^{-2}(\varepsilon - \nu\rho||\bar{x} - x^0||)$ is sure to be > 0 when x^0 is close enough to $\bar{x}$. We have already determined that having $(m', x') = (0,0)$ is incompatible with $p' \neq 0$, so the desired conclusion, supporting the existence of a neighborhood N as in (44), has been reached.

5 Appendix: simpler proof under full indispensability

Suppose that *every* good is indispensable to *every* agent. The suborthants O_i all coincide then with the interior of $\mathbb{R}_+^n$, so that all holdings, initial and final, are positive. The existence of an equilibrium is guaranteed in the usual way, and the stability analysis can be carried out with just the classical implicit function theorem, obviating the need for advanced methodology. The optimality conditions for the utility maximization problems are described solely by equations:

$$\begin{cases} \lambda_i(1, p) - \nabla u_i(m_i, x_i) = (0,0), \\ m_i - m_i^0 - p{\cdot}(x_i^0 - x_i) = 0, \end{cases} \qquad \text{for } i = 1, \ldots, r. \tag{53}$$

Those conditions together with

$$\sum\nolimits_{i=1}^r x_i - \sum\nolimits_{i=1}^r x_i^0 = 0, \tag{54}$$

characterize an equilibrium in the *enhanced*[38] sense of a combination (p, m, x, λ) that depends on the initial holdings (m^0, x^0). Note that not only the goods and prices but also the multipliers λ_i will all be positive here, as follows via indispensability from the first condition in (53), corresponding to the first condition in (24).

Although the implicit function theorem is typically articulated with matrices, we will need to apply it in several ways and can operate in this setting more conveniently with the linearizations of the equations in question, namely

$$\begin{cases} \lambda_i'(1,p) + \lambda_i(0,p') - \nabla^2 u_i(m_i,x_i)(m_i',x_i') = (0,0), \\ m_i' - m_i^{0\prime} - p'{\cdot}(x_i^0 - x_i) - p{\cdot}(x_i^{0\prime} - x_i') = 0, \end{cases} \qquad \text{for } i = 1,\dots,r, \tag{55}$$

$$\textstyle\sum_{i=1}^r x_i' - \sum_{i=1}^r x_i^{0\prime} = 0, \tag{56}$$

where the "primed" elements stand for perturbations. Because all goods are indispensable, the Hessians $\nabla^2 u_i(m_i, x_i)$ are negative definite, not just partially.

To establish shift stability we are interested in applying the implicit function theorem to solve (53)–(54) for (p, m, x, λ) in terms of (m^0, x^0) in the local sense around a reference equilibrium $(\bar p, \bar m, \bar x, \bar\lambda)$ and its own holdings $(\bar m^0, \bar x^0) = (\bar m, \bar x)$. This specializes (55) to

$$\begin{cases} \lambda_i'(1,\bar p) + \bar\lambda_i(0,p') - \nabla^2 u_i(\bar m_i,\bar x_i)(m_i',x_i') = (0,0), \\ m_i' - m_i^{0\prime} - \bar p{\cdot}(x_i^{0\prime} - x_i') = 0, \end{cases} \qquad \text{for } i = 1,\dots,r. \tag{57}$$

The condition we need to verify, corresponding to the nonsingularity of the Jacobian in the implicit function theorem, is the following: when $(m^{0\prime}, x^{0\prime}) = (0,0)$, the only solution (p', m', x', λ') to (56)–(57) is $(p', m', x', \lambda') = (0,0,0,0)$. In other words, we have to show that only $(p', m', x', \lambda') = (0,0,0,0)$ solves

$$\textstyle\sum_{i=1}^r x_i' = 0, \qquad \lambda_i'(1,\bar p) + \bar\lambda_i(0,p') - \nabla^2 u_i(\bar m_i,\bar x_i)(m_i',x_i') = (0,0), \qquad m_i' + \bar p{\cdot}x_i' = 0. \tag{58}$$

This is seen through first multiplying the Hessian condition on the left by (m_i', x_i') to obtain

$$\lambda_i'(m_i' + \bar p{\cdot}x_i') + \bar\lambda_i p'{\cdot}x_i' - (m_i',x_i'){\cdot}\nabla^2 u_i(\bar m_i,\bar x_i)(m_i',x_i') = 0,$$

where the first term vanishes by the third condition in (58), and next through dividing by $\bar\lambda_i$ and adding over i, to get

$$p'{\cdot}\textstyle\sum_{i=1}^r x_i' - \sum_{i=1}^r \dfrac{1}{\bar\lambda_i}(m_i',x_i'){\cdot}\nabla^2 u_i(\bar m_i,\bar x_i)(m_i',x_i') = 0.$$

Now the first term vanishes through the first condition in (58), so the Hessian sum must vanish as well. Since the Hessians are negative-definite, that implies $(m_i', x_i') = (0,0)$ for all i. The second condition in (58) then says $\lambda_i'(1,\bar p) + \bar\lambda_i(0,p') = (0,0)$, which requires both $\lambda_i' = 0$ and $p' = 0$. This ends the argument for shift stability, with the equilibrium mapping coming out as continuously differentiable, not just semidifferentiable.

[38]We use this term when the multipliers are included, as in our other papers.

The argument for tâtonnement stability goes along similar lines but replaces the supply-demand equation (56) by

$$z - \sum_{i=1}^{r} x_i - \sum_{i=1}^{r} x_i^0 = 0, \tag{59}$$

where z is now an additional variable giving the excess demand This equation linearizes to

$$z' - \sum_{i=1}^{r} x_i' - \sum_{i=1}^{r} x_i^{0\prime} = 0. \tag{60}$$

Again we have a reference equilibrium $(\bar{p}, \bar{m}, \bar{x}, \bar{\lambda})$ but the reference initial holdings need not equal $(\bar{m}, \bar{x})$. However, it will be important eventually to have them close enough to $(\bar{m}, \bar{x})$.

Our first step is to demonstrate that, with such closeness, and with p near $\bar{p}$, the conditions (53) can be solved for (m, x, λ) in terms of (m^0, x^0, p). For this, the linearization to work with is

$$\begin{cases} \lambda_i'(1, \bar{p}) + \bar{\lambda}_i(0, p') - \nabla^2 u_i(\bar{m}_i, \bar{x}_i)(m_i', x_i') = (0, 0), \\ m_i' - m_i^{0\prime} - p'{\cdot}(x_i^0 - \bar{x}_i) - \bar{p}{\cdot}(x_i^{0\prime} - x_i') = 0, \end{cases} \quad \text{for } i = 1, \ldots, r. \tag{61}$$

which differs from (57) in having a p' term in the second line because x_i^0 need not equal $\bar{x}_i$. The issue is whether, when $(m_i^{0\prime}, x_i^{0\prime}, p') = (0, 0, 0)$, the only possible solution to the linear equations (61) is $(m', x', \lambda') = (0, 0, 0)$. This is confirmed exactly as before for (57), through having $p' = 0$, but it was good anyway to have the p' term because it provides information about derivatives. Specifically, the rate of change (m', x', λ') of (m, x, λ) relative to a rate of change $(m_i^{0\prime}, x_i^{0\prime}, p')$ away from $(m^0, x^0, \bar{p})$ is obtained by solving these equations. In particular, we get for some $\rho > 0$ a bound of type

$$||(m', x', \lambda')|| \leq \rho ||(m_i^{0\prime}, x_i^{0\prime}, p')||. \tag{62}$$

A second step, with a side purpose, is to observe that, locally around the same reference elements, we can also solve (55) for (p, λ) as a (continuously differentiable) function of (m^0, x^0, m, x). The same linearized equations give the basis for the test. We have to demonstrate that, when $(m_i^{0\prime}, x_i^{0\prime}, m', x') = (0, 0, 0, 0)$, the only solution to (61) is $(p', \lambda') = (0, 0)$. Indeed, we already see this from the first line of (61) through the Hessian term vanishing. The purpose of this step was as follows. The continuous differentiability of the solution function implies, for the partial function going from (m, x) to (p, λ) with (m^0, x^0) as parameter, the existence of a Lipschitz constant that is effective as long as both (m, x) and (m^0, x^0) are close enough to the equilibrium holdings $(\bar{m}, \bar{x})$. That yields bounds

$$||p'|| \leq \kappa ||(m', x')||, \qquad ||\lambda_i'|| \leq \kappa ||(m', x')||. \tag{63}$$

For the final step we fix (m^0, x^0) and focus on $(m_i^{0\prime}, x_i^{0\prime}) = (0, 0)$, thereby reducing (61) to

$$\begin{cases} \lambda_i'(1, \bar{p}) + \bar{\lambda}_i(0, p') - \nabla^2 u_i(\bar{m}_i, \bar{x}_i)(m_i', x_i') = (0, 0), \\ m_i' - p'{\cdot}(x_i^0 - \bar{x}_i) + \bar{p}{\cdot}x_i' = 0, \end{cases} \quad \text{for } i = 1, \ldots, r. \tag{64}$$

The rate of change x_i' at the reference elements is determined by solving this for (x_i', λ_i'), and the rate of change of the excess demand z (at the reference $\bar{z} = 0$) is

$$z' = \sum_{i=1}^{r} x_i'. \tag{65}$$

In (64) we once again use the trick of multiplying the Hessian equation on the left by (m_i', x_i') and then invoking the second line. This yields

$$\lambda_i' p' \cdot (x_i^0 - \bar{x}_i) + \bar{\lambda}_i p' \cdot x_i' - (m_i', x_i') \cdot \nabla u_i(\bar{m}_i, \bar{x}_i)(m_i', x_i') = 0.$$

Next we divide by $\bar{\lambda}_i$ and sum over i, obtaining with (65) that

$$p' \cdot z' = \sum_{i=1}^r \frac{1}{\bar{\lambda}_i}(m_i', x_i') \cdot \nabla u_i(\bar{m}_i, \bar{x}_i)(m_i', x_i') - p' \cdot \sum_{i=1}^r \frac{\lambda_i'}{\bar{\lambda}_i}(x_i^0 - \bar{x}_i).$$

In this expression we have for some $\varepsilon > 0$ a bound of type

$$\sum_{i=1}^r \frac{1}{\bar{\lambda}_i}(m_i', x_i') \cdot \nabla u_i(\bar{m}_i, \bar{x}_i)(m_i', x_i') \leq -\varepsilon ||(m', x')||^2$$

coming from the negative-definiteness of the Hessians (and the positivity of each $\bar{\lambda}_i$, but also for some $\nu > 0$ a bound of type

$$-p' \cdot \sum_{i=1}^r \frac{\lambda_i'}{\bar{\lambda}_i}(x_i^0 - \bar{x}_i) \leq \nu ||(m', x')||^2$$

coming from (63). Putting these bounds together with the one in (62), we arrive at

$$p' \cdot z' \leq -\rho(\varepsilon - \nu ||x^0 - \bar{x}||) ||p'||^2$$

as long as x^0 is near enough to $\bar{x}$ to ensure that $\varepsilon - \nu ||(x^0 - \bar{x})|| > 0$. This means that locally we have $p' \cdot z' \leq -\mu ||p'||^2$ for some $\mu > 0$. Since $z' = Jp'$ for the Jacobian of the excess demand mapping at $\bar{p}$, we get the desired monotonicity property for tâtonnement.

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