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# AN EULER–NEWTON CONTINUATION METHOD FOR TRACKING SOLUTION TRAJECTORIES OF PARAMETRIC VARIATIONAL INEQUALITIES*

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**Abstract.** A finite-dimensional variational inequality parameterized by $t \in [0,1]$ is studied under the assumption that each point of the graph of its generally set-valued solution mapping is a point of strongly regularity. It is shown that there are finitely many Lipschitz continuous functions on $[0,1]$ whose graphs do not intersect each other such that for each value of the parameter the set of values of the solution mapping is the union of the values of these functions. Moreover, the property of strong regularity is uniform with respect to the parameter along any such function graph. An Euler–Newton continuation method for tracking a solution trajectory is introduced and demonstrated to have $l^\infty$ accuracy of order $O(h^4)$, thus generalizing a known error estimate for equations. Two examples of tracking economic equilibrium parametrically illustrate the theoretical results.

**Key words.** variational inequality, strong regularity, Euler–Newton continuation, error estimate, economic equilibrium

**AMS subject classifications.** Primary, 49J53; Secondary, 49K40, 65H20, 90C30

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## 1. Introduction.

For an equation $f(t,u) = 0$ given by a function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, the question of how to track a "solution trajectory" $\bar{u}(t)$ as a function of $t \in [0,1]$ (possibly, but not necessarily, interpreted as time) is important and has received much attention in numerical analysis; see the basic reference [2]. Tracking refers to computing an approximation which starts from knowing $u_0 = \bar{u}(0)$ at $t = 0$ and tries to stay close to $\bar{u}(t)$ as $t$ proceeds toward 1.

In this paper the equation is replaced by the *variational inequality*

$$\text{for every } t \in [0,1] \text{ find } u \in C \text{ such that } \langle f(t,u), v-u\rangle \geq 0 \quad \text{for all} \quad v \in C, \tag{1.1}$$

where $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$, $C$ is a nonempty, closed convex set in $\mathbb{R}^n$, and $\langle \cdot , \cdot \rangle$ is the usual scalar product. In terms of the normal cone mapping

$$N_C : u \mapsto \begin{cases} N_C(u) & \text{for} \quad u \in C, \\ \emptyset & \text{otherwise,} \end{cases}$$

where $N_C(u) = \{v \mid \langle v, u' - u\rangle \leq 0 \text{ for all } u' \in C\}$ for $u \in C$, the variational inequality

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(1.1) can be written as a generalized equation of the form

$$f(t,u) + N_C(u) \ni 0.$$

The equation case corresponds to $C = \mathbb{R}^n$, since the normal cone $N_C(u)$ is then $\{0\}$. The generally set-valued mapping $S : t \mapsto S(t) = \{ u \in C \mid f(t,u) + N_C(u) \ni 0 \}$ is the *solution mapping* to the parameterized variational inequality, and a *solution trajectory* over $[0,1]$ is in this case a function $\bar{u}(\cdot)$ such that $\bar{u}(t) \in S(t)$ for all $t \in [0,1]$, that is, $\bar{u}(\cdot)$ is a *selection* of $S$ over $[0,1]$.

We assume throughout that there exists a bounded set $D \subset \mathbb{R}^n$ such that for each $t \in [0,1]$ the set $S(t)$ is nonempty and contained in $D$ for all $[0,1]$, and also the function $f$ and its derivatives $\nabla_t f$, $\nabla_u f$, $\nabla^2_{tu} f$, and $\nabla^2_{uu} f$ are continuous on an open set containing $[0,1] \times D$.

We introduce and study, in this setting of a parameterized variational inequality, a method of Euler–Newton type which is a straightforward extension of the standard Euler–Newton continuation, or path-following, as described in [2], for solving equations of the form $f(t,u) = 0$ obtained from (1.1) by simply taking $C = \mathbb{R}^n$. That standard scheme is a predictor-corrector method of the following kind. For $N > 1$, let $\{t_i\}_{i=0}^N$ with $t_0 = 0$, $t_N = 1$, be a uniform (for simplicity) grid on $[0,1]$ with step size $h = t_{i+1} - t_i = 1/N$ for $i = 0, 1, \dots, N-1$. Starting from a solution $u_0$ to $f(0,u) = 0$, the method iterates between an Euler predictor step and a Newton corrector step:

$$\left\{ \begin{array}{l} u_{i+1} = v_{i+1} = u_i - h \nabla_u f(t_i, u_i)^{-1} \nabla_t f(t_i, u_i), \\ u_{i+1} = v_{i+1} - \nabla_u f(t_{i+1}, v_{i+1})^{-1} f(t_{i+1}, v_{i+1}). \end{array} \right. \tag{1.2}$$

According to [2, Theorem 5.2.1], if the Jacobian $\nabla_u f(t,u)$ is nonsingular whenever $f(t,u) = 0$, then there exists a continuously differentiable solution trajectory $\bar{u}$ for which the error of the method is of order $O(h^2)$ uniformly along the path. It should be noted that [2] considers an inexact version of the method in which the value of the Jacobian $\nabla_u f(t_i, u_i)$ is approximated by a matrix $A_i$ with accuracy satisfying $\max_{0 \le i \le N} \|A_i - \nabla_u f(t_i, u_i)\| = O(h)$. From the analysis in the present paper, as well as from the proof given in [2], one deduces that if the exact values of the Jacobian are used, then the $l^\infty$ error, that is, the maximum error over the mesh $\{t_i\}$, is of order $O(h^4)$. The inexactness in computing the Jacobian assumed in [2, Theorem 5.2.1] reduces the $l^\infty$ order of the error to three. Since the solution trajectory is a $C^1$ function, if the points $(t_i, u_i)$ are interpolated by a piecewise linear function, then one obtains that the error in the uniform (Chebyshev) norm is of order $O(h^2)$, as claimed in [2, Theorem 5.2.1].

Here we propose an extension of the Euler–Newton continuation method to the variational inequality (1.1), in which both the predictor and corrector steps consist of solving linearized variational inequalities:

$$\left\{ \begin{array}{l} f(t_i, u_i) + h \nabla_t f(t_i, u_i) + \nabla_u f(t_i, u_i)(v_{i+1} - u_i) + N_C(v_{i+1}) \ni 0, \\ f(t_{i+1}, v_{i+1}) + \nabla_u f(t_{i+1}, v_{i+1})(u_{i+1} - v_{i+1}) + N_C(u_{i+1}) \ni 0. \end{array} \right. \tag{1.3}$$

The nonsingularity assumption on the Jacobian is accordingly replaced by the condition that every point $(t,u)$ in the graph of the solution mapping $S$ of (1.1) is strongly regular. Strong regularity, defined at the beginning of section 2 in the form to be used in this paper, is a key concept that goes back to Robinson [11], and it reduces to the Jacobian nonsingularity property in the equation case, where $C = \mathbb{R}^n$.

A crucial phase of our analysis, carried out in section 3, is showing that on the strong regularity assumption for (1.1), there are finitely many functions $\bar{u}_j : [0,1] \to \mathbb{R}^n$, each of which is Lipschitz continuous and whose graphs are isolated from each other, such that for each $t \in [0,1]$ the set $S(t)$ is the union of the values $\bar{u}_j(t)$, that is, $\bar{u}_j$ are the solution trajectories. Moreover the strong regularity along the graphs of the solution trajectories is *uniform* with respect to $t \in [0,1]$ (Theorem 3.2). Based on this, we give a proof in section 4 of our main result (Theorem 4.1): if $\bar{u}$ is a solution trajectory and $u_0$ is chosen to be equal to $\bar{u}(0)$, then the following error estimate holds for the method (1.3):

$$\max_{0 \le i \le N} \|u_i - \bar{u}(t_i)\| = O(h^4). \tag{1.4}$$

Observe that the Euler step of iteration (1.3) does not reduce to the Euler step in (1.2) when $C = \mathbb{R}^n$. However, it reduces to a method which gives the same order of error; we will show this in section 4. We will also discuss there two inexact versions of the method with the same order of error. We should note that a solution trajectory of the variational inequality (1.1) cannot be expected to be smoother than Lipschitz continuous; therefore a piecewise linear (or of higher order) interpolation across $(t_i, u_i)$ will have error of order $O(h)$ in the uniform norm over the interval $[0,1]$.

To our knowledge, a Newton-type continuation for variational inequalities was first considered by Pang [7], who gave conditions under which the method is executable and convergent but did not furnish error estimates. Both analytic and computational results regarding homotopy methods for solving variational inequalities by converting them to equations involving normal mappings were given in [9] and [10]; for a more recent study in this direction see [1]. Much closer to our analysis is a recent paper by Zavala and Anitescu [12], who looked at nonlinear optimization problems in which the data may change in time. Specifically, they studied the variational inequality representing the Karush–Kuhn–Tucker optimality conditions and employed sufficient conditions for strong regularity *locally around a point* in a solution trajectory without necessarily being uniform along it. In addition, the time-stepping method in [12] uses a predictor step but not a corrector step and achieves second-order accuracy.

Solution paths having turning points or bifurcations do not enter the treatment here but have been studied elsewhere in particular situations, as in the book [6], and were also considered in [9] and [10]. Actually, our main assumption, that each point in the graph of the solution mapping is strongly regular, excludes the existence of turning points. Broadening our analysis to cover singularities is a topic requiring further investigations.

There are several other directions for future research as well. First, inexact versions of the Euler–Newton continuation method are to be explored, e.g., in line with the recent paper [3]. Next, our analysis can be extended to differential-variational inequalities in the spirit of [8], where a variational inequality is coupled with a differential equation, to which a combination of a predictor-corrector method and a Runge–Kutta method can be applied. A step further is to apply a predictor-corrector technique to optimal control problems, e.g., for solving the variational inequality appearing in the maximum principle. Finally, developing efficient numerical strategies for solving large-scale practical problems is the ultimate goal for such research.

In the last section of this paper we apply the Euler–Newton method to track parametrically the solution of an economic equilibrium model developed in [4]. The numerical results for two examples confirm the order of convergence in (1.4).

In our analysis we do not use the particular properties of the normal cone mapping $N_C$ in (1.1). The obtained results remain valid if we replace $N_C$ with *any* mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, hence they cover, for instance, systems of equalities and inequalities and also other kinds of generalized equations.

**2. Background on strong regularity.** In this section $X$ is a complete metric space and $Y$ is a linear metric space with a shift-invariant metric. Both metrics are denoted by $\rho$. The closed ball centered at $x$ with radius $r$ is denoted by $\mathbb{B}_r(x)$. Recall that a function $g : X \to Y$ is said to be Lipschitz continuous around $\bar{x} \in \operatorname{dom} g$ if there exist a neighborhood $U$ of $\bar{x}$ and a constant $\mu$ such that

$$\rho(g(x), g(x')) \le \mu\rho(x, x') \quad \text{for all} \quad x, x' \in U.$$

Also, recall that a mapping $F : X \rightrightarrows Y$ with $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is said to have a Lipschitz localization around $\bar{x}$ for $\bar{y}$ when there exist neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{y}$ such that the restricted mapping $U \ni x \mapsto F(x) \cap V$ is single-valued, that is, a function which is moreover Lipschitz continuous around $\bar{x}$. The following basic definition echoes a concept coined by Robinson [11].

Definition 2.1. *A mapping $F : X \rightrightarrows Y$ with $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is said to be strongly regular at $\bar{x}$ for $\bar{y}$ with constant $\kappa$ when $F^{-1}$ has a Lipschitz localization around $\bar{y}$ for $\bar{x}$ with Lipschitz constant $\kappa$.*

*Remark* 2.2. Let $F$ be strongly regular at $\bar{x}$ for $\bar{y}$ with a Lipschitz constant $\kappa$ and neighborhoods $\mathbb{B}_a(\bar{x})$ and $\mathbb{B}_b(\bar{y})$. Then from Definition 2.1 one can deduce that for every positive constants $a' \le a$ and $b' \le b$ such that $\kappa b' \le a'$, the mapping $F$ is strongly regular with a Lipschitz constant $\kappa$ and neighborhoods $\mathbb{B}_{a'}(\bar{x})$ and $\mathbb{B}_{b'}(\bar{y})$. Indeed, in this case any $y \in \mathbb{B}_{b'}(\bar{y})$ will be in the domain of $F^{-1}(\cdot) \cap \mathbb{B}_{a'}(\bar{x})$.

The following result is a particular form of a general paradigm in analysis linked with the implicit function theorem, the Lyusternik–Graves theorem, Robinson's theorem, and beyond; for an extended study on the subject see [5].

Theorem 2.3. *Let $F : X \rightrightarrows Y$ be strongly regular at $\bar{x}$ for $\bar{y}$ with constant $\kappa$ and let $g : X \to Y$ be Lipschitz continuous around $\bar{x}$ with constant $\mu$ such that $\kappa\mu < 1$. Then $g + F$ is strongly regular at $\bar{x}$ for $g(\bar{x}) + \bar{y}$.*

We need this theorem in the following slightly more general form, which we supply with a proof for completeness.

Theorem 2.4. *Let $X$ be a complete metric space and $Y$ be a linear metric space with a shift-invariant metric. For a mapping $F : X \rightrightarrows Y$ and a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, suppose that the mapping*

$$\mathbb{B}_b(\bar{y}) \ni y \mapsto F^{-1}(y) \cap \mathbb{B}_a(\bar{x})$$

*is single-valued, that is, a function which is moreover Lipschitz continuous with Lipschitz constant $\kappa$ on $\mathbb{B}_b(\bar{y})$. Let $\mu > 0$ and $\kappa'$ be such that $\kappa\mu < 1$ and $\kappa' \ge \kappa/(1 - \kappa\mu)$. Then for every positive constants $\alpha$ and $\beta$ such that*

$$\alpha \le a, \quad \mu\alpha + 2\beta \le b, \quad \text{and} \quad 2\kappa\beta \le \alpha(1 - \kappa\mu), \tag{2.1}$$

*and for every function $g : X \to Y$ satisfying*

$$\rho(g(\bar{x}), 0) \le \beta, \tag{2.2}$$

*and*

$$\rho(g(x), g(x')) \le \mu\rho(x', x) \quad \text{for every } x', x \in \mathbb{B}_\alpha(\bar{x}), \tag{2.3}$$

*the mapping $y \mapsto (g + F)^{-1}(y) \cap \mathbb{B}_\alpha(\bar{x})$ is a Lipschitz continuous function on $\mathbb{B}_\beta(\bar{y})$ with Lipschitz constant $\kappa'$.*

*Proof.* Pick $\mu$, $\kappa'$ as required and then $\alpha$, $\beta$ according to (2.1). For any $x \in \mathbb{B}_\alpha(\bar{x})$ and any $y \in \mathbb{B}_\beta(\bar{y})$, from (2.2) and (2.3) we have

$$\begin{aligned}\rho(-g(x)+y,\bar{y}) &\leq \rho(g(x),g(\bar{x})) + \rho(g(\bar{x}),0) + \rho(y,\bar{y}) \\ &\leq \mu\rho(x,\bar{x}) + \beta + \beta \leq \mu\alpha + 2\beta \leq b.\end{aligned}$$

By assumption, the mapping $y \mapsto s(y) := F^{-1}(y) \cap \mathbb{B}_a(\bar{x})$ is a Lipschitz continuous function on $\mathbb{B}_b(\bar{y})$ with Lipschitz constant $\kappa$. Fix $y \in \mathbb{B}_\beta(\bar{y})$ and consider the function $\Phi(x) = s(-g(x)+y)$ on $\mathbb{B}_\alpha(\bar{x})$. Observing that $\bar{x} = s(\bar{y})$, using (2.2) and (2.3), and taking into account (2.1), we get

$$\begin{aligned}\rho(\bar{x},\Phi(\bar{x})) &= \rho(s(-g(\bar{x})+y), s(\bar{y})) \\ &\leq \kappa\rho(\bar{y}, y - g(\bar{x})) \leq \kappa(\rho(g(\bar{x}),0) + \rho(y,\bar{y})) \leq 2\kappa\beta \leq \alpha(1-\kappa\mu).\end{aligned}$$

From (2.3), for any $u, v \in \mathbb{B}_\alpha(\bar{x})$,

$$\begin{aligned}\rho(\Phi(u),\Phi(v)) &= \rho(s(-g(u)+y), s(-g(v)+y)) \\ &\leq \kappa\rho(g(u),g(v)) \leq \kappa\mu\rho(u,v).\end{aligned}$$

Hence, by the standard contraction mapping principle, see, e.g., [5, Theorem 1A.2], there exists a unique fixed point $\hat{x} = \Phi(\hat{x})$ in $\mathbb{B}_\alpha(\bar{x})$. This, translated back to the original setting, means that the mapping $y \mapsto \tilde{s}(y) := (g+F)^{-1}(y) \cap \mathbb{B}_\alpha(\bar{x})$ is single-valued, that is, a function which is moreover Lipschitz defined on $\mathbb{B}_\beta(\bar{y})$. Let $y, y' \in \mathbb{B}_\beta(\bar{y})$. Utilizing the equality $\tilde{s}(y) = s(-g(\tilde{s}(y)) + y)$ we have

$$\begin{aligned}\rho(\tilde{s}(y),\tilde{s}(y')) &= \rho(s(-g(\tilde{s}(y))+y), s(-g(\tilde{s}(y'))+y'))) \\ &\leq \kappa\rho(g(\tilde{s}(y)), g(\tilde{s}(y'))) + \kappa\rho(y,y') \\ &\leq \kappa\mu\rho(\tilde{s}(y),\tilde{s}(y')) + \kappa\rho(y,y').\end{aligned}$$

Hence

$$\rho(\tilde{s}(y),\tilde{s}(y')) \leq \frac{\kappa\rho(y,y')}{1-\kappa\mu} \leq \kappa'\rho(y,y'),$$

that is, $\tilde{s}$ is Lipschitz continuous with Lipschitz constant $\kappa'$. The proof is complete. ∎

Observe that Theorem 2.4 doesn't claim strong regularity of the mapping $g+F$ at $\bar{x}$ for $\bar{y}$ since $(\bar{x},\bar{y})$ may not be in the graph of $g+F$; on the other hand $(\bar{x},\bar{y})$ is required to be "close enough" to that graph. At this point this is only a technical improvement of Theorem 2.3 which, however, becomes important later in the paper. Theorem 2.3 can be easily derived from Theorem 2.4, e.g., by applying Theorem 2.4 to the mapping $\tilde{F}(x) = F(x) - \bar{y}$ and the function $\tilde{g}(x) = g(x) - g(\bar{x})$ and then translating the obtained result to strong regularity of the mapping $g+F$ at $\bar{x}$ for $g(\bar{x}) + \bar{y}$.

**3. Uniform strong regularity.** For any given $(t,u) \in \operatorname{gph} S$, the graph of the solution mapping of (1.1), define the mapping

$$v \mapsto G_{t,u}(v) := f(t,u) + \nabla_u f(t,u)(v-u) + N_C(v). \tag{3.1}$$

A point $(t,u) \in \mathbb{R}^{1+n}$ is said to be a *strongly regular point* for the variational inequality (1.1) when $(t,u) \in \operatorname{gph} S$ and the mapping $G_{t,u}$ is strongly regular at 0 for $u$.

We start with the following version of Robinson's theorem which easily follows from Theorem 2.3.

THEOREM 3.1. *Let* $(\bar{t}, \bar{u})$ *be a strongly regular point for* (1.1). *Then there are open neighborhoods* $T$ *of* $\bar{t}$ *and* $U$ *of* $\bar{u}$ *such that the mapping* $T \cap [0,1] \ni \tau \mapsto S(\tau) \cap U$ *is single-valued and Lipschitz continuous on* $T \cap [0,1]$.

*Proof.* Observe that the Lipschitz constant of the function $v \mapsto g(v) := f(\bar{t}, v) - f(\bar{t}, \bar{u}) - \nabla_u f(\bar{t}, \bar{u})(v - \bar{u})$ can be made arbitrarily small by choosing a sufficiently small neighborhoods of $\bar{u}$. Then apply Theorem 2.3 with $F = G_{\bar{t},\bar{u}}$ and $g$ as defined. ∎

Our main assumption in the paper is that each point in $\operatorname{gph} S$ is strongly regular. This assumption combined with Theorem 3.1 means that for every $(t, u) \in \operatorname{gph} S$ there are neighborhoods $T_{t,u}$ of $t$ and $U_{t,u}$ of $u$ such that the mapping $T_{t,u} \cap [0,1] \ni t \mapsto S(t) \cap U_{t,u}$ is single-valued and Lipschitz continuous on $T_{t,u} \cap [0,1]$. We will now show that for each $t \in [0,1]$ the set of solutions $S(t)$ is actually the union of the values at $t$ of finitely many functions that are Lipschitz continuous and their graphs never intersect each other on $[0,1]$; moreover, along any such function $\bar{u}(\cdot)$ the mapping $G_{t,\bar{u}(t)}$ is strongly regular *uniformly* in $t \in [0,1]$, meaning that the neighborhoods and the constant involved in the definition do not depend on $t$. Sufficient as well as necessary and sufficient conditions for strong regularity are known in the literature; see, e.g., [5, Chapter 2].

From the assumed uniform boundedness of the solution mapping $S$ and the continuity of $f$ and its derivatives, we get, for use in what follows, the existence of a constant $K > 0$ such that

$$\sup_{t \in [0,1], v \in D} (\|\nabla_t f(t,v)\| + \|\nabla_u f(t,v)\| + \|\nabla^2_{uu} f(t,v)\| + \|\nabla^2_{ut} f(t,v)\|) \leq K. \tag{3.2}$$

The main result of this section follows.

THEOREM 3.2. *Suppose that each point in* $\operatorname{gph} S$ *is strongly regular. Then there exist finitely many Lipschitz continuous functions* $\bar{u}_j : [0,1] \to \mathbb{R}^n$, $j = 1, 2, \ldots, M$ *such that for each* $t \in [0,1]$ *one has* $S(t) = \cup_{1 \leq j \leq M} \{\bar{u}_j(t)\}$; *moreover, the graphs of the functions* $\bar{u}_j$ *are isolated from each other in the sense that there exists* $\delta > 0$ *such that* $\|u_{j'}(t) - u_j(t)\| \geq \delta$ *for every* $j' \neq j$ *and every* $t \in [0,1]$. *Also, there exist positive constants* $a$, $b$, *and* $\lambda$ *such that for each such function* $\bar{u}_i$ *and for each* $t \in [0,1]$ *the mapping*

$$\mathbb{B}_b(0) \ni w \mapsto G^{-1}_{t,\bar{u}_i(t)}(w) \cap \mathbb{B}_a(\bar{u}_i(t)) \tag{3.3}$$

*is a Lipschitz continuous function with a Lipschitz constant* $\lambda$.

*Proof.* Let $(t, v) \in \operatorname{gph} S$. Then, according to Theorem 3.1 there exists a neighborhood $T_{t,v}$ of $t$ which is open relative to $[0,1]$ and an open neighborhood $U_{t,v}$ of $v$ such that the mapping $T_{t,v} \ni \tau \mapsto S(\tau) \cap U_{t,v}$ is a function, denoted $u_{t,v}(\cdot)$, which is Lipschitz continuous on $T_{t,v}$ with Lipschitz constant $L_{t,v}$. From the open covering $\{T_{t,v} \times U_{t,v}\}_{(t,v) \in \operatorname{gph} S}$ of the graph of $S$, which is a compact set in $\mathbb{R}^{1+n}$, we extract a finite subcovering $\{T_{t_j,v_j} \times U_{t_j,v_j}\}_{j=1}^M$. Let $L = \max_{1 \leq j \leq M} L_{t_j,v_j}$.

Let $\tau \in [0,1]$ and choose any $\bar{u} \in S(\tau)$. Now we will prove that there exists a Lipschitz continuous function $\bar{u}(\cdot)$ with Lipschitz constant $L$ such that $\bar{u}(t) \in S(t)$ for all $t \in [0,1]$ and also $\bar{u}(\tau) = \bar{u}$.

Assume $\tau < 1$. Then there exists $j \in \{1, \ldots, M\}$ such that $(\tau, \bar{u}) \in T_{t_j,v_j} \times U_{t_j,v_j}$. Define $\bar{u}(t) = u_{t_j,v_j}(t)$ for all $t \in (t'_j, t''_j) := T_{t_j,v_j}$. Then $\bar{u}(\tau) = \bar{u}$ and $\bar{u}(\cdot)$ is Lipschitz continuous on $[t'_j, t''_j]$. If $t''_j < 1$, then there exists some $i \in \{1, \ldots, M\}$ such that $(t_j^{''}, \bar{u}(t_j^{''})) \in T_{t_i,v_i} \times U_{t_i,v_i} := (t_i', t_i^{''}) \times U_{t_i,v_i}$. Then of course $u_{t_i,v_i}(t_j^{''}) = \bar{u}(t_j^{''})$, and we can extend $\bar{u}(\cdot)$ to $[t_j, t_{i''}]$ as $\bar{u}(t) = u_{t_i,v_i}(t)$ for $t \in [t_{j'}, t_{i''}]$. After at most $M$ such steps we extend $\bar{u}(\cdot)$ to $[t_{j'}, 1]$. By repeating the same argument on the interval $[0, \tau]$ we extend $\bar{u}(\cdot)$ on the entire interval $[0,1]$, thus obtaining a Lipschitz continuous selection of $S$. If $\tau = 1$, then we repeat the same argument on $[0,1]$ starting from 1 and going to the left.

Now assume that $\tau, \bar{u}$ and $(\theta, \tilde{u})$ are two points in $\operatorname{gph} S$ and let $\bar{u}(\cdot)$ and $\tilde{u}(\cdot)$ be the functions determined by the above procedure such that $\bar{u}(\tau) = \bar{u}$ and $\tilde{u}(\theta) = \tilde{u}$. Assume that $\bar{u}(0) \neq \tilde{u}(0)$ and the set $\Delta := \{t \in [0,1] \,|\, \bar{u}(t) = \tilde{u}(t)\}$ is nonempty. Since $\Delta$ is closed, $\inf \Delta := v > 0$ is attained, and then we have that $\bar{u}(\nu) = \tilde{u}(\nu)$ and $\bar{u}(t) \neq \tilde{u}(t)$ for $t \in [0, \nu)$. But then, according to Theorem 3.1, $(\nu, \bar{u}(\nu)) \in \operatorname{gph} S$ cannot be a strongly regular point of $S$, a contradiction. Thus, the number of different Lipschitz continuous functions $\bar{u}(\cdot)$ constructed as above from points $(\tau, \bar{u}) \in \operatorname{gph} S$ is not more than the number of points in $S(0)$. Hence there are finitely many Lipschitz continuous functions $\bar{u}_j(\cdot)$ such that for every $t \in [0,1]$ one has $S(t) = \cup_j \{\bar{u}_j(t)\}$. This proves the first part of the theorem.

Choose a Lipschitz continuous function $\bar{u}(\cdot)$ whose values are in the set of values of $S$, that is, $\bar{u}(\cdot)$ is one of the functions $\bar{u}_j(\cdot)$. Let $t \in (0,1)$ and denote $G_t = G_{t,\bar{u}(t)}$ for simplicity. Let $a_t$, $b_t$, and $\lambda_t$ be positive constants such that the mapping

$$\mathbb{B}_{b_t}(0) \ni w \mapsto G_t^{-1}(w) \cap \mathbb{B}_{a_t}(\bar{u}(t)) \tag{3.4}$$

is a Lipschitz continuous function with Lipschitz constant $\lambda_t$. Make $b_t > 0$ smaller if necessary so that

$$b_t \lambda_t < a_t. \tag{3.5}$$

Let $\rho_t \in (0, \delta_t)$ be such that $L\rho_t < a_t/2$. Then, from the Lipschitz continuity of $\bar{u}$ around $t$ we have that $\mathbb{B}_{a_t/2}(\bar{u}(\tau)) \subset \mathbb{B}_{a_t}(\bar{u}(t))$ for all $\tau \in (t - \rho_t, t + \rho_t)$. Make $\rho_t > 0$ smaller if necessary so that

$$K(L+1)\rho_t < 1/\lambda_t. \tag{3.6}$$

We will now apply Theorem 2.4 to show that there exist a neighborhood $O_t$ of $t$ and positive constants $\alpha_t$ and $\beta_t$ such that for each $\tau \in O_t \cap [0,1]$ the mapping

$$\mathbb{B}_{\beta_t}(0) \ni w \mapsto G_\tau^{-1}(w) \cap \mathbb{B}_{\alpha_t}(\bar{u}(t)) \tag{3.7}$$

is a Lipschitz continuous function.

Consider the function

$$\begin{aligned} g_{t,\tau}(v) = {} & f(t, \bar{u}(t)) - f(\tau, \bar{u}(\tau)) + (\nabla_u f(t, \bar{u}(t)) - \nabla_u f(\tau, \bar{u}(\tau)))v \\ & - 2\nabla_u f(t, \bar{u}(t))\bar{u}(t) + \nabla_u f(\tau, \bar{u}(\tau))\bar{u}(\tau). \end{aligned}$$

Note that the Lipschitz constant of $g_{t,\tau}$ is bounded by the expression on the left of (3.6). For each $v$ we have

$$G_t(v) = G_\tau(v) + g_{t,\tau}(v).$$

We wish to apply Theorem 2.4 with $F = G_t$, $\bar{x} = \bar{u}(t)$, $\bar{y} = 0$, $g = g_{t,\tau}$, $a = a_t$, $b = b_t$, $\kappa = \lambda_t$, $\mu = \mu_t := K(L+1)\rho_t$, and

$$\kappa = \lambda_t' := \frac{3\lambda_t}{2(1 - K(L+1)\rho_t\lambda_t)} > \frac{\lambda_t}{1 - \lambda_t\mu_t}. \tag{3.8}$$

For that purpose we need to show that there exist constants $\alpha_t$ and $\beta_t$ that satisfy the inequalities

$$\alpha_t \le a_t, \quad \mu_t\alpha_t + 2\beta_t \le b_t, \quad 2\lambda_t\beta_t \le \alpha_t(1-\lambda_t\mu_t), \quad \|g_{t,\tau}(\bar{u}(t))\| \le \beta_t. \tag{3.9}$$

From the evaluation

$$\begin{aligned} & g_{t,\tau}(\bar{u}(t)) \\ &\quad = \int_0^1 \frac{d}{ds} f(\tau + s(t-\tau), \bar{u}(\tau) + s(\bar{u}(t) - \bar{u}(\tau)))ds - \nabla_u f(\tau, \bar{u}(\tau))(\bar{u}(t) - \bar{u}(\tau)) \\ &\quad = \int_0^1 (t-\tau)\nabla_t f(\tau + s(t-\tau), \bar{u}(\tau) + s(\bar{u}(t) - \bar{u}(\tau)))ds \\ &\qquad + \int_0^1 \left[\nabla_u f(\tau + s(t-\tau), \bar{u}(\tau) + s(\bar{u}(t) - \bar{u}(\tau))) - \nabla_u f(\tau, \bar{u}(\tau))\right](\bar{u}(t) - \bar{u}(\tau))ds \end{aligned}$$

we obtain

$$\|g_{t,\tau}(\bar{u}(t))\| \le K\rho_t + \frac{1}{2}KL\rho_t^2 + KL^2\rho_t^2.$$

Choose $\rho_t$ smaller if necessary such that $\frac{1}{2}L\rho_t + L^2\rho_t < 1$; then

$$\|g_{t,\tau}(\bar{u}(t))\| \le 2K\rho_t.$$

Denoting $A := K(1+L)$ and $B := 2K$ we have

$$\mu_t = A\rho_t \quad \text{and} \quad \|g_{t,\tau}(\bar{u}(t))\| \le B\rho_t.$$

Set $\beta_t := B\rho_t$. We will now show that there exists a positive $\alpha_t$ which satisfies all inequalities in (3.9) and also

$$\lambda_t'\beta_t < \alpha_t. \tag{3.10}$$

Substituting the already chosen $\mu_t$ and $\beta_t$ in (3.9) we obtain that $\alpha_t$ should satisfy

$$\begin{cases} \alpha_t \le a_t, \\ A\rho_t\alpha_t + 2B\rho_t \le b_t, \\ 2\lambda_t B\rho_t \le \alpha_t(1-\lambda_t A\rho_t). \end{cases} \tag{3.11}$$

System (3.11) has a solution $\alpha_t > 0$ provided that

$$\frac{2\lambda_t B\rho_t}{1-\lambda_t A\rho_t} \le \frac{b_t - 2B\rho_t}{A\rho_t} \quad \text{and} \quad \frac{b_t - 2B\rho_t}{A\rho_t} \le a_t.$$

Thus, everything comes down to checking whether this system of inequalities is consistent. But this system is consistent whenever

$$(2B + b_t\lambda_t A)\rho_t \le b_t \le (2B + Aa_t)\rho_t,$$

which in turn always holds because of (3.5). Hence, there exist $\alpha_t$ satisfying (3.11). Moreover, using (3.8) and the third inequality in (3.9) we obtain

$$\lambda_t'\beta_t = \frac{3}{2}\,\frac{\beta_t\lambda_t}{1-\lambda_t\mu_t} < \frac{2\beta_t\lambda_t}{1-\lambda_t\mu_t} \le \alpha_t,$$

hence (3.10) holds.

We are now ready to apply Theorem 2.4, from which we conclude that the mapping in (3.7) is a Lipschitz continuous function with Lipschitz constant $\lambda'_t$. From the open covering $\cup_{t\in[0,1]}(t-\rho_t, t+\rho_t)$ of $[0,1]$ choose a finite subcovering of open intervals $(t_i-\rho_{t_i}, t_i+\rho_{t_i})$, $i=1,2,\ldots,m$. Let $a=\min\{\alpha_{t_i} \mid i=1,\ldots,m\}$, $b=\min\{\beta_{t_i} \mid i=1,\ldots,m\}$, and $\lambda=\max\{\lambda'_{t_i} \mid i=1,\ldots,m\}$. From (3.10) we get $b\le a/\lambda$; then the observation in Remark 2.2 applies, hence for each $\tau\in(t_i-\rho_{t_i}, t_i+\rho_{t_i})\cap[0,1]$ the mapping $\mathbb{B}_b(0)\ni w\mapsto G_\tau^{-1}(w)\cap\mathbb{B}_a(\bar u(\tau))$ is a Lipschitz continuous function with Lipschitz constant $\lambda$. Let $t\in[0,1]$; then $t\in(t_i-\rho_{t_i}, t_i+\rho_{t_i})$ for some $i\in\{1,\ldots,m\}$. Hence the mapping $\mathbb{B}_b(0)\ni w\mapsto G_t^{-1}(w)\cap\mathbb{B}_a(\bar u(t))$ is a Lipschitz continuous function with Lipschitz constant $\lambda$. The proof is complete. ∎

**4. Euler–Newton continuation.** For $N>1$, let $\{t_i\}_{i=0}^N$ with $t_0=0$, $t_N=1$, be a uniform grid on $[0,1]$ with step size $h=t_{i+1}-t_i=1/N$ for $i=0,1,\ldots,N-1$. According to Theorem 3.2, the graph of the solution mapping $S$ consists of the graphs of finitely many Lipschitz continuous functions which are isolated from each other; let $L$ be a Lipschitz constant for all such functions. Choose any of these functions and call it $\bar u(\cdot)$. Also, we know from Theorem 3.2 that there exist positive $a$, $b$, and $\kappa$ such that for each $i=0,1,\ldots,N-1$, the mapping

$$\mathbb{B}_b(0)\ni w\mapsto G_{t_i}^{-1}(w)\cap\mathbb{B}_a(\bar u(t_i)) \tag{4.1}$$

is a Lipschitz continuous function with Lipschitz constant $\kappa$, where we recall that $G_{t_i}=G_{t_i,\bar u(t_i)}$ for $G_{t,u}$ given in (3.1).

THEOREM 4.1. *Suppose that each point in* gph $S$ *is strongly regular and let* $\bar u$ *be a Lipschitz continuous selection of the solution mapping* $S$*. Let* $u_0=\bar u(0)$*. Then there exist positive constants* $c$ *and* $\beta$ *and a natural* $N_0$ *such that for any natural* $N\ge N_0$ *the iteration* (1.3) *generates a unique sequence* $\{u_i\}$ *with* $h=1/N$ *starting from* $u_0$ *and such that* $u_i\in\mathbb{B}_\beta(\bar u(t_i))$ *for* $i=0,1,\ldots,N$*. Moreover, this sequence satisfies*

$$\max_{0\le i\le N}\|u_i-\bar u(t_i)\|\le ch^4. \tag{4.2}$$

*Proof.* Given $a$, $b$, and $\kappa$ as in (4.1), let $\kappa'$, $\mu$, $\alpha$, and $\beta$ be chosen according to Theorem 2.4. Let $K$ be as in (3.2), let

$$c:=\frac{K^3\kappa'^2}{2}(1+L+L^2)^2, \tag{4.3}$$

and chose $N_0$ so large that for $h=1/N$ with $N\ge N_0$ the following inequalities hold:

$$ch^3(2+2L+ch^3)\le 1+L^2, \tag{4.4}$$

$$Kh(1+L+ch^3)\le\mu, \qquad Kh^2\left(1+L+L^2\right)\le\beta, \tag{4.5}$$

$$\kappa' Kh^2(1+L+L^2)\le\mu, \qquad ch^4\le\beta. \tag{4.6}$$

To prove (4.2) we proceed by induction. First, for $i=0$ we have $u_0=\bar u(t_0)$ and there is nothing more to prove. Let for $j=1,2,\ldots,i$ the iterates $u_j$ be already generated by (1.3) uniquely in $\mathbb{B}_\beta(\bar u(t_j))$ and in such a way that

$$\|u_j-\bar u(t_j)\|\le ch^4 \quad \text{for all} \quad j=1,2,\ldots,i.$$

We will prove that (1.3) determines a unique $u_{i+1}\in\mathbb{B}_b(\bar u(t_{i+1}))$ and $u_{i+1}$ satisfies

$$\|u_{i+1}-\bar u(t_{i+1})\|\le ch^4. \tag{4.7}$$

We start with the Euler step. The generalized equation

$$f(t_i,u_i)+\nabla_u f(t_i,u_i)(v-u_i)+h\,\nabla_t f(t_i,u_i)+N_C(v)\ni 0 \tag{4.8}$$

for $v\in\mathbb{B}_\beta(\bar u(t_{i+1}))$ can be written as

$$g(v)+G_{t_{i+1}}(v)\ni 0, \tag{4.9}$$

where, as before, $G_t=G_{t,\bar u(t)}$ with $G_{t,u}$ defined in (3.1), and

$$\begin{aligned} g(v)=\ &f(t_i,u_i)+\nabla_u f(t_i,u_i)(v-u_i)+h\,\nabla_t f(t_i,u_i)\\ &-[f(t_{i+1},\bar u(t_{i+1}))+\nabla_u f(t_{i+1},\bar u(t_{i+1}))(v-\bar u(t_{i+1}))]. \end{aligned}$$

For any $v$, $v'\in\mathbb{R}^n$ we have

$$\begin{aligned} \|g(v)-g(v')\| &= \|[\nabla_u f(t_i,u_i)-\nabla_u f(t_{i+1},\bar u(t_{i+1}))](v-v')\|\\ &\le K\,(h+\|u_i-\bar u(t_{i+1})\|)\,\|v-v'\|\\ &\le K(h+\|u_i-\bar u(t_i)\|+\|\bar u(t_i)-\bar u(t_{i+1})\|)\,\|v-v'\|\\ &\le K(h+ch^4+Lh)\|v-v'\|\le \mu\|v-v'\|, \end{aligned}$$

where we use the first inequality in (4.5). Furthermore,

$$\begin{aligned} &\|g(\bar u(t_{i+1}))\|\\ &=\Bigg\|\int_0^1 \frac{d}{ds}f(t_i+sh,u_i+s(\bar u(t_{i+1})-u_i))\,ds\\ &\quad -\nabla_u f(t_i,u_i))(\bar u(t_{i+1})-u_i)-h\nabla_t f(t_i,u_i)\Bigg\|\\ &\le\Bigg\|\int_0^1 h\,[\nabla_t f(t_i+sh,u_i+s(\bar u(t_{i+1})-u_i))-\nabla_t f(t_i,u_i)]\,ds\Bigg\|\\ &\quad+\Bigg\|\int_0^1[\nabla_u f(t_i+sh,u_i+s(\bar u(t_{i+1})-u_i))-\nabla_u f(t_i,u_i)]\,(\bar u(t_{i+1})-u_i)\,ds\Bigg\|\\ &\le\int_0^1 K(sh^2+sh\|\bar u(t_{i+1})-u_i\|)\,ds+\int_0^1 K(sh\|\bar u(t_{i+1})-u_i\|+s\|\bar u(t_{i+1})-u_i\|^2)ds\\ &\le\frac{Kh^2}{2}+\frac{Kh}{2}\|\bar u(t_{i+1})-u_i\|+\frac{Kh}{2}\|\bar u(t_{i+1})-u_i\|+\frac{K}{2}\|\bar u(t_{i+1})-u_i\|^2\\ &\le\frac{Kh^2}{2}+Kh(\|\bar u(t_{i+1})-\bar u(t_i)\|+\|\bar u(t_i)-u_i\|)+\frac{K}{2}(\|\bar u(t_{i+1})-\bar u(t_i)\|+\|\bar u(t_i)-u_i\|)^2\\ &\le\frac{K}{2}(h^2+2h(Lh+ch^4)+(Lh+ch^4)^2)\le Kh^2(1+L+L^2), \end{aligned}$$

where in the last inequality we use (4.4). This implies that $\|g(\bar u(t_{i+1}))\|\le\beta$ due to the second relation in (4.5). Applying Theorem 2.4 we obtain the existence of a unique in $\mathbb{B}_\beta(\bar u(t_{i+1}))$ solution $v_{i+1}$ of (4.9), hence of (4.8), and moreover the function

$$\mathbb{B}_\beta(0)\ni y\mapsto\xi(y):=(g+G_{t_{i+1}})^{-1}(y)\cap\mathbb{B}_\alpha(\bar u(t_{i+1}))$$

is Lipschitz continuous on $\mathbb{B}_\beta(0)$ with constant $\kappa'$. Observe that $v_{i+1}=\xi(0)$ and $\bar u(t_{i+1})=\xi(g(\bar u(t_{i+1})))$; then

$$\|v_{i+1}-\bar u(t_{i+1})\|=\|\xi(0)-\xi(g(\bar u(t_{i+1})))\|\le\kappa'\|g(\bar u(t_{i+1}))\|\le\kappa'Kh^2\left(1+L+L^2\right). \tag{4.10}$$

The Newton step solves the generalized equation

$$f(t_{i+1}, v_{i+1}) + \nabla_u f(t_{i+1}, v_{i+1})(u - v_{i+1}) + N_C(u) \ni 0 \tag{4.11}$$

for $u \in \mathbb{B}_\beta(\bar{u}(t_{i+1}))$, which can be rewritten as

$$h(u) + G_{t_{i+1}}(u) \ni 0$$

with

$$\begin{aligned} h(v) = {} & f(t_{i+1}, v_{i+1}) + \nabla_u f(t_{i+1}, v_{i+1})(u - v_{i+1}) \\ & - [f(t_{i+1}, \bar{u}(t_{i+1})) + \nabla_u f(t_{i+1}, \bar{u}(t_{i+1}))(u - \bar{u}(t_{i+1}))]. \end{aligned}$$

For any $v$, $v' \in \mathbb{R}^n$ we have

$$\begin{aligned} \|h(v) - h(v')\| &= \|(\nabla_u f(t_{i+1}, v_{i+1}) - \nabla_u f(t_{i+1}, \bar{u}(t_{i+1})))(v - v')\| \\ &\le K\|v_{i+1} - \bar{u}(t_{i+1})\|\|v - v'\| \le \kappa' K^2 h^2 (1 + L + L^2)\|v - v'\| \\ &\le \mu\|v - v'\|, \end{aligned}$$

where we use (4.10) and the first inequality in (4.6). Moreover,

$$\begin{aligned} &\|h(\bar{u}(t_{i+1}))\| \\ &\quad = \|f(t_{i+1}, v_{i+1}) + \nabla_u f(t_{i+1}, v_{i+1})(\bar{u}(t_{i+1}) - v_{i+1}) - f(t_{i+1}, \bar{u}(t_{i+1}))\| \\ &\quad = \left\| \int_0^1 \frac{d}{ds} f(t_{i+1}, v_{i+1} + s(\bar{u}(t_{i+1}) - v_{i+1}))\, ds - \nabla_u f(t_{i+1}, v_{i+1})(\bar{u}(t_{i+1}) - v_{i+1}) \right\| \\ &\quad = \left\| \int_0^1 \left[\nabla_u f(t_{i+1}, v_{i+1} + s(v_{i+1} - \bar{u}(t_{i+1}))) - \nabla_u f(t_{i+1}, v_{i+1})\right] (v_{i+1} - \bar{u}(t_{i+1}))\, ds \right\| \\ &\quad \le \int_0^1 sK\|v_{i+1} - \bar{u}(t_{i+1})\|^2\, ds = \frac{K}{2}\|v_{i+1} - \bar{u}(t_{i+1})\|^2 \\ &\quad \le \frac{K}{2}\left(\kappa' K h^2 \left(1 + L + L^2\right)\right)^2 = ch^4. \end{aligned}$$

In particular, this implies that $\|h(\bar{u}(t_{i+1}))\| \le \beta$ due to the second relation in (4.6). Applying Theorem 2.4 with $g = h$ in the same way as for the estimate (4.10) we obtain that there exists a unique in $\mathbb{B}_\beta(\bar{u}(t_{i+1}))$ solution $u_{i+1}$ of (4.11) which moreover satisfies (4.7). This completes the induction step and the proof of (4.2). ∎

For $C = \mathbb{R}^n$ the Euler step for the method (1.3) becomes

$$v_{i+1} = u_i - \nabla_u f(t_i, u_i)^{-1}(h\nabla_t f(t_i, u_i) + f(t_i, u_i)) \tag{4.12}$$

which is different from the Euler step in the equation case (1.2) given in [2]. We proved in Theorem 4.1 that the method combining the modified Euler step (4.12) with the standard Newton step has error of order $O(h^4)$. It turns out that the error has the same order when we use the method (1.2). This could be shown in various ways; in our case the simplest is to follow the proof of Theorem 4.1. Indeed, if instead of $g$ in (4.9) we use the function

$$\begin{aligned} \bar{g}(v) = {} & \nabla_u f(t_i, u_i)(v - u_i) + h\, \nabla_t f(t_i, u_i) \\ & - [f(t_{i+1}, \bar{u}(t_{i+1})) + \nabla_u f(t_{i+1}, \bar{u}(t_{i+1}))(v - \bar{u}(t_{i+1}))], \end{aligned}$$

then, from the induction hypothesis and the fact that $f(t_i, \bar{u}(t_i)) = 0$, we get

$$\|f(t_i, u_i)\| = \|f(t_i, u_i) - f(t_i, \bar{u}(t_i))\| \le Kch^4.$$

Hence,

$$\begin{aligned}\|\bar{g}(\bar{u}(t_{i+1}))\| &\le \|g(\bar{u}(t_{i+1}))\| + \|\bar{g}(\bar{u}(t_{i+1})) - g(\bar{u}(t_{i+1}))\| \\ &\le \|g(\bar{u}(t_{i+1}))\| + \|f(t_i, u_i)\| \le \|g(\bar{u}(t_{i+1}))\| + Kch^4.\end{aligned}$$

Thus, the estimate for $\|\bar{g}(\bar{u}(t_{i+1})\|$ is of the same order as for $\|g(\bar{u}(t_{i+1})\|$ and hence the final estimate (4.7) is of the same order.

Consider now the following enhanced version of the method (1.3), where the already computed Jacobian $\nabla_u f(t_i, v_i)$ in the preceding corrector step is used in the next prediction step in place of $\nabla_u f(t_i, u_i)$:

$$\text{(4.13)} \qquad \begin{cases} f(t_i, u_i) + h\nabla_t f(t_i, u_i) + \nabla_u f(t_i, v_i)(v_{i+1} - u_i) + N_C(v_{i+1}) \ni 0, \\ f(t_{i+1}, v_{i+1}) + \nabla_u f(t_{i+1}, v_{i+1})(u_{i+1} - v_{i+1}) + N_C(u_{i+1}) \ni 0. \end{cases}$$

The initial value $u_0$ is chosen as in Theorem 4.1. We will now show that the method (4.13) has error of the same order $O(h^4)$ as (1.3). To this end, we will use an induction argument similar to the one used in the proof of Theorem 4.1.

*Proof of $O(h^4)$ convergence for method* (4.13). Given $a$, $b$, and $\kappa$ as in (4.1), let $\kappa'$, $\mu$, $\alpha$, and $\beta$ be chosen according to Theorem 2.4. Let $c$ be as in (4.3) and let

$$\text{(4.14)} \qquad D := \kappa' K \left(1 + L + \frac{1}{2}L^2\right).$$

Choose $N_0$ so large that for $h = 1/N$ with $N \ge N_0$ the following inequalities are satisfied:

$$\text{(4.15)} \qquad ch^3 + cLh^3 + \frac{1}{2}c^2h^6 + DLh + Dch^4 \le \frac{1}{2},$$

$$\text{(4.16)} \qquad Kh(1 + Dh + L) \le \mu,$$

$$\text{(4.17)} \qquad K\left(1 + L + \frac{1}{2}L^2\right)h^2 \le \beta.$$

Let $u_j \in \mathbb{B}_\beta(\bar{u}(t_j))$ and $v_j$ be already defined by (4.13) for $j = 1, 2, \ldots, i$ in such a way that

$$\|u_j - \bar{u}(t_j)\| \le ch^4 \quad \text{and} \quad \|v_j - \bar{u}(t_j)\| \le Dh^2 \quad \text{for} \quad j = 1, 2, \ldots, i.$$

Let us rewrite the variational inequality

$$f(t_i, u_i) + \nabla_u f(t_i, v_i)(v - u_i) + h\,\nabla_t f(t_i, u_i) + N_C(v) \ni 0$$

as the inclusion

$$\text{(4.18)} \qquad g(v) + G_{t_{i+1}}(v) \ni 0$$

with

$$\begin{aligned} g(v) = {} & f(t_i, u_i) + \nabla_u f(t_i, v_i)(v - u_i) + h\,\nabla_t f(t_i, u_i) \\ & - [f(t_{i+1}, \bar{u}(t_{i+1})) + \nabla_u f(t_{i+1}, \bar{u}(t_{i+1}))(v - \bar{u}(t_{i+1}))]. \end{aligned}$$

For any $v$, $v' \in \mathbb{R}^n$ we have

$$\begin{aligned}\|g(v) - g(v')\| &= \|[\nabla_u f(t_i, v_i) - \nabla_u f(t_{i+1}, \bar{u}(t_{i+1}))](v - v')\| \\ &\le K\,(h + \|v_i - \bar{u}(t_{i+1})\|)\,\|v - v'\| \\ &\le K(h + \|v_i - \bar{u}(t_i)\| + \|\bar{u}(t_i) - \bar{u}(t_{i+1})\|)\,\|v - v'\| \\ &\le K(h + Dh^2 + Lh)\|v - v'\| \le \mu\|v - v'\|,\end{aligned}$$

where we use (4.16). Furthermore,

$$\begin{aligned}&\|g(\bar{u}(t_{i+1}))\| \\ &\quad= \left\| \int_0^1 \frac{d}{ds} f(t_i + sh, u_i + s(\bar{u}(t_{i+1}) - u_i))ds \right. \\ &\qquad \left. - \nabla_u f(t_i, v_i)(\bar{u}(t_{i+1}) - u_i) - h\nabla_t f(t_i, u_i) \right\| \\ &\quad\le \left\| \int_0^1 h\left[\nabla_t f(t_i + sh, u_i + s(\bar{u}(t_{i+1}) - u_i)) - \nabla_t f(t_i, u_i)\right] ds \right\| \\ &\qquad + \left\| \int_0^1 \left[\nabla_u f(t_i + sh, u_i + s(\bar{u}(t_{i+1}) - u_i)) - \nabla_u f(t_i, u_i)\right] (\bar{u}(t_{i+1}) - u_i)\, ds \right\| \\ &\qquad + \left\| \int_0^1 \left[\nabla_u f(t_i, u_i) - \nabla_u f(t_i, v_i)\right] (\bar{u}(t_{i+1}) - u_i)\, ds \right\| \\ &\quad\le \int_0^1 K(sh^2 + sh\,\|\bar{u}(t_{i+1}) - u_i\|)\, ds \\ &\qquad + \int_0^1 K(sh\|\bar{u}(t_{i+1}) - u_i\| + s\|\bar{u}(t_{i+1}) - u_i\|^2)\, ds + K\|u_i - v_i\|\|\bar{u}(t_{i+1}) - u_i\| \\ &\quad\le \frac{Kh^2}{2} + 2\frac{Kh}{2}(\|\bar{u}(t_{i+1}) - \bar{u}(t_i)\| + \|\bar{u}(t_i) - u_i\|) \\ &\qquad + \frac{K}{2}(\|\bar{u}(t_{i+1}) - \bar{u}(t_i)\| + \|\bar{u}(t_i) - u_i\|)^2 \\ &\qquad + K\|u_i - v_i\|(\|\bar{u}(t_{i+1}) - \bar{u}(t_i)\| + \|\bar{u}(t_i) - u_i\|) \\ &\quad\le \frac{Kh^2}{2} + Kh^2(L + ch^3) + \frac{Kh^2}{2}(L + ch^3)^2 + KDh^2(Lh + ch^4) \\ &\quad\le K\left(1 + L + \frac{1}{2}L^2\right)h^2,\end{aligned}$$

where in the last inequality we use (4.15). Then from (4.17) we get that $\|g(\bar{u}(t_{i+1}))\| \le \beta$. Hence, Theorem 2.4 implies the existence of a unique in $\mathbb{B}_\beta(\bar{u}(t_{i+1}))$ solution $v_{i+1}$ of (4.18). The function

$$\mathbb{B}_\beta(0) \ni y \mapsto \xi(y) := (g + G_{t,\bar{u}(t_{i+1})})^{-1}(y) \cap \mathbb{B}_\alpha(\bar{u}(t_{i+1}))$$

is Lipschitz continuous with constant $\kappa'$ and $v_{i+1} = \xi(0)$, $\bar{u}(t_{i+1}) = \xi(g(\bar{u}(t_{i+1})))$. Hence,

$$\|v_{i+1} - \bar{u}(t_{i+1})\| = \|\xi(0) - \xi(g(\bar{u}(t_{i+1})))\| \le \kappa'\|g(\bar{u}(t_{i+1}))\| = Dh^2.$$

The rest of the proof involving the Newton step is completely analogous to the one in the proof of Theorem 4.1. ∎

We end this section with an important observation. Consider the following method where we have not just one but *two* corrector (Newton) steps:

$$\begin{cases} f(t_i,u_i)+h\nabla_t f(t_i,u_i)+\nabla_u f(t_i,u_i)(v_{i+1}-u_i)+N_C(v_{i+1})\ni 0,\\ f(t_{i+1},v_{i+1})+\nabla_u f(t_{i+1},v_{i+1})(w_{i+1}-v_{i+1})+N_C(w_{i+1})\ni 0,\\ f(t_{i+1},w_{i+1})+\nabla_u f(t_{i+1},w_{i+1})(u_{i+1}-w_{i+1})+N_C(u_{i+1})\ni 0. \end{cases}$$

By repeating the argument used in the proof of Theorem 4.1 one obtains an estimate for the $l^\infty$ error of order $O(h^8)$. A third Newton step will give $O(h^{16})$! Such a strategy would be perhaps acceptable for relatively small problems, and we use it in the numerical examples given in section 5. For practical problems, however, a trade-off is to be sought between the theoretical accuracy and computational complexity of an algorithm. Also, one should remember that the error in the uniform norm will always be $O(h)$ in general, unless the solution has better smoothness properties than just Lipschitz continuity.

**5. Tracking economic equilibrium parametrically.** In the previous paper [4] a model of economic equilibrium was proposed for exchange of goods in a single time period, where there are $r$ agents, each of which starts with a vector $x_i^0\in\mathbb{R}^n$ of goods and trades them for another goods vectors $x_i\in\mathbb{R}^n$. This is done through a market in which goods have a price vector $p\in\mathbb{R}^n_+$. In addition, agent $i$ has an initial amount of money $m_i^0\in\mathbb{R}_+$ and ends up, after trading, with an amount of money $m_i\in\mathbb{R}_+$. The optimization problem for agent $i$ is to maximize a utility function $u_i(m_i,x_i)$ over a set $\mathbb{R}_+\times U_i$ subject to the budget constraint

$$m_i-m_i^0+\langle p,x_i-x_i^0\rangle\le 0, \tag{5.1}$$

where the sets $U_i\subset\mathbb{R}^n$ are nonempty, closed, and convex and the functions $u_i$ are continuously differentiable, concave, and nondecreasing over $\mathbb{R}_+\times U_i$. In addition to the budget constraints (5.1) there are supply-demand requirements for money and goods of the form

$$\sum_{i=1}^r[m_i-m_i^0]\le 0,\quad \sum_{i=1}^r[x_i-x_i^0]\le 0. \tag{5.2}$$

It is shown in [4, Theorem 1] that under some mild conditions an equilibrium always exists, and moreover, it satisfies a first-order optimality condition for each agent involving the Lagrange functions

$$L_i(p,m_i,x_i,\lambda_i)=-u(m_i,x_i)+\lambda_i(m_i-m_i^0+\langle p,x_i-x_i^0\rangle)$$

with a Lagrange multiplier $\lambda_i\ge 0$, $i=1,\dots,r$, associated with the budget constraint (5.1). If we add to that the supply-demand constraints (5.2) written as complementarity conditions, we obtain a variational inequality for the vectors $p\in\mathbb{R}^n_+$, $m=(m_1,\dots,m_r)\in\mathbb{R}^r_+$, $x=(x_1,\dots,x_r)\in U_1\times U_2\times\cdots U_r$, and $\lambda=(\lambda_0,\dots,\lambda_r)\in\mathbb{R}^r_+$ of the form

$$-g(p,m,x,\lambda,m^0,x^0)\in N_C(p,m,x,\lambda), \tag{5.3}$$

where

$$C=\mathbb{R}^n_+\times\mathbb{R}^r_+\times U_1\times\cdots\times U_r\times\mathbb{R}^r_+, \tag{5.4}$$

and

$$g(p, m, x, \lambda, m^0, x^0) = \begin{pmatrix} \sum_{i=1}^{r}[x_i^0 - x_i] \\ \cdots \\ \lambda_i - \nabla_{m_i} u_i(m_i, x_i) \\ \cdots \\ \lambda_i p - \nabla_{x_i} u_i(m_i, x_i) \\ \cdots \\ m_i^0 - m_i + \langle p, x_i^0 - x_i \rangle \\ \cdots \end{pmatrix}. \tag{5.5}$$

The initial endowments are represented by the vectors $m^0 = (m_1^0, \ldots, m_r^0) \in \mathbb{R}_+^r$ and $x^0 = (x_1^0, \ldots, x_r^0) \in U_1 \times U_2 \times \cdots U_r$. In [4, Theorem 3] it is shown that the equilibrium mapping associated with (5.3) is strongly regular provided that for each agent $i$ the initial goods $x_i^0$ are sufficiently close to the equilibrium vector $\bar{x}_i$, in other words, when the trade starts with amounts of goods not too far from the equilibrium. Note that the first inequality in (5.2) does not appear in (5.3) since at equilibrium that automatically becomes an equality.

In this section we extend the model (5.3) to a parametric framework by considering a market with varying endowments $(m_i^0(t), x_i^0(t))$ for a parameter $t \in [0, 1]$ (possibly, but not necessarily, representing time). For each $t \in [0, 1]$ the endowments $(m_i^0(t), x_i^0(t))$ are traded to obtain an equilibrium vector $(\bar{p}(t), \bar{m}(t), \bar{x}(t))$ with an associated Lagrange multiplier $\bar{\lambda}(t)$ satisfying (5.3). Thus we consider the following problem: given functions $(m_i^0(\cdot), x_i^0(\cdot))$ representing the initial endowments over a period $[0, 1]$, track the associated equilibrium trajectory which solves

$$-g(p(t), m(t), x(t), \lambda(t), m^0(t), x^0(t)) \in N_C(p(t), m(t), x(t), \lambda(t)), \tag{5.6}$$

where $C$ and $g$ are given as in (5.4) and (5.5) with variables replaced by their values at $t$.

In the rest of this section we present some numerical experiments with the Euler–Newton continuation scheme developed in the preceding section for two simple examples of dynamic economic equilibrium based on the model (5.6). We shall not try to find here economic interpretations of the results; our goal is to only illustrate the numerical features of the scheme.

In both examples there are two agents with utility functions

$$u_i(m_i, x_i) = \alpha_i \ln(m_i) + \beta_i \ln(x_i), \quad i = 1, 2,$$

and a single good subject to the constraints

$$x_i \in U_i = [\xi_i, \eta_i], \quad i = 1, 2$$

for some positive $\xi_i$ and $\eta_i$. The variational inequality (5.3) for the vector $(p, m_1, m_2, x_1, x_2, \lambda_1, \lambda_2)$ has the following specific form:

$$\begin{pmatrix} \sum_{i=1}^{2}[x_i^0 - x_i] \\ \lambda_1 - \frac{\alpha_1}{m_1} \\ \lambda_2 - \frac{\alpha_2}{m_2} \\ \lambda_1 p - \frac{\beta_1}{x_1} \\ \lambda_2 p - \frac{\beta_2}{x_2} \\ m_1^0 - m_1 + p \cdot [x_1^0 - x_1] \\ m_2^0 - m_2 + p \cdot [x_2^0 - x_2] \end{pmatrix} \in \begin{pmatrix} N_{\mathbb{R}_+}(p) \\ N_{\mathbb{R}_+}(m_1) \\ N_{\mathbb{R}_+}(m_2) \\ N_{U_1}(x_1) \\ N_{U_2}(x_2) \\ N_{\mathbb{R}_+}(\lambda_1) \\ N_{\mathbb{R}_+}(\lambda_2) \end{pmatrix}.$$

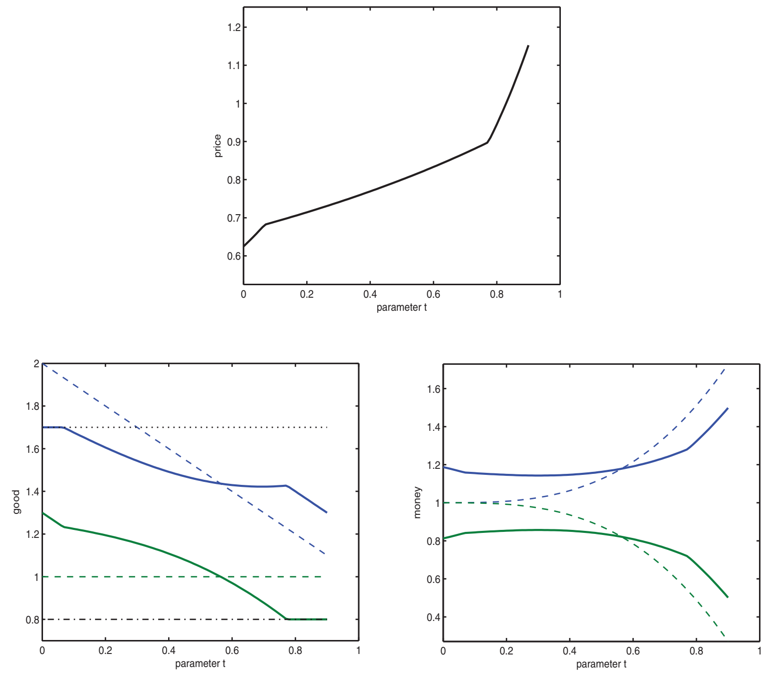


FIG. 5.1. *Graphs of price, goods, and money in Example* 1.

The numerical implementation of the Euler–Newton method (1.3) for this particular case has been done in MATLAB by Anton Belyakov (Vienna University of Technology). Each step of the method reduces to solving linear complementarity problems (LCP). The MATLAB function LCP by Yuval available at http://www.mathworks.com/matlabcentral/fileexchange/20952 has been used for solving these problems. In order to evaluate the error for a given step size $h$ we use a high-accuracy (about $10^{-12}$) approximation of the exact solution obtained by multiple application of the Newton step, as described at the end of section 4. (Five Newton steps turned out to be enough in our tests.) The computations are done for the following data.

*Example* 1. It is assumed that the agents have the same utility functions with $\alpha_i = \beta_i = \alpha = 0.1$ for $i = 1, 2$. The functions for the initial endowments have the form

$$m_1^0(t) = 1 + t^3, \quad m_2^0(t) = 1 - t^3, \quad x_1^0(t) = 2 - t, \quad x_2^0(t) = 1,$$

and the constraints are $U_i = [0.8, 1.7]$, $i = 1, 2$. The solution is presented in Figure 5.1. The first plot gives the trajectory of the equilibrium price, the second shows the evolution of goods, and the third shows the evolution of money. The horizontal (dotted and dash-dotted) lines in the second plot correspond to the lower and the upper bound for the goods of the two agents. The dashed lines in the second and the third plot represent the initial endowments. The solid lines in the second and the third plot represent the evolution of the goods and the money of the agents at equilibrium.

Table 5.1 presents the obtained error for different step sizes $h$. The last column confirms the $O(h^4)$ order of the error and gives a numerical estimation of the constant $C$ in (4.2).

Table 5.1
*The error in Example* 1.

| $h$ | $l^\infty$ error $E(h)$ | $E(h)/h^4$ |
|---|---|---|
| 0.1 | $2.87541 \times 10^{-4}$ | 2.8754136 |
| 0.05 | $2.01561 \times 10^{-5}$ | 3.2249756 |
| 0.025 | $1.31335 \times 10^{-6}$ | 3.3621632 |
| 0.0125 | $8.3478 \times 10^{-8}$ | 3.4192589 |

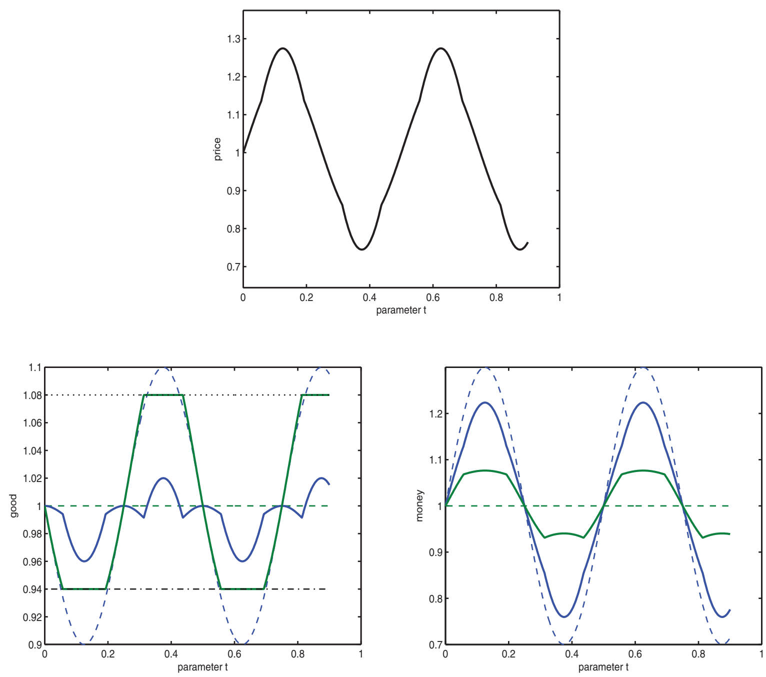


Fig. 5.2. *Graphs of price, goods, and money in Example* 2.

*Example* 2. The utility functions of the two agents are as in Example 1. The functions for the initial endowments in money and goods are

$$m_1^0(t) = 1 + 0.3\sin(4\pi t), \quad m_2^0(t) = 1, \quad x_1^0(t) = 1 - 0.1\sin(4\pi t), \quad x_2^0(t) = 1,$$

and the constraints are $U_i = [0.94, 1.08]$, $i = 1, 2$.

Figure 5.2 presents the evolution of the equilibrium using the kinds of lines as for Example 1. Observe that in the second example the solution trajectory is more involved, e.g., there are more points where the solution is not differentiable (changes of the binding constraints) and the variation of the solution is higher. Related to that might be the larger constant $C$ in the error bound (4.2). Table 5.2 gives the error for different $h$. For $h = 0.1$ no sensible results were obtained. This may be an indication that in this case the number $N_0$ in Theorem 4.1 is larger than $1/(0.1) = 10$.

We also tested on the second example the enhanced version (4.13) of the method, where the Jacobian $\nabla_u f(t_i, v_i)$ from the preceding corrector step is used in place of the current Jacobian $\nabla_u f(t_i, u_i)$. The results are presented in Table 5.3. As proved, the order of the error remains $O(h^4)$, with a slightly different constant.

TABLE 5.2
*The error in Example* 2.

| $h$ | $l^\infty$ error $E(h)$ | $E(h)/h^4$ |
|---|---|---|
| 0.1 | — | — |
| 0.05 | $3.29938 \times 10^{-3}$ | 527.9005 |
| 0.025 | $2.03042 \times 10^{-4}$ | 519.7862 |
| 0.0125 | $1.28470 \times 10^{-5}$ | 526.2256 |

TABLE 5.3
*The error in Example* 2 *for the enhanced version of the method.*

| $h$ | $l^\infty$ error $E(h)$ | $E(h)/h^4$ |
|---|---|---|
| 0.1 | — | — |
| 0.05 | $3.34425 \times 10^{-3}$ | 535.0803 |
| 0.025 | $2.02765 \times 10^{-4}$ | 519.0796 |
| 0.0125 | $1.28461 \times 10^{-5}$ | 526.1765 |

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