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CHARACTERIZATIONS OF FULL STABILITY IN CONSTRAINED OPTIMIZATION*

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Abstract. This paper is mainly devoted to the study of the so-called full Lipschitzian stability of local solutions to finite-dimensional parameterized problems of constrained optimization, which has been well recognized as a very important property from the viewpoints of both optimization theory and its applications. Based on second-order generalized differential tools of variational analysis, we obtain necessary and sufficient conditions for fully stable local minimizers in general classes of constrained optimization problems, including problems of composite optimization, mathematical programs with polyhedral constraints, as well as problems of extended and classical nonlinear programming with twice continuously differentiable data.

Key words. variational analysis, constrained parametric optimization, nonlinear and extended nonlinear programming, full stability of local minimizers, strong regularity, second-order subdifferentials, parametric prox-regularity, amenability

AMS subject classifications. 49J52, 90C30, 90C31

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1. Introduction.

Lipschitzian stability of locally optimal solutions with respect to small parameter perturbations is undoubtedly important in optimization theory, allowing us to recognize robust solutions and support computational work from the viewpoints of justifying numerical algorithms, their convergence properties, stopping criteria, etc. There are several versions of Lipschitzian stability in optimization; see, e.g., the books [1, 3, 5, 13, 23] and the references therein. The focus of this paper is on what is known as *full stability* of locally optimal solutions introduced by Levy, Poliquin, and Rockafellar [6]. This notion emerged as a far-going extension of *tilt stability* of local minimizers in the sense of Poliquin and Rockafellar [18]; see section 3 below for the precise definitions and more discussions. It seems to us that full stability is probably the most fundamental stability notion for locally optimal solutions, from both theoretical and practical points of view, particularly in connection with numerical methodology and applications [6, 18].

In [6], the authors derived necessary and sufficient conditions for fully stable minimizers of parameterized optimization problems written in the unconstrained format with extended-real-valued and prox-regular cost functions. They expressed these conditions in terms of a partial modification of the *second-order subdifferential* (or generalized Hessian) in the sense of Mordukhovich [11], which was previously used in [18] for characterizations of tilt stability. As mentioned in [6], implementing this approach in particular classes of constrained optimization problems important for the

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theory and applications requires the developments of second-order subdifferential calculus for the constructions involved, which was challenging and not available at that time. Such a calculus has been partly developed in the recent paper by Mordukhovich and Rockafellar [16] with applications to tilt stability therein.

The main goal of this paper is to obtain *complete characterizations* of full stability for remarkable classes of constrained optimization problems expressing these characterizations entirely in terms of the problem data. The classes under consideration include general models given in *composite formats* of optimization (particularly with fully amenable compositions), mathematical programs with *polyhedral constraints* (MPPC) on function values, problems of the so-called extended nonlinear programming (ENLP), and consequently for classical problems of *nonlinear programming* (NLP) with $\mathcal{C}^2$ equality and inequality constraints. The key machinery is based on *exact* (equality type) second-order calculus rules for the aforementioned constructions taken partly from [16] and also the new ones derived in this paper.

The rest of the paper is organized as follows. In section 2 we review the basic generalized differential tools of variational analysis used in formulations and proofs of the main results. Section 3 presents definitions of full stability and related notions for optimization problems written in the *unconstrained* extended-real-valued format. We discuss the second-order necessary and sufficient conditions for full stability of local minimizers in this setting [6] and establish relationships between full stability of local minimizers and the new notion of *partial strong metric regularity* (PSMR) of the corresponding subdifferential mappings. Then these conditions are characterized via a certain *uniform second-order growth condition* (USOGC), which is important in what follows.

Section 4 is devoted to deriving *exact chain rules* for partial second-order subdifferentials of extended-real-valued functions belonging to major classes of *fully amenable compositions* with *compatible parameterization*, which are overwhelmingly encountered in finite-dimensional variational analysis and parametric optimization. The pivoting role in these results is played by the *second-order qualification condition* (SOQC), which is a partial specification of the basic one introduced and exploited in [16]. Then these calculus rules and related results from [16] are applied in section 5 to establishing *necessary and sufficient* conditions for full stability of local minimizers in fairly general *composite* models of constrained optimization, particularly those described by parametrically fully amenable compositions.

Section 6 concerns MPPC models with $\mathcal{C}^2$ data and provides, based on the second-order variational analysis developed in sections 4 and 5, complete characterizations of full stability of locally optimal solutions to MPPC under various constraint qualifications. In particular, the *polyhedral constraint qualification* (PCQ) is formulated in this section as an implementation of SOQC in MPPC models governed by fully amenable compositions. It is shown that PCQ is in fact a manifestation of *nondegeneracy* in MPPC and agrees with the classical *linear independence constraint qualification* (LICQ) for NLP being strictly *weaker* than the latter for MPPC. In this section we characterize full stability in MPPC under PCQ via the new *polyhedral* version of the *strong second-order optimality condition* (PSSOC) and also via PSMR and USOGC under the partial version of the *Robinson constraint qualification* (RCQ), which reduces to the partial version of the *Mangasarian–Fromovitz constraint qualification* (MFCQ) in the case of NLP. Another equivalence proved here is between full stability and Robinson's *strong regularity* of the *KKT system* associated with MPPC under PCQ.

The final section 7 presents a characterization of full stability of locally optimal solutions to problems of ENLP, which deal with special classes of outer extended-real-valued functions in composite models of optimization related to Lagrangian duality. This characterization is obtained via an appropriate *extension* of the *strong second-order optimality condition* (ESSOC) and is based on the *complete calculation* of the second-order subdifferential for the so-called dualizing representation in ENLP.

Throughout the paper we use standard notation of variational analysis; cf. [12, 23]. Recall that given a set-valued mapping $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, the symbol

$$\mathop{\mathrm{Lim\,sup}}_{x\to\bar{x}} F(x) := \Big\{ y \in \mathbb{R}^m \Big|\ \exists\, x_k \to \bar{x},\ \exists\, y_k \to y \ \text{ as } \ k \to \infty \ \text{ with } \ y_k \in F(x_k) \ \text{ for all } \ k \in \mathbb{N} := \{1, 2, \ldots\} \Big\} \tag{1.1}$$

signifies the *Painlevé–Kuratowski outer limit* of F as $x \to \bar{x}$. Given a set $\Omega \subset \mathbb{R}^n$ and an extended-real-valued function $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}} := (-\infty, \infty]$ finite at $\bar{x}$, the symbols $x \xrightarrow{\Omega} \bar{x}$ and $x \xrightarrow{\varphi} \bar{x}$ stand for $x \to \bar{x}$ with $x \in \Omega$ and for $x \to \bar{x}$ with $\varphi(x) \to \varphi(\bar{x})$, respectively. As usual, $\mathbb{B}(x, r) = \mathbb{B}_r(x)$ denotes the closed ball of the space in question centered at x with radius $r > 0$.

2. Tools of variational analysis. In this section we briefly overview some basic constructions of generalized differentiation in variational analysis, which are widely used in what follows. The major focus of this paper is on second-order subdifferential (or generalized Hessian) constructions for extended-real-valued functions while, following mainly [12, 23], we start with recalling the corresponding first-order subdifferentials as well as associated objects of variational geometry.

Given $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ finite at $\bar{x}$, its *regular subdifferential* (known also as the presubdifferential and as the Fréchet or viscosity subdifferential) at $\bar{x}$ is

$$\widehat{\partial}\varphi(\bar{x}) := \left\{ v \in \mathbb{R}^n \,\Big|\, \liminf_{x\to\bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle v, x - \bar{x}\rangle}{\|x - \bar{x}\|} \ge 0 \right\}. \tag{2.1}$$

While $\widehat{\partial}\varphi(\bar{x})$ reduces to a singleton $\{\nabla\varphi(\bar{x})\}$ if φ is Fréchet differentiable at $\bar{x}$ and to the classical subdifferential of convex analysis if φ is convex, the set (2.1) may often be empty for nonconvex and nonsmooth functions as, e.g., for $\varphi(x) = -|x|$ at $\bar{x} = 0 \in \mathbb{R}$. Another serious disadvantage of (2.1) is the failure of standard calculus rules inevitably required in the theory and applications of variational analysis, including those to optimization and equilibria.

The picture dramatically changes when we perform a limiting procedure over the mapping $x \mapsto \widehat{\partial}\varphi(x)$ as $x \xrightarrow{\varphi} \bar{x}$ that leads us to the (basic first-order) *subdifferential* of φ at $\bar{x}$ defined by

$$\partial\varphi(\bar{x}) := \mathop{\mathrm{Lim\,sup}}_{x \xrightarrow{\varphi} \bar{x}} \widehat{\partial}\varphi(x) \tag{2.2}$$

and known also as the general, or limiting, or Mordukhovich subdifferential; it was first introduced in [9] in an equivalent way. In contrast to (2.1), the subgradient set (2.2) is often nonconvex (e.g., $\partial\varphi(0) = \{-1, 1\}$ for $\varphi(x) = -|x|$) while enjoying a *full calculus* based on *variational/extremal principles*, which replace separation arguments in the absence of convexity.

We need also another first-order subdifferential construction for $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ finite at $\bar{x}$, which is a complement to (2.2) in the case of non-Lipschitzian functions. The *singular/horizon subdifferential* of φ at $\bar{x}$ is defined by

$$\partial^\infty \varphi(\bar{x}) := \operatorname*{Lim\,sup}_{\substack{x \xrightarrow{\varphi} \bar{x} \\ \lambda \downarrow 0}} \lambda \widehat{\partial} \varphi(x). \tag{2.3}$$

We know that $\partial^\infty \varphi(\bar{x}) = \{0\}$ if and only if φ is locally Lipschitzian around $\bar{x}$, provided that it is lower semicontinuous (l.s.c.) around this point.

Recall further some constructions of variational geometry needed in what follows and associated with the subdifferential ones defined above. Given a set $\emptyset \neq \Omega \subset \mathbb{R}^n$, consider its indicator function $\delta(x;\Omega)$ equal to 0 for $x \in \Omega$ and to ∞ otherwise. For any fixed $\bar{x} \in \Omega$, the *regular normal cone* to Ω at $\bar{x}$ is

$$\widehat{N}(\bar{x};\Omega) := \widehat{\partial}\delta(\bar{x};\Omega) = \left\{ v \in \mathbb{R}^n \,\middle|\, \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle v, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \right\} \tag{2.4}$$

and the (basic, limiting) *normal cone* to Ω at $\bar{x}$ is $N(\bar{x};\Omega) := \partial\delta(\bar{x};\Omega)$. It follows from (2.2) and (2.4) that the normal cone $N(\bar{x};\Omega)$ admits the limiting representation

$$N(\bar{x};\Omega) = \operatorname*{Lim\,sup}_{x \xrightarrow{\Omega} \bar{x}} \widehat{N}(x;\Omega) \tag{2.5}$$

via the Painlevé–Kuratowski outer limit (1.1). If Ω is locally closed around $\bar{x}$, representation (2.5) is equivalent to the original definition by Mordukhovich [9]:

$$N(\bar{x};\Omega) = \operatorname*{Lim\,sup}_{x \to \bar{x}} \Big[\operatorname{cone}\big(x - \Pi(x;\Omega)\big) \Big],$$

where $\Pi(x;\Omega)$ stands for the Euclidean projector of $x \in \mathbb{R}^n$ on Ω and where "cone" signifies the (nonconvex) conic hull of a set. Observe also the *duality/polarity* correspondence

$$\widehat{N}(\bar{x};\Omega) = T(\bar{x};\Omega)^* := \Big\{ v \in \mathbb{R}^n \,\Big|\, \langle v, w \rangle \leq 0 \;\text{ for all }\; w \in T(\bar{x};\Omega) \Big\} \tag{2.6}$$

between the regular normal cone (2.4) and the *tangent cone* to Ω at $\bar{x} \in \Omega$ defined by

$$T(\bar{x};\Omega) := \Big\{ w \in \mathbb{R}^n \,\Big|\, \exists\, x_k \xrightarrow{\Omega} \bar{x},\ \alpha_k \geq 0 \;\text{ with }\; \alpha_k(x_k - \bar{x}) \to w \;\text{ as }\; k \to \infty \Big\} \tag{2.7}$$

and known also as the Bouligand–Severi contingent cone to Ω at this point. Note that the basic normal cone (2.5) *cannot* be tangentially generated in a polar form (2.6), since it is intrinsically nonconvex, while the polar T^* to any set T is always convex. In what follows we may also use the subindex set notation like $N_\Omega(\bar{x})$, $T_\Omega(\bar{x})$, etc., for the constructions involved.

Given further a mapping $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, define its *coderivative* [10] at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ by

$$D^*F(\bar{x},\bar{y})(v) := \Big\{ u \in \mathbb{R}^n \,\Big|\, (u,-v) \in N\big((\bar{x},\bar{y});\operatorname{gph} F\big) \Big\}, \quad v \in \mathbb{R}^m, \tag{2.8}$$

via the normal cone (2.5) to the graph $\operatorname{gph} F$. The set-valued mapping $D^*F(\bar{x},\bar{y})\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is clearly positive-homogeneous; Moreover, if the mapping $F\colon \mathbb{R}^n \to \mathbb{R}^m$

is single-valued (then we omit $\bar{y} = F(\bar{x})$ in the coderivative notation) and *strictly differentiable* at $\bar{x}$ (which is automatic when it is $\mathcal{C}^1$ around this point), then the coderivative (2.8) is also single-valued and reduces to the *adjoint* derivative operator

$$D^*F(\bar{x})(v) = \big\{\nabla F(\bar{x})^* v\big\}, \quad v \in \mathbb{R}^m, \tag{2.9}$$

with the operator symbol * on the right-hand side of (2.9) standing for the matrix transposition in finite dimensions. It is worth noting that the coderivative values in (2.8) are often nonconvex sets due to the intrinsic nonconvexity of the normal cone on the right-hand side therein. Observe furthermore that this nonconvex normal cone is taken to a *graphical* set. Thus its convexification in (2.8), which reduces to the convexified/Clarke normal cone to the set in question, creates serious trouble; see Rockafellar [21] and Mordukhovich [12, subsection 3.2.4] for more details.

Coming back to extended-real-valued functions, let us present their second-order subdifferential constructions, which are at the heart of the variational techniques developed in this paper. Given $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ finite at $\bar{x}$, pick a subgradient $\bar{y} \in \partial\varphi(\bar{x})$ and, following Mordukhovich [11], introduce the second-order subdifferential (or generalized Hessian) of φ at $\bar{x}$ relative to $\bar{y}$ by

$$\partial^2\varphi(\bar{x},\bar{y})(u) := (D^*\partial\varphi)(\bar{x},\bar{y})(u), \quad u \in \mathbb{R}^n, \tag{2.10}$$

via the coderivative (2.8) of the first-order subdifferential mapping (2.2). Observe that for $\varphi \in \mathcal{C}^2$ with the (symmetric) Hessian matrix $\nabla^2\varphi(\bar{x})$ we have

$$\partial^2\varphi(\bar{x})(u) = \big\{\nabla^2\varphi(\bar{x})u\big\} \ \text{ for all } \ u \in \mathbb{R}^n.$$

Referring the reader to the book [12] and the recent paper [16] (as well as the bibliographies therein) for the theory and applications of the second-order subdifferential (2.10), from now on we focus on an appropriate partial counterpart of (2.10) for functions $\varphi\colon \mathbb{R}^n \times \mathbb{R}^d \to \overline{\mathbb{R}}$ of two variables $(x,w) \in \mathbb{R}^n \times \mathbb{R}^d$. Consider the partial first-order subgradient mapping

$$\partial_x\varphi(x,w) := \Big\{\text{set of subgradients } v \text{ of } \varphi_w := \varphi(\cdot,w) \text{ at } x\Big\} = \partial\varphi_w(x), \tag{2.11}$$

take $(\bar{x},\bar{w})$ with $\varphi(\bar{x},\bar{w}) < \infty$, and define the *extended partial* second-order subdifferential of φ with respect to x at $(\bar{x},\bar{w})$ relative to some $\bar{y} \in \partial_x\varphi(\bar{x},\bar{w})$ by

$$\widetilde{\partial}^2_x\varphi(\bar{x},\bar{w},\bar{y})(u) := (D^*\partial_x\varphi)(\bar{x},\bar{w},\bar{y})(u), \quad u \in \mathbb{R}^n. \tag{2.12}$$

This second-order construction was first employed by Levy, Poliquin, and Rockafellar [6] for characterizing full stability of extended-real-valued functions in the unconstrained format of optimization; see section 3. Some amount of calculus for (2.12) has been recently developed in the aforementioned paper by Mordukhovich and Rockafellar [16], while more calculus results are given in section 4 below. Note that the second-order construction (2.12) is different from the standard partial second-order subdifferential

$$\partial^2_x\varphi(\bar{x},\bar{w},\bar{y})(u) := (D^*\partial\varphi_{\bar{w}})(\bar{x},\bar{y})(u) = \partial^2\varphi_{\bar{w}}(\bar{x},\bar{y})(u), \quad u \in \mathbb{R}^n,$$

of $\varphi = \varphi(x,w)$ with respect to x at $(\bar{x},\bar{w})$ relative to $\bar{y} \in \partial_x\varphi(\bar{x},\bar{w})$, even in the classical $\mathcal{C}^2$ setting. Indeed, for such functions φ with $\bar{y} = \nabla_x\varphi(\bar{x},\bar{w})$ we have

$$\begin{aligned} &\partial^2_x\varphi(\bar{x},\bar{w})(u) = \big\{\nabla^2_{xx}\varphi(\bar{x},\bar{w})u\big\}, \ \text{ while} \\ &\widetilde{\partial}^2_x\varphi(\bar{x},\bar{w})(u) = \{(\nabla^2_{xx}\varphi(\bar{x},\bar{w})u, \nabla^2_{xw}\varphi(\bar{x},\bar{w})u)\} \ \text{ for all } \ u \in \mathbb{R}^n. \end{aligned} \tag{2.13}$$

3. Full stability and strong regularity in unconstrained format. Let $\varphi\colon \mathbb{R}^n \times \mathbb{R}^d \to \overline{\mathbb{R}} = (-\infty, \infty]$ be a proper extended-real-valued function of two variables $(x, w) \in \mathbb{R}^n \times \mathbb{R}^d$. Throughout the paper we assume, unless otherwise stated, that φ is l.s.c. around the reference points of its effective domain

$$\operatorname{dom}\varphi := \{(x, w) \in \mathbb{R}^n \times \mathbb{R}^d |\ \varphi(x, w) < \infty\}.$$

Following Levy, Poliquin, and Rockafellar [6], consider the two-parametric unconstrained problem of minimizing the perturbed function φ defined by

$$\text{minimize}\ \ \varphi(x, w) - \langle v, x\rangle\ \ \text{over}\ \ x \in \mathbb{R}^n \tag{3.1}$$

and label it as $\mathcal{P}(w, v)$. In this parameterized optimization problem, the vector $w \in \mathbb{R}^d$ signifies general parameter perturbations (called *basic* perturbations in [6]), while the linear parametric shift of the objective with $v \in \mathbb{R}^n$ in (3.1) represents the so-called tilt perturbations.

Our primary goal is to investigate the following fairly general type of *quantitative/Lipschitzian stability* of local minimizers for the parameterized family $\mathcal{P}(w, v)$ of the optimization problems (3.1) with respect to parameter perturbations (w, v) varying around the given nominal parameter value $(\bar{w}, \bar{v})$ corresponding to the unperturbed problem $\mathcal{P}(\bar{w}, \bar{v})$. Feasible solutions to $\mathcal{P}(w, v)$ are the points $x \in \mathbb{R}^n$ such that the function value $\varphi(x, w)$ is finite.

Let $\bar{x}$ be a feasible solution to the unperturbed problem $\mathcal{P}(\bar{w}, \bar{v})$. For any number $\nu > 0$ we consider the (local) optimal value function

$$m_\nu(w, v) := \inf_{\|x-\bar{x}\|\le\nu} \Big\{\varphi(x, w) - \langle v, x\rangle\Big\}, \quad (w, v) \in \mathbb{R}^d \times \mathbb{R}^n, \tag{3.2}$$

for the perturbed optimization problem (3.1) and then the corresponding parametric family of optimal solution sets to (3.1) given by

$$M_\nu(w, v) := \operatorname{argmin}_{\|x-\bar{x}\|\le\nu}\Big\{\varphi(x, w) - \langle v, x\rangle\Big\}, \quad (w, v) \in \mathbb{R}^d \times \mathbb{R}^n. \tag{3.3}$$

A point $\bar{x}$ is said to be a locally optimal solution to $\mathcal{P}(\bar{w}, \bar{v})$ if $\bar{x} \in M_\nu(\bar{w}, \bar{v})$ for some small $\nu > 0$.

The main focus of this paper is the following notion of Lipschitzian stability for locally optimal solutions to the unperturbed problem $\mathcal{P}(\bar{w}, \bar{v})$ introduced in [6].

DEFINITION 3.1 (full stability). *A point $\bar{x}$ is a* fully stable *locally optimal solution to problem $\mathcal{P}(\bar{w}, \bar{v})$ if there exist a number $\nu > 0$ and neighborhoods W of $\bar{w}$ and V of $\bar{v}$ such that the mapping $(w, v) \mapsto M_\nu(w, v)$ is single-valued and Lipschitz continuous with $M_\nu(\bar{w}, \bar{v}) = \{\bar{x}\}$ and the function $(w, v) \mapsto m_\nu(w, v)$ is likewise Lipschitz continuous on $W \times V$.*

Tilt stability of local minimizers $\bar{x}$ introduced earlier by Poliquin and Rockafellar [18] corresponds to Definition 3.1 under the fixed basic parameter $w = \bar{w}$, i.e., it imposes single-valued Lipschitzian behavior of $v \to M_\nu(\bar{w}, v)$ with respect to tilt perturbations v in (3.1). Observe that in this case the Lipschitz continuity of the optimal value functions $m_\nu(\bar{w}, v)$ is automatic in the finite-dimensional setting under consideration, since it follows from (3.2) that $m_\nu(\bar{w}, v)$ is finite and concave in v. Note also that the idea of considering stability from the viewpoint of single-valued Lipschitzian behavior goes back to Robinson [20], being mainly motivated by applications to numerical algorithms in optimization.

To formulate the main result of [6] on characterizing full stability of local minimizers in problem $\mathcal{P}(\bar{w},\bar{v})$ with an extended-real-valued φ in finite dimensions, we need to recall the following important notions of variational analysis; cf. [6, 17, 23] for more details. An l.s.c. function $\varphi(x,w)$ is prox-regular in x at $\bar{x}$ for $\bar{v}$ with compatible parameterization by w at $\bar{w}$ if $\bar{v}\in\partial_x\varphi(\bar{x},\bar{w})$ and there exist neighborhoods U of $\bar{x}$, W of $\bar{w}$, and V of $\bar{v}$ together with numbers $\varepsilon>0$ and $\gamma\ge 0$ such that

$$\begin{aligned}&\varphi(u,w)\ge\varphi(x,w)+\langle v,u-x\rangle-\frac{\gamma}{2}\|u-x\|^2\ \text{ for all }\ u\in U\\&\quad\text{when }\ v\in\partial_x\varphi(x,w)\cap V,\ x\in U,\ w\in W,\ \varphi(x,w)\le\varphi(\bar{x},\bar{w})+\varepsilon.\end{aligned}\tag{3.4}$$

Furthermore, $\varphi(x,w)$ is called *subdifferentially continuous* at $(\bar{x},\bar{w},\bar{v})$ if it is continuous as a function of (x,w,v) on the partial subdifferential graph $\operatorname{gph}\partial_x\varphi$ at this point. If both these properties hold simultaneously, we say that φ is *continuously prox-regular* in x at $\bar{x}$ for $\bar{v}$ with compatible parameterization by w at $\bar{w}$ or simply that this function is parametrically continuously prox-regular at $(\bar{x},\bar{w},\bar{v})$.

It is known from [6] that the class of parametrically continuously prox-regular functions $\varphi\colon\mathbb{R}^n\times\mathbb{R}^d\to\overline{\mathbb{R}}$ at $(\bar{x},\bar{w},\bar{v})$ with $\bar{v}\in\partial_x\varphi(\bar{x},\bar{w})$ is fairly large, including, in particular, all extended-real-valued functions $\varphi(x,w)$ that are *strongly amenable* in x at $\bar{x}$ with compatible parametrization by w at $\bar{w}$ in the following sense: There are $h\colon\mathbb{R}^n\times\mathbb{R}^d\to\mathbb{R}^m$ and $\theta\colon\mathbb{R}^m\to\overline{\mathbb{R}}$ such that $\varphi(x,w)=\theta(h(x,w))$ and h is $\mathcal{C}^2$ around $(\bar{x},\bar{w})$, while θ is convex, proper, l.s.c., and finite at $h(\bar{x},\bar{w})$ under the first-order qualification condition

$$\partial^\infty\theta\big(h(\bar{x},\bar{w})\big)\cap\ker\nabla_x h(\bar{x},\bar{w})^*=\{0\}.\tag{3.5}$$

The parametric continuous prox-regularity of such functions is proved in [6, Proposition 2.2], where it is shown in addition that the parametric strong amenability of φ formulated above ensures the validity of the *basic* constraint qualification:

$$(0,q)\in\partial^\infty\varphi(\bar{x},\bar{w})\Longrightarrow q=0.\tag{3.6}$$

The strong amenability property and its parametric expansion hold not only in the obvious cases of $\mathcal{C}^2$ and convex functions but in dramatically larger frameworks typically encountered in finite-dimensional variational analysis and optimization; see [7, 6, 18, 23].

The main result of [6, Theorem 2.3] is as follows.

THEOREM 3.2 (characterization of full stability in unconstrained extended-real-valued format). *Let $\bar{x}$ be a feasible solution to the unperturbed problem $\mathcal{P}(\bar{w},\bar{v})$ in* (3.1) *at which the first-order necessary optimality condition $\bar{v}\in\partial_x\varphi(\bar{x},\bar{w})$ and the basic constraint qualification* (3.6) *are satisfied. Assume in addition that φ is parametrically continuously prox-regular at $(\bar{x},\bar{w},\bar{v})$. Then $\bar{x}$ is a fully stable locally optimal solution to $\mathcal{P}(\bar{x},\bar{w})$ if and only if the second-order conditions*

$$(0,q)\in\widetilde{\partial}^2_x\varphi(\bar{x},\bar{w},\bar{v})(0)\Longrightarrow q=0,\tag{3.7}$$

$$[(p,q)\in\widetilde{\partial}^2_x\varphi(\bar{x},\bar{w},\bar{v})(u),\ u\ne 0]\Longrightarrow\langle p,u\rangle>0\tag{3.8}$$

hold via the extended second-order subdifferential mapping (2.12).

In the subsequent sections of the paper we employ Theorem 3.2 to obtain verifiable necessary and sufficient conditions for full stability of local minimizers in favorable classes of constrained optimization problems in terms of the problem data. Achieving it requires the implementation and development of second-order subdifferential calculus as well as *precise calculating* the partial second-order subdifferential constructions for the corresponding functions involved.

We proceed in this section with establishing useful relationships between full stability of local minimizers in the unconstrained format of (3.1) with an extended-real-valued function $\varphi(x, w)$ and an appropriate version of the so-called strong metric regularity of the partial subdifferential mapping $\partial_x \varphi$. Recall [3] that a set-valued mapping $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *strongly metrically regular* at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if the inverse mapping F^{-1} admits a Lipschitzian single-valued localization around $(\bar{x}, \bar{y})$, i.e., there are neighborhood U of $\bar{x}$ and V of $\bar{y}$ and a single-valued Lipschitz continuous mapping $f\colon V \to U$ such that $f(\bar{y}) = \bar{x}$ and $F^{-1}(y) \cap U = \{f(y)\}$ for all $y \in V$. This notion is an abstract version of Robinson's strong regularity for variational inequalities and NLP problems [20]; see more discussions in section 6.

Close relationships (equivalences under appropriate constraint qualifications) between tilt stability and strong regularity have been recently established by Mordukhovich and Rockafellar [16] and Mordukhovich and Outrata [14] in the framework of NLP and by Lewis and Zhang [8] and Drusvyatskiy and Lewis [4] via strong metric regularity of subdifferential mappings for extended-real-valued objective functions in the general unconstrained format of nonparametric optimization. Based on [6], we now extend the latter results to the parametric framework of (3.1) while establishing the equivalence between full stability of locally optimal solutions to (3.1) and an appropriate notion of PSMR for the corresponding partial subdifferential mapping of the function $\varphi(x, w)$ therein. We also establish characterizations of these notions via a certain partial second-order growth condition.

Given a function $\varphi\colon \mathbb{R}^n \times \mathbb{R}^d \to \overline{\mathbb{R}}$, consider its partial first-order subdifferential mapping $\partial_x \varphi\colon \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ and define the *partial inverse* of $\partial_x \varphi$ by

$$S_\varphi(w, v) := \big\{x \in \mathbb{R}^n \big|\ v \in \partial_x \varphi(x, w)\big\}, \tag{3.9}$$

where the subdifferential is understood in the basic sense (2.2).

DEFINITION 3.3 (PSMR). *Given $(\bar{x}, \bar{w}) \in \operatorname{dom} \varphi$ and $\bar{v} \in \partial_x \varphi(\bar{x}, \bar{w})$, we say that the partial subdifferential mapping $\partial_x \varphi\colon \mathbb{R}^n \times \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ is PSMR at $(\bar{x}, \bar{w}, \bar{v})$ if its partial inverse* (3.9) *admits a Lipschitzian single-valued localization around this point.*

Note that the notion introduced in Definition 3.3 is different from the (total) strong metric regularity of $\partial_x \varphi$ at $(\bar{x}, \bar{w}, \bar{v})$ discussed above, since it concerns Lipschitzian localizations of the partial inverse S_φ instead of the inverse mapping $(\partial_x \varphi)^{-1}$.

THEOREM 3.4 (full stability versus PSMR). *Given a function $\varphi\colon \mathbb{R}^n \times \mathbb{R}^d \to \overline{\mathbb{R}}$ with $(\bar{x}, \bar{w}) \in \operatorname{dom} \varphi$, consider the unperturbed problem $\mathcal{P}(\bar{w}, \bar{v})$ in* (3.1) *with some $\bar{v} \in \partial_x \varphi(\bar{x}, \bar{w})$ and let $\bar{x}$ be a locally optimal solution to $\mathcal{P}(\bar{w}, \bar{v})$, i.e., $\bar{x} \in M_\nu(\bar{w}, \bar{v})$ for some number $\nu > 0$ in* (3.3). *Assume that the basic constraint qualification* (3.6) *is satisfied at $(\bar{x}, \bar{w})$. The following assertions hold:*

(i) *If $\partial_x \varphi$ is PSMR at $(\bar{x}, \bar{w}, \bar{v})$, then $\bar{x}$ is a fully stable local minimizer for $\mathcal{P}(\bar{w}, \bar{v})$ and the function φ is prox-regular in x at $\bar{x}$ with compatible parameterization by w at $\bar{w}$.*

(ii) *Conversely, if φ is parametrically continuously prox-regular at $(\bar{x}, \bar{w}, \bar{v})$ and if $\bar{x}$ is a fully stable local minimizer for $\mathcal{P}(\bar{w}, \bar{v})$, then $\partial_x \varphi$ is PSMR at $(\bar{x}, \bar{w}, \bar{v})$.*

Proof. To justify assertion (i), assume that the partial subdifferential mapping $\partial_x \varphi$ is PSMR at $(\bar{x}, \bar{w}, \bar{v})$ and fix the number $\nu > 0$ from the formulation of the theorem. Then it follows from Definition 3.3 that there exist neighborhoods U of $\bar{x}$, V of $\bar{v}$, and W of $\bar{w}$ such that for all $(w, v) \in W \times V$ the localization $S_\varphi(w, v) \cap U$ is single-valued. Without loss of generality suppose $\mathbb{B}_\nu(\bar{x}) \subset U$. We claim that

$$M_\nu(\bar{w}, \bar{v}) = \{\bar{x}\}.$$

Indeed, by the stationary condition in (3.1) and the assumed PSMR property we have

$$\varphi_0(\bar{x}, \bar{w}) - \langle \bar{v}, \bar{x} \rangle < \varphi_0(x, \bar{w}) - \langle \bar{v}, x \rangle \text{ for all } x \in \mathrm{int}\mathbb{B}_\nu(\bar{x}). \tag{3.10}$$

If there is $\widetilde{x} \in M_\nu(\bar{w}, \bar{v})$ with $\widetilde{x} \neq \bar{x}$ and $\|\widetilde{x} - \bar{x}\| = \nu$, then replacing $M_\nu(\bar{w}, \bar{v})$ by $M_{\nu/2}(\bar{w}, \bar{v})$ gives us $M_{\nu/2}(\bar{w}, \bar{v}) = \{\bar{x}\}$. Thus we can suppose that $M_\nu(\bar{w}, \bar{v}) = \{\bar{x}\}$. Invoking now the basic constraint qualification (3.6) and employing [6, Proposition 3.5] ensure the Lipschitz continuity around $(\bar{w}, \bar{v})$ of the optimal value function m_ν from (3.2) and allow us to find $\eta > 0$ with

$$M_\nu(w, v) \subset \mathrm{int}\mathbb{B}_\nu(\bar{x}) \text{ whenever } (w, v) \in \mathrm{int}\mathbb{B}_\eta(\bar{w}) \times \mathrm{int}\mathbb{B}_\eta(\bar{v}).$$

Thus we have under the assumptions made that

$$M_\nu(w, v) \subset S_\varphi(w, v) \cap \mathrm{int}\mathbb{B}_\nu(\bar{x}) \text{ for all } (w, v) \in \mathrm{int}\mathbb{B}_\eta(\bar{w}) \times \mathrm{int}\mathbb{B}_\eta(\bar{v}), \tag{3.11}$$

which in fact holds as equality by the single-valuedness of the right-hand side and the nonemptiness of the left-hand one, implying hence that M_ν is single-valued and Lipschitz continuous around $(\bar{w}, \bar{v})$. This means that $\bar{x}$ is a fully stable local minimizer of $\mathcal{P}(\bar{w}, \bar{v})$ by Definition 3.1.

To complete the proof of assertion (i), it remains to justify the claimed parametric prox-regularity of φ at $(\bar{x}, \bar{w})$. Take any $x \in \mathrm{int}\mathbb{B}_\nu(\bar{x})$, $w \in \mathrm{int}\mathbb{B}_\eta(\bar{w})$, and $v \in \partial_x \varphi(x, w) \cap \mathrm{int}\mathbb{B}_\eta(\bar{v})$ with the positive numbers ν, η found above. Then $x \in M_\nu(w, v)$ by the equality in (3.11), and thus we get from the construction of M_ν in (3.3) that

$$\varphi(u, w) \geq \varphi(x, w) + \langle v, u - x \rangle \text{ whenever } u \in \mathrm{int}\mathbb{B}_\nu(\bar{x}),$$

which obviously implies by (3.4) the desired parametric prox-regularity of φ.

To justify assertion (ii), observe that it follows from the second part of [6, Theorem 2.3] that (3.11) holds as equality with some numbers $\nu, \eta > 0$ provided that φ is parametrically continuously prox-regular at $(\bar{x}, \bar{w}, \bar{v})$. Since $\bar{x}$ is now assumed to be a fully stable local minimizer in $\mathcal{P}(\bar{w}, \bar{v})$, this ensures the single-valued Lipschitzian localization of S_φ around $(\bar{w}, \bar{v}, \bar{x})$ and thus justifies the PSMR property of the partial subdifferential mapping $\partial_x \varphi$ at $(\bar{x}, \bar{w}, \bar{v})$. ☐

Next we derive necessary and sufficient conditions for PSMR from Definition 3.3 and full stability properties in the case of general extended-real-valued functions via a partial version of the so-called uniform second-order (quadratic) growth condition.

DEFINITION 3.5 (USOGC). *Given $\varphi\colon \mathbb{R}^n \times \mathbb{R}^d \to \overline{\mathbb{R}}$ finite at $(\bar{x}, \bar{w})$ and given a partial subgradient $\bar{v} \in \partial_x \varphi(\bar{x}, \bar{w})$, we say that the USOGC holds for φ at $(\bar{x}, \bar{w}, \bar{v})$ if there exist a constant $\eta > 0$ and neighborhoods U of $\bar{x}$, W of $\bar{w}$, and V of $\bar{v}$ such that for any $(w, v) \in W \times V$ there is a point $x_{wv} \in U$ (necessarily unique) satisfying $v \in \partial_x \varphi(x_{wv}, w)$ and*

$$\varphi(u, w) \geq \varphi(x_{wv}, w) + \langle v, u - x_{wv} \rangle + \eta \|u - x_{wv}\|^2 \text{ whenever } u \in U. \tag{3.12}$$

Note that for problems of conic programming with $\mathcal{C}^2$ data this notion appeared in a different, while equivalent, form in [1, Definition 5.16] as the "uniform second-order (quadratic) growth condition with respect to the $\mathcal{C}^2$-smooth parameterization." Its version "with respect to the tilt parameterization" was employed in [1, Theorem 5.36] for characterizing tilt-stable minimizers of conic programs and then in [8, Theorem 6.3] and [4, Theorem 3.3] in more general settings of extended-real-valued functions.

Let us employ USOGC from Definition 3.5 to characterize the fully stable local minimizer of $\mathcal{P}(\bar w,\bar v)$. To achieve this goal, we use the following lemma obtained in [6, Lemma 5.2].

LEMMA 3.6 (uniform second-order growth for convex functions). *Let $f\colon \mathbb{R}^n\to\overline{\mathbb{R}}$ be a proper, l.s.c., and convex function whose conjugate f^* is differentiable on* $\mathrm{int}\mathbb{B}_\nu(\bar v)$ *for some $\bar v\in\mathbb{R}^n$ and $\nu>0$, and let the gradient of f^* be Lipschitz continuous on* $\mathrm{int}\mathbb{B}_\nu(\bar v)$ *with constant $\sigma>0$. Then for any $(x,v)\in(\mathrm{gph}\,\partial f)\cap[\mathrm{int}\mathbb{B}_{\frac{\sigma\nu}{4}}(\bar x)\times\mathrm{int}\mathbb{B}_{\frac{\nu}{2}}(\bar v)]$ with $\bar x:=\nabla f^*(\bar v)$ we have*

$$f(u)\ge f(x)+\langle v,u-x\rangle+\frac{1}{2\sigma}\|u-x\|^2\ \ \textit{whenever}\ \ u\in\mathbb{B}_{\frac{\sigma\nu}{4}}(\bar x). \tag{3.13}$$

Proof. Consider the open set $O:=\{v\in\mathbb{R}^n|\ \mathbb{B}_{\frac{\nu}{2}}(v)\subset\mathrm{int}\mathbb{B}_\nu(\bar v)\}$. Then by [6, Lemma 5.2] for all $v\in\partial f(x)\cap O$ we get the estimate

$$f(u)\ge f(x)+\langle v,u-x\rangle+\frac{1}{2\sigma}\|u-x\|^2\ \ \text{whenever}\ \ \|u-x\|\le\frac{\nu\sigma}{2},$$

which implies (3.13) for the corresponding pairs (x,v). ∎

Recall that a mapping $T:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ is *locally maximal monotone* around $(\bar x,\bar v)$ if there exist neighborhoods $U\times V$ of $(\bar x,\bar v)$ such that every monotone mapping $S:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ with $(\mathrm{gph}\,T)\cap(U\times V)\subset\mathrm{gph}\,S$ satisfies $(\mathrm{gph}\,T)\cap(U\times V)=(\mathrm{gph}\,S)\cap(U\times V)$.

THEOREM 3.7 (relationships between full stability and uniform second-order growth). *Let $\varphi\colon\mathbb{R}^n\times\mathbb{R}^d\to\overline{\mathbb{R}}$ be l.s.c. with $\bar v\in\partial_x\varphi(\bar x,\bar w)$ for some $(\bar x,\bar w)\in\mathrm{dom}\,\varphi$. The following assertions hold:*

- (i) *If $\bar x$ is a fully stable local minimizer of the unperturbed problem $\mathcal{P}(\bar w,\bar v)$ in* (3.1) *and the basic constraint qualification* (3.6) *is satisfied at $(\bar x,\bar w)$, then the USOGC of Definition* 3.5 *holds at $(\bar x,\bar w,\bar v)$.*
- (ii) *Conversely, assume that φ is parametrically continuously prox-regular at $(\bar x,\bar w,\bar v)$ and that the USOGC holds at this point with the mapping $(w,v)\mapsto x_{wv}$ in Definition* 3.5 *being locally Lipschitzian around $(\bar w,\bar v)$. Then $\partial_x\varphi$ is PSMR at $(\bar x,\bar w,\bar v)$.*

Proof. To justify (i), let $\bar x$ be a fully stable locally optimal solution to problem $\mathcal{P}(\bar w,\bar v)$. Then there is a number $\nu>0$ such that the mapping $(w,v)\mapsto M_\nu(w,v)$ from (3.3) is single-valued and Lipschitz continuous on $\mathrm{int}\mathbb{B}_\nu(\bar w)\times\mathrm{int}\mathbb{B}_\nu(\bar v)$ with some constant $\sigma>0$. For any fixed $w\in\mathrm{int}\mathbb{B}_\nu(\bar w)$ consider the function $\varphi_w(\cdot)=\varphi(\cdot,w)$ and define

$$\bar\varphi_w:=\varphi_w+\delta_{\mathbb{B}_\nu(\bar x)},\quad g_w:=\bar\varphi_w^*,\ \ \text{and}\ \ h_w:=g_w^*.$$

We easily get from (3.3) and the definition of g_w that

$$M_\nu(w,v)=\mathrm{argmin}_{x\in\mathbb{B}_\nu(\bar x)}\big\{\varphi(x,w)-\langle v,x\rangle\big\}\in\partial g_w(v)\ \ \text{for}\ \ v\in\mathrm{int}\mathbb{B}_\nu(\bar v). \tag{3.14}$$

Indeed, it follows from the constructions above that the function g_w is convex and is expressed as

$$g_w(v)=\mathrm{argmax}_{x\in\mathbb{B}_\nu(\bar x)}\big\{\langle v,x\rangle-\varphi_w(x)\big\}.$$

This readily implies the relationships

$$\begin{aligned} g_w(v') - g_w(v) &\geq \langle v', M_\nu(w,v)\rangle - \varphi_w(M_\nu(w,v)) - \langle v, M_\nu(w,v)\rangle + \varphi_w(M_\nu(w,v)) \\ &= \langle v' - v, M_\nu(w,v)\rangle \quad \text{for all} \quad v' \in \mathbb{R}^n, \end{aligned}$$

which yields that (3.14) holds. Consider further the mapping $T_w(\cdot) := M_\nu(w,\cdot)$ and show that it is monotone on $\text{int}\mathbb{B}_\nu(\bar{v})$. To check it, pick $x_i \in T_w(v_i)$ with $v_i \in \text{int}\mathbb{B}_\nu(\bar{v})$ as $i = 1, 2$ and get from (3.14) that

$$\begin{aligned} \langle x_1 - x_2, v_1 - v_2\rangle &= \langle x_1, v_1\rangle - \langle x_2, v_1\rangle - \langle x_1, v_2\rangle + \langle x_2, v_2\rangle \\ &= \big[g_w(v_1) - \langle x_2, v_1\rangle + \varphi_w(x_2)\big] + \big[g_w(v_2) - \langle x_1, v_2\rangle + \varphi_w(x_1)\big] \geq 0. \end{aligned}$$

Since T_w is (Lipschitz) continuous, it is locally maximal monotone around $(\bar{v}, \bar{x})$ relative to $\text{int}\mathbb{B}_\nu(\bar{v}) \times \text{int}\mathbb{B}_\nu(\bar{x})$; see [23, Example 12.7]. Remembering next that the subdifferential mappings for convex functions are also maximal monotone, we conclude from (3.14) that

$$\partial g_w(v) = T_w(v) \quad \text{for all} \quad v \in \text{int}\mathbb{B}_\nu(\bar{v}).$$

Thus g_w is Fréchet differentiable on $\text{int}\mathbb{B}_\nu(\bar{v})$ and its gradient mapping ∇g_w is Lipschitz continuous with constant σ on this set. Now we are in a position of applying Lemma 3.6 to the function $f := h_w$ with $h_w^* = g_w^{**} = g_w$. This gives us the estimate

$$h_w(u) \geq h_w(x) + \langle v, u - x\rangle + \frac{1}{2\sigma}\|u - x\|^2 \quad \text{whenever} \quad u \in \text{int}\mathbb{B}_{\frac{\sigma\nu}{4}}(\bar{x}) \tag{3.15}$$

for all $(x, v) \in (\text{gph}\,\partial h_w) \cap [\text{int}\mathbb{B}_{\frac{\sigma\nu}{4}}(\bar{x}) \times \text{int}\mathbb{B}_{\frac{\nu}{2}}(\bar{v})]$. Observe that since the Lipschitz constant σ does not depend on the w, the estimate in (3.15) is uniform with respect to w in the selected neighborhood of $\bar{w}$. Also we can assume without loss of generality that $\text{int}\mathbb{B}_{\frac{\sigma\nu}{4}}(\bar{x}) \subset \text{int}\mathbb{B}_\nu(\bar{x})$.

Take now $x \in (\partial h_w)^{-1}(v) = \partial g_w(v) = T_w(v)$ and get from the single-valuedness of the set $T_w(v)$ by its construction above that

$$h_w(T_w(v)) = h_w(x) = \varphi_w(x) = \varphi(x, w).$$

This allows us to deduce from (3.15) that

$$\varphi(u, w) \geq \varphi(x, w) + \langle v, u - x\rangle + \frac{1}{2\sigma}\|u - x\|^2 \tag{3.16}$$

whenever $(x, v) \in (\text{gph}\,\partial h_w) \cap [\text{int}\mathbb{B}_{\frac{\sigma\nu}{4}}(\bar{x}) \times \text{int}\mathbb{B}_{\frac{\nu}{2}}(\bar{v})]$ and $u \in \text{int}\mathbb{B}_{\frac{\sigma\nu}{4}}(\bar{x})$.

To conclude the proof of assertion (i), we need to justify the possibility of replacing the set $\text{gph}\,\partial h_w$ by that of $\text{gph}\,\partial\varphi_w$ in estimate (3.16). Remember that $(\text{gph}\,\partial\varphi_w) \cap [\text{int}\mathbb{B}_{\frac{\sigma\nu}{4}}(\bar{x}) \times \text{int}\mathbb{B}_{\frac{\nu}{2}}(\bar{v})] = (\text{gph}\,\partial T_w) \cap [\text{int}\mathbb{B}_{\frac{\sigma\nu}{4}}(\bar{x}) \times \text{int}\mathbb{B}_{\frac{\nu}{2}}(\bar{v})]$. Take $(x, v) \in (\text{gph}\,\partial\varphi_w) \cap [\text{int}\mathbb{B}_{\frac{\sigma\nu}{4}}(\bar{x}) \times \text{int}\mathbb{B}_{\frac{\nu}{2}}(\bar{v})]$. This ensures therefore that

$$x = T_w(v) = \partial g_w(v) = (\partial h_w)^{-1}(v),$$

and so $(x, v) \in (\text{gph}\,\partial h_w) \cap [\text{int}\mathbb{B}_{\frac{\sigma\nu}{4}}(\bar{x}) \times \text{int}\mathbb{B}_{\frac{\nu}{2}}(\bar{v})]$. Since $\bar{x}$ is a fully stable local minimizer of (3.1), it follows that $\bar{x} \in M_\gamma(\bar{w}, \bar{v})$ for some $\gamma < \frac{\sigma\nu}{4}$. Taking into account the basic constraint qualification (3.6) together with [6, Proposition 3.5] gives us $M_\gamma(w, v) \in \text{int}\mathbb{B}_\gamma(\bar{x})$ for any $(w, v) \in \text{int}\mathbb{B}_\gamma(\bar{w}) \times \text{int}\mathbb{B}_\gamma(\bar{v})$. Thus for any $(w, v) \in$

$\mathrm{int}\mathbb{B}_\gamma(\bar{w}) \times \mathrm{int}\mathbb{B}_\gamma(\bar{v})$ we can find $x_{wv} = M_\gamma(w,v) \in \mathrm{int}\mathbb{B}_\gamma(\bar{x})$ such that (3.16) holds. This justifies the validity of the USOGC for φ at $(\bar{x},\bar{w},\bar{v})$ and hence ends the proof of (i).

Next we justify assertion (ii) observing by Theorem 3.4 that it suffices to show that the mapping $\partial_x\varphi$ is PSMR at $(\bar{x},\bar{w},\bar{v})$ under the assumptions made. To proceed, fix the neighborhoods U of $\bar{x}$, W of $\bar{w}$, and V of $\bar{v}$ for which the second-order growth condition (3.12) holds and thus gives us the single-valued and Lipschitz continuous mapping $s: W \times V \to U$ defined by $s(w,v) := x_{wv}$. Denote $T_w(\cdot) := s(w,\cdot)$ and pick any vectors $v_i \in T_w^{-1}(x_i)$ with $v_i \in V$ and $x_i \in U$ for $i = 1,2$. By (3.12) with $\eta = (2\sigma)^{-1}$ for some $\sigma > 0$ we get the estimates

$$\varphi(x_2,w) \geq \varphi(x_1,w) + \langle v_1, x_2 - x_1\rangle + \frac{1}{2\sigma}\|x_2 - x_1\|^2,$$
$$\varphi(x_1,w) \geq \varphi(x_2,w) + \langle v_2, x_1 - x_2\rangle + \frac{1}{2\sigma}\|x_2 - x_1\|^2,$$

which tell us that the mapping T_w^{-1} is locally *strongly monotone* with constant σ^{-1}; see [23, Definition 12.53]. Hence T_w is locally monotone relative to V and U and in fact is locally maximal monotone relative to $V \times U$ due to its continuity. Note that if $(v,x) \in \mathrm{gph}\, T_w$, then $v \in \partial\varphi_w(x)$.

Let $F_w : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be the mapping for which $\mathrm{gph}\, F_w^{-1}$ is the intersection of $\mathrm{gph}\, \partial\varphi_w$ and $U \times V$. We have $\mathrm{gph}\, T_w \subset \mathrm{gph}\, F_w$ and thus the inclusions

$$T_w^{-1}(x) \subset F_w^{-1}(x) \subset \partial\varphi_w(x) \ \text{ whenever } \ x \in U. \tag{3.17}$$

It follows from the parametric continuous prox-regularity of φ that the mapping $\partial\varphi_w$ are locally *hypomonotone* whenever $w \in W$ with the same constant $\gamma > 0$ from (3.4), and so the mapping $F_w^{-1} + tI$ is locally strongly monotone with constant $t - \gamma$ for any fixed $t > \gamma$; see [23, Example 12.28]. Since T_w^{-1} is locally strongly monotone with constant σ^{-1}, we keep this property for the mapping $T_w^{-1} + tI$ with constant $\sigma^{-1} + t$. Hence the mappings $(F_w^{-1}+tI)^{-1}$ and $(T_w^{-1}+tI)^{-1}$ are single-valued on their domains. Furthermore, it follows from (3.17) that $\mathrm{gph}\,(T_w^{-1} + tI)^{-1} \subset \mathrm{gph}\,(F_w^{-1} + tI)^{-1}$. Now we claim that

$$T_w(v) = F_w(v) \in U \ \text{ for any } \ w \in W \ \text{ and } \ v \in V. \tag{3.18}$$

To justify this, by using (3.17) we get $T_w(v) \subset F_w(v)$ for all $v \in V$. To prove the reverse inclusion, let R be the maximal extension of T_w. Since the mapping T_w is locally maximal monotone relative to $V \times U$, we have $(\mathrm{gph}\, T_w) \cap (V \times U) = (\mathrm{gph}\, R) \cap (V \times U)$. Denote $O := J_t(V \times U)$ with the bilinear mapping $J_t(v,u) := (v + tu, u)$ and observe that O is a neighborhood of $(\bar{v} + t\bar{x}, \bar{x})$. It is easy to see that

$$(\mathrm{gph}\,(T_w^{-1} + tI)^{-1}) \cap O = (\mathrm{gph}\,(R^{-1} + tI)^{-1}) \cap O \subset (\mathrm{gph}\,(F_w^{-1} + tI)^{-1}) \cap O. \tag{3.19}$$

Pick $(u,v) \in U \times V$ such that $u \in F_w(v)$. This tells us that $(v + tu, u) \in (\mathrm{gph}\,(F_w^{-1} + tI)^{-1}) \cap O$. Employing Minty's theorem [23, Theorem 12.12] for the maximal monotone mapping R, we have $\mathrm{dom}\,(R^{-1}+tI)^{-1}) = \mathbb{R}^n$, which says that $(R^{-1}+tI)^{-1})(v+tu) \neq \emptyset$. Taking into account that the mapping $(F_w^{-1} + tI)^{-1}$ is single-valued on its domain and applying (3.19) ensure that $u = (F_w^{-1}+tI)^{-1})(v+tu) = (R^{-1}+tI)^{-1})(v+tu)$. This implies that $u = (T_w^{-1} + tI)^{-1})(v + tu)$ due to (3.19), which justifies the reverse inclusion. Recalling finally definition (3.9) of the partial inverse S_φ, we easily deduce from (3.18) that

$$S_\varphi(w,v) \cap U = \{s(w,v)\} \ \text{ whenever } \ (w,v) \in W \times V$$

for the mapping s defined at the beginning of the proof of (ii). This means that s is a Lipschitzian single-valued localization of S_φ, and thus $\partial_x\varphi$ is PSMR at $(\bar{x},\bar{w},\bar{v})$ by Definition 3.3. $\square$

The only assumption that seems to be restrictive in Theorem 3.7 is the Lipschitz continuity of the mapping $(w,v)\mapsto x_{wv}$. We show in section 6 that it holds for a broad class of MPPC under the classical Robinson qualification condition.

4. Exact second-order chain rules for partial subdifferentials. This section is devoted to deriving exact (i.e., the equality-type) chain rules for the extended partial second-order subdifferential (2.12) of parametric compositions given in the form

$$\varphi(x,w)=(\theta\circ h)(x,w):=\theta(h(x,w))\ \text{ with }\ x\in\mathbb{R}^n\ \text{ and }\ w\in\mathbb{R}^d, \tag{4.1}$$

where $h\colon\mathbb{R}^n\times\mathbb{R}^d\to\mathbb{R}^m$ and $\theta\colon\mathbb{R}^m\to\overline{\mathbb{R}}$ finite at $\bar{z}:=h(\bar{x},\bar{w})$. Let $\bar{v}\in\partial_x\varphi(\bar{x},\bar{w})$ be a first-order partial subgradient, which is fixed in what follows. Assuming that the mapping h is continuously differentiable around $(\bar{x},\bar{w})$ and its derivative ∇h with respect to both variable (x,w) is strictly differentiable at this point and then imposing the *full rank condition*

$$\operatorname{rank}\nabla_x h(\bar{x},\bar{w})=m \tag{4.2}$$

on the corresponding partial Jacobian matrix, the *exact second-order chain rule*

$$\begin{aligned}\widetilde{\partial}^2_x\varphi(\bar{x},\bar{w},\bar{v})(u)=&\Big(\nabla^2_{xx}\langle\bar{y},h\rangle(\bar{x},\bar{w})u,\nabla^2_{xw}\langle\bar{y},h\rangle(\bar{x},\bar{w})u\Big)\\&+\Big(\nabla_x h(\bar{x},\bar{w}),\nabla_w h(\bar{x},\bar{w})\Big)^*\partial^2\theta(\bar{z},\bar{y})(\nabla_x h(\bar{x},\bar{w})u)\end{aligned} \tag{4.3}$$

is proved [16, Theorem 3.1], where u is any vector from $\mathbb{R}^n$, while $\bar{y}$ is a unique vector satisfying

$$\bar{y}\in\partial\theta(\bar{z})\ \text{ and }\ \nabla_x h(\bar{x},\bar{w})^*\bar{y}=\bar{v}. \tag{4.4}$$

Our goal in this section is to justify the exact second-order chain rule (4.3) for particular classes of outer functions θ in compositions (4.1) without imposing the full rank condition (4.2). In this way we extend the corresponding results of [16] obtained for the full second-order subdifferential (2.10) to its partial counterpart (2.12).

Recall [7] that an extended-real-valued function $\varphi(x,w)$ on $\mathbb{R}^n\times\mathbb{R}^d$ is fully amenable in x at $\bar{x}$ with compatible parameterization by w at $\bar{w}$ if it is strongly amenable with compatible parameterization in the sense above (see the discussion before Theorem 3.2), while the outer function θ in its composite representation (4.1) can be chosen as *piecewise linear-quadratic*, i.e., its graph is the union of finitely many polyhedral sets; see [23, Chapter 13] for more details.

To proceed with deriving the exact second-order chain rule (4.3) for particular classes of fully amenable compositions with compatible parameterization (4.1), we define the set

$$M(\bar{x},\bar{w},\bar{v}):=\Big\{y\in\mathbb{R}^m\,\Big|\,y\in\partial\theta(\bar{z})\ \text{ with }\ \nabla_x h(\bar{x},\bar{w})^*y=\bar{v}\Big\} \tag{4.5}$$

in the notation above. This set is obviously a singleton if the full rank condition (4.2) holds, which is not assumed anymore. Denote by $S(z)$ a subspace of $\mathbb{R}^m$ parallel to the affine hull $\operatorname{aff}\partial\theta(z)$ of the subdifferential $\partial\theta(z)$. It follows from the proof of

[16, Theorem 4.1] that if φ in (4.1) is fully amenable in x at $\bar{x}$ with compatible parameterization by w at $\bar{w}$, then for any sufficiently small neighborhood O of $\bar{z}$ there are finitely many subspaces $S(z)$ such that

$$\partial^2\theta(\bar{z},y)(0)=\bigcup_{z\in O}S(z)\ \text{ whenever }\ y\in M(\bar{x},\bar{w},\bar{v}). \tag{4.6}$$

Consider now a subclass of fully amenable compositions (4.1) with compatible parameterization, where the outer function θ is (convex) piecewise linear, i.e., its epigraph is a polyhedral set; see [23, Theorem 2.49] for this equivalent description. The next theorem establishes the validity of the exact second-order subdifferential chain rule (4.3) for such fully amenable compositions without imposing the full rank condition (4.2).

Theorem 4.1 (exact second-order chain rule for parametric compositions with piecewise linear outer functions). *Let φ in* (4.1) *be fully amenable in x at $\bar{x}$ with compatible parameterization by w at $\bar{w}$, where the outer function θ is piecewise linear. Then for any $\bar{v}\in\partial_x\varphi(\bar{x},\bar{w})$ we have*

$$\partial^2\theta(\bar{z},\bar{y})(0)=S(\bar{z})\ \textit{ whenever }\ \bar{y}\in M(\bar{x},\bar{w},\bar{v}) \tag{4.7}$$

with $\bar{z}=h(\bar{x},\bar{w})$. Furthermore, imposing the SOQC

$$\partial^2\theta\big(\bar{z},\bar{y})(0)\cap\ker\nabla_x h(\bar{x},\bar{w})^*=\{0\} \tag{4.8}$$

ensures that $M(\bar{x},\bar{w},\bar{v})$ in (4.5) *is in fact a singleton $\{\bar{y}\}$ and the second-order chain rule* (4.3) *holds.*

Proof. Fix a neighborhood O of $\bar{z}$ such that representation (4.6) holds with the subgradient $\bar{v}\in\partial_x\varphi(\bar{x},\bar{w})$ fixed above. It follows easily from the piecewise linearity of θ that $\partial\theta(z)\subset\partial\theta(\bar{z})$ for all $z\in O$. This implies that $S(z)\subset S(\bar{z})$ for such vectors z, and thus representation (4.6) reduces to (4.7). Let us deduce from (4.7) and (4.8) that the set $M(\bar{x},\bar{w},\bar{v})$ from (4.5) is a singleton $\{\bar{y}\}$. Indeed, picking any $y_1,y_2\in M(\bar{x},\bar{w},\bar{v})$ gives us that $y_1,y_2\in\partial\theta(\bar{z})$ and that $y_1-y_2\in\ker\nabla_x h(\bar{x},\bar{w})^*$. Since $S(\bar{z})$ is the subspace parallel to $\operatorname{aff}\partial\theta(\bar{z})$, we get $y_1-y_2\in S(\bar{z})$, and thus $y_1=y_2$ by (4.7) and the second-order qualification condition (4.8). Denoting now $L:=S(\bar{z})$ summarizes the situation above as follows:

$$L\cap\ker\nabla_x h(\bar{x},\bar{w})^*=\{0\}\ \text{ with }\ S(z)\subset L\ \text{ for all }\ z\in O. \tag{4.9}$$

To proceed further, let $\dim L=:s\le m$ and observe that for $s=m$ the first relationship in (4.9) yields the full rank condition (4.2), and thus the exact second-order chain rule (4.3) follows in this case from [16, Theorem 3.1]. It remains to consider the case of $s<m$ and proceed similarly to the proof of [16, Lemma 4.2 and Theorem 4.3] with the corresponding modifications and details presented here for completeness and the reader's convenience.

In this case we denote by A the matrix of a linear isometry from $\mathbb{R}^m$ into $\mathbb{R}^s\times\mathbb{R}^{m-s}$ under which $A^*L=\mathbb{R}^s\times\{0\}$. Observe the composite representation $\varphi=\vartheta\circ P$, where $P:=A^{-1}h$ and $\vartheta:=\theta A$. The first-order chain rule of convex analysis gives us

$$\nabla_x P(x,w)=A^{-1}\nabla_x h(x,w)\ \text{ and }\ \partial\vartheta(z')=A^*\partial\theta(z)\ \text{ with }\ Az'=z. \tag{4.10}$$

Since $S(z)$ is the subspace parallel to $\operatorname{aff}\partial\theta(z)$, for each $z\in O$ there is a vector $b_z\in\mathbb{R}^m$ such that $S(z)=\operatorname{aff}\partial\theta(z)+b_z$. This ensures that

$$\begin{aligned} v=(v_1,\ldots,v_m)\in\partial\vartheta(z')=A^*\partial\theta(z)&\subset A^*\partial\theta(\bar z)\\ &\subset A^*L-A^*b_{\bar z}\subset\mathbb{R}^s\times\{0\}-A^*b_{\bar z}. \end{aligned}\tag{4.11}$$

Consider first the case of $b_{\bar z}=0$ above. Then it follows directly from the relationships in (4.11) and (4.9) that $v_{s+1}=\cdots=v_m=0$. Representing now $P(x,w)=(p_1(x,w),\ldots,p_m(x,w))$ and using the full amenability of φ, we have

$$y\in\partial\varphi(x,w)\iff\begin{cases}\exists\, v\in\partial\vartheta(P(x,w))\ \text{ such that}\\ y=\nabla_xP(x,w)^*v=\displaystyle\sum_{i=1}^s\nabla_xp_i(x,w)^*v_i.\end{cases}\tag{4.12}$$

This means that in analyzing the subgradient mapping $\partial\varphi$ locally via ϑ and P it is possible to pass without loss of generality to the submatrix $P_0(x,w):=(p_1(x,w),\ldots,p_s(x,w))$. Let us now show that $\operatorname{rank}\nabla_xP_0(\bar x,\bar w)=s$. Indeed, consider the equation

$$\nabla_xP_0(\bar x,\bar w)^*u=0\tag{4.13}$$

from which we deduce the equalities

$$\nabla_xh(x,w)^*(A^{-1})^*(u,0)=\nabla_xP(\bar x,\bar w)^*(u,0)=0.$$

Since $(u,0)\in\mathbb{R}^s\times\{0\}$, it follows from the kernel condition in (4.9) that $u=0$, and hence (4.13) has only the trivial solution, which means that $\operatorname{rank}\nabla_xP_0(\bar x,\bar w)=s$. By this we reduce the situation in the proof of the theorem in the case of $b_{\bar z}=0$ under consideration to the full rank condition relative to the submatrix $\nabla_xP_0(\bar x,\bar w)$ and thus can apply again the exact second-order chain rule from [16, Theorem 3.1].

Next we consider the remaining case of $b:=b_{\bar z}\neq0$ in (4.11). Defining now the *bar functions* $\overline{\theta}(z):=\theta(z)-\langle b,z\rangle$ and $\overline{\varphi}:=\overline{\theta}\circ h$, observe that they are in the previous case setting; thus we have the exact second-order chain rule (4.3) for $\overline{\varphi}$. To get the result for the original composition φ, we begin with the elementary first-order subdifferential sum rule written as

$$\partial_x\overline{\varphi}(\bar x,\bar w)=\partial_x\varphi(\bar x,\bar w)-\nabla_xh(\bar x,\bar w)^*b.$$

Thus for any $\widetilde v\in\partial_x\overline{\varphi}(\bar x,\bar w)$ there is a subgradient $\bar v\in\partial_x\varphi(\bar x,\bar w)$ such that $\widetilde v=\bar v-\nabla_xh(\bar x,\bar w)^*b$, and so $\bar y\in\partial\theta(\bar x,\bar w)$ with $\widetilde v=\nabla_xh(\bar x,\bar w)^*(\bar y-b)$. This implies that $\bar v=\nabla_xh(\bar x,\bar w)^*\bar y$. Employing further the coderivative sum rule from [12, Theorem 1.62] correspondingly modified for the extended partial subdifferential (2.12) and taking into account this subdifferential representation for $\mathcal{C}^2$ functions (2.13), we get the expression

$$\widetilde\partial^2_x\overline{\varphi}(\bar x,\bar w,\widetilde v)(u)=\widetilde\partial^2_x\varphi(\bar x,\bar w,\bar v)(u)-\Big(\nabla^2_{xx}\langle b,h\rangle(\bar x,\bar w)u,\nabla^2_{xw}\langle b,h\rangle(\bar x,\bar w)u\Big).\tag{4.14}$$

On the other hand, by the justified second-order chain rule (4.3) for $\overline{\varphi}$ in this setting we have

$$\begin{aligned}\widetilde\partial^2_x\overline{\varphi}(\bar x,\bar w,\widetilde v)(u)=&\Big(\nabla^2_{xx}\langle\bar y-b,h\rangle(\bar x,\bar w)u,\nabla^2_{xw}\langle\bar y-b,h\rangle(\bar x,\bar w)u\Big)\\ &+\Big(\nabla_xh(\bar x,\bar w),\nabla_wh(\bar x,\bar w)\Big)^*\partial^2\overline{\theta}(\bar z,\bar y-b)(\nabla_xh(\bar x,\bar w)u)\end{aligned}\tag{4.15}$$

whenever $u \in \mathbb{R}^n$. Substituting finally the obvious relationship

$$\partial^2\overline{\theta}(\bar{z}, \bar{y} - b)(u) = \partial^2\theta(\bar{z}, \bar{y})(u), \quad u \in \mathbb{R}^n,$$

into (4.14) and (4.15), we arrive at the second-order chain rule (4.3) for the composition φ under consideration in the case of $b \neq 0$ and thus complete the proof of the theorem. □

Next we consider a major subclass of piecewise linear-quadratic outer functions in parametric fully amenable compositions given by

$$\theta(z) = \sup_{p \in P} \Big\{ \langle p, z\rangle - \tfrac{1}{2}\langle p, Qp\rangle \Big\}, \tag{4.16}$$

where $P \subset \mathbb{R}^m$ is a nonempty polyhedral set and where $Q \in \mathbb{R}^{m\times m}$ is a symmetric positive-semidefinite matrix ensuring the convexity of (4.16). It has been well recognized that extended-real-valued functions of type (4.16) play a significant role in many aspects of variational analysis, particularly in setting up "penalty expressions" in composite formats of optimization; see [22, 23].

Recall further the classical notion of openness for mappings h between topological spaces: h is open at $\bar{u}$ if for any neighborhood U of $\bar{u}$ there is some neighborhood V of $h(\bar{u})$ such that $V \subset h(U)$. It is well known that the openness property is essentially less demanding than its linear counterpart (openness at a linear rate) around the reference point, which is characterized for smooth mappings by the surjectivity/full rank of their derivatives; see [12, 23]. Note to this end that considering smooth mappings $h\colon \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^m$ of two variables between finite-dimensional spaces, the linear openness of h around $(\bar{x}, \bar{w})$ is equivalent to full rank of the total Jacobian $\nabla h(\bar{x}, \bar{w})$, which is obviously a less restrictive condition than the full rank requirement (4.2) on the partial Jacobian at this point.

The next theorem establishes the exact second-order chain rule for parametric fully amenable compositions with outer functions (4.16). It extends to the parametric case the second-order chain rule from [16, Theorem 4.5] while giving a new proof even in the nonparametric setting.

THEOREM 4.2 (exact second-order chain rule for a major subclass of parametric fully amenable compositions). *Let the composition φ in* (4.1) *be fully amenable in x at $\bar{x}$ with compatible parameterization by w at $\bar{w}$, where the outer function θ belongs to class* (4.16). *Assume that Q is positive-definite and that $h\colon \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^m$ is open at $(\bar{x}, \bar{w})$. Then for any partial subgradient $\bar{v} \in \partial_x\varphi(\bar{x}, \bar{w})$ the set $M(\bar{x}, \bar{w}, \bar{v})$ in* (4.5) *is a singleton $\{\bar{y}\}$ and the second-order chain rule* (4.3) *holds.*

Proof. First we show that the positive-definiteness of Q ensures that the subdifferential mapping $z \mapsto \partial\theta(z)$ is single-valued and Lipschitz continuous around $\bar{z}$. Indeed, it follows from [16, Lemma 4.4] that

$$\partial^2\theta(\bar{z}, y)(0) = \{0\} \ \text{ for any } \ y \in \partial\theta(\bar{z}), \tag{4.17}$$

which implies by (4.6) that $S(z) = \{0\}$ whenever z is sufficiently close to $\bar{z}$ and the validity of the second-order qualification condition (4.8). This justifies the single-valuedness of the subdifferential mapping $z \mapsto \partial\theta(z) = \nabla\theta(z)$ around $\bar{z}$ and ensures, in particular, that $M(\bar{x}, \bar{w}, \bar{v}) =: \{\bar{y}\}$. Moreover, by the underlying relationship (4.17) and definition (2.10) of the second-order subdifferential we have

$$\{0\} = \partial^2\theta(\bar{z}, \bar{y})(0) = (D^*\partial\theta)(\bar{z}, \bar{y})(0) \ \text{ with } \ \bar{y} = \nabla\theta(\bar{z}),$$

and hence the Mordukhovich criterion [23, Theorem 9.40] tells us that the mapping $z \mapsto \nabla\theta(z)$ is in fact locally Lipschitzian around $\bar{z}$.

Observe further that the inclusion $\subset$ in (4.3) is established in [16, Theorem 3.3] in a more general setting. To justify the opposite inclusion $\supset$ in (4.3), take any $(\widetilde{x},\widetilde{w})$ near to $(\bar{x},\bar{w})$, denote $\widetilde{z} := h(\widetilde{x},\widetilde{w})$ and $\widetilde{y} := \nabla\theta(\widetilde{z})$, and then show that

$$\begin{aligned}(4.18)\quad \widehat{\partial}\langle u, \nabla_x\varphi\rangle(\widetilde{x},\widetilde{w}) \supset \Big(\nabla^2_{xx}\langle\widetilde{y},h\rangle(\widetilde{x},\widetilde{w})u, \nabla^2_{xw}\langle\widetilde{y},h\rangle(\widetilde{x},\widetilde{w})u\Big)\\ + \Big(\nabla_x h(\widetilde{x},\widetilde{w}), \nabla_w h(\widetilde{x},\widetilde{w})\Big)^*\widehat{\partial}\langle\nabla_x h(\bar{x},\bar{w})u, \nabla\theta\rangle(\widetilde{z})\end{aligned}$$

for all $u \in \mathbb{R}^n$. Indeed, picking any $p \in \widehat{\partial}\langle\nabla_x h(\bar{x},\bar{w})u, \nabla\theta\rangle(\widetilde{z})$ and fixing an arbitrary number $\gamma > 0$, we get the estimate

$$\begin{aligned}&\langle p, z-\widetilde{z}\rangle - \langle\nabla_x h(\bar{x},\bar{w})u, \nabla\theta(z)-\nabla\theta(\widetilde{z})\rangle\\ &\quad\le \gamma(\|z-\widetilde{z}\| + \|\langle\nabla_x h(\bar{x},\bar{w})u, \nabla\theta(z)-\nabla\theta(\widetilde{z})\rangle\|)\\ &\quad\le (\ell + \ell^2\|\nabla_x h(\bar{x},\bar{w})u\|)\gamma(\|x-\widetilde{x}\| + \|w-\widetilde{w}\|),\end{aligned}$$

where (x,w) is sufficiently close to $(\widetilde{x},\widetilde{w})$, $z = h(x,w)$, and ℓ is a common local Lipschitz constant for h, ∇h, and $\nabla\theta$. With no loss of generality, suppose that $\|\bar{x}-\widetilde{x}\| + \|\bar{w}-\widetilde{w}\| < \gamma$ and $\|x-\widetilde{x}\| + \|w-\widetilde{w}\| < 1$. Then elementary transformations give us the relationships

$$\begin{aligned}&\langle\nabla_x h(\bar{x},\bar{w})u, \nabla\theta(z)-\nabla\theta(\widetilde{z})\rangle\\ &\quad= \langle u, (\nabla_x h(\bar{x},\bar{w}) - \nabla_x h(\widetilde{x},\widetilde{w}))^*(\nabla\theta(z)-\nabla\theta(\widetilde{z}))\rangle\\ &\qquad+ \langle u, \nabla_x h(\widetilde{x},\widetilde{w})^*(\nabla\theta(z)-\nabla\theta(\widetilde{z}))\rangle\\ &\quad\le \|u\|\ell^3(\|\bar{x}-\widetilde{x}\| + \|\bar{w}-\widetilde{w}\|)(\|x-\widetilde{x}\| + \|w-\widetilde{w}\|)\\ &\qquad+ \langle u, \nabla_x\varphi(x,w) - \nabla_x\varphi(\widetilde{x},\widetilde{w})\rangle\\ &\qquad+ \langle u, (\nabla_x h(\widetilde{x},\widetilde{w}) - \nabla_x h(x,w))^*\nabla\theta(\widetilde{z})\rangle\\ &\qquad+ \langle u, (\nabla_x h(\widetilde{x},\widetilde{w}) - \nabla_x h(x,w))^*(\nabla\theta(z)-\nabla\theta(\widetilde{z}))\rangle\\ &\quad\le \|u\|\ell^3\gamma(\|x-\widetilde{x}\| + \|w-\widetilde{w}\|) + \langle u, \nabla_x\varphi(x,w) - \nabla_x\varphi(\widetilde{x},\widetilde{w})\rangle\\ &\qquad+ \langle(\nabla^2_{xx}\langle\widetilde{y},h\rangle(\widetilde{x},\widetilde{w})u, \nabla^2_{xw}\langle\widetilde{y},h\rangle(\widetilde{x},\widetilde{w})u), (\widetilde{x}-x, \widetilde{w}-w)\rangle\\ &\qquad+ \gamma(\|x-\widetilde{x}\| + \|w-\widetilde{w}\|) + \ell^3\|u\|\gamma(\|x-\widetilde{x}\| + \|w-\widetilde{w}\|)\\ &\quad= \langle u, \nabla_x\varphi(x,w) - \nabla_x\varphi(\widetilde{x},\widetilde{w})\rangle + \mu\gamma(\|x-\widetilde{x}\| + \|w-\widetilde{w}\|)\\ &\qquad+ \Big\langle\Big(\nabla^2_{xx}\langle\widetilde{y},h\rangle(\widetilde{x},\widetilde{w})u, \nabla^2_{xw}\langle\widetilde{y},h\rangle(\widetilde{x},\widetilde{w})u\Big), (\widetilde{x}-x, \widetilde{w}-w)\Big\rangle,\end{aligned}$$

where $\mu := 2\|u\|\ell^3 + 1$ and $\widetilde{y} = \nabla\theta(\widetilde{z})$. Similar arguments ensure that

$$\langle\nabla_x h(\widetilde{x},\widetilde{w})^*q, \nabla_w h(\widetilde{x},\widetilde{w})^*q), (x-\widetilde{x}, w-\widetilde{w})\rangle \le \langle q, z-\widetilde{z}\rangle + \gamma(\|x-\widetilde{x}\| + \|w-\widetilde{w}\|)$$

for any $q \in \widehat{\partial}\langle\nabla_x h(\bar{x},\bar{w})u, \nabla\theta\rangle(\widetilde{z})$ and all pairs (x,w) sufficiently close to $(\widetilde{x},\widetilde{w})$. Combining the above estimates gives us

$$\begin{aligned}&\Big\langle(\nabla_x h(\widetilde{x},\widetilde{w})^*w, \nabla_w h(\widetilde{x},\widetilde{w})^*w)\\ &\qquad+ \Big(\nabla^2_{xx}\langle\widetilde{y},h\rangle(\widetilde{x},\widetilde{w})u, \nabla^2_{xw}\langle\widetilde{y},h\rangle(\widetilde{x},\widetilde{w})u\Big), (x-\widetilde{x}, w-\widetilde{w})\Big\rangle\\ &\qquad- \langle u, \nabla_x\varphi(x,w) - \nabla_x\varphi(\widetilde{x},\widetilde{w})\rangle \le \gamma\Big(\mu + 2 + \ell^2\|\nabla h(\bar{x},\bar{w})u\|\Big)\Big(\|x-\widetilde{x}\|\\ &\qquad+ \|w-\widetilde{w}\| + \|\nabla_x\varphi(x,w) - \nabla_x\varphi(\widetilde{x},\widetilde{w})\|\Big),\end{aligned}$$

which ensures (4.18) by taking into account construction (2.1) of the regular subdifferential.

To justify the desired limiting version of (4.18), we proceed as follows. Take any vector

$$q \in \partial^2\theta(\bar{z},\bar{y})(\nabla_x h(\bar{x},\bar{w})u) = \partial\langle\nabla_x h(\bar{x},\bar{w})u, \nabla\theta\rangle(\bar{z})$$

with $u \in \mathbb{R}^n$ and by definition (2.2) find sequences $z_k \to \bar{z}$ and $q_k \to q$ as $k \to \infty$ such that $q_k \in \widehat{\partial}\langle\nabla_x h(\bar{x},\bar{w})u, \nabla\theta\rangle(z_k)$ for all $k \in \mathbb{N}$. By the assumed openness of h at $(\bar{x},\bar{w})$ there are sequences $(x_k,w_k) \to (\bar{x},\bar{w})$ with $z_k = h(x_k,w_k)$. Substituting finally $(x_k,w_k) = (\widetilde{x},\widetilde{w})$ into (4.18) and passing to the limit as $k \to \infty$ complete the proof of the theorem. $\square$

5. Full stability in composite models of optimization. In this section we apply the developed second-order calculus rules to derive necessary and sufficient conditions for full stability in composite models of optimization written in the form

(5.1)
$$\text{minimize } \ \varphi(x) := \varphi_0(x) + \theta(\varphi_1(x),\ldots,\varphi_m(x)) = \varphi_0(x) + \theta(\Phi(x)) \ \text{ over } \ x \in \mathbb{R}^n,$$

where $\varphi_0\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $\theta\colon \mathbb{R}^m \to \overline{\mathbb{R}}$, and $\Phi(x) := (\varphi_1(x),\ldots,\varphi_m(x))$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. Written in the unconstrained form, problem (5.1) is actually a problem of constrained optimization with the cost function φ_0 and the set of feasible solutions given by

$$X := \{x \in \mathbb{R}^n \,|\, (\varphi_1(x),\ldots,\varphi_m(x)) \in Z\} \ \text{ with } \ Z := \{z \in \mathbb{R}^m \,|\, \theta(z) < \infty\}.$$

Observe that the results presented in this section for problem (5.1) can be easily transferred to problem of this type with additional geometric constraints given by $x \in \Omega$ via a polyhedral set $\Omega \subset \mathbb{R}^n$. Indeed the only change needed to be done is replacing the mapping Φ in (5.1) by $x \mapsto (x,\varphi_1(x),\ldots,\varphi_m(x))$ and the set Z above by the convex polyhedron $\Omega \times Z$. As discussed in [22, 23], the composite format (5.1) is a general and convenient framework, from both theoretical and computational viewpoints, to accommodate a variety of particular models in constrained optimization. Note that conventional nonlinear programs with s inequality constraints and $m-s$ equality constraints can be written in the form

$$\text{minimize } \ \varphi_0(x) + \delta_Z(\Phi(x)) \ \text{ over } \ x \in \mathbb{R}^n \tag{5.2}$$

via the indicator functions of the set $Z = \mathbb{R}^s_- \times \{0\}^{m-s}$. Extended versions of nonlinear programs are studied in sections 6 and 7 below.

Following the scheme of section 3, consider now the fully perturbed version $\mathcal{P}(w,v)$ of (5.1) with two parameters $(w,v) \in \mathbb{R}^d \times \mathbb{R}^n$ standing, respectively, for basic and tilt perturbations:

(5.3)
$$\text{minimize } \ \varphi(x,w) - \langle v,x\rangle \ \text{ over } \ x \in \mathbb{R}^n \ \text{ with } \ \varphi(x,w) := \varphi_0(x,w) + (\theta\circ\Phi)(x,w)$$

and $\Phi(x,w) = (\varphi_1(x,w),\ldots,\varphi_m(x,w))$ defined on $\mathbb{R}^n \times \mathbb{R}^d$. Our first characterization of full stability in (5.2) utilizes the exact chain rule (4.3) for the extended second-order subdifferential obtained in [16, Theorem 3.1] under the full rank condition (4.2) on $\Phi = h$. For simplicity we suppose that all the functions φ_i for $i = 0,\ldots,m$ are $\mathcal{C}^2$ around the reference points, although it is sufficient to assume that φ_i are merely smooth with strictly differentiable derivatives. Observe also that such properties are sometimes needed only partially with respect to the decision variable x; see the formulations and proofs below. It is worth noting that in the next theorem we use the second-order subdifferential of $\theta = \theta(z)$ and the special form (2.13) of the extended partial second-order subdifferential for the $\mathcal{C}^2$ functions $\varphi_i = \varphi_i(x,w)$.

THEOREM 5.1 (characterizing fully stable local minimizers for composite problems under full rank condition). *Let $\bar{x}$ be a feasible solution to the unperturbed problem $\mathcal{P}(\bar{w},\bar{v})$ in (5.3) with some $\bar{w} \in \mathbb{R}^d$ and $\bar{v} \in \partial_x\varphi(\bar{x},\bar{w})$, where $\varphi_0, \Phi \in \mathcal{C}^2$ around $(\bar{x},\bar{w})$ under the validity of the full rank condition*

$$\operatorname{rank} \nabla_x\Phi(\bar{x},\bar{w}) = m. \tag{5.4}$$

Assume further that the outer function θ is continuously prox-regular at $\bar{z} := \Phi(\bar{x},\bar{w})$ for the unique vector $\bar{y}$ satisfying the relationships

$$\nabla_x\Phi(\bar{x},\bar{w})^*\bar{y} = \bar{v} - \nabla_x\varphi_0(\bar{x},\bar{w}) \quad \text{and} \quad \bar{y} \in \partial\theta(\bar{z}). \tag{5.5}$$

Then $\bar{x}$ is a fully stable local minimizer for $\mathcal{P}(\bar{w},\bar{v})$ if and only if we have the implication

$$[(p,q) \in \mathcal{T}(\bar{x},\bar{w},\bar{v})(u),\ u \neq 0] \Longrightarrow \langle p,u\rangle > 0 \tag{5.6}$$

for the set-valued mapping $\mathcal{T}(\bar{x},\bar{w},\bar{v})\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n \times \mathbb{R}^d$ defined by

$$\begin{aligned}
&\mathcal{T}(\bar{x},\bar{w},\bar{v})(u)\\
&\quad := \Big(\nabla^2_{xx}\varphi_0(\bar{x},\bar{w})u, \nabla^2_{xw}\varphi_0(\bar{x},\bar{w})u\Big) + \Big(\nabla^2_{xx}\langle\bar{y},\Phi\rangle(\bar{x},\bar{w})u, \nabla^2_{xw}\langle\bar{y},\Phi\rangle(\bar{x},\bar{w})u\Big)\\
&\qquad + \Big(\nabla_x\Phi(\bar{x},\bar{w}), \nabla_w\Phi(\bar{x},\bar{w})\Big)^*\partial^2\theta(\bar{z},\bar{y})(\nabla_x\Phi(\bar{x},\bar{w})u), \quad u \in \mathbb{R}^n.
\end{aligned}$$

Proof. We apply the characterization of full stability from Theorem 3.2 to the function $\varphi(x,w)$ in (5.3). Observe first that the condition $\bar{v} \in \partial_x\varphi(\bar{x},\bar{w})$ on the tilt perturbation can be equivalently written as

$$\bar{v} \in \partial_x\varphi_0(\bar{x},\bar{w}) + \nabla_x\Phi(\bar{x},\bar{w})^*\partial\theta(\bar{z}). \tag{5.7}$$

Indeed, this follows from first-order rules for φ in (5.3) under the full rank assumption on $\nabla_x\Phi(\bar{x},\bar{w})$; see, e.g., [12, Propositions 1.107(ii) and 1.112(i)]. Employing further the calculus of prox-regularity from [19, Theorems 2.1 and 2.2], which can be easily extended to the parametric case under consideration, allows us to conclude that the composite function φ is parametrically continuously prox-regular at $(\bar{x},\bar{w},\bar{v})$.

Let us show next that the basic constraint qualification (3.6) is automatically satisfied, under the assumptions made, for the function φ given in (5.3). Indeed, by the smoothness of φ_0 the constraint qualification (3.6) is clearly equivalent to

$$(0,q) \in \partial^\infty(\theta\circ\Phi)(\bar{x},\bar{w}) \Longrightarrow q = 0. \tag{5.8}$$

Employing in (5.8) the chain rule for (2.3) from [12, Proposition 1.107(ii)] reduces it to the implication

$$\Big[\nabla_x\Phi(\bar{x},\bar{w})^*p = 0,\ \nabla_w\Phi(\bar{x},\bar{w})^*p = q,\ p \in \partial^\infty\theta(\bar{z})\Big] \Longrightarrow q = 0,$$

which obviously holds due to the full rank condition (5.4).

Now we are ready to apply the characterization of full stability from Theorem 3.2 to the function φ in (5.3). Let us first check that condition (3.7) is automatically satisfied in the setting under consideration. To proceed, apply to this composite function φ the second-order sum rule from [12, Proposition 1.121] and then the second-order chain rule from [16, Theorem 3.1], which tell us that (3.7) is equivalent to

$$\Big[(0,q)\in\Big(\nabla_x\Phi(\bar{x},\bar{w}),\nabla_w\Phi(\bar{x},\bar{w})\Big)^*\partial^2\theta(\bar{z},\bar{y})(0)\Big]\Longrightarrow q=0,$$

where the uniqueness of the vector $\bar{y}$ satisfying (5.5) follows from the full rank condition (5.4). The last implication can be rewritten as

$$\Big[\nabla_x\Phi(\bar{x},\bar{w})^*p=0,\ \nabla_w\Phi(\bar{x},\bar{w})^*p=q,\ p\in\partial^2\theta(\bar{z},\bar{y})(0)\Big]\Longrightarrow q=0,$$

which surely holds by the full rank of $\nabla_x\Phi(\bar{x},\bar{w})$ in (5.4). To complete the proof of the theorem, it remains finally to observe that condition (3.8) in Theorem 3.2 reduces to that of (5.6) imposed in this theorem due to the aforementioned second-order sum and chain rules from [12, Proposition 1.121] and [16, Theorem 3.1] applied to the function φ in (5.3). ∎

Note that the case of only the tilt perturbations in (5.3), i.e., when φ_0 and Φ do not depend on w therein, Theorem 5.1 reduces to the characterization of tilt-stable minimizers for (5.1) obtained in [16, Theorem 5.1]. The next result gives characterizations of fully stable locally optimal solutions to $\mathcal{P}(\bar{w},\bar{v})$ in (5.3) for two major classes of parametrically amenable composition in (5.3) that are derived on the basis of the new second-order chain rules from section 4 and extend the corresponding characterizations of tilt stability obtained in [16, Theorem 5.4].

Theorem 5.2 (characterizations of full stability in optimization problems described by parametrically fully amenable compositions). *Let $\bar{x}$ be a feasible solution to the unperturbed problem $\mathcal{P}(\bar{w},\bar{v})$ in* (5.3) *with some $\bar{w}\in\mathbb{R}^d$ and $\bar{v}\in\partial_x\varphi(\bar{x},\bar{w})$. Assume that $\varphi_0\in\mathcal{C}^2$ around $(\bar{x},\bar{w})$ and that the composition $\theta\circ\Phi$ is fully amenable in x at $\bar{x}$ with compatible parameterization by w at $\bar{w}$ and with the outer function θ of one of the following types:*

(a) *either θ is piecewise linear,*

(b) *or θ is of class* (4.16) *under the assumptions of Theorem* 4.2.

Suppose also in case (a) *that the second-order qualification condition* (4.8) *holds with $h=\Phi$, where $\bar{y}$ is the unique vector satisfying* (5.5). *Then $\bar{x}$ is fully stable local minimizer of $\mathcal{P}(\bar{w},\bar{v})$ if and only if condition* (5.6) *is satisfied for the set-valued mapping $\mathcal{T}(\bar{x},\bar{w},\bar{v})$ defined in Theorem* 5.1, *where the second-order subdifferential $\partial^2\theta(\bar{z},\bar{y})$ is calculated by the corresponding formulas in* [16].

Proof. As mentioned in section 3, the assumed parametric amenability of $\theta\circ\Phi$ implies the parametric continuous prox-regularity of this composition at $(\bar{x},\bar{w},\bar{v})$ and the validity of the basic constraint qualification (5.8). These properties stay for the function φ in (5.3) while adding the $\mathcal{C}^2$ function φ_0 to the composition $\theta\circ\Phi$; cf. [19, Theorem 2.2]. Observe further that the partial subgradient $\bar{v}\in\partial_x\varphi(\bar{x},\bar{w})$ satisfies inclusion (5.7) by the first-order chain rule from [12, Corollary 3.43] and [23, Theorem 10.6] held under the qualification condition (3.5) with $h=\Phi$ for amenable compositions. Moreover, the uniqueness of $\bar{y}$ satisfying (5.5) in cases (a) and (b) is proved in Theorems 4.1 and 4.2, respectively.

To now apply Theorem 3.2 to the composite function (5.3) in the settings under consideration, we argue similarly to the proof of Theorem 5.1 that (3.7) is satisfied in these frameworks due to the second-order qualification condition (4.8) with $h=\Phi$.

Employing finally in (5.3) the exact second-order sum rule and chain rule from [12, Proposition 1.121] as well as the above Theorems 4.1 and 4.2 allows us to conclude that condition (3.8) is equivalent to (5.6) for the underlying operator $\mathcal{T}(\bar{x}, \bar{w}, \bar{v})$. This justifies full stability of $\bar{x}$ under the assumptions made and thus completes the proof of the theorem. $\square$

6. Full stability and strong regularity for mathematical programs with polyhedral constraints. This section mainly concerns the study of full stability and strong regularity for local optimal solutions to MPPC by which we understand constrained optimization problems of the following type:

$$\text{minimize } \varphi_0(x) \text{ subject to } \Phi(x) = (\varphi_1(x), \ldots, \varphi_m(x)) \in Z, \tag{6.1}$$

where $Z \subset \mathbb{R}^m$ is a convex polyhedron given by

$$Z := \{z \in \mathbb{R}^m |\ \langle a_j, z\rangle \le b_j \text{ for all } j = 1, \ldots, l\} \tag{6.2}$$

with fixed vectors $a_j \in \mathbb{R}^m$ and numbers $b_j \in \mathbb{R}$ as $l \in \mathbb{N}$, and where all the functions φ_i, $i = 0, \ldots, m$, are $\mathcal{C}^2$ around the reference points. Similarly to the discussion at the beginning of section 5, it is easy to observe that the results of this section can be transferred to MPPC models with additional geometric constraints given by $x \in \Omega$ via a convex polyhedron $\Omega \subset \mathbb{R}^n$.

We can clearly rewrite problem (6.1) in extended-real-valued form (5.1) with $\theta = \delta_Z$, or equivalently as (5.2). Note that conventional problems of NLP

$$\begin{cases} \text{minimize } \varphi_0(x) \text{ subject to } \varphi_i(x) \le 0,\ i = 1, \ldots, s, \\ \text{and } \varphi_i(x) = 0, \quad i = s+1, \ldots, m, \end{cases} \tag{6.3}$$

can be written in form (6.1) with the polyhedral set Z in (6.2) generated by $b_j = 0$ and

$$a_j = \begin{cases} e_j & \text{for } j = 1, \ldots, m, \\ -e_{j-m+s} & \text{for } j = m+1, \ldots, 2m-s, \end{cases} \tag{6.4}$$

where each $e_j \in \mathbb{R}^m$ is a unit vector the jth component of which is 1, while the others are 0.

To study full stability of local minimizers in (6.1), consider the two-parametric version $\mathcal{P}(w, v)$ of this problem that can be written as

$$\text{minimize } \varphi_0(x, w) + \delta_Z(\Phi(x, w)) - \langle v, x\rangle \text{ over } x \in \mathbb{R}^n \tag{6.5}$$

with $\Phi(x, w) := (\varphi_1(x, w), \ldots, \varphi_m(x, w))$. Let $\bar{x}$ be a feasible solution to the unperturbed problem $\mathcal{P}(\bar{w}, \bar{v})$ corresponding to the nominal parameter pair $(\bar{w}, \bar{v})$ with $\bar{w} \in \mathbb{R}^d$, $\Phi(\bar{x}, \bar{w}) \in Z$, and $\bar{v} \in \partial_x \varphi(\bar{x}, \bar{w})$, where

$$\varphi(x, w) := \varphi_0(x, w) + \delta_Z(\Phi(x, w)). \tag{6.6}$$

First we address relationships between full stability of local minimizers for MPPC and the corresponding specification of the PSMR property of the partial subdifferential mapping $\partial_x\varphi$ for φ defined in (6.6). Recall [1, Definition 2.86] that the RCQ with respect to x holds at $(\bar{x},\bar{w})$ with $\Phi(\bar{x},\bar{w})\in Z$ in (6.1) if we have the inclusion

$$0\in \operatorname{int}\Big\{\Phi(\bar{x},\bar{w})+\nabla_x\Phi(\bar{x},\bar{w})\mathbb{R}^n-Z\Big\}. \tag{6.7}$$

It is well known that this condition can be equivalently described as

$$N_Z(\Phi(\bar{x},\bar{w}))\cap\ker\nabla_x\Phi(\bar{x},\bar{w})^*=\{0\}, \tag{6.8}$$

which obviously reduces to the MFCQ with respect to x for NLP. The following result establishes the equivalence between full stability of local minimizers for MPCC and the elaborated PSMR condition for such problems under RCQ.

PROPOSITION 6.1 (equivalence between full stability of local minimizers and PSMR for MPPC under RCQ). *Let $\Phi(\bar{x},\bar{w})\in Z$ for MPPC* (6.1)*, and let RCQ* (6.7) *hold at $(\bar{x},\bar{w})$. Then $\bar{x}$ is a fully stable locally optimal solution to $\mathcal{P}(\bar{w},\bar{v})$ in* (6.5) *with $\bar{v}$ satisfying*

$$\bar{v}\in\nabla_x\varphi_0(\bar{x},\bar{w})+\nabla_x\Phi(\bar{x},\bar{w})^*N_Z(\Phi(\bar{x},\bar{w})) \tag{6.9}$$

if and only if $\bar{x}\in M_\nu(\bar{w},\bar{v})$ for some $\nu>0$ and the subgradient mapping $\partial_x\varphi$ for φ from (6.6) *is PSMR at $(\bar{x},\bar{w},\bar{v})$, where the partial inverse mapping* (3.9) *is equivalently represented locally around $(\bar{x},\bar{w},\bar{v})$ as*

$$S_\varphi(w,v)=\Big\{x\in\mathbb{R}^n\,\Big|\,v\in\nabla_x\varphi_0(x,w)+\nabla_x\Phi(x,w)^*N_Z(\Phi(x,w))\Big\}. \tag{6.10}$$

Proof. Note (see, e.g., [23, Exercises 8.14 and 10.26]) that the convexity of Z and the validity of RCQ at $(\bar{x},\bar{w})$ in the equivalent form (6.8) ensure the exact first-order subdifferential chain rule

$$\partial_x\delta_Z(\Phi(\bar{x},\bar{w}))=\nabla_x\Phi(\bar{x},\bar{w})^*N_Z(\Phi(\bar{x},\bar{w})).$$

Combining it with the elementary sum rule in (6.6) gives us the representation

$$\partial_x\varphi(\bar{x},\bar{w})=\nabla_x\varphi_0(\bar{x},\bar{w})+\nabla_x\Phi(\bar{x},\bar{w})^*N_Z(\Phi(\bar{x},\bar{w})), \tag{6.11}$$

which holds also for (x,w) around $(\bar{w},\bar{v})$. This allows us to describe the stationary condition $\bar{v}\in\partial_x\varphi(\bar{x},\bar{w})$ in the form (6.9) and also justifies the equivalent form (6.10) of the partial inverse (3.9) under RCQ.

Now we employ Theorem 3.4 for (6.6). It follows from [6, Proposition 2.2] that condition (3.6) holds automatically under the assumed RCQ, which justifies the sufficiency part of the proposition.

To obtain the converse implication of the theorem, we use again [23, Proposition 2.2], which ensures the parametric continuous prox-regularity of φ at $(\bar{x},\bar{w},\bar{v})$ under RCQ. It remains to employ the partial subdifferential representation (6.11) to complete the proof. ∎

For our further considerations, recall the following well-known formula (see, e.g., [3, Theorem 2E.3]) for the normal cone to the polyhedral set Z at $\Phi(\bar{x},\bar{w})$:

$$N_Z(\Phi(\bar{x},\bar{w}))=\left\{\sum_{j=1}^{l}\mu_ja_j\,\middle|\,\mu_j\ge 0\ \text{ for }\ j\in I(\Phi(\bar{x},\bar{w})),\ \mu_j=0\ \text{ for }\ j\notin I(\Phi(\bar{x},\bar{w}))\right\}, \tag{6.12}$$

where $I(z) := \{i \in \{1,\ldots,l\}|\ \langle a_j, z\rangle = b_j\}$ signifies the set of *active indices* in the polyhedral description (6.2). The associate description of the tangent cone to Z at $\Phi(\bar{x},\bar{w})$ is

$$T_Z(\Phi(\bar{x},\bar{w})) = \Big\{z \in \mathbb{R}^m \Big|\ \langle a_j, z\rangle \le 0 \ \text{ for }\ j \in I(\Phi(\bar{x},\bar{w}))\Big\}. \tag{6.13}$$

Since our analysis is local, we suppose without loss of generality that all the inequality constraints in (6.1) with the polyhedral set Z in (6.2) are active at $(\bar{x},\bar{w})$, i.e., $I(\Phi(\bar{x},\bar{w})) = \{1,\ldots,l\}$.

Now we formulate yet another constraint qualification in MPPC crucial for the subsequent characterization of fully stable locally optimal solutions to (6.1) with the polyhedral constraint set (6.2) and establish its relationship with Robinson's strong regularity.

DEFINITION 6.2 (polyhedral constraint qualification). *Let $\Phi(\bar{x},\bar{w}) \in Z$ for the polyhedral set Z from* (6.2). *We say that the* PCQ *holds at* $(\bar{x},\bar{w})$ *if*

$$\Big\{z \in \mathbb{R}^m \Big|\ \langle a_j, z\rangle = 0 \ \textit{ for all }\ j = 1,\ldots,l\Big\}^{\perp} \cap \ker \nabla_x\Phi(\bar{x},\bar{w})^* = \{0\}. \tag{6.14}$$

It is worth mentioning that by [23, Lemma 6.45] condition (6.14) can be rephrased as

$$\text{span}\Big\{a_j \Big|\ j = 1,\ldots,l\Big\} \cap \ker \nabla_x\Phi(\bar{x},\bar{w})^* = \{0\}. \tag{6.15}$$

Furthermore, it is not hard to check that for NLP (6.3) with the generating vectors a_j given in (6.4) the introduced PCQ reduces, by taking into account that all the inequality constraints are active, to the classical LICQ with respect to the decision variable x: the partial gradients of the constraint functions at the reference point

$$\nabla_x\varphi_1(\bar{x},\bar{w}),\ldots,\nabla_x\varphi_m(\bar{x},\bar{w}) \ \text{ are linearly independent.} \tag{6.16}$$

Of course, LICQ (6.16) ensures the validity of PCQ from Definition 6.2 in the general MPCC setting. We show in what follows that the usage of PCQ allows us to obtain *strictly better* results in comparison with those (also new), which hold under LICQ in the MPPC framework.

As can be seen from the proof of our major characterizations of full stability in MPPC given in Theorem 6.6, PCQ (6.14) is generated by (actually equivalent to) the second-order qualification condition (4.8) ensuring the validity of the exact second-order chain rule of Theorem 4.1 in the MPCC framework. Prior to deriving characterizations of fully stable local minimizers of MPPC under PCQ, let us discuss its relationship with RCQ, nondegenerate points, and its role in describing the KKT variational system associated with MPPC. Following the pattern of [1, Definition 4.70] and taking into account that the polyhedral set Z in (6.2) is $\mathcal{C}^\infty$-reducible to the positive orthant $\mathbb{R}^l_+$ at any $\bar{z} \in Z$ (see [1, Example 3.139]), we say that $\bar{x} \in \mathbb{R}^n$ is a nondegenerate point of the mapping Φ with respect to the parameter $\bar{w}$ if

$$\nabla_x\Phi(\bar{x},\bar{w})\mathbb{R}^n + \text{lin}\Big\{T_C(\Phi(\bar{x},\bar{w}))\Big\} = \mathbb{R}^m, \tag{6.17}$$

where $T_C(\bar{z})$ is the tangent cone at $\bar{z} \in C$ to the set

$$C := \{z \in \mathbb{R}^m|\ \langle a_j, z\rangle = b_j \ \text{ for all }\ a_j \in \mathcal{M}\}, \tag{6.18}$$

$\operatorname{lin}\Omega$ stands for the largest linear subspace contained in Ω, and $\mathcal{M}$ is the maximal set of independent vectors in $\{a_j|\ j=1,\dots,l\}$. In what follows we use the standard Lagrangian function defined by

$$L(x,w,\lambda) := \varphi_0(x,w) + \sum_{i=1}^{m} \lambda_i \varphi_i(x,w) \ \text{ with } \ \lambda = (\lambda_1,\dots,\lambda_m) \in \mathbb{R}^m. \tag{6.19}$$

Proposition 6.3 (relationships for PCQ). *Let $(\bar{x},\bar{w})$ be such that $\Phi(\bar{x},\bar{w}) \in Z$ in the framework of MPPC* (6.1) *with Z from* (6.2). *Then we have the following assertions:*

(i) *PCQ holds at $(\bar{x},\bar{w})$ if and only if $\bar{x}$ is a nondegenerate point of Φ with respect to $\bar{w}$.*

(ii) *For any $\bar{v}$ satisfying* (6.9) *we have that the KKT system*

$$\begin{aligned}\nabla_x L(\bar{x},\bar{w},\bar{\lambda}) &= \nabla_x \varphi_0(\bar{x},\bar{w}) + \sum_{i=1}^{m} \bar{\lambda}_i \nabla_x \varphi_i(\bar{x},\bar{w}) \\ &= \nabla_x \varphi_0(\bar{x},\bar{w}) + \nabla_x \Phi(\bar{x},\bar{w})^* \bar{\lambda} = \bar{v}\end{aligned} \tag{6.20}$$

admits the unique Lagrange multiplier $\bar{\lambda} = (\bar{\lambda}_1,\dots,\bar{\lambda}_m) \in N_Z((\Phi(\bar{x},\bar{w}))$.

(iii) *PCQ* (6.14) *always implies RCQ* (6.7) *at the same point.*

Proof. To justify (i), observe that the tangent cone to C in (6.18) is actually a subspace given by

$$\begin{aligned}T_C(\bar{z}) &= \big\{z \in \mathbb{R}^m \big|\ \langle a_j, z\rangle = 0 \ \text{ for all } \ a_j \in \mathcal{M}\big\} \\ &= \big\{z \in \mathbb{R}^m \big|\ \langle a_j, z\rangle = 0 \ \text{ for all } \ j = 1,\dots,l\big\}.\end{aligned}$$

Then taking the orthogonal complement of both sides in (6.17), we arrive at the equivalent PCQ condition (6.14) and thus show that assertion (i) holds.

To verify (ii), let λ_1 and λ_2 be two Lagrange multipliers satisfying (6.20). This gives us

$$\lambda_1 - \lambda_2 \in \ker \nabla_x \Phi(\bar{x},\bar{w})^*. \tag{6.21}$$

It easily follows from the construction of the set C in (6.18) that

$$a_j \in C^{\perp} \ \text{ for all } \ j = 1,\dots,l. \tag{6.22}$$

By $\lambda_1,\lambda_2 \in N_Z(\Phi(\bar{x},\bar{w}))$ and the normal cone representation (6.12) we get from (6.22) that $\lambda_1 - \lambda_2 \in C^{\perp}$, which tells us that $\lambda_1 = \lambda_2$ due to PCQ (6.14) and thus justifies assertion (ii).

To proceed finally with the proof of (iii), assume that PCQ holds and then verify the validity of RCQ in the equivalent form (6.8). Let $\bar{y}$ be an element in the left-hand side of (6.8). Employing again the normal cone representation (6.12) gives us numbers $\mu_j \ge 0$ for $j = 1,\dots,l$ such that $\bar{y} = \sum_{j=1}^{l} \mu_j a_j$. Then (6.22) ensures that $\bar{y}$ belongs to left-hand side of (6.14). Now using PCQ (6.14) tells us that $\bar{y} = 0$, and thus RCQ (6.7) is satisfied, which completes the proof of the proposition. ∎

Note that PCQ (6.14) can be equivalently written as

$$\operatorname{span}\{N_Z(\Phi(\bar{x},\bar{w}))\} \cap \ker \nabla_x \Phi(\bar{x},\bar{w})^* = \{0\},$$

which makes it easy to observe that PCQ is *robust* with respect to small perturbations (x,w) of $(\bar{x},\bar{w})$ and then allow us to conclude by Proposition 6.3(ii) that for any triples

(x,w,v) sufficiently close to $(\bar{x},\bar{w},\bar{v})$ and satisfying in (6.20) the corresponding set of Lagrange multipliers is a singleton.

DEFINITION 6.4 (PSSOC). *Let $\bar{\lambda}\in\mathbb{R}^m$ be a vector of Lagrange multipliers in MPPC. We say that the PSSOC holds at $(\bar{x},\bar{w},\bar{v},\bar{\lambda})$ with $\bar{v}$ satisfying* (6.9) *if*

$$\langle u,\nabla_{xx}^2 L(\bar{x},\bar{w},\bar{\lambda})u\rangle>0 \ \text{ for all } \ 0\neq u\in S_Z \tag{6.23}$$

via the Lagrangian function (6.19)*, where the subspace S_Z is defined as*

$$S_Z:=\{u\in\mathbb{R}^n|\ \langle a_j,\nabla_x\Phi(\bar{x},\bar{w})u\rangle=0 \ \text{ whenever } \ j\in I_1(\bar{\lambda})\}. \tag{6.24}$$

Our next goal is to characterize full stability of local minimizers for MPPC and the equivalent PSMR property of $\partial_x\varphi$ under RCQ in terms of the corresponding MPPC specification of USOGC from Definition 3.5 formulated as follows: Given $\varphi\colon\mathbb{R}^n\times\mathbb{R}^d\to\overline{\mathbb{R}}$ in (6.6) with $\Phi(\bar{x},\bar{w})\in Z$ and given $\bar{v}\in\partial_x\varphi(\bar{x},\bar{w})$, we say that the MPPC USOGC holds for Φ at $(\bar{x},\bar{w},\bar{v})$ if there exist $\eta>0$ and neighborhoods U of $\bar{x}$, W of $\bar{w}$, and V of $\bar{v}$ such that for any $(w,v)\in W\times V$ there is a point $x_{wv}\in U$ satisfying $v\in\partial_x\varphi(x_{wv},w)$ and

$$\varphi_0(u,w)\geq\varphi_0(x_{wv},w)+\langle v,u-x_{wv}\rangle+\eta\|u-x_{wv}\|^2 \ \text{ for } \ u\in U,\ \Phi(u,w)\in Z. \tag{6.25}$$

THEOREM 6.5 (characterizing full stability in MPPC via USOGC under PCQ). *Let $(\bar{x},\bar{w})$ be such that $\Phi(\bar{x},\bar{w})\in Z$, let PCQ* (6.14) *hold at $(\bar{x},\bar{w})$, and let $\bar{v}$ be taken from* (6.9)*. Then $\bar{x}$ is a fully stable local minimizer of $\mathcal{P}(\bar{w},\bar{v})$ in* (6.5) *if and only if USOGC* (6.25) *is satisfied at $(\bar{x},\bar{w},\bar{v})$.*

Proof. The necessity part of the theorem follows from Theorem 3.7(i) by taking into representation (6.11) valid under PCQ. We now verify the sufficiency part by employing Proposition 6.1 and showing that the assumed PCQ condition ensures in the MPPC framework that (6.10) admits a Lipschitzian single-valued localization around $(\bar{v},\bar{w})$. Indeed, we prove that the mapping $(w,v)\mapsto x_{wv}$ is a Lipschitzian single-valued localization of (6.10) around $(\bar{v},\bar{w})$. To justify this, observe that employing USOGC (6.25) ensures the existence of positive numbers ν and $\eta=\frac{1}{2\sigma}$ for which the second-order growth condition (6.25) holds with $U:=\mathrm{int}\mathbb{B}_\nu(\bar{x})$, $W:=\mathrm{int}\mathbb{B}_\nu(\bar{w})$, and $V:=\mathrm{int}\mathbb{B}_\nu(\bar{v})$. It easily follows from (6.25) that for all $(w,v)\in W\times V$ the point x_{wv} is a unique minimizer of the cost function in (6.5) over $x\in\mathrm{cl}\,U=\mathbb{B}_\nu(\bar{x})$.

As mentioned in the proof of Proposition 6.1, the function φ in (6.6) is parametrically continuously prox-regular at $(\bar{x},\bar{w},\bar{v})$ under RCQ. Furthermore, it follows from [6, Proposition 3.5] that we can suppose without loss of generality that $x_{wv}\in\mathrm{int}\mathbb{B}_{\frac{\nu}{2}}(\bar{x})$. Observe further that a close look at the proof of Theorem 5.1 reveals that implication (3.7) holds under the assumed PCQ condition. Employing now [6, Propositions 3.2 and 4.3] tells us the mapping

$$\Theta(w):=\mathrm{gph}\,\partial\varphi_w=\mathrm{gph}\,\partial_x\varphi(\cdot,w)=\big\{(x,\nabla_x\varphi_0(x,w)+\nabla_x\Phi(x,w)^*\lambda)\big|\ \lambda\in N_Z(\phi(x,w))\big\}$$

has the Aubin/Lipschitz-like property around $(\bar{w},\bar{x},\bar{v})$ due to the Mordukhovich criterion of [23, Theorem 9.40]. This means that there is $k>0$ such that

$$\Theta(w)\cap(U\times V)\subset\Theta(w')+k\|w-w'\|\mathbb{B}_1(0)\times\mathbb{B}_1(0) \ \text{ for all } \ w,w'\in W. \tag{6.26}$$

Let us now show that the mapping $(w,v)\mapsto x_{wv}$ is Lipschitz continuous around $(\bar{w},\bar{v})$. To proceed, take $w_1,w_2\in W$ and $v_1,v_2\in V$ and observe that USOGC (6.25) ensures the existence of unique minimizers $x_{w_1v_1},x_{w_2v_2}\in\mathrm{int}\mathbb{B}_{\frac{\nu}{2}}(\bar{x})$. This implies that

$(x_{w_1v_1}, v_1) \in \Theta(w_1)$ and $(x_{w_2v_2}, v_2) \in \Theta(w_2)$. Since furthermore $(x_{w_2v_2}, v_2) \in U \times V$, (6.26) gives us a pair $(\widetilde{x}, \widetilde{v}) \in \Theta(w_1)$ such that

$$\|x_{w_2v_2} - \widetilde{x}\| + \|v_2 - \widetilde{v}\| \le k\|w_1 - w_2\|. \tag{6.27}$$

It allows us to suppose that $(x_{w_2v_2}, v_2) \in U \times V$; otherwise we can shrink the neighborhoods above. Using this together with USOGC (6.25) gives us that $M_\nu(w_1, \widetilde{v}) = \{\widetilde{x}\}$. Implementing now USOGC (6.25) with the constant $\eta = (2\sigma)^{-1}$, we arrive at

$$\varphi_0(x_{w_1v_1}, w_1) \ge \varphi_0(\widetilde{x}, w_1) + \langle \widetilde{v}, x_{w_1v_1} - \widetilde{x}\rangle + \frac{1}{2\sigma}\|x_{w_1v_1} - \widetilde{x}\|^2,$$
$$\varphi_0(\widetilde{x}, w_1) \ge \varphi_0(x_{w_1v_1}, w_1) + \langle v_1, \widetilde{x} - x_{w_1v_1}\rangle + \frac{1}{2\sigma}\|x_{w_1v_1} - \widetilde{x}\|^2,$$

which implies in turn the estimate

$$\begin{aligned} \|x_{w_1v_1} - \widetilde{x}\| &\le \sigma\|\widetilde{v} - v_1\| \\ &\le \sigma\|\widetilde{v} - v_2\| + \sigma\|v_2 - v_1\| \\ &\le \sigma k\|w_1 - w_2\| + \sigma\|v_2 - v_1\|. \end{aligned} \tag{6.28}$$

Combining finally (6.27) and (6.28), we conclude that

$$\begin{aligned} \|x_{w_1v_1} - x_{w_2v_2}\| &\le \|\widetilde{x} - x_{w_2v_2}\| + \|x_{w_1v_1} - \widetilde{x}\| \\ &\le \beta(\|w_1 - w_2\| + \|v_1 - v_2\|), \end{aligned}$$

where $\beta := k(\sigma + 1)$. This justifies the required Lipschitz continuity of the mapping $(w, v) \mapsto x_{wv}$ around $(\bar{w}, \bar{v})$ and thus completes the proof of the theorem. ∎

The normal cone description (6.12) allows us to find $\{\bar{\mu}_j |\ j = 1, \ldots, l\}$ such that

$$\bar{\lambda} = \sum_{j=1}^{l} \bar{\mu}_j a_j \ \text{ with } \ \bar{\mu}_j \ge 0 \ \text{ as } \ j = 1, \ldots, l. \tag{6.29}$$

Based on (6.29), consider the two index sets corresponding to the vector $\bar{\lambda}$ in (6.29),

$$I_1(\bar{\lambda}) := \Big\{ j \in \{1, \ldots, l\} \Big| \ \bar{\mu}_j > 0 \Big\} \ \text{ and } \ I_2(\bar{\lambda}) := \Big\{ j \in \{1, \ldots, l\} \Big| \ \bar{\mu}_j = 0 \Big\}, \tag{6.30}$$

and introduce the following polyhedral second-order optimality condition for MPPC.

Note that in the classical NLP case (6.3) corresponding to (6.4) the PSSOC from Definition 6.4 reduces to the partial version of the well-recognized in NLP *strong second-order sufficient optimality condition* (SSOSC) introduced by Robinson [20], i.e.,

$$\begin{cases} \langle u, \nabla^2_{xx} L(\bar{x}, \bar{w}, \bar{\lambda}) u\rangle > 0 \ \text{ whenever } \ u \in \mathbb{R}^n \ \text{ such that } \ \langle \nabla_x \varphi_i(\bar{x}, \bar{w}), u\rangle = 0 \\ \text{for all } \ i = s+1, \ldots, m \ \text{ and } \ i \in \{1, \ldots, s\} \ \text{ with } \ \bar{\lambda}_i > 0. \end{cases} \tag{6.31}$$

The next major result provides a complete characterization of fully stable local minimizers for problem $\mathcal{P}(\bar{w}, \bar{v})$ in (6.5) under PCQ via PSSOC from Definition 6.4 expressed entirely in terms of the problem data at the reference solution point.

THEOREM 6.6 (characterization of full stability in MPPC via PSSOC under PCQ). *Let $\bar{x}$ be a feasible solution to problem $\mathcal{P}(\bar{w}, \bar{v})$ in* (6.5) *for some $\bar{w} \in \mathbb{R}^d$ and $\bar{v}$ from* (6.9). *Assume that PCQ* (6.14) *is satisfied at $(\bar{x}, \bar{w})$. Then we have the following assertions:*

(i) *If $\bar{x}$ is a fully stable locally optimal solution to $\mathcal{P}(\bar{w},\bar{v})$, then PSSOC from Definition* 6.4 *holds at $(\bar{x},\bar{w},\bar{v},\bar{\lambda})$ with the unique multiplier vector $\bar{\lambda}\in N_Z(\Phi(\bar{x},\bar{w}))$ satisfying* (6.20).

(ii) *Conversely, the validity of PSSOC at $(\bar{x},\bar{w},\bar{v},\bar{\lambda})$ with $\bar{\lambda}\in N_Z(\Phi(\bar{x},\bar{w}))$ satisfying* (6.20) *ensures that $\bar{x}$ is a fully stable locally optimal solution to $\mathcal{P}(\bar{w},\bar{v})$ in* (6.5).

Proof. Let $(\bar{x},\bar{w})$ be such that $\Phi(\bar{x},\bar{w})\in Z$. First we show that PCQ (6.14) is equivalent to the second-order qualification condition (4.8) in the framework of MPPC (6.1). Represent problem $\mathcal{P}(\bar{w},\bar{v})$ in the composite form (6.5) with $\theta=\delta_Z$ and observe by the piecewise linearity of δ_Z that we are in the setting of Theorem 5.2(a), where the second-order qualification condition (4.8) is written as

$$\partial^2\delta_Z(\bar{z},\bar{\lambda})(0)\cap\ker\nabla_x\Phi(\bar{x},\bar{w})^*=\{0\}, \tag{6.32}$$

$\bar{z}=\Phi(\bar{x},\bar{w})$, and $\bar{\lambda}\in N_Z(\Phi(\bar{x},\bar{w}))$ is the unique vector satisfying (6.20); this follows from Proposition 6.3(ii). Consider now the critical cone

$$K:=T_Z(\bar{z})\cap\bar{\lambda}^{\perp}=\Big\{z\in T_Z(\bar{z})\Big|\ \langle\bar{\lambda},z\rangle=0\Big\} \tag{6.33}$$

to Z at $\bar{z}$ generated by the tangent cone (6.13) and the Lagrange multiplier $\bar{\lambda}$. By the proof of [2, Theorem 2] (see also [18, Proposition 4.4]) we have

$$q\in\partial^2\delta_Z(\bar{z},\bar{\lambda})(0)\Longleftrightarrow\begin{cases}\text{there exist closed faces}\\ K_1 \text{ and } K_2 \text{ of } K \text{ with } K_1\subset K_2,\\ 0\in K_1-K_2,\ q\in(K_2-K_1)^*,\end{cases} \tag{6.34}$$

where the closed face $C\subset K$ of the polyhedral cone (6.33) is defined by

$$C:=\{z\in K|\ \langle z,y\rangle=0\}\ \text{ for some }\ y\in K^*$$

via the polar cone K^* in question. Picking any $z\in K$ and using (6.13) give us

$$\langle a_j,z\rangle\le 0\ \text{ for all }\ j=1,\ldots,l,$$

which implies in turn by formula (6.29) that $\sum_{j=1}^{l}\bar{\mu}_j\langle a_j,z\rangle=0$. This provides therefore the convenient critical cone representation

$$K=\Big\{z\in\mathbb{R}^m\Big|\ \langle a_j,z\rangle=0\ \text{ for }\ j\in I_1(\bar{\lambda})\ \text{ and }\ \langle a_j,z\rangle\le 0\ \text{ for }\ j\in I_2(\bar{\lambda})\Big\} \tag{6.35}$$

via the index sets (6.30). It follows directly from representation (6.35) that

$$K\cap(-K)=\{z\in\mathbb{R}^m|\ \langle a_j,z\rangle=0\ \text{ for all }\ j=1,\ldots,l\}, \tag{6.36}$$

which readily implies the polar representation

$$(K_2-K_1)^*=(K\cap(-K))^*=(K\cap(-K))^{\perp}\ \text{ with }\ K_1=K_2=K\cap(-K). \tag{6.37}$$

By formula (6.34) for $\partial^2\delta_Z(\bar{z},\bar{\lambda})$ with $u=0$ we have the inclusion

$$\partial^2\delta_Z(\bar{z},\bar{\lambda})(0)\supset(K\cap(-K))^{\perp}.$$

To get further the opposite inclusion $\subset$ therein, take any $q\in\partial^2\delta_Z(\bar{z},\bar{\lambda})(0)$ and by representation (6.34) find some closed faces K_1 and K_2 of the critical cone K such that $K_1 \subset K_2$, $0 \in K_1 - K_2$, and also $q \in (K_2 - K_1)^*$. Since $K \cap (-K)$ is the smallest closed face of the critical cone K, we get that $K \cap (-K) \subset K_1$, $K \cap (-K) \subset K_2$, and hence

$$[K \cap (-K)] - [K \cap (-K)] \subset K_2 - K_1,$$

which shows us together with (6.37) that

$$q \in (K_2 - K_1)^* \subset (K \cap (-K))^{\perp} \text{ and thus } \partial^2 \delta_Z(\bar{z}, \bar{\lambda})(0) \subset (K \cap (-K))^{\perp}.$$

Combining this with the inclusion $\supset$ proved above ensures the equality

$$\partial^2 \delta_Z(\bar{z}, \bar{\lambda})(0) = (K \cap (-K))^{\perp}. \tag{6.38}$$

Substituting it into (6.32), we arrive at the polyhedral constraint qualification (6.14), which is thus equivalent to the second-order qualification condition (4.8) in the MPPC framework. Theorem 5.2(i) tells us that condition (5.6) is necessary and sufficient for full stability of the given local minimizer $\bar{x}$ in $\mathcal{P}(\bar{w}, \bar{v})$, where the mapping $\mathcal{T}(\bar{x}, \bar{w}, \bar{v})$ is defined in Theorem 5.1.

After these preparations, we proceed with the justification of assertion (i) of the theorem. Since a fully stable local minimizer for $\mathcal{P}(\bar{w}, \bar{v})$ is obviously a usual local minimizer for this problem, it follows from the first-order necessary optimality conditions for $\mathcal{P}(\bar{w}, \bar{v})$ under PCQ (6.14) that there is a unique vector $\bar{\lambda} \in N_Z((\Phi(\bar{x}, \bar{w}))$ satisfying (6.20). It is clear that all the assumptions of Theorem 5.2(i) are satisfied in our MPPC setting under the imposed PCQ.

Consider the set-valued mapping $\mathcal{T}(\bar{x}, \bar{w}, \bar{v}) = (\mathcal{T}_1(\bar{x}, \bar{w}, \bar{v}), \mathcal{T}_2(\bar{x}, \bar{w}, \bar{v})) \colon \mathbb{R}^n \rightrightarrows \mathbb{R}^{2n}$ given by

$$\begin{cases} \mathcal{T}_1(\bar{x}, \bar{w}, \bar{v})(u) = \nabla^2_{xx} L(\bar{x}, \bar{w}, \bar{\lambda})u + \nabla_x \Phi(\bar{x}, \bar{w})^* \partial^2 \delta_Z(\bar{z}, \bar{\lambda})(\nabla_x \Phi(\bar{x}, \bar{w})u), \\ \mathcal{T}_2(\bar{x}, \bar{w}, \bar{v})(u) = \nabla^2_{xw} L(\bar{x}, \bar{w}, \bar{\lambda})u + \nabla_w \Phi(\bar{x}, \bar{w})^* \partial^2 \delta_Z(\bar{z}, \bar{\lambda})(\nabla_x \Phi(\bar{x}, \bar{w})u) \end{cases} \tag{6.39}$$

for all $u \in \mathbb{R}^n$, where $\bar{z} := \Phi(\bar{x}, \bar{w})$. Theorem 5.2(i) tells us that condition (5.6) holds for the mapping $\mathcal{T}(\bar{x}, \bar{w}, \bar{v})$ in (6.39). This means that

$$\langle p, u \rangle > 0 \text{ whenever } p \in \nabla^2_{xx} L(\bar{x}, \bar{w}, \bar{\lambda})u + \nabla_x \Phi(\bar{x}, \bar{w})^* \delta^2_Z(\bar{z}, \bar{\lambda})(\nabla_x \Phi(\bar{x}, \bar{w})u), \ u \neq 0,$$

which is equivalent to the relationship

$$\langle u, \nabla^2_{xx} L(\bar{x}, \bar{w}, \bar{\lambda})u \rangle + \langle q, \nabla_x \Phi(\bar{x}, \bar{w})u \rangle > 0 \tag{6.40}$$

for all $q \in \partial^2 \delta_Z(\bar{z}, \bar{\lambda})(\nabla_x \Phi(\bar{x}, \bar{w})u)$ with $u \neq 0$. To complete the proof of (i), we need to show that (6.40) implies the validity of PSSOC at $(\bar{x}, \bar{w}, \bar{v}, \bar{\lambda})$, which requires calculating the second-order subdifferential $\partial^2 \delta_Z(\bar{z}, \bar{\lambda})(\nabla_x \Phi(\bar{x}, \bar{w})u)$. Consider again the critical cone (6.33). Similarly to (6.34) we have

(6.41)

$$q \in \partial^2 \delta_Z(\bar{z}, \bar{\lambda})(\nabla_x \Phi(\bar{x}, \bar{w})u) \Longleftrightarrow \begin{cases} \text{there exist closed faces} \\ K_1 \text{ and } K_2 \text{ of } K \text{ with } K_1 \subset K_2, \\ \nabla_x \Phi(\bar{x}, \bar{z})u \in K_1 - K_2, \ q \in (K_2 - K_1)^*. \end{cases}$$

Taking two closed faces K_1 and K_2 of K and using (6.35) ensure that

$$\langle a_j, z \rangle = 0 \text{ for all } z \in K_1 - K_2 \text{ and } j \in I_1(\bar{\lambda}). \tag{6.42}$$

Now fix $0 \neq u \in S_Z$ and pick any $q \in \partial^2\delta_Z(\bar{z},\bar{\lambda})(\nabla_x\Phi(\bar{x},\bar{w})u)$ generated by the vector u under consideration. Then by (6.41) we find closed faces $K_1 \subset K_2$ of K such that

$$\nabla_x\Phi(\bar{x},\bar{w})u \in K_1 - K_2 \quad\text{and}\quad q \in (K_2 - K_1)^*,$$

which yields by (6.42) the relationship

$$\langle a_j, \nabla_x\Phi(\bar{x},\bar{w})u\rangle = 0 \quad\text{for}\quad j \in I_1(\bar{\lambda}). \tag{6.43}$$

Define next the vector $\widetilde{q} \in \mathbb{R}^m$ by the summation

$$\widetilde{q} := \sum_{j \in I_1(\bar{\lambda})} a_j$$

and observe by (6.42) that $\widetilde{q} \in (K_2 - K_1)^*$ whenever K_1 and K_2 are from (6.34). It yields

$$\widetilde{q} \in \partial^2\delta_Z(\bar{z},\bar{\lambda})(\nabla_x\Phi(\bar{x},\bar{w})u) \quad\text{and}\quad \langle\widetilde{q}, \nabla_x\Phi(\bar{x},\bar{w})u\rangle = 0.$$

Letting now $q := \widetilde{q}$ in (6.40) gives us that $\langle u, \nabla^2_{xx}L(\bar{x},\bar{w},\bar{\lambda})u)\rangle > 0$. This verifies PSSOC at $(\bar{x},\bar{w},\bar{v},\bar{\lambda})$ from Definition 6.4 and completes the proof of assertion (i).

To justify the converse assertion (ii), assume that PSSOC holds at $(\bar{x},\bar{w},\bar{v},\bar{\lambda})$ with the multiplier $\bar{\lambda} \in N_Z(\Phi(\bar{x},\bar{w}))$ satisfying (6.20) under the validity of PCQ (6.14) at $(\bar{x},\bar{w})$. To show that $\bar{x}$ is a fully stable locally optimal solution to problem $\mathcal{P}(\bar{w},\bar{v})$ in (6.5), we need to check the validity of the second-order condition (5.6) for the mapping $\mathcal{T}(\bar{x},\bar{w},\bar{v})$ defined in (6.39). To proceed, take arbitrary vectors $u \neq 0$ and $q \in \mathcal{Q} := \partial^2\delta_Z(\bar{z},\bar{\lambda})(\nabla_x\Phi(\bar{x},\bar{w})u)$. Employing again (6.41) tells us that there are two closed faces $K_1 \subset K_2$ of the critical cone K such that

$$\nabla_x\Phi(\bar{x},\bar{w})u \in K_1 - K_2 \quad\text{and}\quad q \in (K_2 - K_1)^*,$$

which ensures the inequality

$$\langle q, \nabla_x\Phi(\bar{x},\bar{w})\rangle \geq 0. \tag{6.44}$$

It follows from (6.43) that $u \in S_Z$ in (6.24). Finally using (6.44) together with (6.23) yields

$$\begin{aligned}\langle u, \nabla^2_{xx}L(\bar{x},\bar{w},\bar{\lambda})u\rangle + \langle q, \nabla_x\Phi(\bar{x},\bar{w})u\rangle &\geq \langle u, \nabla^2_{xx}L(\bar{x},\bar{w},\bar{\lambda})u\rangle + 0\\ &= \langle u, \nabla^2_{xx}L(\bar{x},\bar{w},\bar{\lambda})u\rangle > 0,\end{aligned}$$

which implies (6.40) and shows therefore that condition (5.6) holds for the data of (6.5). Thus we get that $\bar{x}$ is a fully stable local minimizer of $\mathcal{P}(\bar{w},\bar{v})$ and we complete the proof of the theorem. ◻

The following corollary of Theorem 6.6 is a new result that provides a characterization of tilt stability in the general framework of MPPC (6.1).

COROLLARY 6.7 (characterization of tilt stability in MPPC via PSSOC under PCQ). *Let $\bar{x}$ be a feasible solution to problem $\mathcal{P}(\bar{v})$ in* (6.5) *with $\varphi_i = \varphi_i(x)$ for all $i = 0,\ldots,m$, and let $\bar{v}$ satisfy* (6.9)*. Assume that PCQ* (6.14) *is satisfied at this point. Then we have the following assertions:*

(i) *If $\bar{x}$ is a tilt-stable local minimizer of $\mathcal{P}(\bar{v})$, then PSSOC from Definition* 6.4 *holds at $(\bar{x},\bar{v},\bar{\lambda})$, where $\Phi = \Phi(\bar{x})$ and $L = L(\bar{x},\bar{\lambda})$ with the unique multiplier $\bar{\lambda} \in N_Z(\Phi(\bar{x}))$ that is determined from the relationships in* (6.20).

(ii) *Conversely, the validity of PSSOC at* $(\bar{x}, \bar{v}, \bar{\lambda})$ *with* $\bar{\lambda} \in N_Z(\Phi(\bar{x}))$ *satisfying* (6.20) *ensures that* $\bar{x}$ *is a tilt-stable local minimizer of the unperturbed problem* $\mathcal{P}(\bar{v})$.

Proof. The proof immediately follows from Theorem 6.6 and the definition of tilt stability. ∎

In the case of the conventional NLP (6.3) corresponding to the choice of a_j in (6.4) the characterization of tilt stability in Corollary 6.7 goes back to [16, Theorem 5.2].

The second corollary of Theorem 6.6 presented below gives a complete characterization, entirely in terms of the problem data, of full stability of locally optimal solutions to nonlinear programs described by $\mathcal{C}^2$ functions. This is a new result in classical NLP.

COROLLARY 6.8 (characterization of full stability in NLP via partial SSOSC under LICQ). *Let* $\bar{x}$ *be a feasible solution to problem* $\mathcal{P}(\bar{w}, \bar{v})$ *corresponding to NLP in* (6.3) *with some vectors* $\bar{w} \in \mathbb{R}^d$ *and* $\bar{v} \in \mathbb{R}^n$ *from* (6.9). *Assume that LICQ* (6.16) *holds at* $(\bar{x}, \bar{w})$. *Then* $\bar{x}$ *is a fully stable local minimizer for* $\mathcal{P}(\bar{w}, \bar{v})$ *if and only if the partial SSOSC* (6.31) *holds at* $(\bar{x}, \bar{w}, \bar{v}, \bar{\lambda})$ *with the unique Lagrange multiplier* $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_m) \in \mathbb{R}^s_+ \times \mathbb{R}^{m-s}$ *satisfying* (6.20).

Proof. The proof follows directly from Theorem 6.6 with Z specified in (6.4) due to the facts discussed above that PCQ reduces to LICQ and PSSOC reduces to SSOSC in NLP models. ∎

As mentioned above, the PCQ condition reduces to LICQ in the case of NLP, in fact, even if

$$\operatorname{span}\left\{a_j \middle| \, j = 1, \ldots, l\right\} = \mathbb{R}^n.$$

Furthermore, since LICQ implies PCQ in the general MPPC framework, the results of Theorem 6.6 and Corollary 6.7 definitely hold for full and tilt stability in MPPC with the replacement of PCQ by LICQ. However, the following example shows that in other MPPC settings the imposed PCQ may be satisfied and thus ensures the required stability while LICQ fails. This occurs even in the case of tilt stability.

Example 6.9 (tilt stability for MPPC without LICQ). It is sufficient to present an example of the constraint system $\Phi(x) \in Z$ in (6.1) with a convex polyhedron Z of type (6.2) for which the qualification condition (6.14) is satisfied at some $\bar{x}$ while the Jacobian matrix $\nabla\Phi(\bar{x})$ is not of full rank. Then it is easy to find a cost function $\varphi_0 = \varphi(x)$ such that $\bar{x}$ is a local minimizer for the corresponding MPPC (6.1). To proceed, construct the mapping $\Phi = (\varphi_1, \varphi_2, \varphi_3)\colon \mathbb{R}^3 \to \mathbb{R}^3$ with $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ by

$$\varphi_1(x) := x_1 + x_2, \quad \varphi_2(x) := x_1 + x_3, \quad \varphi_3(x) := x_1^2 + x_2^2 + x_3^2$$

and consider the convex polyhedron $Z \subset \mathbb{R}^3$ in (6.2) formed by

$$a_1 = (1, 1, 0) \text{ and } a_2 = (1, 0, 1) \text{ with } b_1 = b_2 = 0.$$

It follows from the proof of Theorem 6.6 that

$$\dim(K \cap (-K)) = \dim\{z \in \mathbb{R}^3 | \, \langle a_j, z\rangle = 0 \text{ for } i = 1, 2\} = 1.$$

Since a_1 and a_2 are linearly independent in $\mathbb{R}^3$ and $\dim(K \cap (-K))^{\perp} = 2$, we get that

$$\partial^2 \delta_Z(\Phi(0), \bar{\lambda})(0) = (K \cap (-K))^{\perp} = \operatorname{span}\{a_1, a_2\} = \operatorname{span}\{(1, 1, 0), (1, 0, 1)\}$$

for each $\bar{\lambda} \in N_Z(0,0,0)$. On the other hand, direct calculations show that

$$\nabla\Phi(0,0,0)^* = \begin{pmatrix} \nabla\varphi_1(0,0,0) \\ \nabla\varphi_2(0,0,0) \\ \nabla\varphi_3(0,0,0) \end{pmatrix}^* = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

which yields that $\mathrm{Im}\nabla\Phi(0,0,0)^* = \mathrm{span}\{(1,0,0),(0,1,0)\}$ and hence $\ker\nabla\Phi(0,0,0)^* = \mathrm{span}\{(0,0,1)\}$. Thus we have the relationships

$$\begin{aligned}\partial^2\delta_Z(\Phi(0),\bar{\lambda})(0) \cap \ker\nabla\Phi(0,0,0)^* &= \mathrm{span}\{(1,1,0),(1,0,1)\} \cap \mathrm{span}\{(0,0,1)\} \\ &= \{(0,0,0)\}.\end{aligned}$$

Therefore PCQ (6.14) holds while $\mathrm{rank}\nabla\Phi(0,0,0) = 2$, and hence LICQ (6.16) is not satisfied.

Finally in this section, we establish relationships between full stability of local minimizers for MPPC and Robinson's notion of strong regularity for the associated parametric KKT system (6.45) involving Lagrange multipliers. Following the idea of [20], we say that the canonically perturbed KKT system

$$-v + \nabla_x L(x,w,\lambda) = 0, \quad \lambda \in N_Z(\Phi(x,w)) \tag{6.45}$$

is strongly regular at $(\bar{w},\bar{v},\bar{x},\lambda)$ if its solution map $S_{KKT}\colon (w,v) \mapsto (x,\lambda)$ is single-valued and Lipschitz continuous when (w,v,x,λ) varies around $(\bar{w},\bar{v},\bar{x},\bar{\lambda})$; see [1, 2, 9] for more details.

The equivalence between tilt stability and strong regularity in NLP first derived in [16, Corollary 5.3] and then in [14, Corollary 3.7] with different proofs. In what follows we extend this equivalence to full stability of general MPPC (and hence NLP) models by replacing LICQ by PCQ in the MPPC setting.

Theorem 6.10 (equivalence between full stability and strong regularity for MPPC under PCQ). *Let $\Phi(\bar{x},\bar{w}) \in Z$. Then $\bar{x}$ is a fully stable locally optimal solution to problem $\mathcal{P}(\bar{w},\bar{v})$ from* (6.5) *with $\bar{v}$ satisfying* (6.9) *and PCQ* (6.14) *holds at $(\bar{x},\bar{w})$ if and only if $\bar{x} \in M_\nu(\bar{w},\bar{v})$ for some $\nu > 0$ and the KKT system* (6.45) *is strongly regular at $(\bar{w},\bar{v},\bar{x},\bar{\lambda})$, where $\bar{\lambda}$ is the unique solution to* (6.45) *corresponding to the triple $(\bar{x},\bar{w},\bar{v})$.*

Proof. Assume first that the KKT system (6.45) is strongly regular at $(\bar{w},\bar{v},\bar{x},\bar{\lambda})$. It follows from the necessity part of [1, Theorem 5.24] that the nondegeneracy condition (6.17) is satisfied. Employing this together with Proposition 6.3(i) gives us PCQ (6.14). Let us now show that the partial subdifferential mapping $\partial_x\varphi$ for φ in (6.6) is PSMR at $(\bar{x},\bar{w},\bar{v})$. Then, by taking into account that PCQ implies RCQ (6.7) due to Proposition 6.3(iii), we can conclude from Proposition 6.1 that $\bar{x}$ is a fully stable local minimizer of the unperturbed problem $\mathcal{P}(\bar{w},\bar{v})$ in (6.5).

To proceed, find by the assumed strong regularity of (6.45) a number $\nu > 0$ such that for all $(w,v) \in \mathrm{int}\mathbb{B}_\nu(\bar{w}) \times \mathrm{int}\mathbb{B}_\nu(\bar{v})$ the mapping $S_{KKT} : (w,v) \mapsto (x_{wv},\lambda_{wv})$ is locally single-valued and Lipschitz continuous with constant $\ell > 0$. Consider the neighborhoods $U := \mathrm{int}\mathbb{B}_{2\ell\nu}(\bar{x})$, $W := \mathrm{int}\mathbb{B}_\nu(\bar{w})$, and $V := \mathrm{int}\mathbb{B}_\nu(\bar{v})$ in Definition 3.3 of PSMR for φ in (6.6). It follows from the aforementioned properties of S_{KKT} that the localization of the partial inverse S_φ in (6.10) relative to $W \times V$ and U is single-valued and Lipschitz continuous. Hence the mapping $\partial_x\varphi$ from (6.11) is PSMR at $(\bar{x},\bar{w},\bar{v})$, which therefore justifies the "if" part of the theorem.

To prove the converse implication of the theorem, let $\bar{x}$ be a fully stable locally optimal solution to $\mathcal{P}(\bar{w},\bar{v})$ in (6.5). It follows from Proposition 6.3(ii) that the assumed PCQ (6.14) gives the single-valuedness of the mapping S_{KKT} on some neighborhoods $W \times V$ of $(\bar{w}, \bar{v})$, and so it remains to justify the Lipschitz continuity of $S_{KKT}\colon (w,v) \mapsto (x_{wv}, \lambda_{wv})$. In fact it is shown in the proof of Theorem 6.5 that the mapping $(w,v) \mapsto x_{wv}$ is Lipschitz continuous around $(\bar{w}, \bar{v})$ with constant $\ell > 0$. Let us now check that the mapping $(w,v) \mapsto \lambda_{wv}$ is Lipschitz continuous around $(\bar{w}, \bar{v})$ as well. Since RCQ (6.7) holds due to PCQ (6.14), the Lagrange multipliers λ_{wv} in (6.45) are uniformly bounded (w,v) sufficiently close to $(\bar{w}, \bar{v})$. Without loss of generality suppose that there is $\rho < \infty$ such that

$$\|\lambda_{wv}\| \le \rho \ \text{ for all } \ (w,v) \in W \times V.$$

Take arbitrary vectors $w_1, w_2 \in W$ and $v_1, v_2 \in V$ and suppose that $\ell > 0$ is the Lipschitz constant for the mapping $\nabla_x \varphi$ and $\nabla_x \Phi$ as well. By (6.45) we have the equality

$$\begin{aligned} \nabla_x \Phi(x_{w_2v_2}, w_2)^*(\lambda_{w_2v_2} - \lambda_{w_1v_1}) = {} & \Big(\nabla_x \Phi(x_{w_1v_1}, w_1) - \nabla_x \Phi(x_{w_2v_2}, w_2)\Big)^* \lambda_{w_1v_1} \\ & + \nabla_x \varphi_0(x_{w_1v_1}, w_1) - \nabla_x \varphi_0(x_{w_2v_2}, w_2) + v_2 - v_1. \end{aligned} \tag{6.46}$$

Remember from the proof of Theorem 4.1 that there is a linear isometry A from $\mathbb{R}^m$ into $\mathbb{R}^s \times \mathbb{R}^{m-s}$ under which $A^*L = \mathbb{R}^s \times \{0\}$ with $L = S(\Phi(\bar{x}, \bar{w}))$ and $s = \dim L$, where $S(\Phi(\bar{x}, \bar{w}))$ is the subspace parallel to $\operatorname{aff} N_Z(\Phi(\bar{x}, \bar{w}))$. Consider the composite representation $\delta_Z \circ \Phi = \vartheta \circ P$ with $P := A^{-1}\Phi$ and $\vartheta := \delta_Z A$. Similarly to (4.10) we get the calculations

$$\nabla_x P(x,w) = A^{-1} \nabla_x \Phi(x,w) \ \text{ and } \ \partial\vartheta(z') = A^* N_Z(z) \ \text{ with } \ Az' = z. \tag{6.47}$$

Employing (6.47) gives us the inclusions

$$\zeta_1 = (\zeta_{11}, \ldots, \zeta_{1m}) \in \partial\vartheta(z'_1) \ \text{ and } \ \zeta_2 = (\zeta_{21}, \ldots, \zeta_{2m}) \in \partial\vartheta(z'_2) \tag{6.48}$$

with $Az'_1 = \Phi(x_{w_1v_1}, w_1)$ and $Az'_2 = \Phi(x_{w_2v_2}, w_2)$ such that

$$\zeta_1 = A^* \lambda_{w_1v_1} \quad \text{and} \quad \zeta_2 = A^* \lambda_{w_2v_2}. \tag{6.49}$$

Using (6.46) together with (6.49) leads us to the equality

$$\begin{aligned} \nabla_x P(x_{w_2v_2}, w_2)^*(\zeta_2 - \zeta_1) = {} & \Big(\nabla_x \Phi(x_{w_1v_1}, w_1) - \nabla_x \Phi(x_{w_2v_2}, w_2)\Big)^* \lambda_{w_1v_1} \\ & + \nabla_x \varphi_0(x_{w_1v_1}, w_1) - \nabla_x \varphi_0(x_{w_2v_2}, w_2) + v_2 - v_1. \end{aligned} \tag{6.50}$$

By the subdifferential representation (4.12) we have

$$\begin{aligned} \nabla_x P(x_{w_2v_2}, w_2)^*(\zeta_2 - \zeta_1) &= \sum_{i=1}^{s} \nabla_x P(x_{w_2v_2}, w_2)^*(\zeta_{2i} - \zeta_{1i}) \\ &= \nabla_x P_0(x_{w_2v_2}, w_2)^*(\zeta'_2 - \zeta'_1), \end{aligned} \tag{6.51}$$

where P_0 is defined as in the proof of Theorem 4.1 and where $\zeta'_1 = (\zeta_{11}, \ldots, \zeta_{1s})$ and $\zeta'_2 = (\zeta_{21}, \ldots, \zeta_{2s})$.

It follows from the proof of Theorem 4.1 that $\operatorname{rank} \nabla_x P_0(\bar{x}, \bar{w}) = s$. Let us show now that we can always reduce the situation to the square case of $s = n$. Indeed, if $s < n$ we introduce a linear transformation $\widetilde{P}\colon \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^{n-s}$ such that the mapping

$$\overline{P}(x,w) := (P_0(x,w), \widetilde{P}(x,w))\colon \mathbb{R}^n \times \mathbb{R}^d \longrightarrow \mathbb{R}^n$$

has full rank. This can be done, e.g., by choosing an orthogonal basis $\{b_1,\ldots,b_{n-s}\}$ in the $(n-s)$-dimensional space $\{u \in R^n|\ \nabla_x P_0(\bar{x},\bar{w})u = 0\}$ and then letting $\widetilde{P}(x,w) := (\langle b_1, x\rangle,\ldots,\langle b_{n-s}, x\rangle)$. Furthermore, define $\overline{\vartheta}(z,q) := \vartheta(z)$ for all $z \in \mathbb{R}^m$ and $q \in \mathbb{R}^{n-m}$ and let $z := (P_0(x,w), \langle b_1, x\rangle, \ldots, \langle b_{m-s}, x\rangle)$. Employing the elementary subdifferential chain rule gives us

$$\begin{aligned}\partial_x(\overline{\vartheta}\circ\overline{P})(x,w) &= \nabla_x\overline{P}(x,w)^*\partial\overline{\vartheta}(\overline{P}(x,w))\\ &= (\nabla_x P_0(x,w)^*, b_1,\ldots,b_{n-s})(\partial\vartheta(z), 0^{n-m})\\ &= (\nabla_x P_0(x,w)^*, b_1,\ldots,b_{m-s})\partial\vartheta(z).\end{aligned} \tag{6.52}$$

By the proof of Theorem 4.1 we have $\partial\vartheta(z) \subset \mathbb{R}^s \times \{0\}^{m-s}$, which allows us to represent $\zeta_1 = (\zeta_1', 0^{m-s})$ and $\zeta_2 = (\zeta_2', 0^{m-s})$. Using this together with (6.52) and (6.51) ensures the existence of $\zeta_1'' \in \partial\overline{\vartheta}(z_1'')$ and $\zeta_2'' \in \partial\overline{\vartheta}(z_2'')$ such that $z_1'' = \overline{P}(x_{w_1v_1}, w_1)$, $z_2'' = \overline{P}(x_{w_2v_2}, w_2)$, and

$$\begin{aligned}\nabla_x P_0(x_{w_2v_2},w_2)^*(\zeta_2'-\zeta_1') &= (\nabla_x P_0(x_{w_2v_2},w_2)^*, b_1,\ldots,b_{m-s})(\zeta_2-\zeta_1)\\ &= \nabla_x\overline{P}(x_{w_2v_2},w_2)^*(\zeta_2''-\zeta_1''),\end{aligned}$$

and so we get $\zeta_1'' = (\zeta_1, 0^{n-m})$ and $\zeta_2'' = (\zeta_2, 0^{n-m})$. Substituting (6.51) into (6.50) and invoking the classical inverse function theorem for the mapping $\overline{P}$ invertible in x give us the estimates

$$\begin{aligned}\|\zeta_2''-\zeta_1''\| &\le \Big\|\Big(\nabla_x\overline{P}(x_{w_2v_2},w_2)^*\Big)^{-1}\Big\|\Big(\|\nabla_x\Phi(x_{w_1v_1},w_1)-\nabla_x\Phi(x_{w_2v_2},w_2)\|\cdot\|\lambda_{w_1v_1}\|\\ &\quad+\|\nabla_x\varphi_0(x_{w_1v_1},w_1)-\nabla_x\varphi_0(x_{w_2v_2},w_2)\|+\|v_2-v_1\|\Big)\\ &\le \gamma\Big[\rho\ell\Big(\|x_{w_2v_2}-x_{w_1v_1}\|+\|w_2-w_1\|\Big)\\ &\quad+\ell\Big(\|x_{w_2v_2}-x_{w_1v_1}\|+\|w_2-w_1\|\Big)+\|v_2-v_1\|\Big],\end{aligned} \tag{6.53}$$

where $\gamma > 0$ is the upper bound of $\|(\nabla_x\overline{P}(x,w)^*)^{-1}\|$ for all the pairs (x,w) sufficiently close to $(\bar{x},\bar{w})$. Also the equalities in (6.49) imply the relationship

$$\|\lambda_{w_2v_2}-\lambda_{w_1v_1}\| \le \|(A^*)^{-1}\|\cdot\|\zeta_2-\zeta_1\| = \|(A^*)^{-1}\|\cdot\|\zeta_2''-\zeta_1''\|. \tag{6.54}$$

Finally taking into account the local Lipschitz continuity of the mapping $(w,v) \mapsto x_{wv}$ together with the estimates in (6.53) and (6.54), we conclude that the mapping $(w,v) \mapsto \lambda_{wv}$ is Lipschitz continuous around $(\bar{w},\bar{v})$ as well. This completes the proof of the theorem. ◻

The equivalence results obtained in Theorems 6.5 and 6.10 allow us to employ the PSSOC characterization of full stability in Theorem 6.6 to establish new necessary and sufficient conditions for PSMR of $\partial_x\varphi$ in (6.11) and Robinson's strong regularity of the KKT system (6.45) under PCQ.

COROLLARY 6.11 (characterizing PSMR and strong regularity in MPPC under PCQ). *Let $\Phi(\bar{x},\bar{w}) \in Z$ for MPPC in* (6.1)*, let PCQ* (6.14) *hold at* $(\bar{x},\bar{w})$*, let $\bar{v}$ be taken from* (6.9)*, and let $\bar{\lambda} \in N_Z(\Phi(\bar{x},\bar{w}))$ be a unique multiplier satisfying* (6.20).

Then the validity of PSSOC at $(\bar{x}, \bar{w}, \bar{v}, \bar{\lambda})$ *from Definition* 6.4 *is necessary and sufficient for the PSMR property of* $\partial_x \varphi$ *at* $(\bar{x}, \bar{w}, \bar{v})$ *with* φ *from* (6.6) *as well as for strong regularity of the KKT system* (6.45) *at* $(\bar{w}, \bar{v}, \bar{x}, \bar{\lambda})$.

Proof. The conclusion follows immediately by combining the characterization of Theorem 6.6 with the equivalences in Theorems 6.5 and 6.10. ∎

Note that for the classical problems of NLP the result of Corollary 6.11 concerning strong regularity under LICQ is well known in mathematical programming; see [1, 2] and the references therein. It is equally well recognized that strong regularity of the KKT system associated with NLP implies LICQ. The following example largely related to Example 6.9 shows that in the MPPC case we do not have LICQ as a consequence of strong regularity. Note to this end that, as follows from Proposition 6.3(i) and the necessity part of [1, Theorem 5.24], strong regularity does imply PCQ.

Example 6.12 (strong regularity in MPPC without LICQ). Consider the constraint mapping $\Phi(x) = \big(\varphi_1(x), \varphi_2(x), \varphi_3(x)\big) \colon \mathbb{R}^3 \to \mathbb{R}^3$ with $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and the convex polyhedron Z defined as in Example 6.9. Take further the cost function

$$\varphi_0(x) := x_1^2 + x_2^2 + x_3^2 - x_1 - x_2 \tag{6.55}$$

and show first that $\bar{x} := (0, 0, 0)$ is a tilt-stable local minimizer of the corresponding unperturbed problem $\mathcal{P}(\bar{v})$. Using the calculations in Example 6.9, we get the equation

$$\nabla \varphi_0(\bar{x}) + \nabla \Phi(\bar{x})^* \bar{\lambda} = \bar{v},$$

which for the vector of Lagrange multipliers $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3)$ is written as

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^* \begin{pmatrix} \bar{\lambda}_1 \\ \bar{\lambda}_2 \\ \bar{\lambda}_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \tag{6.56}$$

The solution of this equation is $\bar{\lambda} = (1, 0, \bar{\lambda}_3)$, where $\bar{\lambda}_3$ is an arbitrary real number. Since we have the additional condition $\bar{\lambda} \in N_Z(\Phi(\bar{x}))$, where the normal cone is calculated by

$$N_Z(\Phi(\bar{x})) = \big\{\mu_1 a_1 + \mu_2 a_2 \big|\ \mu_1,\ \mu_2 \geq 0\big\},$$

we have the unique Lagrange multiplier $\bar{\lambda}$ with $\bar{\lambda}_3 = 1$. Let us now check the validity of PSSOC at $(\bar{x}, \bar{v}, \bar{\lambda})$. To proceed, observe that the subspace S_Z from (6.24) reduces in this case to

$$S_Z = \big\{u := (u_1, u_2, u_3) |\ u_1 + u_2 = 0\big\},$$

while the Hessian of the Lagrangian function is

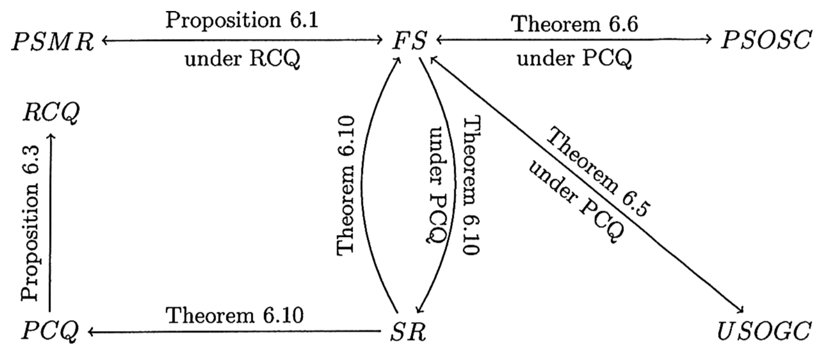
$$\begin{aligned} \nabla^2 L(\bar{x}, \bar{\lambda}) &= \nabla^2 \varphi_0(\bar{x}) + \bar{\lambda}_1 \nabla^2 \varphi_1(\bar{x}) + \bar{\lambda}_2 \nabla^2 \varphi_2(\bar{x}) + \bar{\lambda}_3 \nabla^2 \varphi_3(\bar{x}) \\ &= 2I + 0 + 0 + 2I = 4I, \end{aligned} \tag{6.57}$$

where I stands for the 3×3 identity matrix. Employing (6.57) justifies the validity of PSSOC due to

$$\langle u, \nabla^2 L(\bar{x}, \bar{\lambda}) u \rangle = 4\|u\|^2 > 0 \quad \text{whenever} \quad 0 \neq u \in S_Z.$$

It is shown in Example 6.9 that PCQ holds in this setting, and thus Theorem 6.6 tells us that $\bar{x}$ is a tilt-stable local minimizer of $\mathcal{P}(\bar{v})$. Finally, Theorem 6.10 ensures strong regularity of the KKT system (6.45) at $(\bar{v}, \bar{x}, \bar{\lambda})$, while we know from Example 6.9 that LICQ is not satisfied for $\mathcal{P}(\bar{v})$ at this point.

Summarizing the results obtained above for full stability of local minimizers in the context of MPPC, we see that its PSSOC characterization and the equivalence to Robinson's strong regularity require PCQ, while its USOGC characterization and the equivalence to PSMR hold under the less restrictive RCQ, which reduces to MFCQ in the case of NLP. These relationships are depicted in the following diagram, where FS and SR stand for full stability and strong regularity, respectively, while the other abbreviations have been defined above.



7. Full stability in ENLP. The last section is devoted to full stability of optimization problems written in the composite format (5.1) with the outer function $\theta\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ defined by

$$\theta(z) := \sup_{p\in P}\Big\{\langle p, z\rangle - \vartheta(p)\Big\}, \tag{7.1}$$

where $\vartheta\colon \mathbb{R}^m \to \mathbb{R}$ is a smooth function convex on the polyhedral set $\emptyset \neq P \subset \mathbb{R}^m$ given by

$$P := \Big\{p \in \mathbb{R}^m \Big|\ \langle a_j, p\rangle \le b_j \ \text{ for all } \ j = 1, \ldots, l\Big\} \tag{7.2}$$

with fixed vectors $a_j \in \mathbb{R}^m$ and numbers $b_j \in \mathbb{R}$ as $l \in \mathbb{N}$. We see that θ in (7.1) is convex, proper, and l.s.c. Note that the function θ from (4.16) is a special case of (7.1) with $\vartheta(p) = \frac{1}{2}\langle p, Qp\rangle$, where Q is a symmetric and positive-semidefinite matrix. Note also that standard NLP problems can be modeled in the ENLP form with $\vartheta(p) = 0$; see [22].

Composite optimization problems of type (5.1) with functions θ given by (7.1) are introduced by Rockafellar [22] (see also [23]) under the name of ENLP. It is argued in [22, 23] that model (4.1) with term (7.1) provides a very convenient framework for developing both theoretical and computational aspects of optimization in broad classes of constrained problems, including stochastic programming, robust optimization, etc. The special expression (7.1) for the extended-real-valued function θ, known as a dualizing representation, is significant with respect to the theory and applications of Lagrange multipliers in ENLP.

As in section 6, we denote by $I(p)$ the set of active indices $j \in \{1, \ldots, l\}$ in the polyhedral description (7.2) at $p \in P$ (i.e., such j that $\langle a_j, p\rangle = b_j$) and we have the following representation of the normal cone to the convex polyhedron P at the given point $\bar{p} \in P$:

$$N_P(\bar{p}) = \left\{ \sum_{j=1}^{l} \mu_j a_j \,\middle|\, \mu_j \ge 0 \text{ for } j \in I(\bar{p}) \ \text{ and } \ \mu_j = 0 \text{ for } j \notin I(\bar{p}) \right\}. \tag{7.3}$$

The next results, which is of its own interest, provides the exact calculation of the second-order subdifferential for the function θ defined in (7.1). It extends to the case of general convex and $\mathcal{C}^2$ functions ϑ in (7.1) the one from [16, Lemma 4.4] for quadratic functions.

PROPOSITION 7.1 (calculation of the second-order subdifferential for dualizing representations). *Let θ be an extended-real-valued function defined in* (7.1) *under the assumptions above, and let $\bar{z} \in \operatorname{dom}\theta$. Pick some $\bar{p} \in \partial\theta(\bar{z})$ and suppose that ϑ is $\mathcal{C}^2$ around $\bar{p}$. Then we have the following formula for calculating the second-order subdifferential of θ at $(\bar{z},\bar{p})$:*

$$q \in \partial^2\theta(\bar{z},\bar{p})(u) \Longleftrightarrow \begin{cases} \text{there exist closed faces } K_1 \text{ and } K_2 \text{ of } K \text{ with} \\ K_1 \subset K_2,\ q \in K_2 - K_1,\ \nabla^2\vartheta(\bar{p})^* q - u \in (K_2 - K_1)^* \end{cases} \tag{7.4}$$

for all $u \in \mathbb{R}^m$, where $K = T_P(\bar{p}) \cap (\bar{z} - \nabla\vartheta(\bar{p}))^{\perp}$ is the corresponding critical cone with the tangent cone $T_P(\bar{p})$ to the convex polyhedron (7.2) *at $\bar{p} \in P$ computed by*

$$T_P(\bar{p}) = \left\{ p \in \mathbb{R}^m \,\middle|\, \langle a_j, p\rangle \le 0 \ \textit{ for all } \ i \in I(\bar{p}) \right\}. \tag{7.5}$$

Proof. It follows from the form of the dualizing representation θ in (7.1) and the definition of conjugate functions in convex analysis that

$$\theta^*(p) = \vartheta(p) + \delta_P(p), \quad p \in \mathbb{R}^m, \tag{7.6}$$

where θ^* is the convex conjugate function of θ and where δ_P is the indicator function of the polyhedron P; see, e.g., [22, Proposition 1]. Since $\partial\theta^* = (\partial\theta)^{-1}$, we have

$$q \in \partial^2\theta(\bar{z},\bar{p})(u) \Longleftrightarrow -u \in \partial^2\theta^*(\bar{p},\bar{z})(-q) \ \text{ whenever } \ u, q \in \mathbb{R}^m. \tag{7.7}$$

Furthermore, it follows from [23, Proposition 11.3] and representation (7.6) that

$$\partial\theta(z) = \operatorname{argmax}_{p\in P}\Big\{\langle z,p\rangle - \vartheta(p)\Big\}, \quad z \in \mathbb{R}^m. \tag{7.8}$$

Basic convex analysis tells us that the maximum of the concave function $\langle z,p\rangle - \vartheta(p)$ over the convex set P is attained at $p \in P$ if and only if $z - \nabla\vartheta(p) \in N_P(p)$. This yields by (7.8) that

$$\partial\theta^*(p) = (\partial\theta)^{-1}(p) = \nabla\vartheta(p) + N_P(p), \quad p \in P, \tag{7.9}$$

and hence $\bar{z} \in \partial\theta^*(\bar{p}) \Longleftrightarrow [\bar{z} - \nabla\vartheta(\bar{p}) \in N_P(\bar{p})]$. Taking into account definition (2.10) of the second-order subdifferential and applying the coderivative sum rule from [12, Theorem 1.62] to the sum in (7.9), we get the expression

$$\partial^2\theta^*(\bar{p},\bar{z})(-q) = D^*N_P(\bar{p},\bar{z} - \nabla\vartheta(\bar{p}))(-q) - \nabla^2\vartheta(\bar{p})^* q, \quad q \in \mathbb{R}^m,$$

where the last term on the right-hand side is due to (2.11) with the symmetric Hessian $\nabla^2\vartheta(\bar{p})$ for the $\mathcal{C}^2$ function ϑ. This ensures the following description of the second-order subdifferential of the conjugate function θ^* to the dualizing representation:

$$-u \in \partial^2\theta^*(\bar{p},\bar{z})(-q) \Longleftrightarrow [\nabla^2\vartheta(\bar{p})^* q - u \in \partial^2\delta_P(\bar{p},\bar{z} - \nabla\vartheta(\bar{p}))(-q)].$$

Employing finally the calculation of $\partial^2\delta_P$ obtained in (6.34) and using relationship (7.7), we arrive at the second-order subdifferential representation (7.4), where the tangent cone formula (7.5) follows from [3, Theorem 2E.3]. This completes the proof of the proposition. $\square$

To study full stability of local minimizers in the framework of ENLP, consider the two-parametric problem $\mathcal{P}(w,v)$ written as

$$\begin{cases} \text{minimize } \ \varphi(x,w)-\langle v,x\rangle \ \text{ over } \ x\in\mathbb{R}^n \ \text{ with} \\ \varphi(x,w):=\varphi_0(x,w)+\theta(\Phi(x,w)),\ \theta(z):=\sup\limits_{p\in P}\big\{\langle p,z\rangle-\vartheta(p)\big\}, \end{cases} \tag{7.10}$$

$\Phi(x,w):=(\varphi_1(x,w),\ldots,\varphi_m(x,w))$, and the polyhedral set P defined in (7.2). We keep the assumptions of Proposition 7.1 regarding the function ϑ in (7.10) and suppose in what follows that all the functions $\varphi_0,\ldots,\varphi_m$ are $\mathcal{C}^2$ around the reference point $(\bar x,\bar w)$. We also impose the LICQ condition (6.16) at $(\bar x,\bar w)$, which amounts to the full rank of the partial Jacobian $\nabla_x\Phi(\bar x,\bar w)$. Under the imposed LICQ, the stationarity condition $\bar v\in\partial_x\varphi(\bar x,\bar w)$ on the tilt perturbation $\bar v$ in (7.10) is equivalent (by the first-order subdifferential sum and chain rules from [12, 23]) to

$$\bar v\in\nabla_x\varphi_0(\bar x,\bar w)+\nabla_x\Phi(\bar x,\bar w)^*\partial\theta(\Phi(\bar x,\bar w)). \tag{7.11}$$

Define further the extended Lagrangian function for the perturbed ENLP (7.10) by

$$\mathcal{L}(x,w,p):=\varphi_0(x,w)+\Phi(x,w)^*p-\vartheta(p) \ \text{ with } \ p\in\mathbb{R}^m, \tag{7.12}$$

where the vector $p=(p_1,\ldots,p_m)$ signifies Lagrange multipliers. The following definition is the ENLP counterpart of the classical SSOSC (6.31) in nonlinear programming.

DEFINITION 7.2 (extended strong second-order optimality condition). *Let $\bar p\in\mathbb{R}^m$ be a vector of Lagrange multipliers in ENLP. We say that the ESSOC holds at $(\bar x,\bar w,\bar v,\bar p)$ in problem $\mathcal{P}(\bar w,\bar v)$ from* (7.10) *with $\bar v$ satisfying* (7.11) *if*

$$\langle u,\nabla^2_{xx}\mathcal{L}(\bar x,\bar w,\bar p)u\rangle>0 \ \textit{ for all } \ 0\neq u\in\mathcal{S}, \tag{7.13}$$

where the subspace $\mathcal{S}\subset\mathbb{R}^n$ is given by

$$\mathcal{S}:=\Big\{u\in\mathbb{R}^n\Big|\ \nabla_x\Phi(\bar x,\bar w)u\in\{p\in\mathbb{R}^m|\ \langle a_j,p\rangle=0 \ \textit{ for all } \ j=1,\ldots,l\}^\perp\Big\}. \tag{7.14}$$

As we mentioned in (6.15), employing [23, Lemma 6.45], $\mathcal{S}$ can be expressed as

$$\mathcal{S}:=\Big\{u\in\mathbb{R}^n\Big|\ \nabla_x\Phi(\bar x,\bar w)u\in\operatorname{span}\{a_j\in\mathbb{R}^m|\ j=1,\ldots,l\}\}. \tag{7.15}$$

Now we are ready to formulate and prove the main result of this section on characterizing full stability of local minimizers in ENLP via ESSOC from Definition 7.2. Recall that standard NLP problems can be modeled in the ENLP form with $\vartheta(p)=0$, and thus the next theorem is an extension of [16, Theorem 5.2].

THEOREM 7.3 (characterizing full stability of locally optimal solutions to ENLP via ESSOC). *Let $\bar x$ be a feasible solution to problem $\mathcal{P}(\bar w,\bar v)$ in* (7.10) *for some $\bar w\in\mathbb{R}^d$ and $\bar v$ satisfying* (7.11). *Assume that LICQ* (6.16) *holds at $(\bar x,\bar w)$ and determine the unique vector $\bar p\in\mathbb{R}^m$ of Lagrange multiplies from*

$$\nabla_x\Phi(\bar x,\bar w)^*\bar p=\bar v-\nabla_x\varphi_0(\bar x,\bar w). \tag{7.16}$$

Then we have the following assertions:

(i) *If $\bar{x}$ is a fully stable locally optimal solution to $\mathcal{P}(\bar{w},\bar{v})$, then ESSOC holds at $(\bar{x},\bar{w},\bar{v},\bar{p})$.*

(ii) *Conversely, the validity of ESSOC at $(\bar{x},\bar{w},\bar{v},\bar{p})$ with $\nabla^2\vartheta(\bar{p})=0$ yields that $\bar{x}$ is a fully stable locally optimal solution to problem $\mathcal{P}(\bar{w},\bar{v})$.*

Proof. Observe first that since the assumed LICQ amounts to the full rank of the partial Jacobian $\nabla_x\Phi(\bar{x},\bar{w})$, (7.16) for $\bar{p}$ admits a unique solution if any.

To prove (i), we take into account that every fully stable locally optimal solution to $\mathcal{P}(\bar{w},\bar{v})$ is a usual local minimizer for this problem and, applying the classical stationary conditions in (7.10) to $\bar{x}$, ensure the existence of the (unique) Lagrange multiplier $\bar{p}\in\nabla_x\Phi(\bar{x},\bar{w})^*\partial\theta(\Phi(\bar{x},\bar{w}))$ satisfying (7.16). Since the function θ from (7.1) is proper, l.s.c., and convex, it is continuously prox-regular at $\bar{z}$ (see [23, Example 13.30]), and hence we can apply Theorem 5.1 to problem $\mathcal{P}(\bar{w},\bar{v})$ from (7.10). The aforementioned theorem formulated via the data of problem (7.10) ensures the validity of condition (5.6) for the set-valued mapping $\mathcal{T}(\bar{x},\bar{w},\bar{v})=(\mathcal{T}_1(\bar{x},\bar{w},\bar{v}),\mathcal{T}_2(\bar{x},\bar{w},\bar{v}))\colon\mathbb{R}^n\rightrightarrows\mathbb{R}^{2n}$ with $\mathcal{T}_i(\bar{x},\bar{w},\bar{v})$, $i=1,2$, defined by

$$
\begin{cases}\mathcal{T}_1(\bar{x},\bar{w},\bar{v})(u):=\nabla^2_{xx}\mathcal{L}(\bar{x},\bar{w},\bar{p})u+\nabla_x\Phi(\bar{x},\bar{w})^*\partial^2\theta(\bar{z},\bar{p})(\nabla_x\Phi(\bar{x},\bar{w})u),\\ \mathcal{T}_2(\bar{x},\bar{w},\bar{v})(u):=\nabla^2_{xw}\mathcal{L}(\bar{x},\bar{w},\bar{p})u+\nabla_w\Phi(\bar{x},\bar{u})^*\partial^2\theta(\bar{z},\bar{p})(\nabla_x\Phi(\bar{x},\bar{w})u)\end{cases}\tag{7.17}
$$

via the extended Lagrangian (7.12). To justify assertion (i) of this theorem, we need to show that condition (5.6) for the mapping $\mathcal{T}(\bar{x},\bar{w},\bar{v})$ given in (7.17) implies the fulfillment of ESSOC from Definition 7.2. In the notation above, condition (5.6) amounts to saying that

$$
\langle u,\nabla^2_{xx}\mathcal{L}(\bar{x},\bar{w},\bar{p})u\rangle+\langle q,\nabla_x\Phi(\bar{x},\bar{w})u\rangle>0\ \text{ if }\ q\in\partial^2\theta(\bar{z},\bar{p})(\nabla_x\Phi(\bar{x},\bar{w})u),\ u\neq 0.\tag{7.18}
$$

Employing Proposition 7.1 to calculate the second-order subdifferential $\partial^2\theta(\bar{z},\bar{p})\times(\nabla_x\Phi(\bar{x},\bar{w})u)$, we get

$$
q\in\partial^2\theta(\bar{z},\bar{p})(\nabla_x\Phi(\bar{x},\bar{w})u)\Longleftrightarrow\begin{cases}\text{there exist closed faces } K_1 \text{ and } K_2\\ \text{of } K \text{ with } K_1\subset K_2,\ q\in K_2-K_1,\\ \nabla^2\vartheta(\bar{p})^*q-\nabla_x\Phi(\bar{x},\bar{w})u\in(K_2-K_1)^*\end{cases}\tag{7.19}
$$

with the critical cone $K=T_P(\bar{p})\cap(\bar{z}-\nabla\vartheta(\bar{p}))^\perp$. Fix $0\neq u\in\mathcal{S}$ in (7.15) and pick $q\in\partial^2\theta(\bar{z},\bar{p})(\nabla_x\Phi(\bar{x},\bar{w})u)$. It follows from (7.15) that

$$
\nabla_x\Phi(\bar{x},\bar{w})u\in\Big\{p\in\mathbb{R}^m\,\Big|\ \langle a_j,p\rangle=0\ \text{ for all }\ j=1,\ldots,l\Big\}^\perp.
$$

Similarly to the proof of Theorem 6.6 we observe the representations

$$
K\cap(-K)=\Big\{p\in\mathbb{R}^m|\ \langle a_j,p\rangle=0\ \text{ for all }\ j=1,\ldots,l\Big\}\ \text{ and}
$$
$$
\Big([K\cap(-K)]-[K\cap(-K)]\Big)^*=(K\cap(-K))^\perp,
$$

which immediately imply the inclusions

$$
0\in[K\cap(-K)]-[K\cap(-K)]\ \text{and}\ -\nabla_x\Phi(\bar{x},\bar{w})u\in\Big([K\cap(-K)]-[K\cap(-K)]\Big)^*.\tag{7.20}
$$

Combining these inclusions with (7.19) shows that

$$
0\in\partial^2\theta(\bar{z},\bar{p})(\nabla_x\Phi(\bar{x},\bar{w})u)\ \text{ for all }\ 0\neq u\in\mathcal{S}
$$

with $\bar{z} := \Phi(\bar{x}, \bar{w})$. Letting now $q = 0$ in (7.18) gives us inequality (7.13) from Definition 7.2, and hence the desired ESSOC at $(\bar{x}, \bar{w}, \bar{v}, \bar{p})$ is satisfied, which justifies assertion (i).

To prove the converse assertion (ii), assume that ESSOC holds at $(\bar{x}, \bar{w}, \bar{v}, \bar{p})$ and that $\nabla^2\vartheta(\bar{p}) = 0$. Let us show that condition (7.18) holds, which thus tells us that $\bar{x}$ is a fully stable local minimizer for $\mathcal{P}(\bar{w}, \bar{v})$ in (7.10) by Theorem 5.1 and the considerations above. To proceed, fix $0 \neq u \in \mathbb{R}^n$ and pick any $q \in \partial^2\theta(\bar{z}, \bar{p})(\nabla_x\Phi(\bar{x}, \bar{w})u)$. Employing (7.19) with $\nabla^2\vartheta(\bar{p}) = 0$ gives us two closed faces $K_1 \subset K_2$ of the critical cone K defined above such that

$$q \in K_2 - K_1, \quad -\nabla_x\Phi(\bar{x}, \bar{w})u \in (K_2 - K_1)^*, \text{ and thus} \tag{7.21}$$

$$\langle q, \nabla_x\Phi(\bar{x}, \bar{w})u\rangle \geq 0 \;\text{ for all }\; q \in \partial^2\theta(\bar{z}, \bar{p})(\nabla_x\Phi(\bar{x}, \bar{w})u). \tag{7.22}$$

Since $K \cap (-K)$ is the smallest closed face of K, we have

$$(K_2 - K_1)^* \subset \Big([K \cap (-K)] - [K \cap (-K)]\Big)^* = (K \cap (-K))^{\perp}.$$

This ensures by (7.21) that $-\nabla_x\Phi(\bar{x}, \bar{w})u \in (K \cap (-K))^{\perp}$ and hence shows by (7.15) that $u \in \mathcal{S}$. Finally, using (7.22) together with (7.13) gives us the relationships

$$\begin{aligned}\langle u, \nabla^2_{xx}\mathcal{L}(\bar{x}, \bar{w}, \bar{p})u\rangle + \langle q, \nabla_x\Phi(\bar{x}, \bar{w})u\rangle &\geq \langle u, \nabla^2_{xx}\mathcal{L}(\bar{x}, \bar{w}, \bar{p})u\rangle + 0\\ &= \langle u, \nabla^2_{xx}\mathcal{L}(\bar{x}, \bar{w}, \bar{\bar{p}})u\rangle > 0,\end{aligned}$$

which justify (7.18) and thus complete the proof of the theorem. ∎

Remark 7.4 (ENLP without LICQ). If the function θ in the ENLP model under consideration is of the piecewise linear-quadratic form (4.16) with a symmetric and positive-definite matrix Q and if the mapping Φ is open at $(\bar{x}, \bar{w})$, then applying Theorem 5.2(b) allows us to characterize fully stable local minimizers of (7.10) similarly to Theorem 7.3 but without LICQ (6.16). It follows from the proof of Theorem 7.3 by replacing the application of Theorem 5.1 therein with that of Theorem 5.2 in case (b).

Remark 7.5 (some open questions). The condition $\nabla^2(\vartheta(\bar{p}) = 0$ in Theorem 7.3(ii), which obviously does not hold for quadratic functions $\vartheta(p) = \langle p, Qp\rangle$, $Q \neq 0$, is definitely restrictive, while it is essential in the proof of Theorem 7.3. The removal of it is an issue for our future research. We also plan to develop similar characterizations of full stability for remarkable classes of problems in conic programming. For tilt stability some results in this direction have been recently obtained in [15].

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