FULL STABILITY IN FINITE-DIMENSIONAL OPTIMIZATION

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Abstract. The paper is devoted to full stability of optimal solutions in general settings of finite-dimensional optimization with applications to particular models of constrained optimization problems including those of conic and specifically semidefinite programming. Developing a new technique of variational analysis and generalized differentiation, we derive second-order characterizations of full stability, in both Lipschitzian and Hölderian settings, and establish their relationships with the conventional notions of strong regularity and strong stability for a large class of problems of constrained optimization with twice continuously differentiable data.

Key words: constrained optimization; full stability; variational analysis; generalized differentiation; conic programming; semidefinite programming; strong regularity; strong stability

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1 Introduction

The concept of *full Lipschitzian stability* of local minimizers in general optimization problems was introduced by Levy, Poliquin and Rockafellar [16] to single out those local solutions, which exhibit "nice" stability properties under appropriate parameter perturbations. Roughly speaking, the properties postulated in [16] require that the local minimizer in question does not lose its uniqueness and evolves "proportionally" (in some Lipschitzian way) with respect to a certain class of two-parametric perturbations; see Section 3 for the precise formulations. The full stability notion of [16] extended the previous one of *tilt stability* introduced by Poliquin and Rockafellar [34], where such a behavior was considered with respect to one-parametric linear/tilt perturbations. Both stability notions in [16, 34] were largely motivated by their roles in the justification of numerical algorithms, particularly the stopping criteria, convergence properties, and robustness.

The first second-order characterizations of tilt stability were obtained by Poliquin and Rockafellar [34] via the second-order subdifferential/generalized Hessian of Mordukhovich [18] in the general framework of extended-real-valued prox-regular functions and by Bonnans and Shapiro [3] via a certain uniform second-order growth condition in the framework of conic programs with $\mathcal{C}^2$-smooth data. More recent developments on tilt stability for various classes of optimization problems in both finite and infinite dimensions can be found in [6, 7, 9, 17, 22, 24, 26, 27, 29].

Much less has been done for full stability. In the pioneering work by Levy, Poliquin and Rockafellar [16] this notion was characterized in terms of a partial modification of the second-order subdifferential from [18] for a class of parametrically prox-regular functions in the unconstrained format of optimization with extended-real-valued objectives. The calculus rules for this partial second-order subdifferential developed by Mordukhovich and Rockafellar [29] allowed them in the joint work with Sarabi [30] to derive constructive second-order characterizations of fully stable minimizers for various classes of constrained optimization problems in finite dimensions including those in nonlinear and extended nonlinear programming and mathematical programs with polyhedral constraints. Quite recently [25] Mordukhovich and Nghia have obtained new characterizations of Lipschitzian and Hölderian (see Section 3) full stability in infinite-dimensional (mainly Hilbert) spaces with applications to nonlinear programming, mathematical programs with polyhedric constraints, and optimal control of elliptic equations.

In this paper we develop a *new approach* to both Lipschitzian and Hölderian full stability by taking into account specific features of finite-dimensional spaces and obtain in this way new second-order characterizations of both types of full stability in general nonsmooth optimization settings as well as for particular classes of constrained optimization problems with $\mathcal{C}^2$-smooth data (e.g., for semidefinite programming). Our approach is significantly different and simpler than that in [16] developed in the Lipschitzian case and allows us to derive not only *qualitative* but also *quantitative* (with precise modulus formulas) characterizations of full stability in general frameworks. Furthermore, for a large class of *mathematical programs* with $\mathcal{C}^2$-*smooth data* (including those of conic programming) satisfying the classical *Robinson constraint qualification* (RCQ) we show that the continuity of the stationary mapping in Kojima's *strong stability* can be strengthened to Hölder continuity with order $\frac{1}{2}$ by using Hölderian full stability. If in addition the constraint are $\mathcal{C}^2$-*reducible* and the optimal point is (partially) *nondegenerate* in the sense of [3], then we establish the equivalence of *Lipschitzian full stability* to Robinson's *strong regularity* of the associated variational inequality. Using finally our general results obtained and the recent coderivative calculations by Ding, Sun and Ye [5] gives us complete characterizations of full stability and related properties for problems of *semidefinite programming* expressed entirely in terms of the initial data.

The rest of the paper is organized as follows. Section 2 presents those preliminaries from variational analysis and generalized differentiation, which are widely used for the statements and proofs of the main results. In Section 3 we formulate the basic notions of Hölderian and Lipschitzian full stability and focus on second-order characterizations of the Hölderian version for the general class of parametrically prox-regular extended-real-valued functions. These characterizations are obtained in terms of a certain second-order growth condition as well as via second-order subdifferential constructions with precise relationships between the corresponding moduli. The major results of Section 4 establish various qualitative and quantitative characterizations of Lipschitzian full stability in the general framework of Section 3. They are expressed in terms the (partial) second-order subdifferentials and imply, in particular, the aforementioned result of [16] derived by a different and essentially involved approach. In contrast to [16], our approach does not appeal to tangential approximations of sets and functions while operating instead with intrinsically nonconvex-valued normal and coderivative mappings, which satisfy comprehensive calculus rules. This leads us to more direct and simple proofs with a variety of quantitative and qualitative characterizations of full and tilt stability.

Section 5 addresses the conventional class of $\mathcal{C}^2$-smooth parametric optimization problems with constraints written in the form $g(x,p) \in \Theta$, where Θ is a closed and convex subset of a finite-dimensional space. Imposing the classical RCQ, we prove that Lipschitzian full stability agrees with Robinson's strong regularity *provided that* Θ is $\mathcal{C}^2$-reducible and the optimal solution is nondegenerate. Furthermore, we establish complete characterizations of all these properties via verifiable conditions involving the second-order subdifferential (or the generalized Hessian) $\partial^2 \delta_\Theta$ of the indicator function δ_Θ of Θ. In Section 6 these results are specified for semidefinite programs, where $\Theta = S_+^m$ is the cone of all the $m \times m$ symmetric positive semidefinite matrices and the second-order construction $\partial^2 \delta_\Theta$ is calculated entirely in terms of the program data. Section 7 contains concluding remarks and discusses some topics of future research.

Our notation is standard in variational analysis and optimization (see, e.g., [20, 38]) except the symbols specified in the text. Everywhere $I\!R^n$ stands for the n-dimensional Euclidian space with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$. We denote by $I\!B$ the closed unit ball in the space in question and by $I\!B_\eta(x) := x + \eta I\!B$ the closed ball centered at x with radius $\eta > 0$. Given a set-valued mapping $F \colon I\!R^n \rightrightarrows I\!R^m$, the symbol

$$\underset{x \to \bar{x}}{\mathrm{Lim\,sup}}\, F(x) := \left\{ y \in I\!R^m \,\middle|\, \begin{array}{l} \exists \text{ sequences } x_k \to \bar{x},\ y_k \to y \text{ such that} \\ y_k \in F(x_k) \text{ for all } k \in I\!N := \{1, 2, \dots\} \end{array} \right\} \tag{1.1}$$

signifies the *Painlevé-Kuratowski outer limit* of $F(x)$ as $x \to \bar{x}$. For a linear operator/matrix A the notation A^* stands for the adjoint operator/matrix transposition.

2 Preliminaries from Variational Analysis

Let $f : I\!R^n \to \overline{I\!R} := (-\infty, \infty]$ be an extended-real-valued function, we always assume that f is *proper*, i.e., $\mathrm{dom}\, f := \{x \in X \,|\, f(x) < \infty\} \neq \emptyset$. The *regular subdifferential* of f at $\bar{x} \in \mathrm{dom}\, f$ (known also as the

presubdifferential and as the Fréchet or viscosity subdifferential) is

$$\widehat{\partial} f(\bar{x}) := \left\{ v \in \mathbb{R}^n \,\middle|\, \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle v, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}. \tag{2.1}$$

The *limiting subdifferential* (known as the general/basic or Mordukhovich subdifferential) and the *singular subdifferential* (known also as the horizon subdifferential) of f at $\bar{x}$ are defined respectively via (1.1) by

$$\partial f(\bar{x}) := \limsup_{x \xrightarrow{f} \bar{x}} \widehat{\partial} f(x) \quad \text{and} \quad \partial^\infty f(\bar{x}) := \limsup_{x \xrightarrow{f} \bar{x}, \lambda \downarrow 0} \lambda \widehat{\partial} f(x), \tag{2.2}$$

where $x \xrightarrow{f} \bar{x}$ signifies that $x \to \bar{x}$ with $f(x) \to f(\bar{x})$. Observe that both regular and limiting subdifferentials reduce to the subdifferential of convex analysis for convex functions and that $\partial^\infty f(\bar{x}) = \{0\}$ when f is locally Lipschitzian around $\bar{x}$. The latter condition becomes a characterization of Lipschitzian continuity around $\bar{x}$ if f is lower semicontinuous (l.s.c.) around $\bar{x}$.

Given a set $\Omega \subset \mathbb{R}^n$ with its indicator function $\delta_\Omega(x)$ equal to 0 for $x \in \Omega$ and to ∞ otherwise, the regular and limiting *normal cones* to Ω at $\bar{x} \in \Omega$ are defined via (2.1) and (2.2) by, respectively,

$$\widehat{N}(\bar{x}; \Omega) := \widehat{\partial} \delta_\Omega(\bar{x}) \quad \text{and} \quad N(\bar{x}; \Omega) := \partial \delta_\Omega(\bar{x}) \tag{2.3}$$

with the notation $\widehat{N}_\Omega(\bar{x})$ and $N_\Omega(\bar{x})$ also used below. The constructions in (2.3) can be rewritten as

$$N(\bar{x}; \Omega) = \limsup_{x \xrightarrow{\Omega} \bar{x}} \widehat{N}(x; \Omega) \quad \text{with} \quad \widehat{N}(x; \Omega) = \left\{ v \in \mathbb{R}^n \,\middle|\, \limsup_{u \xrightarrow{\Omega} x} \frac{\langle v, u - x \rangle}{\|u - x\|} \leq 0 \right\},$$

where the symbol $x \xrightarrow{\Omega} \bar{x}$ signifies that $x \to \bar{x}$ with $x \in \Omega$.

Given a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, we associate with it the *domain* and *graph* by

$$\operatorname{dom} F := \left\{ x \in \mathbb{R}^n \,\middle|\, F(x) \neq \emptyset \right\} \quad \text{and} \quad \operatorname{gph} F := \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \,\middle|\, y \in F(x) \right\}.$$

The *regular* (resp. *limiting*) *coderivative* of F at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is defined via (2.3) by

$$\widehat{D}^* F(\bar{x}, \bar{y})(w) := \left\{ z \in \mathbb{R}^n \,\middle|\, (z, -w) \in \widehat{N}\big((\bar{x}, \bar{y}); \operatorname{gph} F\big) \right\} \quad \text{for all} \quad w \in \mathbb{R}^m, \tag{2.4}$$

$$D^* F(\bar{x}, \bar{y})(w) := \left\{ z \in \mathbb{R}^n \,\middle|\, (z, -w) \in N\big((\bar{x}, \bar{y}); \operatorname{gph} F\big) \right\} \quad \text{for all} \quad w \in \mathbb{R}^m. \tag{2.5}$$

If F is single-valued around $(\bar{x}, \bar{y})$, we omit $\bar{y} = F(\bar{x})$ in the coderivative notation (2.4) and (2.5).

It has been well recognized that the coderivative constructions (2.4) and (2.5) are appropriate tools for the study and characterizations of well-posedness and stability properties that play a major role in many (particularly variational) aspects of nonlinear analysis; see, e.g., [20, Chapter 4] and [38, Chapter 9] for more details. Recall that $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *Lipschitz-like* with modulus $\ell > 0$ around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ (known also as the Aubin or pseudo-Lipschitz property) if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$F(x) \cap V \subset F(u) + \ell \|x - u\| \mathbb{B} \quad \text{for all} \quad x, u \in U. \tag{2.6}$$

This property is fully characterized by the *Mordukhovich criterion* from [38, Theorem 9.40] (known also as the coderivative criterion; see [19, Corollary 5.4]): F is Lipschitz-like (has the Aubin property) around $(\bar{x}, \bar{y})$ if and only if we have the condition

$$D^* F(\bar{x}, \bar{y})(0) = \{0\} \tag{2.7}$$

provided that the graph of F is locally closed around $(\bar{x}, \bar{y})$.

The main generalized differential constructions used in this paper are *second-order subdifferentials* (or *generalized Hessians*) of extended-real-valued functions defined by the scheme of [18] as a coderivative of a first-order subgradient mapping. The basic one from [18] is constructed as follows. Given $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, fix a limiting subgradient $\bar{v} \in \partial f(\bar{x})$ from (2.2) and define $\partial^2 f(\bar{x}, \bar{v}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ by

$$\partial^2 f(\bar{x}, \bar{v})(w) := (D^* \partial f)(\bar{x}, \bar{v})(w), \quad w \in \mathbb{R}^n, \tag{2.8}$$

via the limiting coderivative (2.5) of ∂f. If f is $\mathcal{C}^2$-smooth around $\bar{x}$, we have

$$\partial^2 f(\bar{x})(w) = \left\{ \nabla^2 f(\bar{x})w \right\} \quad \text{for all} \quad w \in I\!\!R^n,$$

i.e., the second-order construction (2.8) reduces to the classical (symmetric) Hessian operator. In the general nonsmooth case the mapping $\partial^2 f(\bar{x}, \bar{v})$ is set-valued and positive homogeneous enjoying well-developed calculus rules, that are mainly based on variational/extremal principles of variational analysis; see, e.g., the book [20] and more recent papers [21, 29, 32] with the references therein. Various modifications of this construction and their partial counterparts were considered in [12, 16, 22, 24, 25, 29, 30]. In what follows we employ the second-order constructions of this type generated by both coderivatives (2.4) and (2.5) of some partial first-order subgradient/normal cone mappings and prefer using directly the coderivative-of-subdifferential notation instead of formally introducing such second-order constructions.

Let us next recall significant concepts of prox-regularity and subdifferential continuity of extended-real-valued functions taken from [16]; cf. also their nonparametric versions in [33, 38]. Given $f : I\!\!R^n \times I\!\!R^d \to \overline{I\!\!R}$ finite at $(\bar{x}, \bar{p})$ and given a partial limiting subgradient $\bar{v} \in \partial_x f(\bar{x}, \bar{p})$, where $\partial_x f$ stands for the *partial limiting subdifferential* of f with respect to x. We say that f is *prox-regular* in x at $\bar{x}$ for $\bar{v}$ with *compatible parameterization* by p at $\bar{p}$ if there are neighborhoods U of $\bar{x}$, V of $\bar{v}$, and P of $\bar{p}$ along with some numbers $\varepsilon > 0$ and $r > 0$ such that

$$\begin{aligned} f(x, p) &\geq f(u, p) + \langle v, x - u \rangle - \tfrac{r}{2}\|x - u\|^2 \quad \text{for all} \quad x \in U, \\ &\text{when} \quad v \in \partial_x f(u, p) \cap V, \ u \in U, p \in P, \quad \text{and} \quad f(u, p) \leq f(\bar{x}, \bar{p}) + \varepsilon. \end{aligned} \tag{2.9}$$

Further, f is *subdifferentially continuous* in x at $\bar{x}$ for $\bar{v}$ with compatible parameterization by p at $\bar{p}$ if the function $(x, p, v) \mapsto f(x, p)$ is continuous relative to $\mathrm{gph}\,\partial_x f$ at $(\bar{x}, \bar{p}, \bar{v})$. We simply call f is *parametrically continuously prox-regular* at $(\bar{x}, \bar{p})$ for $\bar{v}$ when f is prox-regular and subdifferentially continuous in x at $\bar{x}$ for $\bar{v}$ with compatible parameterization by p at $\bar{p}$. If in addition that the *basic constraint qualification* formulated below (3.4) holds at $(\bar{x}, \bar{p})$, then the graph $\mathrm{gph}\,\partial_x f$ is locally closed around $(\bar{x}, \bar{p}, \bar{v})$; see [16, Proposition 3.2].

In the sequel we also need the following notions of monotonicity related to the limiting subdifferential of prox-regular functions. The mapping $T : I\!\!R^n \rightrightarrows I\!\!R^n$ is said to be *monotone* if

$$\langle y - v, x - u \rangle \geq 0 \quad \text{whenever} \quad (x, y), (u, v) \in \mathrm{gph}\,T.$$

The mapping T is *strongly monotone* if its shift $T - rI$ is monotone for some $r > 0$. We say that $T : I\!\!R^n \rightrightarrows I\!\!R^n$ is *maximally monotone* if $T = S$ for any monotone mapping $S : I\!\!R^n \rightrightarrows I\!\!R^n$ with $\mathrm{gph}\,T \subset \mathrm{gph}\,S$. Given a neighborhood $U \times V \subset I\!\!R^n \times I\!\!R^n$, the mapping T is called to be *monotone relative* to $U \times V$ if its localization relative to $U \times V$ is monotone. Recall also that $\widehat{T}$ is a *localization* of T relative to $U \times V$ if $\mathrm{gph}\,\widehat{T} = \mathrm{gph}\,T \cap (U \times V)$. We use notion of a *single-valued localization* to indicate a localization that is single-valued on its domain (not necessary being a neighborhood). Finally, T is *maximally monotone relative* to $U \times V$ if $\mathrm{gph}\,T \cap (U \times V) = \mathrm{gph}\,S \cap (U \times V)$ for any monotone mapping $S : I\!\!R^n \rightrightarrows I\!\!R^n$ satisfying the inclusion $\mathrm{gph}\,T \cap (U \times V) \subset \mathrm{gph}\,S$.

3 Second-Order Characterizations of Hölderian Full Stability

Here we define the notions of Lipschitzian and Hölderian full stability of local minimizers in the general setting of extended-real-valued functions and derive second-order characterizations of the Hölderian one.

Given $f : I\!\!R^n \times I\!\!R^d \to \overline{I\!\!R}$, consider the problem

$$\mathcal{P} \quad \text{minimize} \ f(x, \bar{p}) \ \text{over} \ x \in I\!\!R^n \tag{3.1}$$

and its two-parametric perturbations constructed as

$$\mathcal{P}(v, p) \quad \text{minimize} \ f(x, p) - \langle v, x \rangle \ \text{over} \ x \in I\!\!R^n \tag{3.2}$$

with the *basic* parameter perturbation $p \in I\!\!R^d$ and the *tilt* one $v \in I\!\!R^n$.

Throughout the paper we always assume that the function f in (3.2) is *lower semicontinuous* on $\mathbb{R}^n \times \mathbb{R}^d$. Given $(\bar{x}, \bar{p}) \in \text{dom} f$, $(v, p) \in \mathbb{R}^n \times \mathbb{R}^d$, and $\gamma > 0$, associate with these data the following objects:

$$
\begin{aligned}
m_\gamma(v, p) &:= \inf \left\{ f(x, p) - \langle v, x \rangle \,\middle|\, \|x - \bar{x}\| \leq \gamma \right\}, \\
M_\gamma(v, p) &:= \text{argmin} \left\{ f(x, p) - \langle v, x \rangle \,\middle|\, \|x - \bar{x}\| \leq \gamma \right\}, \\
S(v, p) &:= \left\{ x \in \mathbb{R}^n \,\middle|\, v \in \partial_x f(x, p) \right\}.
\end{aligned}
\tag{3.3}
$$

As in [16], we say that the *basic constraint qualification* (BCQ) holds at $(\bar{x}, \bar{p})$ if

$$
\left[(0, q) \in \partial^\infty f(\bar{x}, \bar{p}) \right] \implies q = 0.
\tag{3.4}
$$

By the Mordukhovich criterion (2.7) this is equivalent to the fact that the set-valued mapping

$$
F : p \mapsto \text{epi}\, f_p \text{ is Lipschitz-like around } \left(\bar{p}, (\bar{x}, f(\bar{x}, \bar{p})) \right),
\tag{3.5}
$$

where the notation $f_p(\cdot) := f(\cdot, p)$ is employed throughout the whole paper; see, e.g., [38, Proposition 10.16]. The following rather straightforward lemma taken from [16, Proposition 3.1] is a useful consequence of BCQ.

Lemma 3.1 (consequence of BCQ). *The validity of BCQ (3.4) ensures the existence of neighborhoods U of $\bar{x}$ and P of $\bar{p}$ along with a number $\varepsilon > 0$ such that*

$$
\left.
\begin{aligned}
&x_1 \in U, p_1, p_2 \in P \\
&f(x_1, p_1) \leq f(\bar{x}, \bar{p}) + \varepsilon
\end{aligned}
\right\} \implies \exists x_2 \ \ with \ \ \left\{
\begin{aligned}
&\|x_1 - x_2\| \leq c\|p_1 - p_2\|, \\
&f(x_2, p_2) \leq f(x_1, p_1) + c\|p_1 - p_2\|,
\end{aligned}
\right.
\tag{3.6}
$$

where $c > 0$ is a modulus of the Lipschitz-like property in (3.5).

Now we are ready to formulate the two main stability properties studied in this paper. The first (Lipschitzian) was introduced in [16] with the modulus modification given in [22] while its Hölderian counterpart has been recently introduced earlier in [25].

Definition 3.2 (Lipschitzian and Hölderian full stability). *Given $f \colon \mathbb{R}^n \times \mathbb{R}^d \to \overline{\mathbb{R}}$ and a point $\bar{x} \in \text{dom} f$ in (3.1) with some nominal basic parameter $\bar{p} \in \mathbb{R}^d$, we say that:*

(i) The point $\bar{x}$ is a LIPSCHITZIAN FULLY STABLE LOCAL MINIMIZER of $\mathcal{P}(\bar{v}, \bar{p})$ in (3.2) corresponding to $\bar{p}$ and some tilt parameter $\bar{v} \in \mathbb{R}^n$ with a modulus pair $(\kappa, \ell) \in \mathbb{R}^2_> := \{(a, b) \in \mathbb{R}^2 \mid a > 0, b > 0\}$ if there are a number $\gamma > 0$ and a neighborhood $V \times P$ of $(\bar{v}, \bar{p})$ such that the mapping $(v, p) \mapsto M_\gamma(v, p)$ is single-valued on $V \times P$ with $M_\gamma(\bar{v}, \bar{p}) = \bar{x}$ satisfying the Lipschitz condition

$$
\|M_\gamma(v_1, p_1) - M_\gamma(v_2, p_2)\| \leq \kappa \|v_1 - v_2\| + \ell \|p_1 - p_2\| \ \ for \ all \ \ v_1, v_2 \in V, \ p_1, p_2 \in P
\tag{3.7}
$$

and that the function $(v, p) \mapsto m_\gamma(v, p)$ is also Lipschitz continuous around $(\bar{v}, \bar{p})$.

(ii) The point $\bar{x}$ is a HÖLDERIAN FULLY STABLE LOCAL MINIMIZER of problem $\mathcal{P}(\bar{v}, \bar{p})$ with a modulus pair $(\kappa, \ell) \in \mathbb{R}^2_>$ if there is a number $\gamma > 0$ such that the mapping M_γ is single-valued on some neighborhood $V \times P$ of $(\bar{v}, \bar{p})$ with $M_\gamma(\bar{v}, \bar{p}) = \bar{x}$ and

$$
\|M_\gamma(v_1, p_1) - M_\gamma(v_2, p_2)\| \leq \kappa \|v_1 - v_2\| + \ell \|p_1 - p_2\|^{\frac{1}{2}} \ \ for \ all \ \ v_1, v_2 \in V, \ p_1, p_2 \in P.
\tag{3.8}
$$

It is worth mentioning that we always have $\bar{v} \in \partial_x f(\bar{x}, \bar{p})$ in Definition 3.2 due to the (generalized) Fermat stationary condition for the local minimizer $\bar{x}$ in $\mathcal{P}(\bar{v}, \bar{p})$. Observe also that when BCQ (3.4) holds at $(\bar{x}, \bar{p})$, the function m_γ is locally Lipschitzian automatically provided that $M_\gamma(\bar{v}, \bar{p}) = \bar{x}$ for some $\gamma > 0$; see [16, Proposition 3.5]. It happens, in particular, when the parameter p is absent. In this case both stability properties in Definitions 3.2 reduce to *tilt stability* of the local minimizer $\bar{x}$ introduced in [34].

Since BCQ (3.4) is assumed in all the results of the paper and the condition $M_\gamma(\bar{v}, \bar{p}) = \bar{x}$ is imposed in Definition 3.2(i), we will not discuss further the local Lipschitz continuity of m_γ but focus on the study of the Lipschitzian (3.7) and Hölderian (3.8) properties of M_γ when $\gamma > 0$ is sufficiently small. It follows from Theorem 3.4 and Theorem 4.1 given below that these two properties agree when the graphical mapping $p \mapsto \text{gph}\, \partial_x f(\cdot, p)$ is Lipschitz-like around $(\bar{p}, (\bar{x}, \bar{v}))$. However, in the general case the Hölderian full stability in Definition 3.2 is *strictly weaker* than the Lipschitzian one and the exponent $r = \frac{1}{2}$ for the parameter p in (3.8) cannot be improved; see the discussion in Section 5.

The rest of this section is devoted to deriving new second-order characterizations of Hölderian full stability. We begin with formulating the following lemma of convex analysis taken from [16, Lemma 5.2].

Lemma 3.3 (convex functions with smooth conjugates). *Let* $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be a convex function such that its Fenchel conjugate* h^* *is differentiable on a certain open set* $V \subset \mathbb{R}^n$ *and the gradient mapping* ∇h^* *is Lipschitz continuous on* V *with some constant* $\kappa > 0$. *Let* $\lambda > 0$ *and* $V_\lambda := \{v \in \mathbb{R}^n | \ \mathbb{B}_\lambda(v) \subset V\}$. *Then we have the quadratic growth condition*

$$h(x) \geq h(u) + \langle v, x - u \rangle + \frac{1}{2\kappa} \|x - u\|^2 \quad \text{whenever} \ \ v \in \partial h(u) \cap V_\lambda, \ \|x - u\| \leq \lambda \kappa.$$

Our first major result gives qualitative and quantitative characterizations of Hölderian full stability for parametrically continuously prox-regular functions via the uniform second-order growth condition formulated in the following theorem. This condition is an extended version of that from [3, Definition 5.16] introduced for $\mathcal{C}^2$-smooth conic programs with respect to the $\mathcal{C}^2$-*smooth parametrization*; see also Definition 5.1 below. The proof of implication **(ii)**$\Longrightarrow$**(i)** in this result has some similarity with that of [3, Theorem 5.17] for problem of conic programming. The obtained characterization and the relationship between the moduli in **(i)** and **(ii)** of Theorem 3.4 are improvements of the corresponding results by Mordukhovich and Nghia [25, Theorem 4.5], which are given in infinite-dimensional spaces. When the parameter p is absent (i.e., we have the tilt stability setting), this goes back to [24, Theorem 3.2] and partly to [6, Theorem 3.3], where the relationship between the moduli is not specified.

Theorem 3.4 (Hölderian full stability via uniform second-order growth). *Assume that BCQ* (3.4) *holds at* $(\bar{x}, \bar{p}) \in \text{dom} f$ *and that the function* f *is parametrically continuously prox-regular at* $(\bar{x}, \bar{p})$ *for* $\bar{v} \in \partial_x f(\bar{x}, \bar{p})$. *Then the following assertions are equivalent:*

(i) *The point* $\bar{x}$ *is a Hölderian fully stable local minimizer of* $\mathcal{P}(\bar{v}, \bar{p})$ *with a modulus pair* $(\kappa, \ell) \in \mathbb{R}^2_>$.

(ii) *There are neighborhoods* U *of* $\bar{x}$, V *of* $\bar{v}$, *and* P *of* $\bar{p}$ *such that the mapping* S *from* (3.3) *admits a single-valued localization* ϑ *relative to* $V \times P \times U$ *such that for any triple* $(v, p, u) \in \text{gph} \vartheta = \text{gph} S \cap (V \times P \times U)$ *we have the* UNIFORM SECOND-ORDER GROWTH CONDITION

$$f(x, p) \geq f(u, p) + \langle v, x - u \rangle + \frac{1}{2\kappa} \|x - u\|^2 \text{ whenever} \ \ x \in U. \tag{3.9}$$

Proof. To justify **(i)**$\Longrightarrow$**(ii)**, suppose that $\bar{x}$ is a Hölderian fully stable local minimizer of problem $\mathcal{P}(\bar{v}, \bar{p})$ with some modulus pair $(\kappa, \ell) \in \mathbb{R}^2_>$. Thus there is a number $\gamma > 0$ such that the mapping M_γ in (3.3) satisfies (3.8) on some neighborhood $V \times P$ of $(\bar{v}, \bar{p})$ with $M_\gamma(\bar{v}, \bar{p}) = \bar{x}$. It follows that

$$\|M_\gamma(v, p) - \bar{x}\| = \|M_\gamma(v, p) - M_\gamma(\bar{v}, \bar{p})\| \leq \kappa \|v - \bar{v}\| + \ell \|p - \bar{p}\|^{\frac{1}{2}} \ \text{ for all } \ (v, p) \in V \times P.$$

By shrinking V and P, the latter allows us to suppose that $M_\gamma(V \times P) \subset U$ with $U := \text{int} \ \mathbb{B}_\gamma(\bar{x})$. Fix $p \in P$ and observe that $\widehat{M}_p(\cdot) := M_\gamma(\cdot, p)$ is monotone. Indeed, for any $v_1, v_2 \in \mathbb{R}^n$ we have

$$\begin{aligned} \langle v_1 - v_2, \widehat{M}_p(v_1) - \widehat{M}_p(v_2) \rangle \ &= \langle v_1, \widehat{M}_p(v_1) - \widehat{M}_p(v_2) \rangle + \langle v_2, \widehat{M}_p(v_2) - \widehat{M}_p(v_1) \rangle \\ &\geq \left[f_p(\widehat{M}_p(v_1)) - f_p(\widehat{M}_p(v_2)) \right] + \left[f_p(\widehat{M}_p(v_2)) - f_p(\widehat{M}_p(v_1)) \right] = 0. \end{aligned}$$

Note from (3.8) that $\widehat{M}_p$ is Lipschitz continuous on V with constant κ, and so it is maximally monotone relative to $V \times U$ (since $\widehat{M}_p : V \to U$). Consider next the Fenchel conjugate of $f_p + \delta_{\mathbb{B}_\gamma(\bar{x})}$ given by

$$g_p(v) := (f_p + \delta_{\mathbb{B}_\gamma(\bar{x})})^*(v) = \sup_{x \in \mathbb{B}_\gamma(\bar{x})} \left\{ \langle v, x \rangle - f_p(x) \right\} \ \text{ for all } \ v \in \mathbb{R}^n,$$

which is a proper l.s.c. convex function. For any $v \in V$ we get from (3.3) the representation $g_p(v) = \langle v, \widehat{M}_p(v) \rangle - f_p(\widehat{M}_p(v))$ and observe that

$$g_p(w) - g_p(v) \geq \langle w, \widehat{M}_p(v) \rangle - f_p(\widehat{M}_p(v)) - (\langle v, \widehat{M}_p(v) \rangle - f_p(\widehat{M}_p(v))) = \langle w - v, \widehat{M}_p(v) \rangle, \quad w \in \mathbb{R}^n.$$

This ensures that $\widehat{M}_p(v) \in \partial g_p(v)$. Since g_p is convex, its subdifferential ∂g_p is monotone. This together with the maximal monotonicity of M relative to $(V \times U)$ implies that $\text{gph} \partial g_p \cap (V \times U) = \text{gph} \widehat{M}_p \cap (V \times U)$, and thus $\partial g_p(v) = \widehat{M}_p(v)$ for all $v \in V$. Hence ∂g_p is single-valued and Lipschitz continuous on V. It thus follows that g_p is differentiable and Lipschitz continuous with constant κ on V.

Denoting $h_p := g_p^*$, we deduce from the biconjugate theorem of convex analysis [38, Theorem 11.1] that $h_p^* = g_p$. Since h_p^* is differentiable and Lipschitz continuous with constant κ on V, we can use Lemma 3.3. Let $\lambda > 0$ be so small that $\mathbb{B}_\lambda(\bar{v}) \subset V$. Thus the set $V_\lambda := \{v \in \mathbb{R}^n | \ \mathbb{B}_\lambda(v) \subset V\}$ is a neighborhood of $\bar{v}$. Let $U_1 \subset U$ be some neighborhood of $\bar{x}$ satisfying $\|x - u\| \leq \kappa\lambda$ for all $x, u \in U_1$. Due to the Hölder continuity of M_γ in (3.12), we find neighborhoods $V_1 \subset V_\lambda$ of $\bar{v}$ and $P_1 \subset P$ of $\bar{p}$ such that

$$M_\gamma(v, p) \subset U_1 \ \text{ whenever } \ (v, p) \in V_1 \times P_1.$$

With $u = \nabla h_p^*(v) = \nabla g_p(v) = \widehat{M}_p(v)$ for all $(v, p) \in V_1 \times P_1$ we have $v \in \partial h_p(u) \cap V_\lambda$ and $u \in U_1$. Applying Lemma 3.3 gives us the estimate

$$h_p(x) \geq h_p(u) + \langle v, x - u \rangle + \frac{1}{2\kappa}\|x - u\|^2 \ \text{ for all } \ x \in U_1, \ (u, v) \in \text{gph}\, \partial h_p \cap (U_1 \times V_1), \ p \in P_1. \quad (3.10)$$

Given any $(v, p) \in V_1 \times P_1$, observe from the Fermat stationary rule for problem $\mathcal{P}(v, p)$ at the local minimizer $\widehat{M}_p(v) \in U_1$ that $(\partial h_p)^{-1}(v) = \nabla g_p(v) = \widehat{M}_p(v) \in (\partial f_p)^{-1}(v)$. Since f is parametrically continuously prox-regular at $(\bar{x}, \bar{p})$ for $\bar{v}$, we assume without loss of generality that

$$f_p(x) \geq f_p(u) + \langle v, x - u \rangle - \frac{r}{2}\|x - u\|^2 \ \text{ for all } \ x \in U_1, \ (u, p, v) \in \text{gph}\, \partial_x f \cap (U_1 \times P_1 \times V_1) \quad (3.11)$$

with some $r > 0$. Fixing $p \in P_1$, let T_p be a localization of ∂f_p relative to $U_1 \times V_1$, and let $I : \mathbb{R}^n \to \mathbb{R}^n$ be the identity mapping. We get from the above inequality that $T_p + sI$ is strongly monotone for any $s > r$, which implies that $(T_p + sI)^{-1}$ is single-valued on its domain.

Define further $W := J(U_1 \times V_1)$ with $J(u, v) := (v + su, u)$ for all $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$. This shows that W is a neighborhood of $(\bar{v} + s\bar{x}, \bar{x})$. It is easy to deduce from the inclusion $(\partial h_p)^{-1}(v) = \widehat{M}_p(v) \in \partial f_p^{-1}(v)$ as $(v, p) \in V_1 \times P_1$ the following relationships:

$$\text{gph}\,(\partial h_p + sI)^{-1} \cap W = \text{gph}\,(\widehat{M}_p^{-1} + sI)^{-1} \cap W \subset \text{gph}\,(\partial f_p + sI)^{-1} \cap W. \quad (3.12)$$

Claim 1. *Let $\alpha, \beta > 0$ satisfy $\mathbb{B}_\alpha(\bar{v} + s\bar{x}) \times \mathbb{B}_\beta(\bar{x}) \subset W$ and $\mathbb{B}_\beta(\bar{x}) \subset U_1$, and let $\ell > 0$ be taken from (3.8). Then there are constants $\varepsilon_1, \delta_1, \nu_1 > 0$ with $\mathbb{B}_{\nu_1}(\bar{x}) \subset V_1$ and*

$$\left(\frac{1}{2\kappa} + \frac{s}{2}\right)^{-1}\left[\varepsilon_1 + (s\kappa + 1)\nu_1 + s\ell\delta_1^{\frac{1}{2}}\right] + \kappa\nu_1 + \ell\delta_1^{\frac{1}{2}} < \beta, \ \text{ and } \ \varepsilon_1 < \alpha \quad (3.13)$$

such that $(\partial h_p + sI)^{-1}(v)$ exists and belongs to $\text{int}\,\mathbb{B}_\beta(\bar{x})$ for all $v \in \text{int}\,\mathbb{B}_{\varepsilon_1}(\bar{v} + s\bar{x})$ and $p \in \text{int}\,\mathbb{B}_{\delta_1}(\bar{p})$.

To justify this claim, take any $v \in \text{int}\,\mathbb{B}_{\varepsilon_1}(\bar{v} + s\bar{x})$ and $p \in \text{int}\,\mathbb{B}_{\delta_1}(\bar{p})$. Fix also $\widetilde{u} := M_\gamma(\widetilde{v}, p)$ with $\widetilde{v} \in \mathbb{B}_{\nu_1}(\bar{v}) \subset V_1$. We get from (3.8) and (3.13) that

$$\|u - \bar{x}\| \leq \kappa\|\widetilde{v} - \bar{v}\| + \ell\|p - \bar{p}\|^{\frac{1}{2}} \leq \kappa\nu_1 + \ell\delta_1^{\frac{1}{2}} < \beta, \quad (3.14)$$

which gives us $u \in \mathbb{B}_\beta(\bar{x}) \subset U_1$ and thus $(\widetilde{u}, \widetilde{v}) \in \text{gph}\, \partial h_p \cap (U_1 \times V_1)$ as appeared in (3.10). Consider next the following optimization problem

$$\inf_{x \in \mathbb{B}_\beta(\bar{x})} \left\{ h_p(x) + \frac{s}{2}\|x\|^2 - \langle v, x \rangle \right\} \quad (3.15)$$

and suppose that $u \in \mathbb{B}_\beta(\bar{x}) \subset U_1$ is a solution to this problem, which always exists due to the l.s.c. property of the cost function. It follows from (3.15) and (3.10) that

$$
\begin{aligned}
h_p(\widetilde{u}) + \tfrac{s}{2}\|\widetilde{u}\|^2 - \langle v, \widetilde{u} \rangle \ &\geq h_p(u) + \tfrac{s}{2}\|u\|^2 - \langle v, u \rangle \\
&\geq h_p(\widetilde{u}) + \langle \widetilde{v}, u - \widetilde{u} \rangle + \tfrac{1}{2\kappa}\|u - \widetilde{u}\|^2 + \tfrac{s}{2}\|u\|^2 - \langle v, u \rangle,
\end{aligned}
$$

which implies in turn the relationships

$$
\begin{aligned}
\left(\tfrac{1}{2\kappa} + \tfrac{s}{2}\right)\|u - \widetilde{u}\|^2 \ &\leq \langle v - \widetilde{v}, u - \widetilde{u} \rangle + \tfrac{s}{2}\|\widetilde{u}\|^2 - \tfrac{s}{2}\|u\|^2 + \tfrac{s}{2}\|u - \widetilde{u}\|^2 \\
&= \langle v - \widetilde{v} - \tfrac{s}{2}(u + \widetilde{u}) + \tfrac{s}{2}(u - \widetilde{u}), \widetilde{u} - u \rangle \\
&= \langle v - \widetilde{v} - s\widetilde{u}, u - \widetilde{u} \rangle \leq \|v - \widetilde{v} - s\widetilde{u}\| \cdot \|u - \widetilde{u}\|.
\end{aligned}
$$

This together with (3.14) and (3.13) tells us that

$$
\begin{aligned}
\|u - \bar{x}\| \;&\le\; \|u - \widetilde{u}\| + \|\widetilde{u} - \bar{x}\| \le \left(\tfrac{1}{2\kappa} + \tfrac{s}{2}\right)^{-1} \|v - \widetilde{v} - s\widetilde{u}\| + \|\widetilde{u} - \bar{x}\| \\
&\le \left(\tfrac{1}{2\kappa} + \tfrac{s}{2}\right)^{-1} \left[\|v - \bar{v} - s\bar{u}\| + \|\widetilde{v} - \bar{v}\| + s\|\widetilde{u} - \bar{u}\|\right] + \|\widetilde{u} - \bar{x}\| \\
&\le \left(\tfrac{1}{2\kappa} + \tfrac{s}{2}\right)^{-1} \left[\varepsilon_1 + \nu_1 + s(\kappa\nu_1 + \ell\delta_1^{\frac{1}{2}})\right] + \kappa\nu_1 + \ell\delta_1^{\frac{1}{2}} < \beta
\end{aligned}
$$

Employing the Fermat stationary rule for problem (3.15) and then the subdifferential sum rule, we arrive at

$$
0 \in \partial\left(h_p + \frac{s}{2}\|\cdot\|^2 - \langle v, \cdot\rangle\right)(u) \subset \partial h_p(u) + su - v,
$$

which yields that $u \in (\partial h_p + sI)^{-1}(v)$ and thus completes the proof of the claim.

Pick further any $v \in \operatorname{int} I\!\!B_{\varepsilon_1}(\bar{v} + s\bar{x})$ and $p \in \operatorname{int} I\!\!B_{\delta_1}(\bar{p})$ and get from Claim 1 the existence of $u \in (\partial h_p + sI)^{-1}(v) \cap (\operatorname{int} I\!\!B_\beta(\bar{x}))$. It follows from the discussion after (3.11) that $(\partial f + sI)^{-1}(v) \cap (\operatorname{int} I\!\!B_\beta(\bar{x}))$ is either singleton or empty. Furthermore, (3.12) tells us that $u = (\partial h_p + sI)^{-1}(v) = (\partial f_p + sI)^{-1}(v) \cap (\operatorname{int} I\!\!B_\beta(\bar{x}))$. Hence we have

$$
\operatorname{gph}(\partial h_p + sI)^{-1} \cap W_1 = \operatorname{gph}(\partial f_p + sI)^{-1} \cap W_1 \ \text{ with } \ W_1 := \operatorname{int} I\!\!B_{\varepsilon_1}(\bar{v} + s\bar{x}) \times \operatorname{int} I\!\!B_\beta(\bar{x}),
$$

which implies in turn the equality

$$
\operatorname{gph} \partial h_p \cap W_2 = \operatorname{gph} \partial f_p \cap W_2 \ \text{ with } \ W_2 := J^{-1}(W_1).
$$

Since W_2 is a neighborhood of $(\bar{x}, \bar{v})$, we find a neighborhood $U_2 \times V_2 \subset W_2 \cap (U_1 \times V_1)$ of $(\bar{x}, \bar{v})$ such that

$$
\operatorname{gph} \nabla g_p^{-1} \cap (U_2 \times V_2) = \operatorname{gph} \partial h_p \cap (U_2 \times V_2) = \operatorname{gph} \partial f_p \cap (U_2 \times V_2). \tag{3.16}
$$

Define now the mapping ϑ from $I\!\!R^n \times I\!\!R^d$ into $I\!\!R^n$ by $\operatorname{gph} \vartheta := \operatorname{gph} S \cap (V_2 \times P_1 \times U_2)$, where S is taken from (3.3). Pick any triple $(v, p, u) \in \operatorname{gph} \vartheta$ and deduce from (3.16) that $u = \nabla g_p(v) = \widehat{M}_p(v)$. Therefore

$$
h_p(u) = g_p^*(u) = \langle u, v\rangle - g_p(v) = \langle u, v\rangle - \left(\langle v, \widehat{M}_p(v)\rangle - f_p(\widehat{M}_p(v))\right) = f_p(u).
$$

Combining this with (3.10) gives us for any $x \in U_2$ and $(v, p, u) \in \operatorname{gph} \vartheta$ that

$$
f_p(x) \ge (f_p + \delta_{I\!\!B_\gamma(\bar{x})})^{**}(x) = h_p(x) \ge f_p(u) + \langle v, x - u\rangle + \frac{1}{2\kappa}\|x - u\|^2,
$$

which readily ensures the single-valuedness of ϑ and inequality (3.9), and thus justifies **(ii)**.

To verify next **(ii)**$\Longrightarrow$**(i)** under BCQ (3.4), we shrink neighborhoods U, V, P in **(ii)** if necessary so that (3.6) holds on them with some constants $\varepsilon, c > 0$. It is clear from (3.9) that $M_\gamma(\bar{v}, \bar{p}) = \bar{x}$ for any $\gamma > 0$ satisfying $I\!\!B_\gamma(\bar{x}) \subset U$. We split the rest of the proof into the following two claims having their own interest.

Claim 2. *We have* $S(v, p) \cap U = \vartheta(v, p) = M_\gamma(v, p)$ *for all* $v \in I\!\!B_\delta(\bar{v}) \subset V$ *and* $p \in I\!\!B_\delta(\bar{p}) \subset P$ *when* $\delta, \gamma > 0$ *are sufficiently small and* $2c\delta < \gamma$.

To justify it, pick $v \in I\!\!B_\delta(\bar{v})$ and $p \in I\!\!B_\delta(\bar{p})$. By choosing $(x_1, p_1) = (\bar{x}, \bar{p})$ and $p_2 = p$ in (3.6) we find $x_2 = x$ with $\|x - \bar{x}\| \le c\|p - \bar{p}\| \le c\delta < \gamma$ such that $f(x, p) \le f(\bar{x}, \bar{p}) + c\|p - \bar{p}\| \le f(\bar{x}, \bar{p}) + c\delta$. Take any $u \in M_\gamma(v, p)$, the latter yields

$$
f(\bar{x}, \bar{p}) + c\delta \ge f(x, p) \ge f(u, p) - \langle v, u - x\rangle, \tag{3.17}
$$

which implies in turns the estimates

$$
f(u, p) \le f(\bar{x}, \bar{p}) + c\delta + \|v\|(\|u - \bar{x}\| + \|x - \bar{x}\|) \le f(\bar{x}, \bar{p}) + c\delta + (\|\bar{v}\| + \delta)(\gamma + c\delta).
$$

Thus $f(u, p) \le f(\bar{x}, \bar{p}) + \varepsilon$ whenever $\gamma, \delta > 0$ are sufficiently small. Using (3.6) again (by choosing now $(x_1, p_1) = (u, p)$ and $p_2 = \bar{p}$ therein), we find $w \in I\!\!R^n$ such that $\|w - u\| \le c\|\bar{p} - p\| \le c\delta < \gamma$ and that

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$f(w, \bar{p}) \le f(u, p) + c\|\bar{p} - p\|$. It follows that $\|w - \bar{x}\| \le \|w - u\| + \|u - \bar{x}\| \le 2\gamma$. Hence $w \in U$ when $\gamma > 0$ is sufficiently small. This together with (3.9) gives us that

$$f(u, p) + c\delta \ge f(w, \bar{p}) \ge f(\bar{x}, \bar{p}) + \langle \bar{v}, w - \bar{x} \rangle + \frac{1}{2\kappa}\|w - \bar{x}\|^2,$$

which ensures together with (3.17) the estimates

$$\begin{aligned}
2c\delta \quad &\ge -\langle v, u - x \rangle + \langle \bar{v}, w - \bar{x} \rangle + \frac{1}{2\kappa}\|w - \bar{x}\|^2 \\
&\ge -\langle v, u - w + \bar{x} - x \rangle + \langle \bar{v} - v, w - \bar{x} \rangle + \frac{1}{2\kappa}\|w - \bar{x}\|^2 \\
&\ge -\|v\|(\|u - w\| + \|\bar{x} - x\|) - \|\bar{v} - v\|\|w - \bar{x}\| + \frac{1}{2\kappa}\|w - \bar{x}\|^2 \\
&\ge -(\|\bar{v}\| + \delta)(c\delta + c\delta) - \delta\|w - \bar{x}\| + \frac{1}{2\kappa}\|w - \bar{x}\|^2.
\end{aligned}$$

When δ is sufficiently small, we get from the obtained inequalities that $\|w - \bar{x}\| < \gamma - c\delta$, which gives us in turn the estimates $\|u - \bar{x}\| \le \|w - \bar{x}\| + \|u - w\| < \gamma - c\delta + c\delta = \gamma$. Since $u \in M_\gamma(v, p)$ and $u \in \text{int } \mathbb{B}_\gamma(\bar{x})$, the Fermat rule tells us that $v \in \partial f_p(u)$, or equivalently $u \in (\partial f_p)^{-1}(v) \cap U = \vartheta(v, p)$. Note from (3.9) that if $\vartheta(v, p)$ exists, it must be a singleton. It follows that $\vartheta(v, p) = M_\gamma(v, p)$ for all $v \in \mathbb{B}_\delta(\bar{v})$ and $p \in \mathbb{B}_\delta(\bar{p})$, which completes the proof of Claim 2.

Claim 3. *We may find* $\delta, \gamma > 0$ *with* $2c\delta < \gamma$ *sufficiently small such that Claim 2 holds and that there is a positive number* ℓ *for which*

$$\|M_\gamma(v_1, p_1) - M_\gamma(v_2, p_2)\| \le \kappa\|v_1 - v_2\| + \ell\|p_1 - p_2\|^{\frac{1}{2}} \tag{3.18}$$

whenever $v_1, v_2 \in V_1 := \text{int } \mathbb{B}_\delta(\bar{v})$ *and* $p_1, p_2 \in P_1 := \text{int } \mathbb{B}_\delta(\bar{p})$, *where* $\kappa > 0$ *is taken from* (3.9).

Indeed, define $u_i := M_\gamma(v_i, p_i) = \vartheta(v_i, p_i)$, $i = 1, 2$, which exist as in Claim 2, and get from (3.6) that there are some x_i such that $\|x_i - \bar{x}\| \le c\|p_i - \bar{p}\| \le c\delta < \gamma$ and $f(x_i, p_i) \le f(\bar{x}, \bar{p}) + c\|p_i - \bar{p}\|$. It follows that

$$\begin{aligned}
f(\bar{x}, \bar{p}) + c\delta \quad &\ge f(x_i, p_i) \ge f(u_i, p_i) + \langle v_i, x_i - u_i \rangle \ge f(u_i, p_i) - \|v_i\|\|x_i - u_i\| \\
&\ge f(u_i, p_i) - \|v_i\|(\|u_i - \bar{x}\| + \|x_i - \bar{x}\|) \ge f(u_i, p_i) - (\|\bar{v}\| + \delta)(\gamma + c\delta),
\end{aligned}$$

which yields $f(u_i, p_i) \le f(\bar{x}, \bar{p}) + \varepsilon$ for $\delta, \gamma > 0$ sufficiently small. Employing (3.6) gives us w_1, w_2 with

$$\begin{cases} \|u_2 - w_1\| \le c\|p_2 - p_1\| \le 2c\delta, \ \|u_1 - w_2\| \le c\|p_1 - p_2\| \le 2c\delta \\ f(w_1, p_1) \le f(u_2, p_2) + c\|p_1 - p_2\|, \ f(w_2, p_2) \le f(u_1, p_1) + c\|p_1 - p_2\|. \end{cases} \tag{3.19}$$

Hence we deduce the estimate $\|w_1 - \bar{x}\| \le \|u_2 - w_1\| + \|u_2 - \bar{x}\| \le 2c\delta + \gamma$, which implies that $w_1 \in U$ and simultaneously $w_2 \in U$ when δ and γ are sufficiently small. This together with (3.9) ensures that

$$\begin{cases} f(w_1, p_1) \ge f(u_1, p_1) + \langle v_1, w_1 - u_1 \rangle + \frac{1}{2\kappa}\|w_1 - u_1\|^2, \\ f(w_2, p_2) \ge f(u_2, p_2) + \langle v_2, w_2 - u_2 \rangle + \frac{1}{2\kappa}\|w_2 - u_2\|^2. \end{cases}$$

Summing up these two inequalities and combining it with (3.19) yields

$$\begin{aligned}
2c\|p_1 - p_2\| \quad &\ge \langle v_1, w_1 - u_1 \rangle + \frac{1}{2\kappa}\|w_1 - u_1\|^2 + \langle v_2, w_2 - u_2 \rangle + \frac{1}{2\kappa}\|w_2 - u_2\|^2 \\
&\ge \langle v_1 - v_2, u_2 - u_1 \rangle + \langle v_1, w_1 - u_2 \rangle + \langle v_2, w_2 - u_1 \rangle + \frac{1}{2\kappa}(\|w_1 - u_2\| - \|u_1 - u_2\|)^2 \\
&\quad + \frac{1}{2\kappa}(\|w_2 - u_1\| - \|u_1 - u_2\|)^2 \\
&\ge -\|v_1 - v_2\| \cdot \|u_1 - u_2\| - (\|\bar{v}\| + \delta)\|w_1 - u_2\| - (\|\bar{v}\| + \delta)\|w_2 - u_1\| \\
&\quad - \frac{1}{\kappa}(\|w_1 - u_2\| + \|w_2 - u_1\|)\|u_1 - u_2\| + \frac{1}{\kappa}\|u_1 - u_2\|^2 \\
&\ge -\|v_1 - v_2\| \cdot \|u_1 - u_2\| - (\|\bar{v}\| + \delta)c\|p_1 - p_2\| - (\|\bar{v}\| + \delta)c\|p_1 - p_2\| \\
&\quad - \frac{1}{\kappa}(c\|p_1 - p_2\| + c\|p_1 - p_2\|)\|u_1 - u_2\| + \frac{1}{\kappa}\|u_1 - u_2\|^2 \\
&\ge -\left(\|v_1 - v_2\| + \frac{2c}{\kappa}\|p_1 - p_2\|\right)\|u_1 - u_2\| - (\|\bar{v}\| + \delta)2c\|p_1 - p_2\| + \frac{1}{\kappa}\|u_1 - u_2\|^2,
\end{aligned}$$

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which implies in turn that

$$\frac{1}{\kappa}\|u_1 - u_2\|^2 - \left(\|v_1 - v_2\| + \frac{2c}{\kappa}\|p_1 - p_2\|\right)\|u_1 - u_2\| - 2c(\|\bar{v}\| + \delta + 1)\|p_1 - p_2\| \leq 0.$$

Therefore we arrive at the following estimates:

$$
\begin{aligned}
\|u_1 - u_2\| \;&\leq\; \frac{\kappa}{2}\Bigg[\|v_1 - v_2\| + \frac{2c}{\kappa}\|p_1 - p_2\| + \\
&\qquad \sqrt{\left(\|v_1 - v_2\| + \frac{2c}{\kappa}\|p_1 - p_2\|\right)^2 + \frac{8c}{\kappa}(\|\bar{v}\| + \delta + 1)\|p_1 - p_2\|}\Bigg] \\
&\leq \kappa\|v_1 - v_2\| + 2c\|p_1 - p_2\| + \sqrt{2c\kappa(\|\bar{v}\| + \delta + 1)}\|p_1 - p_2\|^{\frac{1}{2}} \\
&\leq \kappa\|v_1 - v_2\| + 2c\sqrt{2\delta}\|p_1 - p_2\|^{\frac{1}{2}} + \sqrt{2c\kappa(\|\bar{v}\| + \delta + 1)}\|p_1 - p_2\|^{\frac{1}{2}} \\
&\leq \kappa\|v_1 - v_2\| + \left(2c\sqrt{2\delta} + \sqrt{2c\kappa(\|\bar{v}\| + \delta + 1)}\right)\|p_1 - p_2\|^{\frac{1}{2}}.
\end{aligned}
$$

This clearly justifies the Hölderian condition (3.18) with

$$\ell := \left(2c\sqrt{2\delta} + \sqrt{2c\kappa(\|\bar{v}\| + \delta + 1)}\right)$$

and thus completes the proof of Claim 3 and of the whole theorem. $\triangle$

Remark 3.5 (specification of the argminimum set). We are going to use several times in the rest of the paper the following fact proved in Claim 2 above: For any parameter pair (v,p) near $(\bar{v}, \bar{p})$ and any sufficiently small number $\gamma > 0$ the argminimum set $M_\gamma(v,p)$ from (3.3) reduces to the single-valued localization $\vartheta(p,v)$ of the partial subdifferential inverse $S(v,p)$ also defined in (3.3) provided that the equivalent conditions in **(i)** and **(ii)** of Theorem 3.4 are satisfied.

The next consequence of Theorem 3.4 shows that Hölderian full stability is equivalent to the Hölderian continuity of a localization of the mapping S in (3.3), which is closely related to Hölderian continuity in [41] for the case of variational inequalities.

Corollary 3.6 (Hölderian localization). *Assume that BCQ (3.4) holds at $(\bar{x}, \bar{p}) \in \operatorname{dom} f$ and that f is parametrically continuously prox-regular at $(\bar{x}, \bar{p})$ for $\bar{v} \in \partial_x f(\bar{x}, \bar{p})$. The following assertions are equivalent:*
(i) The point $\bar{x}$ is a Hölderian fully stable local minimizer of $\mathcal{P}(\bar{v}, \bar{p})$ with a modulus pair $(\kappa, \ell) \in \mathbb{R}^2_>$.
(ii) The point $\bar{x}$ is a local minimizer of $\mathcal{P}(\bar{v}, \bar{p})$ and there exists a neighborhood $U \times P \times V$ of $(\bar{x}, \bar{p}, \bar{v})$ such that the partial subdifferential inverse mapping S from (3.3) admits a single-valued localization ϑ relative to $V \times P \times U$ satisfying the following Hölderian condition:

$$\|\vartheta(v_1, p_1) - \vartheta(v_2, p_2)\| \leq \kappa\|v_1 - v_2\| + \ell\|p_1 - p_2\|^{\frac{1}{2}} \quad as \ \ v_1, v_2 \in V, \ p_1, p_2 \in P. \tag{3.20}$$

Proof. When **(i)** holds, there is some $\gamma > 0$ such that $\bar{x} = M_\gamma(\bar{v}, \bar{p})$ by Definition 3.2(ii). Moreover, Remark 3.5 tells us that $\vartheta(v,p) = M_\gamma(v,p)$ for (v,p) near $(\bar{v}, \bar{p})$. This together with (3.8) verifies **(ii)**.

Conversely, supposing that **(ii)** is satisfied gives us some $\gamma > 0$ such that $\bar{x} \in M_\gamma(\bar{v}, \bar{p})$. Then we choose $\delta \in (0, \gamma)$ with $\mathbb{B}_\delta(\bar{x}) \subset U$ and for any $\widehat{x} \in M_\delta(\bar{v}, \bar{p})$ get $\widehat{x} \in M_\gamma(\bar{v}, \bar{p})$ due to

$$f(\widehat{x}, \bar{p}) - \langle \bar{v}, \widehat{x} \rangle \leq f(\bar{x}, \bar{p}) - \langle \bar{v}, \bar{x} \rangle \leq f(\widehat{x}, \bar{p}) - \langle \bar{v}, \widehat{x} \rangle.$$

By the Fermat rule we have $\bar{v} \in \partial_x f(\bar{x}, \bar{p})$ and also $\bar{v} \in \partial_x f(\widehat{x}, \bar{p})$. It follows therefore that $\widehat{x}, \bar{x} \in \vartheta(\bar{v}, \bar{p})$, which yields $\widehat{x} = \bar{x} = M_\delta(\bar{v}, \bar{p})$ by the single-valuedness of ϑ. Employing BCQ and the aforementioned result from [16, Proposition 3.5] allows us to find neighborhoods $V_1 \subset V$ of $\bar{v}$ and $P_1 \subset P$ of $\bar{p}$ such that $\emptyset \neq M_\delta(v,p) \subset \operatorname{int} \mathbb{B}_\delta(\bar{x})$ for all $(v,p) \in V_1 \times P_1$. By the Fermat rule again we deduce that $M_\delta(v,p) \subset \vartheta(v,p)$. Since ϑ is single-valued, this implies that $M_\delta(v,p) = \vartheta(v,p)$ for all $(v,p) \in V_1 \times P_1$. Combining it with (3.20) verifies (3.8) and thus completes the proof of the corollary. $\triangle$

Next we derive a quantitative (with modulus) second-order subdifferential characterization of Hölderian full stability via the regular coderivative of the limiting subdifferential. The tilt stability (p-independent) version of this result has been recently established in [24, Theorem 3.4].

Theorem 3.7 (characterization of Hölderian full stability via the regular coderivative of the limiting subdifferential). *Assume that BCQ (3.4) is satisfied at $(\bar{x}, \bar{p}) \in \mathrm{dom}\, f$ and that f is parametrically continuously prox-regular at $(\bar{x}, \bar{p})$ for $\bar{v} \in \partial_x f(\bar{x}, \bar{p})$. The following assertions are equivalent:*

(i) *The point $\bar{x}$ is a Hölderian fully stable local minimizer of $\mathcal{P}(\bar{v}, \bar{p})$ with a modulus pair $(\kappa, \ell) \in I\!R_>^2$.*

(ii) *There is a constant $\eta > 0$ such that for all $w \in I\!R^n$ we have*

$$\langle z, w \rangle \geq \frac{1}{\kappa} \|w\|^2 \quad \text{whenever } z \in \widehat{D}^* \partial f_p(u, v)(w) \text{ with } (u, p, v) \in \mathrm{gph}\, \partial_x f \cap I\!B_\eta(\bar{x}, \bar{p}, \bar{v}). \tag{3.21}$$

Proof. To justify **(i)**$\Longrightarrow$**(ii)**, find from **(i)** a constant $\gamma > 0$ such that M_γ satisfies the Hölderian condition (3.8). Theorem 3.4 ensures the existence of a single-valued localization ϑ of S relative to a neighborhood $V \times P \times U$ of $(\bar{v}, \bar{p}, \bar{x})$ satisfying the second-order growth condition (3.9). By Corollary 3.6 we may suppose that ϑ also satisfies (3.20). Furthermore, it follows from (3.9) that

$$\begin{aligned}
\langle y - v, x - u \rangle &= \langle y, x - u \rangle + \langle v, u - x \rangle \\
&\geq f_p(u) - f_p(x) + \frac{1}{2\kappa} \|x - u\|^2 + f_p(x) - f_p(u) + \frac{1}{2\kappa} \|x - u\|^2 \\
&= \frac{1}{\kappa} \|x - u\|^2 \quad \text{for any } (x, p, y), (u, p, v) \in \mathrm{gph}\, \partial_x f \cap (U \times P \times V).
\end{aligned} \tag{3.22}$$

To verify (3.21), pick $z \in \widehat{D}^* \partial f_p(u, v)(w)$ with $(u, p, v) \in \mathrm{gph}\, \partial_x f \cap (U \times P \times V)$, $w \in I\!R^n$ and get from (2.4) that for any $\varepsilon > 0$ there is some $\delta > 0$ with $I\!B_\delta(u, v) \subset U \times V$ such that

$$\langle z, x - u \rangle - \langle w, y - v \rangle \leq \varepsilon \big(\|x - u\| + \|y - v\| \big) \quad \text{whenever } (x, y) \in \mathrm{gph}\, \partial f_p \cap I\!B_\delta(u, v). \tag{3.23}$$

When $t > 0$ is small enough, define $u_t := \vartheta(v_t, p)$ with $v_t := v + t(z - 2\kappa^{-1}w) \in V$ and get from Hölder continuity of M_γ that $(u_t, v_t) \to (u, v)$ as $t \downarrow 0$. Note that $(u_t, v_t) \in \mathrm{gph}\, \partial f_p$ and suppose without loss of generality that $(u_t, v_t) \in I\!B_\delta(u, v)$ for all $t > 0$. Replacing (x, y) in (3.23) by (u_t, v_t) and using (3.22) yield

$$\begin{aligned}
\varepsilon \big(\|u_t - u\| + \|v_t - v\| \big) &\geq \langle z, u_t - u \rangle - \langle w, v_t - v \rangle \\
&= \langle t^{-1}(v_t - v) + 2\kappa^{-1}w, u_t - u \rangle - t\langle w, z - 2\kappa^{-1}w \rangle \\
&\geq (\kappa t)^{-1} \|u_t - u\|^2 + 2\kappa^{-1} \langle w, u_t - u \rangle - t\langle w, z - 2\kappa^{-1}w \rangle \\
&\geq (\kappa t)^{-1} \|u_t - u\|^2 - 2\kappa^{-1} \|w\| \cdot \|u_t - u\| + t\kappa^{-1} \|w\|^2 - t\langle w, z - \kappa^{-1}w \rangle \\
&\geq -t\langle w, z - \kappa^{-1}w \rangle = -t\langle z, w \rangle + t\kappa^{-1} \|w\|^2.
\end{aligned} \tag{3.24}$$

Note from (3.20) that $\vartheta(\cdot, p)$ is Lipschitz continuous on V with modulus κ. Thus we have

$$\begin{aligned}
\varepsilon \big(\|u_t - u\| + \|v_t - v\| \big) &= \varepsilon \big(\|\vartheta(v_t, p) - \vartheta(v, p)\| + \|v_t - v\| \big) \leq \varepsilon \big(\kappa \|v_t - v\| + \|v_t - v\| \big) \\
&= \varepsilon(\kappa + 1) \|v_t - v\| = \varepsilon(\kappa + 1)t \|z - 2\kappa^{-1}w\|,
\end{aligned}$$

which together with (3.24) yields $\langle z, w \rangle + \varepsilon(\kappa + 1)\|z - 2\kappa^{-1}w\| \geq \kappa^{-1}\|w\|^2$, and so $\langle z, w \rangle \geq \kappa^{-1}\|w\|^2$ while taking $\varepsilon \downarrow 0$. This ensures (3.21) and thus completes the first part of the proof.

To verify now **(ii)**$\Longrightarrow$**(i)**, assume that BCQ (3.4) holds at $(\bar{x}, \bar{p})$ and that (3.21) is satisfied with some numbers $\eta, \kappa > 0$. Since f is parametrically continuously prox-regular at $(\bar{x}, \bar{p})$ for $\bar{v} \in \partial_x f(\bar{x}, \bar{p})$, there are $r, \varepsilon > 0$ with $\varepsilon < \eta$ such that

$$f_p(x) \geq f_p(u) + \langle v, x - u \rangle - \frac{r}{2} \|x - u\|^2 \quad \text{if } x \in I\!B_\varepsilon(\bar{x}), \ (u, p, v) \in \mathrm{gph}\, \partial_x f \cap I\!B_\varepsilon(\bar{x}, \bar{p}, \bar{v}). \tag{3.25}$$

For $g_p(x, p) := f(x, p) + \frac{s}{2} \|x - \bar{x}\|^2$ with any $s > r$ as $x \in I\!R^n$, we have $\partial g_p(x) = \partial f_p(x) + s(x - \bar{x})$. Define further $W := J\big(I\!B_\varepsilon(\bar{x}, \bar{p}, \bar{v}) \big)$ with $J(u, p, v) := (u, p, v + s(u - \bar{x}))$ for $(u, p, v) \in I\!R^n \times I\!R^d \times I\!R^n$ and observe that W contains the ball $I\!B_{2\delta}(\bar{x}, \bar{p}, \bar{v})$ for some $\delta > 0$ sufficiently small. It is easy to deduce from (3.25) that

$$g_p(x) \geq g_p(u) + \langle v, x - u \rangle + \frac{s - r}{2} \|x - u\|^2 \quad \text{if } x \in I\!B_\delta(\bar{x}), \ (u, p, v) \in \mathrm{gph}\, \partial_x g \cap I\!B_\delta(\bar{x}, \bar{p}, \bar{v}). \tag{3.26}$$

Furthermore, applying the coderivative sum rule from [20, Theorem 1.62(i)] gives us the inclusion

$$z - sw \in \widehat{D}^* \partial f_p \big(u, v - s(u - \bar{x}) \big)(w) \quad \text{if } z \in \widehat{D}^* \partial g_p(u, v)(w) \text{ with } (u, p, v) \in \mathrm{gph}\, \partial_x g \cap I\!B_\delta(\bar{x}, \bar{p}, \bar{v}).$$

Taking into account that $(u, p, v - s(u - \bar{x})) = J^{-1}(u, p, v) \subset J^{-1}(W) = I\!B_\varepsilon(\bar{x}, \bar{p}, \bar{v}) \subset I\!B_\eta(\bar{x}, \bar{p}, \bar{v})$, the obtained inclusion together with (3.21) implies that $\langle z - sw, w \rangle \geq \kappa^{-1} \|w\|^2$. Thus

$$\|z\| \cdot \|w\| \geq \langle z, w \rangle \geq (s + \kappa^{-1}) \|w\|^2,$$

which ensures in turn the estimate

$$\|z\| \geq (s + \kappa^{-1}) \|w\| \quad \text{if} \ \ z \in \widehat{D}^* \partial g_p(u, v)(w) \ \ \text{with} \ \ (u, p, v) \in \operatorname{gph} \partial_x g \cap I\!B_\delta(\bar{x}, \bar{p}, \bar{v}). \qquad (3.27)$$

This allows us to justify in the following claim the second-order growth condition for the shifted function g_p.

Claim. *When g satisfies* (3.26) *and* (3.27), *there is a neighborhood $U \times P \times V$ of $(\bar{x}, \bar{p}, \bar{v})$ such that*

$$g_p(x) \geq g_p(u) + \langle v, x - u \rangle + \frac{s + \kappa^{-1}}{2} \|x - u\|^2 \ \ \text{if} \ \ x \in U, \ (u, p, v) \in \operatorname{gph} \partial_x g \cap (U \times P \times V). \qquad (3.28)$$

Indeed, observe that $\partial^\infty g(\bar{x}, \bar{p}) = \partial^\infty f(\bar{x}, \bar{p})$ due to the subdifferential sum rule from [20, Proposition 1.107(iii)]. Thus the assumed BCQ (3.4) holds also for the function g at $(\bar{x}, \bar{p})$. Applying Theorem 3.4 to the function g, which satisfies inequality (3.26), gives us some $\gamma, \ell > 0$ such that the mapping $M_\gamma^g(v, p) := \operatorname{argmin}\{g(x, p) - \langle v, x \rangle \mid x \in I\!B_\gamma(\bar{x})\}$ is single-valued and that

$$\|M_\gamma^g(v_1, p_1) - M_\gamma^g(v_2, p_2)\| \leq \frac{1}{s - r} \|v_1 - v_2\| + \ell \|p_1 - p_2\|^{\frac{1}{2}} \ \ \text{for all} \ \ v_1, v_2 \in V_1, \ p_1, p_2 \in P_1, \qquad (3.29)$$

where $V_1 \subset I\!B_\delta(\bar{v})$ and $P_1 \subset I\!B_\delta(\bar{p})$ are some neighborhoods of $\bar{v}$ and $\bar{p}$, respectively. Defining now

$$S^g(v, p) := \left\{ u \in I\!R^n \mid v \in \partial_x g(u, p) \right\} \ \ \text{for} \ \ v \in I\!R^n, p \in I\!R^d$$

we deduce from Remark 3.5 that S^g admits a single-valued localization ϑ^g relative to $\operatorname{int} I\!B_\beta(\bar{v}) \times \operatorname{int} I\!B_\beta(\bar{p}) \times \operatorname{int} I\!B_\gamma(\bar{x}) \subset V_1 \times P_1 \times \operatorname{int} I\!B_\gamma(\bar{x})$ for some $\beta \in (0, \delta)$ such that $\vartheta^g = M_\gamma^g$ on $\operatorname{int} I\!B_\beta(\bar{v}) \times \operatorname{int} I\!B_\beta(\bar{p})$. Pick any $p \in P_2 := \operatorname{int} I\!B_\beta(\bar{p})$ and denote $\vartheta_p^g(\cdot) := \vartheta^g(\cdot, p)$. Then the mean value inequality from [20, Corollary 3.50] tells us that

$$\|\langle z, \vartheta_p^g(v_1) \rangle - \langle z, \vartheta_p^g(v_2) \rangle\| \leq \|v_1 - v_2\| \sup \left\{ \|w\| \mid w \in \widehat{\partial} \langle z, \vartheta_p^g(\cdot) \rangle(v), \ v \in V_2 \right\} \qquad (3.30)$$

whenever $z \in I\!B$ and $v_1, v_2 \in V_2 := \operatorname{int} I\!B_{\frac{\beta}{2}}(\bar{v})$. Note that

$$\widehat{\partial} \langle z, \vartheta_p^g(\cdot) \rangle(v) \subset \widehat{D}^* \vartheta_p^g(v)(z) = \widehat{D}^* \partial g_p^{-1}(v)(z), \quad z \in I\!B,$$

which gives us together with (3.27) and (3.30) that

$$\|\vartheta_p^g(v_1) - \vartheta_p^g(v_2)\| = \sup_{z \in I\!B} \|\langle z, \vartheta_p^g(v_1) - \vartheta_p^g(v_2) \rangle\| \leq (s + \kappa^{-1})^{-1} \|v_1 - v_2\|, \ v_1, v_2 \in V_2. \qquad (3.31)$$

Since $\vartheta^g = M_\gamma^g$ on $V_2 \times P_2$, for any $p_1, p_2 \in P_2$ and $v_1, v_2 \in V_2$ we get from (3.29) and (3.31) that

$$\begin{aligned} \|M_\gamma^g(v_1, p_1) - M_\gamma^g(v_2, p_2)\| &\leq \|M_\gamma^g(v_1, p_1) - M_\gamma^g(v_2, p_1)\| + \|M_\gamma^g(v_2, p_1) - M_\gamma^g(v_2, p_2)\| \\ &\leq (s + \kappa^{-1})^{-1} \|v_1 - v_2\| + \ell \|p_1 - p_2\|^{\frac{1}{2}}. \end{aligned}$$

Employing now Theorem 3.4 allows us to find a neighborhood $U \times P \times V$ of $(\bar{x}, \bar{p}, \bar{v})$ such that the growth condition (3.28) holds. This verifies the Claim.

To complete the proof, it suffices to deduce (3.9) from (3.28). Indeed, since $f_p(x) = g_p(x) - \frac{s}{2} \|x - \bar{x}\|^2$, we have $\partial f_p(x) = \partial g_p(x) - s(x - \bar{x})$, which yields

$$f_p(x) \geq f_p(u) + \langle v, x - u \rangle + \frac{1}{2\kappa} \|x - u\|^2 \ \ \text{for all} \ \ (u, p, v) \in \operatorname{gph} \partial_x f \cap Z, \ x \in U,$$

where $Z := J^{-1}(U \times P \times V)$ is a neighborhood of $(\bar{x}, \bar{p}, \bar{v})$. This completes the proof of this theorem by employing once again Theorem 3.4. $\triangle$

4   Second-Order Characterizations of Lipschitzian Full Stability

In this section we study the notion of Lipschitzian full stability with the modulus specification formulated in Definition 3.2(i). The following theorem characterizes this notion in terms of the uniform second-order growth condition (3.9) and the second-order subdifferential condition (3.21) with the precise modulus correspondence. In comparison with Theorems 3.4 and 3.7 this result indeed shows that Hölderian full stability becomes Lipschitzian under the additional condition (4.1) below, which is condition (b) in [16, Theorem 2.3]. By the Mordukhovich criterion (2.7) the latter condition exactly means that $G : p \mapsto \mathrm{gph}\, \partial_x f(\cdot, p)$ is Lipschitz-like around the point $(\bar{p}, (\bar{x}, \bar{v}))$ with $\bar{v} \in \partial_x f(\bar{x}, \bar{p})$; see, e.g., [16, Proposition 4.3].

Theorem 4.1 (second-order characterizations of Lipschitzian vs. Hölderian full stability). *Assume that BCQ (3.4) holds at $(\bar{x}, \bar{p}) \in \mathrm{dom}\, f$ and that f is parametrically continuously prox-regular at $(\bar{x}, \bar{p})$ for $\bar{v} \in \partial_x f(\bar{x}, \bar{p})$. Then the following assertions are equivalent:*
   (i) *The point $\bar{x}$ is a Lipschitzian fully stable local minimizer of $\mathcal{P}(\bar{v}, \bar{p})$ with a modulus pair $(\kappa, \ell) \in I\!\!R^2_>$.*
   (ii) *The uniform second-order growth condition (3.9) holds at $(\bar{x}, \bar{p}, \bar{v})$ together with the condition*

$$(0, q) \in D^* \partial_x f(\bar{x}, \bar{p}, \bar{v})(0) \Longrightarrow q = 0. \tag{4.1}$$

   (iii) *Both conditions (4.1) and (3.21) are satisfied.*

Proof. Implication **(iii)**$\Longrightarrow$**(ii)** is straightforward from Theorem 3.4 and Theorem 3.7. To justify implications **(i)**$\Longrightarrow$**(ii)**, we only need to prove that **(i)**$\Longrightarrow$(4.1) due to Theorems 3.4. To proceed, suppose that $\bar{x}$ is a Lipschitzian fully stable local minimizer of $\mathcal{P}(\bar{v}, \bar{p})$ and then find a number $\gamma > 0$ so small that the mapping M_γ is single-valued and Lipschitz continuous around $(\bar{v}, \bar{p})$ with $M_\gamma(\bar{v}, \bar{p}) = \bar{x}$. By Remark 3.5 there is a neighborhood $U \times V \times P$ of $(\bar{x}, \bar{v}, \bar{p})$ such that $M_\gamma(v, p) = S(v, p) \cap U$ for all $(v, p) \in V \times P$. This together with (3.7) implies that the mapping S in (3.3) is Lipschitz-like around $(\bar{v}, \bar{p})$. The Mordukhovich criterion (2.7) tells us that $D^* S(\bar{v}, \bar{p}, \bar{x})(0) = (0, 0) \in I\!\!R^n \times I\!\!R^d$. Note further from the construction of S that if $(0, q) \in D^* \partial_x f(\bar{x}, \bar{p}, \bar{v})(0)$, then $(0, q) \in D^* S(\bar{v}, \bar{p}, \bar{x})(0)$ and so $q = 0$. This verifies condition (4.1) and thus completes the proof of the first part of the theorem.

It remains to justify implication **(ii)**$\Longrightarrow$ **(i)**. Suppose that both the uniform second-order growth condition (3.9) and the coderivative condition (4.1) holds. By the Mordukhovich criterion (2.7) condition (4.1) is equivalent to the fact that the mapping $G : p \mapsto \mathrm{gph}\, \partial_x f(\cdot, p)$ is Lipschitz-like around $(\bar{p}, \bar{x}, \bar{v})$. Thus there exist a neighborhood $U_1 \times P_1 \times V_1 \subset U \times P \times V$ of $(\bar{x}, \bar{p}, \bar{v})$ and a constant $c > 0$ such that

$$G(p_1) \cap (U_1 \times V_1) \subset G(p_2) + c\|p_1 - p_2\| I\!\!B \ \ \text{for all} \ \ p_1, p_2 \in P_1, \tag{4.2}$$

where U, P, V are taken from (3.9). By Remark 3.5 we assume that $\mathrm{gph}\, M_\gamma \cap (V_1 \times P_1 \times U_1) = \mathrm{gph}\, \vartheta \cap (V_1 \times P_1 \times U_1)$ for ϑ in (3.9). Picking any $(v_1, p_1, u_1), (v_2, p_2, u_2) \in \mathrm{gph}\, \vartheta \cap (V_1 \times P_1 \times U_1) = \mathrm{gph}\, S \cap (V_1 \times P_1 \times U_1)$, and using (4.2) give us a pair $(u, v) \in G(p_2)$ such that

$$\|u_1 - u\| + \|v_1 - v\| \le c\|p_1 - p_2\|, \tag{4.3}$$

since $(u_1, v_1) \in G(p_1) \cap (U_1 \times V_1)$. Shrinking the neighborhood U_1, V_1, P_1 allows us to get $(u, v) \in U \times V$ and deduce from (3.9) that

$$
\begin{aligned}
f(u, p_2) &\ge f(u_2, p_2) + \langle v_2, u - u_2 \rangle + \frac{1}{2\kappa} \|u - u_2\|^2, \\
f(u_2, p_2) &\ge f(u, p_2) + \langle v, u_2 - u \rangle + \frac{1}{2\kappa} \|u_2 - u\|^2.
\end{aligned}
$$

Adding these two inequalities shows that

$$\|v - v_2\| \cdot \|u - u_2\| \ge \langle v - v_2, u - u_2 \rangle \ge \frac{1}{\kappa} \|u - u_2\|^2,$$

which being combined with (4.3) ensures the estimates

$$
\begin{aligned}
\|u_1 - u_2\| &\le \|u_1 - u\| + \|u - u_2\| \le c\|p_1 - p_2\| + \kappa\|v - v_2\| \\
&\le c\|p_1 - p_2\| + \kappa\|v_1 - v_2\| + \kappa\|v - v_1\| \\
&\le c\|p_1 - p_2\| + \kappa\|v_1 - v_2\| + c\kappa\|p_1 - p_2\| = \kappa\|v_1 - v_2\| + c(\kappa + 1)\|p_1 - p_2\|.
\end{aligned}
$$

Therefore we arrive at the relationship

$$\|M_\gamma(v_1, p_1) - M_\gamma(v_2, p_2)\| \le \kappa \|v_1 - v_2\| + c(\kappa+1)\|p_1 - p_2\| \quad \text{for all} \ \ v_1, v_2 \in V_1, \ p_1, p_2 \in P_1.$$

which confirms the Lipschitz property (3.7) with the modulus pair $(\kappa, c(\kappa+1))$ and thus completes the proof of implication **(ii)** $\Longrightarrow$**(i)** and the whole theorem. $\triangle$

As a consequence of Theorem 4.1 we show that Lipschitzian full stability is equivalent to the Lipschitz continuity of a localization of the mapping S in (3.3). Without considering the modulus pair this result has been recently established in [30, Theorem 3.5].

Corollary 4.2 (Lipschitzian localization). *Assume that BCQ (3.4) holds at $(\bar{x}, \bar{p}) \in \operatorname{dom} f$ and that f is parametrically continuously prox-regular at $(\bar{x}, \bar{p})$ for $\bar{v} \in \partial_x f(\bar{x}, \bar{p})$. The following assertions are equivalent:*
(i) The point $\bar{x}$ is a Lipschitzian fully stable local minimizer of the perturbed problem $\mathcal{P}(\bar{v}, \bar{p})$ with the modulus pair $(\kappa, \ell) \in I\!\!R_>^2$.
(ii) We have $\bar{x} \in M_\gamma(\bar{v}, \bar{p})$ for some $\gamma > 0$, and there is a neighborhood $U \times P \times V$ of $(\bar{x}, \bar{p}, \bar{v})$ such that the mapping S from (3.3) admits a single-valued localization ϑ with respect to $V \times P \times U$ satisfying the Lipschitz continuity condition

$$\|\vartheta(v_1, p_1) - \vartheta(v_2, p_2)\| \le \kappa \|v_1 - v_2\| + \ell \|p_1 - p_2\| \quad \text{for all} \ \ v_1, v_2 \in V, \ p_1, p_2 \in P. \tag{4.4}$$

Proof. The proof is similar to Corollary 3.6 by applying Theorem 4.1 instead of Theorem 3.4. $\triangle$

The next approximation lemma is helpful in the proof of the pointwise characterizations of Lipschitzian full stability established in Theorem 4.4 and Corollary 4.5 below.

Lemma 4.3 (coderivative approximation). *Let condition (4.1) hold, and let*

$$\|z\| \ge \mu \|w\| \quad \text{whenever} \ \ (z, q) \in D^* \partial_x f(\bar{x}, \bar{p}, \bar{v})(w) \tag{4.5}$$

with some $\mu > 0$. Then for any $\delta \in (0, \mu)$ there exists $\eta > 0$ such that

$$\|z\| \ge (\mu - \delta)\|w\| \quad \text{whenever} \ \ z \in \widehat{D}^* \partial f_p(u, v)(w) \ \ \text{with} \ \ (u, p, v) \in \operatorname{gph} \partial_x f \cap I\!\!B_\eta(\bar{x}, \bar{p}, \bar{v}). \tag{4.6}$$

Proof. Assuming (4.5), we first show that for any $\delta \in (0, \mu)$ there is $\nu > 0$ satisfying

$$\|z\| \ge (\mu - \delta)\|w\| \quad \text{if} \ \ (z, q) \in \widehat{D}^* \partial_x f(u, p, v)(w) \ \ \text{with} \ \ (u, p, v) \in \operatorname{gph} \partial_x f \cap I\!\!B_\nu(\bar{x}, \bar{p}, \bar{v}). \tag{4.7}$$

Arguing by contradiction, find sequences $(u_k, p_k, v_k) \xrightarrow{\operatorname{gph} \partial_x f} (\bar{x}, \bar{p}, \bar{v})$ and $(z_k, q_k) \in \widehat{D}^* \partial_x f(u_k, p_k, v_k)(w_k)$ such that $\|z_k\| < (\mu - \delta)\|w_k\|$, which clearly implies that $w_k \ne 0$. Denoting $\bar{z}_k := z_k \|w_k\|^{-1}$, $\bar{q}_k := q_k \|w_k\|^{-1}$, and $\bar{w}_k := w_k \|w_k\|^{-1}$ gives us $(\bar{z}_k, \bar{q}_k) \in \widehat{D}^* \partial_x f(u_k, p_k, v_k)(\bar{w}_k)$ as $k \in I\!\!N$. Since (4.1) holds, the mapping $G : p \mapsto \operatorname{gph} \partial_x f(\cdot, p)$ is Lipschitz-like with some modulus $\ell > 0$. Then the result of [20, Theorem 1.43] tells us that $\|\bar{q}_k\| \le \ell(\|\bar{z}_k\| + \|\bar{w}_k\|)$ for all k. It follows that $\|\bar{w}_k\| = 1$, $\|\bar{z}_k\| \le \mu - \delta$, and $\|\bar{q}_k\| \le \ell(\mu - \delta + 1)$. By passing to a subsequence, suppose that $(\bar{z}_k, \bar{q}_k, \bar{w}_k)$ converges to $(\bar{z}, \bar{q}, \bar{w})$ as $k \to \infty$. Hence $\|\bar{w}\| = 1$ and $(\bar{z}, \bar{q}) \in D^* \partial_x f(\bar{x}, \bar{p}, \bar{v})(\bar{w})$ with $\|\bar{z}\| \le (\mu - \delta)$, which contradicts (4.5) and thus verifies condition (4.7).

To justify further (4.6), take any $z \in \widehat{D}^* \partial f_p(u, v)(w)$ with $(u, p, v) \in \operatorname{gph} \partial_x f \cap I\!\!B_\eta(\bar{x}, \bar{p}, \bar{v})$ for some $\eta \in (0, \nu)$. Due to the homogeneity of $\widehat{D}^*$ we assume without loss of generality that $\|z\| + \|w\| \le \frac{1}{2\ell}$. Defining $\Omega_1 := \operatorname{gph} G$ and $\Omega_2 := \{p\} \times I\!\!R^n \times I\!\!R^n$, observe that $(0, z, -w) \in \widehat{N}((p, u, v); \Omega_1 \cap \Omega_2)$. It follows from the fuzzy intersection rule in [20, Lemma 3.1] that for any $\varepsilon > 0$ there are $\lambda \ge 0$, $(p_i, u_i, v_i) \in \Omega_i \cap I\!\!B_\varepsilon(p, u, v)$, and $(q_i, z_i, -w_i) \in \widehat{N}((p_i, u_i, v_i); \Omega_i) + \varepsilon I\!\!B$ as $i = 1, 2$ such that

$$\lambda(0, z, -w) = (q_1, z_1, -w_1) + (q_2, z_2, -w_2) \quad \text{and} \quad \max\{\lambda, \|(q_2, z_2, -w_2)\|\} = 1. \tag{4.8}$$

The construction of Ω_2 yields $\widehat{N}((p_2, u_2, v_2); \Omega_2) \subset I\!\!R^d \times \{0\} \times \{0\}$ and thus $\|z_2\| + \|w_2\| \le \varepsilon$. Moreover, there is $(\bar{q}_1, \bar{z}_1, -\bar{w}_1) \in \widehat{N}((p_1, u_1, v_1); \Omega_1)$ satisfying $\|q_1 - \bar{q}_1\| + \|z_1 - \bar{z}_1\| + \|w_1 - \bar{w}_1\| \le \varepsilon$. The Lipschitz-like

property of G with modulus ℓ ensures by [20, Theorem 1.43] that $\|\bar{q}_1\| \leq \ell(\|\bar{z}_1\| + \|\bar{w}_1\|)$. This together with (4.8) gives us the relationships

$$
\begin{aligned}
\|q_2\| \ &= \|q_1\| \leq \|\bar{q}_1\| + \varepsilon \leq \varepsilon + \ell(\|\bar{z}_1\| + \|\bar{w}_1\|) \leq \varepsilon + \ell(\|z_1\| + \|w_1\| + \varepsilon) \\
&\leq \ell(\|\lambda z - z_2\| + \|\lambda w - w_2\|) + (\ell+1)\varepsilon \leq \ell(\lambda\|z\| + \|z_2\| + \lambda\|w\| + \|w_2\|) + (\ell+1)\varepsilon \\
&\leq \ell\big(\lambda(\|z\| + \|w\|) + \varepsilon\big) + (\ell+1)\varepsilon \leq \ell(\|z\| + \|w\|) + (2\ell+1)\varepsilon < \frac{1}{2} + (2\ell+1)\varepsilon.
\end{aligned}
$$

When $\varepsilon > 0$ is sufficiently small, we have $\|q_2\| < 1 - \varepsilon$ and so $\|(q_2, z_2, -w_2)\| < 1$. It follows from (4.8) that $\lambda = 1$. Combining this with (4.7) and (4.8) implies that

$$
\begin{aligned}
\|z\| \ &= \|z_1 + z_2\| \geq \|\bar{z}_1\| - \|\bar{z}_1 - z_1\| - \|z_2\| \geq (\mu - \delta)\|\bar{w}_1\| - \varepsilon - \varepsilon \\
&\geq (\mu - \delta)(\|w\| - \|w_1 - \bar{w}_1\| - \|w_2\|) - 2\varepsilon \geq (\mu - \delta)\|w\| - 2\varepsilon(\mu - \delta) - 2\varepsilon.
\end{aligned}
$$

Letting finally $\varepsilon \downarrow 0$ shows that $\|z\| \geq (\mu - \delta)\|w\|$ and thus ends the proof of the lemma. $\qquad \triangle$

Now we are ready to derive the main result of this section that provides a complete pointwise characterization of Lipschitzian full stability via the limiting coderivative of the partial subgradient mapping $\partial_x f$. It not only recovers the qualitative criterion (4.1), (4.10) of full stability from [16, Theorem 2.3] obtained by a different approach, but also establishes new quantitative information about Lipschitzian moduli.

Theorem 4.4 (pointwise characterization of Lipschitzian fully stable minimizers via the limiting coderivative of the subdifferential). *Suppose that BCQ (3.4) holds at $(\bar{x}, \bar{p}) \in \operatorname{dom} f$ and that f is parametrically continuously prox-regular at $(\bar{x}, \bar{p})$ for $\bar{v} \in \partial_x f(\bar{x}, \bar{p})$. Consider the following statements:*

(i) *The point $\bar{x}$ is a Lipschitzian fully stable local minimizer of the unperturbed problem $\mathcal{P}(\bar{v}, \bar{p})$ with a modulus pair $(\kappa, \ell) \in \mathbb{R}^2_>$.*

(ii) *Condition (4.1) is satisfied and there is some $\mu > 0$ such that*

$$
\langle z, w \rangle \geq \mu\|w\|^2 \quad \text{whenever} \quad (z, q) \in D^*\partial_x f(\bar{x}, \bar{p}, \bar{v})(w). \tag{4.9}
$$

Then implication (i) $\Longrightarrow$ (ii) *holds with $\mu = \kappa^{-1}$ while implication* (ii) $\Longrightarrow$ (i) *is satisfied with any $\kappa > \mu^{-1}$. Furthermore, the validity of* (i) *with some modulus pair $(\kappa, \ell) \in \mathbb{R}^2_>$ is equivalent to the fulfillment of condition (4.1) together with the positive-definiteness condition*

$$
\langle z, w \rangle > 0 \quad \text{whenever} \quad (z, q) \in D^*\partial_x f(\bar{x}, \bar{p}, \bar{v})(w), \ w \neq 0. \tag{4.10}
$$

Proof. Assuming (i) implies by Theorem 4.1 that both conditions (4.1) and (3.21) hold. Observe that for any $(z, q) \in \widehat{D}^*\partial_x f(x, p, v)(w)$ we have

$$
\limsup_{(u_1, p_1, v_1) \xrightarrow{\operatorname{gph} \partial_x f} (u, p, v)} \frac{\langle z, u_1 - u \rangle + \langle q, p_1 - p \rangle - \langle w, v_1 - v \rangle}{\|u_1 - u\| + \|p_1 - p\| + \|v_1 - v\|} \leq 0.
$$

Choosing $p_1 = p$ in the inequality above gives us $z \in \widehat{D}^*\partial f_p(x, v)(w)$. Hence it follows from (3.21) that

$$
\langle z, w \rangle \geq \kappa^{-1}\|w\|^2 \quad \text{whenever} \quad (z, q) \in \widehat{D}^*\partial_x f(u, p, v)(w) \ \text{ with } \ (u, p, v) \in \operatorname{gph} \partial_x f \cap \mathbb{B}_\eta(\bar{x}, \bar{p}, \bar{v}).
$$

Letting now $\eta \downarrow 0$ and using definition (2.4), we arrive at (4.9) with $\mu = \kappa^{-1}$, which verifies (ii).

To justify the converse implication (ii)$\Longrightarrow$(i), we proceed similarly to the proof of (ii)$\Longrightarrow$(i) in Theorem 3.7 with some modifications. Since f parametrically continuously prox-regular at $(\bar{x}, \bar{p}, \bar{v})$, inequality (3.25) holds for some $r, \varepsilon > 0$. Defining $g(x, p) := f(x, p) + \frac{s}{2}\|x - \bar{x}\|^2$ for $x \in \mathbb{R}^n$, $p \in \mathbb{R}^d$ with some fixed $s > r$, we have $\partial_x g(x, p) = \partial_x f(x, p) + s(x - \bar{x})$. Moreover, the quadratic growth condition (3.26) is satisfied for $g_p(x) := g(x, p)$ with some $\delta > 0$. Note further that $\partial^\infty f(\bar{x}, \bar{p}) = \partial^\infty g(\bar{x}, \bar{p})$ and that $D^*\partial_x g(\bar{x}, \bar{p}, \bar{v})(w) = D^*\partial_x f(\bar{x}, \bar{p}, \bar{v})(w) + (sw, 0)$ by [20, Theorem 1.62(ii)]. Since BCQ and condition (4.1) hold for the function f, both these conditions hold at the same point for the function g as well. By Theorem 4.1 condition (3.26) ensures that $\bar{x}$ is a Lipschitzian fully stable local minimizer of problem $\mathcal{P}(\bar{v}, \bar{p})$ with replacing f by g.

It follows from [20, Theorem 1.62(ii)] that the inclusion $(z,q) \in D^*\partial_x g(\bar{x}, \bar{p}, \bar{v})(w)$ yields $(z - sw, q) \in D^*\partial_x f(\bar{x}, \bar{p}, \bar{v})(w)$. Furthermore, by (4.9) we have $\langle z - sw, w \rangle \geq \mu\|w\|^2$, which implies that

$$\|z\| \cdot \|w\| \geq \langle z, w \rangle \geq (s + \mu)\|w\|^2.$$

Lemma 4.3 ensures that for any $\lambda \in (0, s + \mu)$ there is some $\eta > 0$ such that

$$\|z\| \geq (s + \mu - \lambda)\|w\| \quad \text{whenever} \quad z \in \widehat{D}^*\partial g_p(u,v)(w) \quad \text{with} \quad (u,p,v) \in \mathrm{gph}\,\partial_x g \cap I\!\!B_\eta(\bar{x}, \bar{p}, \bar{v}).$$

Since the function g satisfies (3.26) and (3.27), we employ the Claim in the proof of Theorem 3.7 to find neighborhoods U of $\bar{x}$, P of $\bar{p}$, and V of $\bar{v}$ for which the second-order growth condition

$$g(x,p) \geq g(u,p) + \langle v, x - u \rangle + \frac{s + \mu - \lambda}{2}\|x - u\|^2 \quad \text{if} \quad x \in U, \ (u,p,v) \in \mathrm{gph}\,\partial_x g \cap (U \times P \times V).$$

is satisfied. This implies, with $W := J^{-1}(U \times P \times V)$ and $J(u,p,v) := (u, p, v + s(u - \bar{x}))$, that

$$f(x,p) \geq f(u,p) + \langle v, x - u \rangle + \frac{\mu - \lambda}{2}\|x - u\|^2 \quad \text{for all} \quad x \in U_1, \ (u,p,v) \in \mathrm{gph}\,\partial_x f \cap W. \qquad (4.11)$$

For any $\kappa > \mu^{-1}$ there exists some $\lambda \in (0, s + \mu)$ satisfying $\kappa > (\mu - \lambda)^{-1} > 0$. Theorem 4.1 together with (4.11) tells us that $\bar{x}$ is a Lipschitzian fully stable local minimizer of $P(\bar{v}, \bar{p})$ with the modulus pair $((\mu - \lambda)^{-1}, \ell)$ for some $\ell > 0$. This verifies implication (ii)$\Longrightarrow$(i).

Next we prove the equivalence between (i) with *some* modulus pair $(\kappa, \ell) \in I\!\!R^2_>$ and the validity of (4.10) together with (4.1). Note that (i) readily yields both conditions (4.1) and (4.10) by implication (i)$\Longrightarrow$(ii) proved above. To justify the converse, observe first that the validity of (4.10) and (4.1) ensures the condition

$$(0, q) \in D^*\partial_x f(\bar{v}, \bar{p}, \bar{x})(w) \Longrightarrow (q, w) = 0,$$

which shows that $D^*S(\bar{v}, \bar{p}, \bar{x})(0) = (0, 0)$ for the mapping S from (3.3). By the Mordukhovich criterion (2.7) this tells that S is Lipschitz-like around $(\bar{v}, \bar{p}, \bar{x})$ with some modulus $\ell > 0$. Moreover, arguing as in the proof of (ii)$\Longrightarrow$(i) above when $\mu = 0$ shows that for each $\lambda \in (0, \min\{(5\ell)^{-1}, s\})$ there are neighborhoods U_1 of $\bar{x}$ and W_1 of $(\bar{x}, \bar{p}, \bar{v})$ such that condition (4.11) holds with $\mu = 0$. Define $h(x,p) := f(x,p) + \lambda\|x - \bar{x}\|^2$ with $\partial h(x,p) = \partial f(x,p) + 2\lambda(x - \bar{x})$. It is similar to (3.26) that condition (4.11) with $\mu = 0$ implies the existence of $\delta > 0$ so small that the quadratic growth condition

$$h(x,p) \geq h(u,p) + \langle v, x - u \rangle + \frac{\lambda}{2}\|x - u\|^2 \text{ if } x \in I\!\!B_\delta(\bar{x}), (u,p,v) \in \mathrm{gph}\,\partial_x h \cap I\!\!B_\delta(\bar{x}, \bar{p}, \bar{v}) \qquad (4.12)$$

is satisfied for h. Observe further that for any $(z,q) \in D^*\partial_x h(\bar{x}, \bar{p}, \bar{v})(w)$ we get from [20, Theorem 1.62(ii)] that $(z - 2\lambda w, q) \in D^*\partial_x f(\bar{x}, \bar{p}, \bar{v})(w)$ whenever $w \in I\!\!R^n$, which reads as $(-w, q) \in \widehat{D}^*S(\bar{v}, \bar{p}, \bar{x})(-z + 2\lambda w)$. Since the mapping S is Lipschitz-like around $(\bar{v}, \bar{x})$ with modulus $\ell > 0$, we deduce from [20, Theorem 1.44] that $\ell\|z - 2\lambda w\| \geq \|w\| + \|q\|$. This ensures the fulfillment of the inequalities

$$\ell\|z\| \geq \ell\|z - 2\lambda w\| - 2\ell\lambda\|w\| \geq \|w\| + \|q\| - 2\ell\lambda\|w\| \geq (1 - 2\ell\lambda)(\|w\| + \|q\|),$$

which in turn allow us to arrive at the estimate

$$\|z\| \geq \frac{1 - 2\ell\lambda}{\ell}\|w\| \quad \text{for all} \quad (z,q) \in D^*\partial_x h(\bar{x}, \bar{p}, \bar{v})(w).$$

Employing this inequality together with Lemma 4.3 gives us a number $\eta > 0$ such that

$$\|z\| \geq \frac{1 - 3\ell\lambda}{\ell}\|w\| \quad \text{whenever} \quad z \in \widehat{D}^*\partial h_p(u,v)(w) \quad \text{and} \quad (u,p,v) \in \mathrm{gph}\,\partial_x h \cap I\!\!B_\eta(\bar{x}, \bar{p}, \bar{v}).$$

This together with (4.12) shows that the function h satisfies (3.26) and (3.27). Applying the Claim in the proof of Theorem 3.7 to h ensures the existence of neighborhoods U_2 of $\bar{x}$, P_2 of $\bar{p}$, and V_2 of $\bar{v}$ such that

$$h(x,p) \geq h(u,p) + \langle v, x - u \rangle + \frac{1 - 3\ell\lambda}{2\ell}\|x - u\|^2 \quad \text{for all} \quad x \in U_2, \ (u,p,v) \in \mathrm{gph}\,\partial_x h \cap (U_2 \times P_2 \times V_2).$$

Since $f(x,p) = h(x,p) - \lambda\|x - \bar{x}\|^2$ and $\partial_x f(x,p) = \partial_x h(x,p) - 2\lambda(x - \bar{x})$, this easily implies that

$$f(x,p) \geq f(u,p) + \langle v, x - u \rangle + \frac{1 - 5\ell\lambda}{2\ell}\|x - u\|^2 \ \text{ for all } \ x \in U_2, \ (u,v) \in \text{gph}\,\partial_x f \cap W_2,$$

where $W_2 := J_\lambda^{-1}(U_2 \times P_2 \times V_2)$ and $J_\lambda(u,p,v) := (u,p,v + 2\lambda(u - \bar{x}))$ for all $(u,p,v) \in I\!\!R^n \times I\!\!R^d \times I\!\!R^n$. Applying finally Theorem 4.1 with taking into account the choice of $\lambda < (5\ell)^{-1}$ verifies that $\bar{x}$ is the Lipschitzian fully stable local minimizer of $\mathcal{P}(\bar{v}, \bar{p})$, which completes the proof of the theorem. $\triangle$

The following consequence of Theorem 4.4 is useful for our applications in Section 6.

Corollary 4.5 (another form of the pointwise characterization of Lipschitzian full stability). *In the setting of Theorem* 4.4 *we have the equivalent statements:*
   (i) *The point $\bar{x}$ is a Lipschitzian fully stable local minimizer of problem $\mathcal{P}(\bar{v}, \bar{p})$.*
   (ii) *Condition* (4.1) *is satisfied together with the inequality*

$$\inf\left\{\langle z, w\rangle \middle| (z,q) \in D^*\partial_x f(\bar{x}, \bar{p}, \bar{v})(w)\right\} > 0 \ \text{ for all } \ w \neq 0, \tag{4.13}$$

where we use the convention that $\inf \emptyset := \infty$.

Proof. It is proved in Theorem 4.4 that **(i)** implies the existence of some $\mu > 0$ for which we have condition (4.9) that immediately implies (4.13). Conversely, the validity of (4.13) readily yields (4.10). Together with (4.1) it gives **(i)** by Theorem 4.4 and thus completes the proof of this corollary. $\triangle$

5 Full Stability, Strong Regularity, and Strong Stability in Constrained Optimization

This section concerns the study of the corresponding counterparts of both Hölderian and Lipschitzian full stability of local solutions to the following large class of problems in constrained optimization:

$$\widehat{\mathcal{P}} \quad \begin{cases} \text{minimize } \varphi(x, \bar{p}) \text{ subject to } x \in I\!\!R^n, \\ g(x, \bar{p}) \in \Theta, \end{cases} \tag{5.1}$$

where the cost function $\varphi : I\!\!R^n \times I\!\!R^d \to I\!\!R$ and the constrained mapping $g : I\!\!R^n \times I\!\!R^d \to Y$ are $\mathcal{C}^2$-smooth around the reference point $(\bar{x}, \bar{p})$, where Y is a finite-dimensional Euclidean space[4], and where Θ is a closed and convex subset of Y. Besides standard nonlinear programs (NLP), model (5.1) encompasses various problems of *conic programming* [3, 23] when the set Θ is a cone, mathematical programs with *polyhedral constraints* (MPPC) designated in [30] when Θ is a polyhedral set, etc. It is worth noting that, despite describing (5.1) in the classical *smooth and convex* terms, the progress in the study of full stability and related issues achieved in this and the subsequent sections are based on the results and methods of *nonsmooth variational analysis* developed above.

In accordance with the the scheme of Section 3 the two-parameter perturbation of $\widehat{\mathcal{P}}$ in (5.1) reads as

$$\widehat{\mathcal{P}}(v,p) \quad \begin{cases} \text{minimize } \varphi(x, p) - \langle v, x \rangle \text{ subject to } x \in I\!\!R^n, \\ g(x, p) \in \Theta \end{cases} \tag{5.2}$$

for any $(v,p) \in I\!\!R^n \times I\!\!R^d$. It can be written in the equivalent unconstraint format

$$\widehat{\mathcal{P}}(v,p) \quad \text{minimize } f(x,p) - \langle v, x \rangle \ \text{ with } \ f(x,p) := \varphi(x,p) + \delta_\Theta\big(g(x,p)\big), \ (x,p) \in I\!\!R^n \times I\!\!R^d. \tag{5.3}$$

To proceed with the study of full stability and related properties, recall that the *Robinson constraint qualification* (RCQ) holds in $\widehat{\mathcal{P}}$ at the point $\bar{x}$ with $g(\bar{x}, \bar{p}) \in \Theta$ if

$$0 \in \text{int}\left\{g(\bar{x}, \bar{p}) + \nabla_x g(\bar{x}, \bar{p})I\!\!R^n - \Theta\right\}. \tag{5.4}$$

---

[4]We may write $Y = I\!\!R^s$ with some s while prefer using the symbol Y in order to cover, e.g., the case of $Y = S^m$ in Section 6.

As well known, RCQ (5.4) reduces to the classical *Mangasarian-Fromovitz constraint qualification* (MFCQ) for NLP problems. If x is a local minimizer of $\widehat{\mathcal{P}}$ and RCQ is satisfied at x, then x is the *stationary point* meaning that there is some *Lagrange multiplier* $\lambda \in Y^*$, the dual space of Y, such that

$$0 \in \nabla_x L(x, \bar{p}, \lambda) \ \text{ and } \ \lambda \in N_\Theta(g(x, \bar{p})), \tag{5.5}$$

where $L(\cdot, \cdot, \cdot)$ is the usual *Lagrangian function* defined by

$$L(x, p, \lambda) := \varphi(x, p) + \langle \lambda, g(x, p) \rangle \ \text{ with } \ (x, p) \in \mathbb{R}^n \times \mathbb{R}^d \ \text{ and } \ \lambda \in Y^*. \tag{5.6}$$

The system in (5.5) can be written as the form of Robinson's *generalized equation* (GE) [35]:

$$0 \in \left[\begin{array}{c} \nabla_x L(x, \bar{p}, \lambda) \\ -g(x, \bar{p}) \end{array} \right] + \left[\begin{array}{c} 0 \\ N_\Theta^{-1}(\lambda) \end{array} \right]. \tag{5.7}$$

Note that x is a stationary point of $\widehat{\mathcal{P}}(v, p)$ if and only if $v \in \partial_x f(x, p)$ for (x, p) near $(\bar{x}, \bar{p})$ due to the validity of RCQ (5.4). Since RCQ is always satisfied in all the results below concerning the stability around $(\bar{x}, \bar{p})$, from now on we suppose without loss of generality that the latter equivalence holds for all x.

Let $\Phi : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ and $G : \mathbb{R}^n \times \mathbb{R}^k \to Y$. The pair $(\Phi(x, q), G(x, q))$ provides a $\mathcal{C}^2$-smooth *parameterization* of $(\varphi(x, \bar{p}), g(x, \bar{p}))$ at $\bar{q} \in \mathbb{R}^k$ if both mappings Φ and G are twice continuously differentiable with $\Phi(x, \bar{q}) = \varphi(x, \bar{p})$ and $G(x, \bar{q}) = g(x, \bar{p})$. Consider the following parametric optimization problem:

$$\widetilde{\mathcal{P}}(q) \quad \left\{ \begin{array}{l} \text{minimize } \Phi(x, q) \text{ subject to } x \in \mathbb{R}^n, \\ G(x, q) \in \Theta. \end{array} \right. \tag{5.8}$$

Observe that problem $\widehat{\mathcal{P}}(v, p)$ in (5.2) is a special form of $\widetilde{\mathcal{P}}(q)$ when $\Phi(x, q) = \varphi(x, p) - \langle v, p \rangle$ and $G(x, q) = g(x, p)$ for $q = (v, p) \in \mathbb{R}^n \times \mathbb{R}^d$ and $\bar{q} = (0, \bar{p})$. The next definition is taken from [3, Definition 5.16].

Definition 5.1 (uniform quadratic growth condition). *Let $\bar{x}$ be a stationary point of problem $\widehat{\mathcal{P}}$. The* UNIFORM QUADRATIC GROWTH CONDITION (UQGC) *holds at $\bar{x}$ with respect to a $\mathcal{C}^2$-smooth parameterization $(\Phi(x, q), G(x, q))$ of $(\varphi(x, \bar{p}), g(x, \bar{p}))$ at some $\bar{q} \in \mathbb{R}^k$ if there exist $\ell > 0$ and neighborhoods U of $\bar{x}$ and Q of $\bar{q}$ such that for any $q \in Q$ and any stationary $\bar{x}(q) \in U$ of $\widetilde{\mathcal{P}}(q)$ we have*

$$\Phi(x, q) \geq \Phi(\bar{x}(q), q) + \ell \|x - \bar{x}(q)\|^2 \ \text{ for all } \ x \in U, \ G(x, q) \in \Theta. \tag{5.9}$$

We say that UQGC (5.9) *holds at $\bar{x}$ if it holds for every $\mathcal{C}^2$-smooth parameterization of $(\varphi(x, \bar{p}), g(x, \bar{p}))$.*

Our uniform second-order growth condition (3.9) for the function $f(x, p)$ defined in (5.3) can be viewed as the above UQGC at $\bar{x}$ with respect to the $\mathcal{C}^2$-smooth parameterization $(\varphi(x, p) - \langle v, p \rangle, g(x, p))$. It is shown in [3, Theorem 5.24] that under RCQ (5.4) the defined UQGC is equivalent to Kojima's strong stability [15] formulated in the first part of the following definition taken from [3, Definition 5.33].

Definition 5.2 (strong stability). *We say that a stationary point $\bar{x}$ of problem $\widehat{\mathcal{P}}$ is* STRONGLY STABLE *with respect to a $\mathcal{C}^2$-smooth parameterization $(\Phi(x, q), G(x, q))$ of $(\varphi(x, \bar{p}), g(x, \bar{p}))$ at some $\bar{q}$ if there is a neighborhood $U \times Q$ of $(\bar{x}, \bar{q})$ such that whenever $q \in Q$ the parametric problem $\widetilde{\mathcal{P}}(q)$ has a unique stationary point $\bar{x}(q) \in U$ for which the mapping $q \mapsto \bar{x}(q)$ is continuous on Q. If this holds for any $\mathcal{C}^2$-smooth parameterization of $(\varphi(x, \bar{p}), g(x, \bar{p}))$, we say that $\bar{x}$ is strongly stable. In the conditions above the mapping $q \mapsto \bar{x}(q)$ in Lipschitz continuous on Q, we speak about* LIPSCHITZIAN STRONG STABILITY *of $\bar{x}$.*

Next we show that the continuity of the function $\bar{x}(q)$ in Definition 5.2 can be strengthened to *Hölderian continuity* with degree $\frac{1}{2}$ provided that $\bar{x}$ is a *local minimizer* of problem $\widehat{\mathcal{P}}$ under the validity of RCQ (5.4) at $\bar{x}$. This Hölder continuity can be treated as a natural counterpart of Hölderian full stability in the problem under consideration. In the case of NLP ($\Theta = \{0\} \times \mathbb{R}_-^l$), our result agrees with that by Gfrerer [11, Corollary 3.2] due to the fact that Kojima's strong stability is characterized by Robinson's *strong second-order sufficient condition* (SSOSC) [35]. Note further that the Hölder exponent $\frac{1}{2}$ is shown to the best possible for NLP; see the example in [11] modifying the original one from [36]. The construction of that example also helped us to distinguish between the exact versions of Lipschitzian and Hölderian full stability from Section 3 in the NLP setting; see [25, Example 4.4].

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Theorem 5.3 (strong stability and Hölder continuity). *Let $\bar{x}$ be a local minimizer of problem $\widehat{\mathcal{P}}$, and suppose that RCQ (5.4) holds at $\bar{x}$. Then the point $\bar{x}$ is strongly stable in the sense of Definition 5.2 if and only if for every $\mathcal{C}^2$-smooth parameterization $(\Phi(x,q), G(x,q))$ of $(\varphi(x,\bar{p}), g(x,\bar{p}))$ at some $\bar{q} \in \mathbb{R}^k$ there exist a neighborhood $U \times Q$ of $(\bar{x}, \bar{q})$ and a constant $\kappa > 0$ such that for every $q \in Q$ the parametric problem $\widetilde{\mathcal{P}}(q)$ has a unique stationary point $\bar{x}(q) \in U$ satisfying the Hölder continuity property*

$$\|\bar{x}(q_1) - \bar{x}(q_2)\| \leq \kappa \|q_1 - q_2\|^{\frac{1}{2}} \quad \text{whenever} \ q_1, q_2 \in Q. \tag{5.10}$$

Proof. It is obvious that $\bar{x}$ is strongly stable if the function $\bar{x}(q)$ in Definition 5.2 satisfies the Hölderian continuity property (5.10). Conversely, suppose that the stationary point $\bar{x}$ is strongly stable. Take any $\mathcal{C}^2$-smooth parameterization $(\Phi(x,q), G(x,q))$ of $(\varphi(x,\bar{p}), g(x,\bar{p}))$ at some $\bar{q} \in \mathbb{R}^k$ with $(x,q) \in \mathbb{R}^n \times \mathbb{R}^k$. Define $\Psi(x,w) := \Phi(x,q) - \langle v, x \rangle$ and $\mathcal{G}(x,w) := G(x,p)$ with $w = (q,v) \in \mathbb{R}^k \times \mathbb{R}^n$. Note that $(\Psi, \mathcal{G})$ is also a $\mathcal{C}^2$-smooth parameterization of $(\varphi(x,\bar{p}), g(x,\bar{p}))$ at $\bar{w} := (\bar{q}, 0)$. Since $\bar{x}$ is strongly stable, it follows from [3, Theorem 5.34] that UQGC (5.9) holds at $\bar{x}$ with respect to the parameterization $(\Psi, \mathcal{G})$. By Definition 5.1 there exist $\ell > 0$ and neighborhoods U of $\bar{x}$ and $W = Q \times V$ of $\bar{w} = (\bar{q}, 0)$ such that for any $(q,v) \in Q \times V$ and any stationary point $u \in U$ of the parametric problem $\widetilde{\mathcal{P}}(w)$ we have

$$\Phi(x,q) - \langle v, x \rangle \geq \Phi(u,q) - \langle v, u \rangle + \ell \|x - u\|^2 \quad \text{whenever} \ x \in X, \ G(x,q) \in \Theta. \tag{5.11}$$

Denoting $F(x,q) := \Phi(x,q) + \delta_\Theta(G(x,q))$, observe from [16, Proposition 2.2] that this function is parametrically continuously prox-regular at $(\bar{x}, \bar{q})$ for $\bar{v} = 0 \in \partial_x F(\bar{x}, \bar{p})$ and that BCQ (3.4) holds for this function at $(\bar{x}, \bar{q})$ due to the validity of RCQ. Furthermore (5.11) tells us that the uniform second-order growth condition in (3.9) is satisfied for the function F around $(\bar{x}, \bar{p}, 0) \in \text{gph} \, \partial_x F$. Applying Theorem 3.4 and Corollary 3.6 allows us to find $(\ell_1, \ell_2) \in \mathbb{R}^2_>$ and a neighborhood $U_1 \times Q_1 \times V_1 \subset U \times Q \times V$ of $(\bar{x}, \bar{q}, 0)$ such that for any $(u_i, q_i, v_i) \in \text{gph} \, \partial_x F \cap (U_1 \times P_1 \times V_1)$ with $i = 1, 2$ we have

$$\|u_1 - u_2\| \leq \ell_1 \|v_1 - v_2\| + \ell_2 \|q_1 - q_2\|^{\frac{1}{2}}.$$

Put $v_1 = v_2 = 0$ and note that $u_1 = \bar{x}(q_1)$ and $u_2 = \bar{x}(q_2)$, which gives us the estimate

$$\|\bar{x}(q_1) - \bar{x}(q_2)\| \leq \ell_2 \|q_1 - q_2\|^{\frac{1}{2}} \quad \text{for all} \ q_1, q_2 \in Q_1.$$

This ensures (5.10) and thus completes the proof of the theorem. $\triangle$

Observe from the proof of Theorem 5.3 that when $\bar{x}$ is a local minimizer of problem $\widehat{\mathcal{P}}$, Kojima's strong stability of $\bar{x}$ implies Hölderian full stability at the same point. However, the converse implication is not valid even in the NLP setting. Indeed, it is shown by Mordukhovich and Nghia [25] that, under MFCQ and the well-known *constant rank constraint qualification* for NLP problems, Hölderian full stability and its Lipschitzian counterpart are the same due to the validity of (4.1) (see [25, Proposition 5.2]) and can be characterized by a condition *strictly weaker* than SSOSC. Since SSOSC is equivalent to strong stability in this framework, we conclude that Hölderian full stability can not generally imply strong stability.

Another significant notion of variational analysis is Robinson's *strong regularity* for generalized equations introduced by his landmark paper [35]. We formulate it for the generalized equation (5.7) under consideration.

Definition 5.4 (strong regularity). *Let $(\bar{x}, \bar{\lambda})$ be a solution to the generalized equation (5.7). We say that $(\bar{x}, \bar{\lambda})$ is* STRONGLY REGULAR *if there exist neighborhoods U of $0 \in \mathbb{R}^n \times Y$ and V of $(\bar{x}, \bar{\lambda}) \in \mathbb{R}^n \times Y^*$ such that for every $\delta \in U$ the system*

$$\delta \in \begin{bmatrix} 0 \\ -g(\bar{x}, \bar{p}) \end{bmatrix} + \begin{bmatrix} \nabla^2_{xx} L(\bar{x}, \bar{p}, \bar{\lambda})(x - \bar{x}) + \nabla_x g(\bar{x}, \bar{p})^*(\lambda - \bar{\lambda}) \\ -\nabla_x g(\bar{x}, \bar{p})(x - \bar{x}) \end{bmatrix} + \begin{bmatrix} 0 \\ N_\Theta^{-1}(\lambda) \end{bmatrix} \tag{5.12}$$

has a unique solution in V denoted by $\zeta(\delta)$ and that the mapping $\zeta : U \to V$ is Lipschitz continuous.

It can be deduced from [3, Theorem 5.24] that the strong stability of $(\bar{x}, \bar{\lambda})$ in (5.12) above is *equivalent* to UQGC (5.9) under the following two assumptions:

(A1) The set Θ is $\mathcal{C}^2$-*reducible* to a closed convex set K at $\bar{y} := g(\bar{x}, \bar{p})$, and the reduction is *pointed*. This means that there exist a neighborhood W of $\bar{y}$ and a $\mathcal{C}^2$-smooth mapping $h : W \to \mathbb{R}^k$ such that $\nabla h(\bar{y})$ is surjective, $\Theta \cap W = \{y \in W \mid h(y) \in K\}$, and the tangent cone $T_K(h(\bar{y}))$ is pointed.

(A2) The point $(\bar{x}, \bar{p})$ is *partially nondegenerate* for g with respect to Θ, i.e.,

$$\nabla_x g(\bar{x}, \bar{p}) I\!R^n + \lin\big(T_\Theta\big(g(\bar{x}, \bar{p})\big)\big) = Y, \tag{5.13}$$

where $\lin\big(T_\Theta(g(\bar{x}, \bar{p}))\big)$ is the largest linear subspace of the space Y that is contained in the classical tangent cone $T_\Theta(g(\bar{x}, \bar{p}))$ of convex analysis.

Note that the reducibility condition **(A1)** is satisfied for a great variety of convex sets Θ arising in important classes of problems in constrained optimization. This includes polyhedral sets [3, Example 3.139], the second-order (Lorentz, ice-cream) cone [2, Lemma 15], the cone of positive semidefinite symmetric matrices [3, Example 3.140], etc. In contrast, the nondegeneracy condition **(A2)** is rather restrictive. In particular, for NLP problems it reduces to the classical *linear independence constraint qualification* (LICQ), in the case of MPPC problems (when Θ is a convex polyhedral) it agrees with the polyhedral constraint qualification (PCQ) introduced and studied in [30]; see also [3] for the versions of **(A2)** for other classes of problems in conic programming. Observe that for the general class of problems $\widehat{\mathcal{P}}$ in (5.1) the nondegeneracy condition **(A2)** implies the Robinson constraint qualification (5.4) but clearly not vice versa.

Before deriving the main result of this section we present the following lemma, which is based on the second-order chain rule obtained recently in [21]. This lemma will allow us to make a bridge between general characterizations of Lipschitzian full stability in Section 4 and their applications to the class of constrained problem (5.1) with new links to strong stability and strong regularity.

Lemma 5.5 (limiting coderivative of partial subgradient mappings). *Let both conditions* **(A1)** *and* **(A2)** *be satisfied at* $\bar{x}$, *which is a stationary point of problem* $\widehat{\mathcal{P}}$ *from* (5.1) *in the sense that* $0 \in \partial_x f(\bar{x}, \bar{p})$ *the partial subgradient mapping of the function* f *in* (5.3)*. Then for all* $w \in I\!R^n$ *the limiting coderivative of the partial subgradient mapping* $\partial f_x(\bar{x}, \bar{p})$ *is represented by*

$$D^* \partial_x f(\bar{x}, \bar{p}, 0)(w) = \Big(\nabla^2_{xx} L(\bar{x}, \bar{p}, \bar{\lambda})w, \nabla^2_{xp} L(\bar{x}, \bar{p}, \bar{\lambda})w\Big) + \nabla g(\bar{x}, \bar{p})^* D^* N_\Theta(\bar{y}, \bar{\lambda})\big(\nabla_x g(\bar{x}, \bar{p})w\big) \tag{5.14}$$

with $\bar{y} := g(\bar{x}, \bar{p})$, *where* L *is the Lagrangian* (5.6)*, and where* $\bar{\lambda} \in Y^*$ *is a unique solution of the system*

$$\nabla_x g(\bar{x}, \bar{p})^* \lambda = -\nabla_x \varphi(\bar{x}, \bar{p}) \quad and \quad \lambda \in N_\Theta(\bar{y}). \tag{5.15}$$

Consequently, the coderivative condition (4.1) *is satisfied for this function* f *with* $\bar{v} = 0$.

Proof. Applying the simple subdifferential sum rule to the function f in (5.3), we get from the stationary condition $0 \in \partial_x f(\bar{x}, \bar{p})$ that $0 \in \nabla_x \varphi(x, p) + \partial_x \delta_\Theta(g(x, p))$. Furthermore, the coderivative sum rule from [20, Theorem 1.62] and the second-order subdifferential definition (2.8) give us

$$D^* \partial_x f(\bar{x}, \bar{p}, 0)(w) = \Big(\nabla^2_{xx} \varphi(\bar{x}, \bar{p})w, \nabla^2_{xp} \varphi(\bar{x}, \bar{p})w\Big) + D^* \partial_x (\delta_\Theta \circ g)\big(\bar{x}, \bar{p}, -\nabla_x \varphi(\bar{x}, \bar{p})\big)(w) \tag{5.16}$$

for all $w \in I\!R^n$. The assumed conditions **(A1)** and **(A2)** allow us to apply the second-order chain rule from [21, Theorem 3.6] to the composite function $\delta_\Theta \circ g$ and get in this way the equality

$$D^* \partial_x (\delta_\Theta \circ g)\big(\bar{x}, \bar{p}, -\nabla_x \varphi(\bar{x}, \bar{p})\big)(w) = \begin{array}{l}\Big(\nabla^2_{xx} \langle \bar{\lambda}, g\rangle(\bar{x}, \bar{p})w, \nabla^2_{xp} \langle \bar{\lambda}, g\rangle(\bar{x}, \bar{p})w\Big) \\ +\nabla g(\bar{x}, \bar{p})^* D^* N_\Theta(\bar{y}, \bar{\lambda})\big(\nabla_x g(\bar{x}, \bar{p})w\big)\end{array}$$

for all $w \in I\!R^n$, where $\bar{\lambda}$ solves the KKT system (5.15). This together with (5.16) justifies (5.14).

It remains to verify the validity of (4.1) for the function f with $\bar{v} = 0$. To proceed, pick any vector q with $(0, q) \in D^* \partial_x f(\bar{x}, \bar{p}, 0)(0)$ and get from (5.14) a unique vector $\bar{\lambda} \in Y^*$ satisfying (5.15) such that

$$(0, q) \in \nabla g(\bar{x}, \bar{p})^* D^* N_\Theta(\bar{y}, \bar{\lambda})(0).$$

This allows us to find $z \in D^* N_\Theta(\bar{y}, \bar{\lambda})(0)$ satisfying $0 = \nabla_x g(\bar{x}, \bar{p})^* z$ and $q = \nabla_p g(\bar{x}, \bar{p})^* z$. By the inclusion $\gph N_\Theta \supset \Theta \times \{0\}$, we get that $z \in N_\Theta(\bar{y})$ from $z \in D^* N_\Theta(\bar{y}, \bar{\lambda})(0)$. Since Θ is a closed convex set, it

follows that $\langle z, y \rangle \le 0$ for all $y \in \operatorname{lin}\big(T_\Theta(\bar{y})\big) \subset T_\Theta(\bar{y})$. Due to (5.13) there exist $x \in \mathbb{R}^n$ and $y \in \operatorname{lin}\big(T_\Theta(\bar{y})\big)$ satisfying $\nabla_x g(\bar{x}, \bar{p})x + y = z$. It leads us to

$$\|z\|^2 = \langle z, \nabla_x g(\bar{x}, \bar{p})x + y \rangle = \langle \nabla_x g(\bar{x}, \bar{p})^* z, x \rangle + \langle z, y \rangle \le 0 + 0 = 0,$$

which yields $z = 0$ and thus $q = 0$. This justifies (4.1) and completes the proof of the lemma. $\triangle$

Now we are ready to characterize Lipschitzian full stability of local minimizers in $\widehat{\mathcal{P}}$, which we understand in the sense of Definition 3.2(i) for problem $\widehat{\mathcal{P}}(0, \bar{p})$ in (5.3) with the extended-real-valued objective. The next major theorem not only provides a constructive second-order characterization of Lipschitzian full stability in $\widehat{\mathcal{P}}$ under assumptions **(A1)** and **(A2)** but also establishes its equivalence in this setting to the above notions of strong regularity and Lipschitzian strong stability and thus characterizes these notions as well. Note that the equivalence between assertions **(iii)** and **(iv)** of this theorem has been recently derived in [27, Theorem 6.10] for the case of tilt stability in conic programming when the parameter p is absent.

Theorem 5.6 (equivalence between strong regularity and Lipschitzian full and strong stability for nondegenerate local minimizers and their second-order characterization). *Let $\bar{x}$ be a stationary point of problem $\widehat{\mathcal{P}}$ in (5.1) under the validity of RCQ (5.4), let $\bar{\lambda} \in Y^*$ be the corresponding Lagrange multiplier from (5.5), and let $\bar{y} := g(\bar{x}, \bar{p})$. Assume that the reducibility condition **(A1)** holds at $\bar{x}$. Then the following assertions are equivalent:*

(i) The pair $(\bar{x}, \bar{\lambda})$ is a strongly regular solution to GE (5.7), and $\bar{x}$ is a local minimizer of problem $\widehat{\mathcal{P}}$.

*(ii) The nondegeneracy condition **(A2)** holds, and the point $\bar{x}$ is a Lipschitzian strongly stable local minimizer of problem $\widehat{\mathcal{P}}$.*

*(iii) The nondegeneracy condition **(A2)** holds, and the point $\bar{x}$ is a Lipschitzian fully stable local minimizer of problem $\widehat{\mathcal{P}}$.*

*(iv) The nondegeneracy condition **(A2)** holds together with the second-order subdifferential condition*

$$\langle \nabla_{xx}^2 L(\bar{x}, \bar{p}, \bar{\lambda})w, w \rangle + \inf\Big\{\langle z, \nabla_x g(\bar{x}, \bar{p})w \rangle \,\big|\, z \in D^* N_\Theta(\bar{y}, \bar{\lambda})\big(\nabla_x g(\bar{x}, \bar{p})w\big)\Big\} > 0, \quad w \ne 0. \tag{5.17}$$

Proof. Since $\bar{x}$ is a stationary point of $\widehat{\mathcal{P}}$ at which RCQ (5.4) holds, we deduce from [16, Proposition 2.2] that the function f in (5.3) is parametrically continuously prox-regular at $(\bar{x}, \bar{p})$ for $0 \in \partial_x f(\bar{x}, \bar{p})$ and that BCQ (3.4) holds at $(\bar{x}, \bar{p})$. Observe that implication **(ii)**$\Longrightarrow$**(i)** follows from [3, Theorem 5.35].

To verify next implication **(i)**$\Longrightarrow$**(iii)**, suppose that the point $(\bar{x}, \bar{\lambda})$ is strongly regular for the generalized equation (5.7) and get from [3, Theorem 5.24] that **(A2)** and UQGC (5.9) are satisfied at $\bar{x}$. Defining $\Phi(x, q) := \varphi(x, p) - \langle v, p \rangle$ and $G(x, q) := g(x, p)$ with $q = (v, p)$, note that $(\Phi(x, q), G(x, q))$ is a $\mathcal{C}^2$-smooth parameterization of $(\varphi(x, \bar{p}), g(x, \bar{p}))$ at $\bar{q} := (0, \bar{p})$. Then this UQGC allows us to find $\ell > 0$ as well as neighborhoods $V \times P$ of $\bar{q} = (0, \bar{p})$ and U of $\bar{x}$ such that for any $q = (v, p) \in V \times P$ there is a unique stationary point $\bar{x}(q) \in U$ of problem $\widetilde{\mathcal{P}}(q)$ satisfying

$$\Phi(x, q) \ge \Phi(\bar{x}(q), q) + \ell\|x - \bar{x}(q)\|^2 \quad \text{for all} \ \ x \in U, \ G(x, q) \in \Theta. \tag{5.18}$$

Picking any $(u, p, v) \in \operatorname{gph}\partial_x f \cap (U \times P \times V)$, we have $u = \bar{x}(q)$. It gives us by (5.18) that

$$\varphi(x, p) - \langle v, x \rangle \ge \varphi(u, p) - \langle v, u \rangle + \ell\|x - u\|^2 \quad \text{for all} \ \ x \in U, \ g(x, p) \in \Theta.$$

This clearly implies the inequality

$$f(x, p) \ge f(u, p) + \langle v, x - u \rangle + \ell\|x - u\|^2 \quad \text{for all} \ \ x \in U,$$

which ensures in turn the uniform second-order growth condition (3.9). Taking into account that the coderivative condition (4.1) holds by Lemma 5.5 and then employing Theorem 4.1, we arrive at **(iii)**.

Let us now verify implication **(iii)**$\Longrightarrow$**(iv)**. Assuming **(iii)**, we deduce inequality (4.13) from Corollary 4.5. This together with the second-order representation (5.14) from Lemma 5.5 gives us that

$$
\begin{aligned}
0 \ &< \ \inf\Big\{\langle z, w \rangle \,\big|\, z \in \nabla_{xx}^2 L(\bar{x}, \bar{p}, \bar{\lambda})w + \nabla_x g(\bar{x}, \bar{p})^* D^* N_\Theta(\bar{y}, \bar{\lambda})\big(\nabla_x g(\bar{x}, \bar{p})w\big)\Big\} \\
&= \ \langle \nabla_{xx}^2 L(\bar{x}, \bar{p}, \bar{\lambda})w, w \rangle + \inf\Big\{\langle \nabla_x g(\bar{x}, \bar{p})^* z, w \rangle \,\big|\, z \in D^* N_\Theta(\bar{y}, \bar{\lambda})\big(\nabla_x g(\bar{x}, \bar{p})w\big)\Big\} \\
&= \ \langle \nabla_{xx}^2 L(\bar{x}, \bar{p}, \bar{\lambda})w, w \rangle + \inf\Big\{\langle z, \nabla_x g(\bar{x}, \bar{p})w \rangle \,\big|\, z \in D^* N_\Theta(\bar{y}, \bar{\lambda})\big(\nabla_x g(\bar{x}, \bar{p})w\big)\Big\}
\end{aligned}
$$

for any $w \neq 0$, which shows that condition (5.17) in **(iv)** holds.

To complete the proof of the theorem, it remains to verify implication **(iv)**$\Longrightarrow$**(ii)**. To this end we suppose that condition (5.17) holds and take any $\mathcal{C}^2$-smooth parameterization $(\Phi(x,q), G(x,q))$ of $(\varphi(x,\bar{p}), g(x,\bar{p}))$ at some $\bar{q} \in \mathbb{R}^k$. Observe that $\nabla_x \Phi(\bar{x}, \bar{q}) = \nabla_x \varphi(\bar{x}, \bar{p})$, $\nabla_{xx}^2 \Phi(\bar{x}, \bar{q}) = \nabla_{xx}^2 \varphi(\bar{x}, \bar{p})$, $\nabla_x G(\bar{x}, \bar{q}) = \nabla_x g(\bar{x}, \bar{p})$, and $\nabla_{xx}^2 G(\bar{x}, \bar{q}) = \nabla_{xx}^2 g(\bar{x}, \bar{p})$. By replacing φ by Φ and g by G, we get both conditions **(A1)** and **(A2)** for the pair (Φ, G) at $(\bar{x}, \bar{q})$. Letting $F(x,q) := \Phi(x,q) + \delta_\Theta(G(x,q))$ and combining (5.17) with the second-order representation (5.14) from Lemma 5.5 give us that (4.1) is fulfilled for F at $(\bar{x}, \bar{q}, 0)$ and that

$$\inf \left\{ \langle z, w \rangle \,\middle|\, (z,q) \in D^* \partial_x F(\bar{x}, \bar{q}, 0)(w) \right\} > 0 \;\; \text{for all} \;\; w \neq 0.$$

Unifying this with Corollary 4.5 and Corollary 4.2 allows us to find a neighborhood $(U \times Q \times V)$ of $(\bar{x}, \bar{q}, 0)$ and a constant $\kappa > 0$ such that the mapping S in (3.3), while replacing f by F therein, admits a localization ϑ with respect to $Q \times V \times U$ that satisfies the Lipschitz continuity condition

$$\|\vartheta(v_1, q_1) - \vartheta(v_2, q_2)\| \leq \kappa \big(\|v_1 - v_2\| + \|q_1 - q_2\|\big) \;\; \text{for all} \;\; v_1, v_2 \in V \;\; \text{and} \;\; q_1, q_2 \in Q. \tag{5.19}$$

Define $\bar{x}(q) := \vartheta(0, q)$ for all $q \in Q$ and observe that $\bar{x}(q)$ is a unique stationary point of problem $\widetilde{\mathcal{P}}(q)$ in (5.8). Furthermore, for any $q_1, q_2 \in Q$ we get from (5.19) that

$$\|\bar{x}(q_1) - \bar{x}(q_2)\| \leq \kappa \|q_1 - q_2\|,$$

which ensures the Lipschitz continuity of the function $\bar{x}(q)$ and thus verifies Lipschitzian strong stability in Definition 5.2. This completes the proof of the theorem. $\triangle$

Observe that another characterization of strong regularity from Definition 5.4 for the class of problems modeled as $\widehat{\mathcal{P}}$ in (5.1) via a second-order condition different from (5.17) has been obtained by Bonnans and Shapiro [3, Theorem 5.64] under a certain "strong extended polyhedricity condition," which is not assumed here. Our results in Theorem 5.6 establish the *equivalence* between *all the properties* considered there for the general class of problems $\widehat{\mathcal{P}}$ with new *second-order characterization* (5.17) involving the construction $D^* N_\Theta$ for the underlying convex set Θ. Calculating this second-order object for particular cases of Θ, we arrive at characterizations of the listed properties entirely in terms of the initial data of the mathematical programs. Let us discuss several remarkable classes in mathematical programming, important from both viewpoints of optimization theory and applications, in comparison with known results in this direction. Note that for all the classes discussed below we have the validity of the reducibility condition **(A1)**

• *Nonlinear programming with $\mathcal{C}^2$-smooth data* (NLP). By using the Mordukhovich criterion (2.7) and the calculation of the second-order construction $D^* N_\Theta$ for the orthant $\Theta = \{0\} \times \mathbb{R}_-^l$, Dontchev and Rockafellar [8] proved the equivalence of strong regularity to the simultaneous fulfillment of the *LICQ* and *SSOSC* conditions; see also the discussions and references therein on related results in this vein. It has been recently shown in [30] that condition (5.17) reduces for NLPs to the classical SSOSC being equivalent under the validity of LICQ to Lipschitzian full stability of local minimizers for nonlinear programs.

• *Mathematical programs with polyhedral constraints* (MPPC). Based on the second-order calculus rules from [29] and the coderivative calculations from [8], Mordukhovich, Rockafellar and Sarabi [30] established for this class of optimization problems (5.1) with a polyhedral set Θ a complete characterization of Lipschitzian full stability via the *polyhedral second-order optimality condition* (PSSOC) as well as its equivalence to strong regularity under the *polyhedral constraint qualification*, which is an analog of **(A2)** in the MPPC setting. The aforementioned PSSOC is a MPCC counterpart of the classical SSOSC obtained in the scheme of (5.17).

• *Extended nonlinear programming* (ENLP). The same paper [30] presents a second-order characterization of Lipschitzian full stability for the class of ENLP problems introduced by Rockafellar [37] via a certain duality representation. The characterization is given in terms of the *extended strong second-order optimality condition*, which is an ENLP counterpart of SSOSC obtained in the scheme of (5.17).

• *Second-order cone programming* (SOCP). This subclass of conic programs corresponds to (5.1), where Θ is a product of the *second-order/Lorentz/ice-cream cones*; see [1] for more details and applications. Developing the approach of [30] and invoking the coderivative calculations for the metric projection onto the second-order cone from Outrata and Sun [32], constructive characterizations of Lipschitzian full stability on nondegenerate solutions to SOCPs were established in [28] via an SOCP counterpart of the strong

second-order sufficient optimality condition employed by Bonnans and Ramírez [2] in characterizing strong regularity.

• *Semidefinite programming* (SDP). This major class of conic programs, corresponding to (5.1) with $\Theta = S_+^m$, has been highly recognized in optimization theory and applications; see, e.g., [39, 40] and the references therein. In [39] Sun obtained a characterization of strong regularity of the GE (5.7) associated with SDPs via a counterpart of SSOSC in this setting under the nondegeneracy condition **(A2)**. In Section 6 we show that this SDP version of SSOSC is indeed the same as our condition (5.17) and thus derive from Theorem 5.6 a constructive second-order characterization of full (as well as strong) Lipschitzian stability of locally optimal solutions to semidefinite programs entirely via the their initial data.

• *Other classes of mathematical programs.* Besides the classes of mathematical programs listed above, the second-order construction D^*N_Θ in (5.17) has been constructively calculated for the underlying sets Ω in (5.1), which are not in the discussed forms; see, e.g., [4, 10, 13, 14, 25, 29, 31] and the references therein. These results can be incorporated in the framework of (5.17) and thus allow us to provide via Theorem 5.6 complete characterizations of the equivalent stability properties **(i)**–**(iii)** entirely in terms of initial data of the corresponding mathematical programs under the nondegeneracy condition **(A1)**.

We conclude this section with a convenient second-order condition ensuring the validity of the equivalent stability properties in Theorem 5.6 and therefore their implementations for the particular classes of mathematical programs discussed above. Note that a related result in this vein for Robinson's strong regularity can be extracted from [3, Theorem 5.27 and Corollary 5.29] but under an additional assumption that Y^* has a "lattice structure" that is not the case here; cf. [3, Example 3.57].

Corollary 5.7 (sufficient second-order condition for the equivalent stability properties in mathematical programming). *Let $\bar{x}$ be a stationary point of problem $\widehat{\mathcal{P}}$ in* (5.1)*, and let the conditions* **(A1)** *and* **(A2)** *be satisfied. Assume in addition the second-order condition*

$$\langle \nabla_{xx}^2 L(\bar{x}, \bar{p}, \bar{\lambda})w, w \rangle > 0 \quad \text{whenever} \quad \nabla_x g(\bar{x}, \bar{p})w \in \operatorname{dom} D^*N_\Theta(g(\bar{x}, \bar{p}), \bar{\lambda}), \ w \neq 0, \tag{5.20}$$

where $\bar{\lambda} \in Y^*$ solves the KKT system in (5.15)*. Then all the properties* **(i)**–**(iii)** *of Theorem* 5.6 *hold.*

Proof. When $z \in D^*N_\Theta(g(\bar{x}, \bar{p}), \bar{\lambda})(\nabla_x g(\bar{x}, \bar{p})w)$, we have $\langle z, \nabla_x g(\bar{x}, \bar{p})w \rangle \geq 0$ by the maximal monotonicity of N_Θ and [34, Theorem 2.1]. This together with (5.20) verifies (5.17) and implies therefore that $\bar{x}$ is a Lipschitzian fully stable local minimizer of problem $\widehat{\mathcal{P}}$ due to Theorem 5.6. The other stability/regularity properties of that theorem follows from the established equivalence relationships. $\triangle$

6 Applications to Semidefinite Programming

In this section we develop constructive and nontrivial implementations of the results of Theorem 5.6 for problems of *semidefinite programming* formulated as follows:

$$\check{\mathcal{P}} \quad \begin{cases} \text{minimize} \ \varphi(x, \bar{p}) \text{ subject to } x \in \mathbb{R}^n, \\ g(x, \bar{p}) \in \Theta := S_+^m, \end{cases} \tag{6.1}$$

where $\varphi : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^d \to Y := S^m$ are $\mathcal{C}^2$-smooth mappings, where S^m is the space of $m \times m$ symmetric matrices, and where S_+^m is the cone of all the $m \times m$ positive semidefinite matrices in S^m. Note that the cone S_+^m satisfies the reducibility assumption **(A1)** in Section 5; see, e.g., [3, Example 3.140]. The Robinson constraint qualification (5.4) is written for (6.1) as

$$0 \in \operatorname{int} \left\{ g(\bar{x}, \bar{p}) + \nabla_x g(\bar{x}, \bar{p})\mathbb{R}^n - S_+^m \right\} \tag{6.2}$$

and the partial nondegeneracy condition (5.13) reduces to

$$\nabla_x g(\bar{x}, \bar{p})\mathbb{R}^n + \operatorname{lin}\left(T_{S_+^m}(g(\bar{x}, \bar{p})) \right) = S^m. \tag{6.3}$$

The main goal of this section is to derive a complete characterization of Lipschitzian full stability of local minimizers for (6.1) *entirely in terms of the initial data* (φ, g, S_+^m) of this problem.

Let $A, B \in S^m$ and $\lambda_1(A), \ldots, \lambda_m(A)$ be m eigenvalues of the matrix A with $\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_m(A)$. Denote $\lambda(A) := (\lambda_1(A), \ldots, \lambda_m(A)) \in I\!\!R^m$ and by $\Lambda(A) := \text{diag}(\lambda(A))$ the diagonal matrix whose i-th diagonal entry is $\lambda_i(A)$. Recall the *eigenvalue decomposition* of A is given by

$$A = P \begin{pmatrix} \Lambda_\alpha & 0 & 0 \\ 0 & \Lambda_\beta & 0 \\ 0 & 0 & \Lambda_\gamma \end{pmatrix} P^* \text{ with } P = [P_\alpha \ \ P_\beta \ \ P_\gamma], \tag{6.4}$$

where $\alpha := \{i|\ \lambda_i(A) > 0\}$, $\beta := \{i|\ \lambda_i(A) = 0\}$, $\gamma := \{i|\ \lambda_i(A) < 0\}$, and where P is some $m \times m$ orthogonal matrix. Furthermore, we use the *Frobenius inner product* between A and B defined by

$$\langle A, B \rangle := \text{Tr}(AB),$$

where "Tr" denotes the trace of a matrix; thus the norm of $A \in S^m$ is $\|A\| = \sqrt{\text{Tr}(AA)}$. With these constructions it is well known that the dual space of S^m reduces to S^m.

The next condition is taken from Sun [39, Definition 3.2]. Since we use this condition simultaneously with the nondegeneracy assumption **(A2)** in Section 5, which guarantees the uniqueness of Lagrange multipliers, it makes sense to formulate it under **(A2)** as follows.

Definition 6.1 (strong second-order sufficient condition for SDPs.) *Let $\bar{x}$ be a stationary point of problem $\check{\mathcal{P}}$, and let the partial nondegeneracy condition (6.3) be satisfied. We say that the* SDP-STRONG SECOND-ORDER SUFFICIENT CONDITION *(SDP-SSOSC) holds at $\bar{x}$ if*

$$\langle \nabla_{xx}^2 L(\bar{x}, \bar{p}, \bar{\lambda})w, w \rangle - 2\langle \bar{\lambda}, d(w)g(\bar{x}, \bar{p})^\dagger d(w) \rangle > 0 \ \ \text{for all} \ \ w \in \text{app}(\bar{\lambda}) \setminus \{0\}, \tag{6.5}$$

where $\bar{\lambda}$ is the corresponding unique Lagrange multiplier, $d(w) := \nabla_x g(\bar{x}, \bar{p})w$, $g(\bar{x}, \bar{p})^\dagger$ is the MOORE-PENROSE PSEUDOINVERSE *of $g(\bar{x}, \bar{p})$, and where* $\text{app}(\bar{\lambda})$ *is defined by*

$$\text{app}(\bar{\lambda}) := \left\{ w \in I\!\!R^n |\ P_\beta^* d(w) P_\gamma = 0, \ P_\gamma^* d(w) P_\gamma = 0 \right\} \tag{6.6}$$

with the matrix P taken from (6.4) for $A = g(\bar{x}, \bar{p}) + \bar{\lambda}$.

As discussed in [39, p. 768], the choice of an orthogonal matrix P satisfying the decomposition (6.4) with $A = g(\bar{x}, \bar{p}) + \bar{\lambda}$ does not affect the set $\text{app}(\bar{\lambda})$ in (6.6).

The following calculation of the second-order construction $D^* N_{S_+^m}$ is a reformulation of the recent result from Ding, Sun and Ye [5, Theorem 3.1].

Lemma 6.2 (second-order subdifferential calculation for SDPs). *For any $(X, Y) \in \text{gph} N_{S_+^m}$ consider the the eigenvalue decomposition (6.4) of the matrix $A = X + Y$. Then we have $Z \in D^* N_{S_+^p}(X, Y)(D)$ if and only if $Z = P\widetilde{Z}P^*$ and $D = P\widetilde{D}P^*$ with*

$$\textbf{(i)} \quad \widetilde{Z} = \begin{pmatrix} 0 & 0 & \widetilde{Z}_{\alpha\gamma} \\ 0 & \widetilde{Z}_{\beta\beta} & \widetilde{Z}_{\beta\gamma} \\ \widetilde{Z}_{\gamma\alpha} & \widetilde{Z}_{\gamma\beta} & \widetilde{Z}_{\gamma\gamma} \end{pmatrix} \quad and \quad \widetilde{D} = \begin{pmatrix} \widetilde{D}_{\alpha\alpha} & \widetilde{D}_{\alpha\beta} & \widetilde{D}_{\alpha\gamma} \\ \widetilde{D}_{\beta\alpha} & \widetilde{D}_{\beta\beta} & 0 \\ \widetilde{D}_{\gamma\alpha} & 0 & 0 \end{pmatrix}, \tag{6.7}$$

$$\textbf{(ii)} \quad \widetilde{Z}_{\beta\beta} \in D^* N_{S_+^{|\beta|}}(0,0)(\widetilde{D}_{\beta\beta}) \ \ and \ \ \Sigma_{\alpha\gamma} \circ \widetilde{Z}_{\alpha\gamma} - (E_{\alpha\gamma} - \Sigma_{\alpha\gamma}) \circ \widetilde{D}_{\alpha\gamma} = 0, \tag{6.8}$$

where α, β, γ are taken from (6.4), $|\beta|$ is the cardinality of the set β, E is a $m \times m$ matrix whose all the unit entries, "$\circ$" is the Hadamard product, and where the matrix Σ is defined by

$$\Sigma_{ij} := \frac{\max\{\lambda_i(A), 0\} - \max\{\lambda_j(A), 0\}}{\lambda_i(A) - \lambda_j(A)}, \quad i, j = 1, \ldots, m, \tag{6.9}$$

with the convention that $0/0 := 1$.

Proof. Note that $Z \in D^* N_{S^p_+}(X, Y)(D)$ if and only if $(Z, -D) \in N_{\text{gph}\, N_{S^p_+}}(X, Y)$. Employing [5, Theorem 3.1] verifies claimed representations in the lemma. $\triangle$

The next result is new and plays a crucial role in deriving the main theorem of this section presented below. This lemma provides a precise calculation of the second-order subdifferential condition (5.17) from Theorem 5.6 for the SDP model and shows that it reduces to the SDP-SSOSC condition from Definition 6.1.

Lemma 6.3 (second-order subdifferential condition for SDPs). *Let $\bar{x}$ be a stationary point of problem (6.1), and let $\bar{\lambda}$ is a unique Lagrange multiplier of the corresponding KKT system (5.5) under the validity of the partial nondegeneracy condition (6.3). Then we have $\operatorname{dom} D^* N_{S^m_+}(g(\bar{x}, \bar{p}), \bar{\lambda})(d(\cdot)) = \operatorname{app}(\bar{\lambda})$ and*

$$\inf\left\{\langle Z, d(w)\rangle \,\middle|\, Z \in D^* N_{S^m_+}(g(\bar{x}, \bar{p}), \bar{\lambda})(d(w))\right\} = -2\langle\bar{\lambda}, d(w)g(\bar{x}, \bar{p})^\dagger d(w)\rangle \ \ \text{if } w \in \operatorname{app}(\bar{\lambda}) \qquad (6.10)$$

with $d(w) := \nabla_x g(\bar{x}, \bar{p})w$. Consequently, the second-order subdifferential condition (5.17) from Theorem 5.6 agrees with the SDP-SSOSC condition from Definition 6.1.

Proof. We split the proof of this lemma into following two main steps.

Step 1. *We have that $\operatorname{dom} D^* N_{S^m_+}(g(\bar{x}, \bar{p}), \bar{\lambda})(d(\cdot)) \subset \operatorname{app}(\bar{\lambda})$ and that the inequality "$\geq$" holds in (6.10).*

To show it, pick any $w \in \operatorname{dom} D^* N_{S^m_+}(g(\bar{x}, \bar{p}), \bar{\lambda})(d(\cdot))$ and find $Z \in D^* N_{S^m_+}(g(\bar{x}, \bar{p}), \bar{\lambda})(d(w))$. Let $A := g(\bar{x}, \bar{p}) + \bar{\lambda}$, and let P be an orthogonal matrix satisfying (6.4). With $D := d(w)$ it follows from Lemma 6.2 that $Z = P\widetilde{Z}P^*$ and $D = P\widetilde{D}P^*$, where $\widetilde{Z}, \widetilde{D}$ are taken from (6.7). We get $\widetilde{D} = P^* D P$ and so

$$P^*_\beta D P_\gamma = 0 \ \text{ and } \ P^*_\gamma D P_\gamma = 0,$$

which verifies that $w \in \operatorname{app}(\bar{\lambda})$ due to its expression in (6.6). It gives us the inclusion $\operatorname{dom} D^* N_{S^m_+}(g(\bar{x}, \bar{p}), \bar{\lambda})(d(\cdot)) \subset \operatorname{app}(\bar{\lambda})$. Furthermore, observe from (6.7) that

$$\begin{aligned}
\langle Z, D\rangle &= \operatorname{Tr}(P\widetilde{Z}^* P^* P\widetilde{D}P^*) = \operatorname{Tr}(\widetilde{Z}^*\widetilde{D}P^*P) = \operatorname{Tr}(\widetilde{Z}^*\widetilde{D}) \\
&= \operatorname{Tr}(\widetilde{Z}^*_{\gamma\alpha}\widetilde{D}_{\gamma\alpha}) + \operatorname{Tr}(\widetilde{Z}^*_{\beta\beta}\widetilde{D}_{\beta\beta}) + \operatorname{Tr}(\widetilde{Z}^*_{\alpha\gamma}\widetilde{D}_{\alpha\gamma}) \\
&= \operatorname{Tr}(\widetilde{Z}^*_{\beta\beta}\widetilde{D}_{\beta\beta}) + 2\operatorname{Tr}(\widetilde{Z}^*_{\alpha\gamma}\widetilde{D}_{\alpha\gamma}).
\end{aligned} \qquad (6.11)$$

It follows from (6.9) that for any $i \in \alpha$ and $j \in \gamma$ we have $\Sigma_{ij} = \frac{\lambda_i(A)}{\lambda_i(A) - \lambda_j(A)}$, and thus (6.8) implies that

$$\frac{\lambda_i(A)}{\lambda_i(A) - \lambda_j(A)}\widetilde{Z}_{ij} + \frac{\lambda_j(A)}{\lambda_i(A) - \lambda_j(A)}\widetilde{D}_{ij} = 0,$$

which ensures therefore the equalities

$$\operatorname{Tr}(\widetilde{Z}^*_{\alpha\gamma}\widetilde{D}_{\alpha\gamma}) = \sum_{i\in\alpha, j\in\gamma}\widetilde{Z}_{ij}\widetilde{D}_{ij} = \sum_{i\in\alpha, j\in\gamma} -\frac{\lambda_j(A)}{\lambda_i(A)}\widetilde{D}^2_{ij}. \qquad (6.12)$$

By the spectral decomposition (6.4) and the fact that $\bar{\lambda} \in N_{S^m_+}(g(\bar{x}, \bar{p}))$, which actually means that $-\bar{\lambda} \in S^m_+$ and $\langle\bar{\lambda}, g(\bar{x}, \bar{p})\rangle = 0$, we get the representations

$$g(\bar{x}, \bar{p}) = P\begin{pmatrix} \Lambda_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}P^* \ \text{ and } \ \bar{\lambda} = P\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Lambda_\gamma \end{pmatrix}P^*. \qquad (6.13)$$

Hence the Moore-Penrose matrix $g(\bar{x}, \bar{p})^\dagger$ is formulated in this case as

$$g(\bar{x}, \bar{p})^\dagger = P\begin{pmatrix} \Lambda_\alpha^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}P^*.$$

This together with (6.13) gives us that

$$
\begin{aligned}
\langle \lambda, d(w)g(\bar{x},\bar{p})^\dagger d(w)\rangle \quad &= \operatorname{Tr}\left[P\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \Lambda_\gamma\end{pmatrix}P^*DP\begin{pmatrix}\Lambda_\alpha^{-1} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}P^*D\right]\\
&= \operatorname{Tr}\left[\widetilde{D}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \Lambda_\gamma\end{pmatrix}\widetilde{D}\begin{pmatrix}\Lambda_\alpha^{-1} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right]\\
&= \operatorname{Tr}\left[\begin{pmatrix}0 & 0 & \widetilde{D}_{\alpha\gamma}\Lambda_\gamma\\ 0 & 0 & \widetilde{D}_{\beta\gamma}\Lambda_\gamma\\ 0 & 0 & \widetilde{D}_{\gamma\gamma}\Lambda_\gamma\end{pmatrix}\begin{pmatrix}\widetilde{D}_{\alpha\alpha}\Lambda_\alpha^{-1} & 0 & 0\\ \widetilde{D}_{\beta\alpha}\Lambda_\alpha^{-1} & 0 & 0\\ \widetilde{D}_{\gamma\alpha}\Lambda_\alpha^{-1} & 0 & 0\end{pmatrix}\right]\\
&= \operatorname{Tr}\left[\widetilde{D}_{\alpha\gamma}\Lambda_\gamma\widetilde{D}_{\gamma\alpha}\Lambda_\alpha^{-1}\right] = \sum_{i\in\alpha, j\in\gamma}\frac{\lambda_j(A)}{\lambda_i(A)}\widetilde{D}_{ij}^2.
\end{aligned}
$$

We obtain from this representation as well as (6.11) and (6.12) that

$$
\langle Z, D\rangle = \langle \widetilde{Z}_{\beta\beta}, \widetilde{D}_{\beta\beta}\rangle - 2\langle \lambda, d(w)g(\bar{x},\bar{p})^\dagger d(w)\rangle. \tag{6.14}
$$

Taking into account that the mapping $N_{S_+^{|\beta|}}$ is maximally monotone, it follows from (6.8) and [34, Theorem 2.1] that $\langle \widetilde{Z}_{\beta\beta}, \widetilde{D}_{\beta\beta}\rangle \geq 0$. This together with (6.14) verifies the inequality "$\geq$" in (6.10) for any $w \in \operatorname{dom} D^*N_{S_+^m}(g(\bar{x},\bar{p}),\bar{\lambda})(d(\cdot))$ and thus completes the proof of Step 1.

Step 2. *We have that* $\operatorname{app}(\bar{\lambda}) \subset \operatorname{dom} D^*N_{S_+^m}(g(\bar{x},\bar{p}),\bar{\lambda})(d(\cdot))$ *and that the inequality "$\leq$" holds in* (6.10).

To verify this, pick $w \in \operatorname{app}(\bar{\lambda})$ and define $D := d(w)$. It follows from (6.6) that $\widetilde{D} := P^*DP$ is of form (6.7). Observe from [5, Proposition 3.3], by choosing $\Xi_1 = E$ therein, that $0 \in D^*N_{S_+^{|\beta|}}(0,0)(\widetilde{D}_{\beta\beta})$. By (6.8) find a matrix $\widetilde{Z}$ of form (6.7) satisfying (6.8) and $\widetilde{Z}_{\beta\beta} = 0$. With $Z := P\widetilde{Z}P^*$ it follows from Lemma 6.2 that $Z \in D^*N_{S_+^m}(g(\bar{x},\bar{p}),\bar{\lambda})(D)$. Thus we have $w \in \operatorname{dom} D^*N_{S_+^m}(g(\bar{x},\bar{p}),\lambda)(d(\cdot))$ and deduce from (6.14) that $\langle Z, D\rangle = -2\langle \lambda, d(w)g(\bar{x},\bar{p})^\dagger d(w)\rangle$. This also verifies the inequality "$\leq$" in (6.10) and thus completes the verification of the assertions claimed in Step 2.

Combining finally Step 1 and Step 2 allows us to obtain $\operatorname{dom} D^*N_{S_+^m}(g(\bar{x},\bar{p}),\bar{\lambda})(d(\cdot)) = \operatorname{app}(\bar{\lambda})$ and justify equality (6.5). Hence the second-order subdifferential condition (5.17) agrees with the SDP-SSOSC condition from Definition 6.1, which therefore completes the proof of the lemma. $\triangle$

This lemma together with our major results in Theorem 5.6 allows us not only to recover the equivalence between Robinson's strong regularity and the SDP-SSOSC condition from [39, Theorem 4.1] but also characterize Lipschitzian full stability and strong stability in the SDP framework. Note that ignoring the basic parametric perturbation p provides a complete characterization of *tilt stability* for SDPs entirely via their initial data, which is also new in the literature.

Theorem 6.4 (second-order characterization of Lipschitzian full stability and equivalent properties for SDPs). *Let $\bar{x}$ be a stationary point of problem $\breve{\mathcal{P}}$ in (6.1), and let $\bar{\lambda}$ be the corresponding Lagrange multiplier from (5.5) under the validity of RCQ (6.2). The following assertions are equivalent:*

(i) The point $(\bar{x},\bar{\lambda})$ is strongly regular for (5.12), and $\bar{x}$ is a local minimizer of problem $\breve{\mathcal{P}}$.

(ii) The partial nondegeneracy condition (6.3) holds, and the point $\bar{x}$ is Lipschitzian strongly stable local minimizer of problem $\breve{\mathcal{P}}$.

(iii) The partial nondegeneracy condition (6.3) holds, and the point $\bar{x}$ is a Lipschitzian fully stable local minimizer of problem $\breve{\mathcal{P}}$.

(iv) Both conditions (6.3) and SDP-SSOSC from Definition 6.1 hold.

Proof. It follows directly by combining Theorem 5.6 and Lemma 6.3. $\triangle$

7 Concluding Remarks

This paper demonstrates that *full stability* of locally optimal solutions, in both Lipschitzian and Hölderian frameworks, is a meaningful concept of optimization, which admits verifiable characterizations via second-order constructions of variational analysis in general settings of finite-dimensional optimization with extended-real-valued objectives. Furthermore, for a broad class of constrained optimization problems with $\mathcal{C}^2$-smooth data, including those of conic programming, *Lipschitzian full stability* is proved to be *equivalent* to the well-recognized properties of *strong regularity* of the associated generalized equations and *Lipschitzian strong stability* of local minimizers *provided* the validity of the *reducibility* condition **(A1)** and the *nondegeneracy* condition **(A2)** formulated in Section 5. As a specific application of our general results, we derive a complete second-order characterization of the aforementioned equivalent stability properties entirely in terms of the initial data for the major class of *semidefinite programming* problems.

Observe that, while **(A1)** is unconditionally fulfilled for a variety of problems arising in optimization theory and applications, the nondegeneracy assumption **(A2)** is rather restrictive corresponding to the classical LICQ in nonlinear programming. Theorem 5.6 tells us that the nondegeneracy condition is *necessary* for the validity for *strong regularity* in the general framework (5.1) of constrained optimization. This clearly indicates that full stability is a broader concept that strong regularity even in the most classical settings.

To relax nondegeneracy in the study of (Lipschitzian) *full stability* for the class of (5.1) and/or its specifications is among the main goals of our future research. Note to this end that quite recently Mordukhovich and Nghia [25] have obtained a characterization of full stability for nonlinear programs with $\mathcal{C}^2$-smooth data via a new uniform second-order sufficient optimality condition (defined in a neighborhood of the reference local minimizer) under the validity of both Mangasarian-Fromovitz and constant rank constraint qualifications; see also [22, 24, 26] for previous developments in this direction dealing with tilt stability of NLPs. Important and challenging issues are to establish counterparts of these results for more general classes of mathematical programs and also to derive *pointwise* second-order conditions for full or tilt stability without nondegeneracy for the constrained optimization problems under consideration.

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