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OPTIMIZATION AND DECENTRALIZATION IN THE MATHEMATICS OF ECONOMIC EQUILIBRIUM

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ABSTRACT. A market in which goods can be bought and sold is considered to be in equilibrium if the prices are such that no agent is interested in buying or selling anything and is content to hang on to whatever holdings are already in hand. A classical question in economics is whether, in a situation where agents instead have holdings they may wish to change, there could exist prices under which individually optimal buying and selling would lead to such equilibrium. Although answers to that are well known, little attention has been paid to the implicit concept of a market, which turns out not to reflect market activity as ordinarily understood. Rather than promoting decentralization of decision-making, as is often touted, it appears to require intensive coordination by a central authority. These issues are laid out here along with results which indicate the limitations to realizing the equilibrium through independent economically motivated, budget-balanced trading by agents one-on-one. Nonetheless, an economic mechanism is uncovered which could help toward that goal.

1. INTRODUCTION

Prices have long been a mystery of sorts in economic theory. Surely they are induced by the activities of individual agents in accordance with their individual preferences among goods that might be bought or sold, but how might that actually work?

Real markets are, of course, a phenomenon of everyday life, and they never stand still. Nonetheless, it has proven valuable for fundamental insights to contemplate a single-time model in which agents have holdings they may wish to modify and are able to do so through "market interactions" that lead to their mutual satisfaction, making supply meet demand. The classical approach to this was pioneered by the French economist Walras in the 1870s. We will begin by reviewing it in modern terms, but then proceed to a critique of the

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tacit notion of a market that it employs. Our main contribution will be results that help to bring out the extent to which the "market interactions" can be decentralized into a pattern of agents in pairs dealing directly with each other as in a normal market.

There will be m agents, indexed by $i = 1, \ldots, m$ and n goods, indexed by $j = 1, \ldots, n$. The holdings of agent i are indicated by a vector $x_i = (x_{i1}, \ldots, x_{in})$, where x_{ij} is the quantity held of good j. These vectors will be in the positive orthant $\mathbb{R}^n_{++}$. Each agent i has a utility function u_i on $\mathbb{R}^n_{++}$. that expresses preferences. It will be assumed that

$$u_i \text{ is differentiable, concave and strictly increasing in all arguments,} \tag{1.1}$$

and furthermore that

$$\text{for any } x_i^0 \in \mathbb{R}^n_{++} \text{ the set } \left\{ x_i \in \mathbb{R}^n_{++} \,\middle|\, u_i(x_i) \geq u_i(x_i^0) \right\} \text{ is strictly convex and closed in } \mathbb{R}^n. \tag{1.2}$$

Weaker assumptions could serve for some purposes, but our eventual aim is to show that even under favorable conditions such as these, certain shortcomings in the economic picture persist.

Another key ingredient of the classical model is a price vector $p = (p_1, \ldots, p_n)$ in which p_j is the price at which good j can be bought or sold. Under p, the value of a vector x_i of holdings of agent i is

$$p \cdot x_i = p_1 x_{i1} + \cdots + p_n x_{in}. \tag{1.3}$$

Prices will always be positive here: $p \in \mathbb{R}^n_{++}$.

Optimization problems. *Faced with the price vector p, an agent i with holdings x_i^0 will seek to*

$$\textit{maximize } u_i(x_i) \textit{ subject to } p \cdot x_i = p \cdot x_i^0. \tag{1.4}$$

In other words, an agent, in considering other holdings x_i that might be exchanged at the p prices for the current holdings x_i^0 within budget, will choose holdings having the highest utility.

Note that, under the assumptions in (1.1) and (1.2), a solution to this problem will always exist and be unique. The existence follows from the continuity of u_i in (1.1) and the compactness of the set of feasible vectors x_i, which is due to (1.2) and the positivity of the prices. The uniqueness follows from the strict convexity of the set in (1.2). Another observation is that

$$\bar{x}_i \text{ solves the optimization problem (1.4)} \iff \exists \lambda_i > 0 \text{ such that } \lambda_i \nabla u_i(\bar{x}_i) = p, \tag{1.5}$$

and then in particular

$$(1.6) \qquad \nabla u_i(\bar{x}_i) \cdot [\bar{x}_i - x_i^0] > 0, \quad \text{unless } \bar{x}_i = x_i^0.$$

This invokes the first-order necessary condition for optimality in light of (1.2) and its sufficiency under (1.1); the positivity of the multiplier λ_i for the budget constraint is inevitable from the positivity of the components of $\nabla u_i(\bar{x}_i)$ under (1.1). In (1.6) that positivity again comes into play, together with the concavity of u_i.

Equilibrium. *The combination of a price vector p and agent holdings $\bar{x}_i$ constitutes an equilibrium if the optimization problem for each agent i in the case of $x_i^0 = \bar{x}_i$ is solved by $\bar{x}_i$, or in other words, if for such prices and holdings none of the agents is interested in any changes. More broadly, such a configuration of prices p and holdings $\bar{x}_i$ furnishes an equilibrium reachable from given out-of-equilibrium holdings x_i^0 if the optimization problem for each agent i is solved by $x_i = \bar{x}_i$ and*

$$(1.7) \qquad \sum_{i=1}^{m} \bar{x}_i = \sum_{i=1}^{m} x_i^0 \qquad \textit{(total demand = total supply)}.$$

This definition has been worded carefully to head off a kind confusion in the economics literature in which the combination of p and the holdings x_i^0 is spoken of as the equilibrium without mention of the holdings $\bar{x}_i$. An equilibrium in economics should be a thing in itself, independent of how it may have been reached, as in uses of the term in other subjects. Observe that the possible initial holdings from which a partilar equilibrium with prices p and holdings $\bar{x}_i$ is reachable are any x_i^0s that satisfy, together with (1.7), the budget equations

$$(1.8) \qquad p \cdot x_i^0 = p \cdot \bar{x}_i.$$

Our assumptions on the utility functions guarantee that for any initial holdings there will exist at least one associated equilibium, although it need not be unique [1]. Beyond existence, however, there is a long-standing question without a satisfactory answer. How might a reachable equilibrium be determined? This poses a difficult challenge even in the realm of mathematics and numerical computation, but more importantly in the context of economic activity and social organization.

The standard attempt at an answer is a process called tâtonnement, again going back to Walras, but since pursued by many others [11], [7], [8], [10], [2], [6]. Some sort of market organizer, as a theoretical entity, proposes prices to agents, gets their feedback in demands, and then adjusts the prices up or down to force the demand to be more in tune with the supply. This could go on and on, and in modern versions is often posed as a differential equation which can be shown in some circumstances to converge to the prices that bring about an equilibrium. However, this iterative process is not one of actual buying and selling, but only of information. And the trouble with it is that no such "negotiations" are observed in the real world.

Beyond that question of price determination, there is another issue to be raised in criticism of the traditional framework, and we may be drawing attention to here for the first time. How would the proposed market operate in response to the revealed equilibrium prices? This is not so obvious as it might seem.

In the markets we all know, an agent can buy or sell a quantity of some good at a price in money. The money is traded for the good. But despite appearances, money has not really been involved in the classical model just described. Its prices are merely "relative": their ratios reveal the relative values of different goods. Nothing would change if they were all rescaled by a common positive factor. This does allow for some particular good that everyone appreciates, say gold, to serve as "numéraire" in the sense of the prices being scaled so that its value per unit is 1 (making all other prices be in units of gold). However, that does not mean that a purchase of butter will entail handing over some gold.

In fact, markets in the Walrasion framework are nothing like markets in any ordinary sense. Rather, they manifest as "clearing houses" requiring a scheme of coordination that remains unspecified. An agent, after solving the optimization for prices p at initial holdings x_i^0 to get $\bar{x}_i$, presents the vector $\bar{x}_i - x_i^0$ as the desired adjustment. The positive components of that vector are tied to goods to be bought and the negative components to goods to be sold. The budget constraint forces the value of sales to match the value of purchases. The manner in which the wishes of the agents are to be fulfilled is left undescribed.

Although the equilibrium price vector p promotes decentralized decision making — the optimization problems of the agents are independent of each other — the transfer of the goods among the agents that is supposed to result from those decisions could, in contrast, require a central entity to issue commands and make sure they are obeyed. Might it be possible to fill in some details about the exchange and see a picture of agents trading directly with each other in their own perceived self-interest? That is what we are about to explore.

2. Trading between agents

By a *Walrasian exchange* under a price vector p will be meant a collection of vectors

$$\Delta_i \in \mathbb{R}^n \text{ such that } p \cdot \Delta_i = 0 \text{ and } \sum_{i=1}^{m} \Delta_i = 0. \tag{2.1}$$

The Walrasian exchange associated with an equilibrium in which holdings x_i^0 are adjusted to holdings $\bar{x}_i$ has $\Delta_i = \bar{x}_i - x_i^0$; then (2.1) corresponds to the combination of (1.7) and (1.8). However, we will also be interested in Walrasian exchanges that might not be directly associated with an equilibrium and in particular might be *bilateral*, i.e., with $\Delta_i = 0$ for all but two agents. This would fit with the question, taken up below, of whether the Walrasian exchange associated with attaining an equilibrium can be depicted as the superposition of a number of budget-balanced exchanges between agents acting in pairs. The condition that $p \cdot \Delta_i = 0$ says, of course, that the total value of the positive components of Δ_i equals the total value of the negative components; purchases are balanced by sales.

At first, it may seem that bilateralism is able to characterize the attainment of an equilibrium in an easy way. All that the "market" needs to do is bring prospective buyers and sellers together, hardly intervening beyond providing a bulletin board for them to communicate their wishes and then be able to get together on their own. That would truly stand for full decentralization, but as will be seen, such an interpretation of the model is flawed.

Theorem 2.1 (buyer meets seller). *Any Walrasian exchange can be depicted as the result of finitely many transactions in which an agent i_1 wishing to buy some of good j meets an agent i_2 wishing to sell some of good j, and a quantity of that good is accordingly exchanged at the price p_j.*

Proof. Because the sum in (2.1) vanishes, there must exist for any agent i_1 and good j with $\Delta_{i_1 j} > 0$ another agent i_2 with $\Delta_{i_2 j} < 0$. Let q be the lower of the positive values $\Delta_{i_1 j}$ and $-\Delta_{i_2 j}$. Modify Δ_{i_1} by subtracting the quantity q from its jth component and modify Δ_{i_2} by adding q to its jth component. These modified vectors along with the unmodified ones will together still add to 0 as in (2.1), but at least one component in one of them will have been reduced to 0. This can be repeated until all components of all vectors are 0, which must be reached in finitely many iterations, inasmuch as the the number of nonzero components steadily decreases. □

The flaw with the portrayal in Theorem 2.1 is that the disarmingly simple transaction between a buyer and seller of a single good that it relies on is not *budget-balanced*. We are confronted head on with the fact that, in the Walrasian model, *no money passes from buyer to seller.* The "seller" gives something to the "buyer" but gets nothing in return. This is not a legitimate *market* transaction. Anyway, it goes against the nature of the vector p, which provides just relative prices, not absolute prices. There is no practical sense in saying that a quantity of good j is exchanged at price p_j when p could arbitrarily be rescaled up or down and only the ratios within it are operational.

In a more serious attempt at interpreting a Walrasian exchange, we can draw on the old notion of barter as a substitute for money-based buying and selling.

Simple barter using relative prices. *Given the relative prices in p, an agent i_1 transfers a quantity $q_1 > 0$ of a good j_1 to an agent i_2, who in turn transfers a quantity $q_2 > 0$ of a good j_2 to i_1. The quantities are related by*

$$p_{j_1} q_1 = p_{j_2} q_2. \tag{2.2}$$

Here obviously only the ratio of p_{j_1} to p_{j_2} matters in determining the proportions that are exchanged. This can count as a genuine market transaction, so we can ask whether perhaps, in improving on Theorem 2.1, a Walrasian exchange might be depicted as the result of finitely many simple barter transactions. The following example demonstrats, however, that this may not be fully possible, at least in the way one might wish.

Example 1. *Consider three agents $i = 1, 2, 3$, and three goods $j = 1, 2, 3$. The agents have identical utility functions*

$$u_i(x_{i1}, x_{i2}, x_{i3}) = 3x_{i1}^{1/3} x_{i2}^{1/3} x_{i3}^{1/3}, \tag{2.3}$$

which meet the prescriptions in (1.1) and (1.2) and have

$$\nabla u_i(x_{i1}, x_{i2}, x_{i3}) = \frac{1}{3} u_i(x_{i1}, x_{i2}, x_{i3})(x_{i1}^{-1}, x_{i2}^{-1}, x_{i3}^{-1}), \tag{2.4}$$

and in particular

$$\nabla u_i(\bar{x}_i) = (1, 1, 1) \text{ in the case of } \bar{x}_i = (1, 1, 1). \tag{2.5}$$

The holdings $\bar{x}_i$ in (2.5), together with $p = (1, 1, 1)$, therefore constitute an equilibrium. Moreover this is an equilibrium reachable from the initial holdings

$$x_1^0 = (1.5, 0.5, 1.0), \quad x_2^0 = (1.0, 1.5, 0.5), \quad x_3^0 = (0.5, 1.0, 1.5), \tag{2.6}$$

the corresponding Walrasian exchange being

$$\Delta_1 = (-0.5, 0.5, 0.0), \quad \Delta_2 = (0.0, -0.5, 0.5), \quad \Delta_3 = (0.5, 0.0, -0.5). \tag{2.7}$$

In this exchange, agent 1 has no direct interest in anything other than divesting from good 1 while acquiring good 2, and similarly for agents 2 and 3 but with different goods and no overlap. From that angle it would be impossible to reduce the Walrasian exchange to a collection of simple barter transactions. Yet *such a reduction is possible if the dictates about buying and selling are relaxed. Namely, the holdings x_i^0 can be transformed into the equilibrium holdings $\bar{x}_i$ by the combination of two simple barter transactions that respect the price vector $p = (1, 1, 1)$:*

(2.8)

agent 1 gives 0.5 of good 1 to agent 2, getting 0.5 of good 2 in return,
agent 2 gives 0.5 of good 1 to agent 3, getting 0.5 of good 3 in return.

In the first of the transactions in (2.8), agent 2 accepts a quantity of good 1 despite having no immediate interest in acquiring any, but then turns around to employ it in reaching equilibrium. This has the fruitful interpretation that good 1 is playing the role of *money*, which can naturally go up and down in the course adjustments to holdings, as long as the ups and downs balance out in the end.

It will be worthwhile to pursue this idea more generally. Under our assumptions (1.1) and (1.2), any good could act as "numéraire" with p scaled so that its price is 1. *Let that good now be good $j = 1$, regarding it as money,* and focus on the *normalized* price vector p having $p_1 = 1$. Consider the following special version of the simple barter transactions introduced above.

Money-paired barter using normalized prices. *Given the normalized prices in p an agent i_1 receives from an agent i_2 a quantity q of a good j in return for transfering to agent i_2 a quantity of good 1 as money, namely $p_j q$.*

This nicely recaptures the common market activities of buying and selling in which money has to change hands in tune with current money-based prices. Although technially still "barter" in this context of money being a good, it offers a more appealing way for a Walrasian exchange to be executed.

Theorem 2.2 (buyer meets seller with money). *Under normalized prices with numéraire good $j = 1$ being money, any Walrasian exchange can be depicted as the result of finitely many money-paired barter transactions in which an agent i_1, wishing to acquire some of good $j \neq 1$, meets an agent i_2 wishing to divest some of good j, and a quantity of that good is exchanged accordingly at the money price p_j.*

Proof. Let $\Delta \in \mathbb{R}^{m \times n}$ be the matrix having as its rows the vectors Δ_i of a Walrasian exchange. For each good j and pair of agents i_1 and i_2, let $T_{i_1, i_2, j} \in \mathbb{R}^{m \times n}$ be the matrix that corresponds to the money-paired barter transaction in which i_1 gets a unit of j from i_2 in return for giving i_2 the money amount

p_j:

$$(2.9)\qquad T_{i_1 i_2 j} \text{ has } \begin{cases} -p_j & \text{in position } i_1 1, \text{ but } 1 \text{ in position } i_1 j, \\ p_j & \text{in position } i_2 1, \text{ but } -1 \text{ in position } i_2 j \\ 0 & \text{in every other position.} \end{cases}$$

In terms of assigning to each good $j \neq 1$ the sets

$$(2.10)\qquad I_j^+ = \{ i \,|\, \Delta_{ij} > 0 \}, \qquad I_j^- = \{ i \,|\, \Delta_{ij} < 0 \},$$

let K be the convex cone generated in $I\!R^{m\times n}$ by the matrices

$$(2.11)\qquad T_{i_1 i_2 j} \text{ such that } i_1 \in I_j^+ \text{ and } i_2 \in I_j^-.$$

The claim of the theorem is that Δ lies in K.

The cone K is polyhedral, hence closed, so if Δ were not in it, there would be have to exist a matrix V in $I\!R^{m\times n}$ (as a vector space with the usual inner product) such that

$$(2.12)\qquad V \cdot T \leq 0 \text{ for all } T \in K, \text{ but } V \cdot \Delta > 0.$$

The first part of (2.12) is equivalent to $V \cdot T_{i_1 i_2 j} \geq 0$ for all $T_{i_1 i_2 j}$ in (2.11). By (2.9), we have $V \cdot T_{i_1 i_2 j} = [V_{i_1 j} - V_{i_1 1} p_j] - [V_{i_2 j} - V_{i_2 1} p_j]$, hence $V_{i_1 j} - V_{i_1 1} p_j \leq V_{i_2 j} - V_{i_2 1} p_j$. It follows that for each good $j \neq 1$ there must be a value q_j such that

$$(2.13)\qquad V_{i_1 j} - V_{i_1 1} p_j \leq q_j \leq V_{i_2 j} - V_{i_2 1} p_j \text{ for all } i_1 \in I_j^+ \text{ and all } i_2 \in I_j^-.$$

Then $[V_{ij} - V_{i1} p_j]\Delta_{ij} \geq q_j \Delta_{ij}$, both when $i \in I_j^+$ and when $i \in I_j^-$, so that

$$\begin{aligned} V \cdot \Delta &= \sum\nolimits_{i=1,j=1}^{m,n} V_{ij}\Delta_{ij} \\ &= \sum\nolimits_{i=1}^{m} V_{i1}\Delta_{i1} + \sum\nolimits_{j\neq 1} \Big[\sum\nolimits_{i\in I_j^+} V_{ij}\Delta_{ij} + \sum\nolimits_{i\in I_j^-} V_{ij}\Delta_{ij} \Big] \\ &\geq \sum\nolimits_{i=1}^{m} V_{i1}\Delta_{i1} + \sum\nolimits_{j\neq 1} \Big[\sum\nolimits_{i\in I_j^+} (q_j + V_{i1}p_j)\Delta_{ij} \\ &\qquad + \sum\nolimits_{i\in I_j^-} (q_j + V_{i1}p_j)\Delta_{ij} \Big] \\ &= \sum\nolimits_{i=1}^{n} V_{i1} \Big[\sum\nolimits_{j=1}^{n} p_j \Delta_{ij} \Big] + \sum\nolimits_{j\neq 1} q_j \Big[\sum\nolimits_{i=1}^{m} \Delta_{ij} \Big] = 0, \end{aligned}$$

because of (2.1). But this is impossible, in view of the second part of (2.12). □

It deserves emphasis in Theorem 2.2 that the money good $j = 1$ is not only treated differently in being made part of every barter transaction, but also in not being subjected to the restrictions imposed in (2.11) on the goods $j \neq 1$.

An agent i is permitted in these transactions to acquire money, or give it up, regardless of whether Δ_{i1} is positive, negative or zero. This reflects what we witnessed in Example 1, where agent 1, without having a desire to end up with less of the good $j = 1$, nonetheless bartered some of it way in one transaction, yet bartered it back in another.

3. Motivation shortfall and an iterative scheme

It may seem that Theorem 2.2 meets the challenge ot demonstrating how a Walrasian exchange can always be realized as resulting from a collection of two-agent barter transactions that respect equilibrium prices, and moreover in an agreeable manner with money being essential at every step. But there is a loose end.

In the equilibrium context of $\Delta_i = \bar{x}_i - x_i^0$, with $\bar{x}_i$ solving the optimization problem of agent i for x_i^0 and p, the transactions in question do serve in passing from x_i^0 to $\bar{x}_i$, as each agent is independently motivated to accomplish. The transactions do fit in that way with a desired plan. But can they also be viewed as individually attractive to the agents apart from that plan? If not, the specter of centralized coordination and even social enforcement looms again.

The question in this case concerns the effect of a transaction on an agent's utility. Suppose that an agent i has holdings x_i and switches them to holdings $x_i + td_i$ for some $t > 0$ and $d_i \in I\!R^n$. For this to be advantageous, there should be an increase in utility: $u_i(x_i + td_i) > u_i(x_i)$. That will be true, at least for small t, as long as

$$0 < \frac{d}{dt} u_i(x_i + td_i)\Big|_{t=0} = \nabla u_i(x_i) \cdot d_i. \tag{3.1}$$

Our focus, in thinking about Theorem 2.2, is on the situation where an agent i_1 holding $x_{i_1}^0$ has as d_{i_1} the special vector with a 1 in a position $j \neq 1$ but $-p_j$ in initial position, while an agent i_2 holding $\bar{x}_{i_2}^0$ has $d_{i_2} = -d_{i_1}$. Then, in the simplifying notation

$$\nabla u_i(x_i^0) = g_i = (g_{i1}, g_{i2}, \ldots, g_{in}) \in I\!R^n_{++}, \tag{3.2}$$

the question revolves around

$$\nabla u_{i_1}(x_{i_1}^0) \cdot d_{i_1} = -g_{i_1 1} p_j + g_{i_1 j}, \qquad \nabla u_{i_2}(x_{i_2}^0) \cdot d_{i_2} = g_{i_2 1} p_j - g_{i_2 j}. \tag{3.3}$$

For this transaction to be *mutually attractive*, at least initially in the sense of (3.1), both expressions in (3.3) should be positive, which is the same as the condition that

$$\frac{g_{i_1 j}}{g_{i_1 1}} > p_j > \frac{g_{i_2 j}}{g_{i_2 1}}. \tag{3.4}$$

But Theorem 2.2 says nothing about this. What is the situation in Example 1?

Example 2. *The prices p_j in Example 1 are all 1 and the utility gradient vectors for the three agents at their holdings x_i^0 are*

$$g_1 = c_1(2,6,3), \quad g_2 = c_2(3,2,6), \quad g_3 = c_3(6,3,2), \quad \text{where } c_i = u_i(\bar{x}_i)/9 > 0.$$

For the two transactions in (2.8) enable equilibrium to be reached, we do have the attractiveness condition in (3.4) fulfilled:

$$\frac{g_{12}}{g_{11}} = \frac{6}{3} > 1 > \frac{2}{3} = \frac{g_{22}}{g_{21}}, \qquad \frac{g_{23}}{g_{21}} = \frac{6}{3} > 1 > \frac{3}{6} = \frac{g_{32}}{g_{31}}.$$

This is nice as an example, but in general we are in the dark. For all we know, there could be a situation in which, in the notation (3.2) and with respect to the normalized price vector p,

$$\text{(3.5)} \qquad \text{either } \frac{g_{ij}}{g_{i1}} > p_j \text{ for all } i \text{ and } j \neq 1, \text{ or } \frac{g_{ij}}{g_{i1}} < p_j \text{ for all } i \text{ and } j \neq 1.$$

In the first case no agent perceives an immediate interest in a money purchase of any good j at the given prices, whereas in the second case it is sales that lack incentive. Could this ever be possible?

Without an answer to that, we can turn to the backup question of whether in the Walras framework there anyway ought to be at least one bilateral, but perhaps *multigood*, trade that both agents would find advantageous. To provide some insights, we now show — back in the earlier setting without the designation of a money good or even the barter pattern of pairs of agents trading in two goods at a time — that there must always exist at least such trade.

Theorem 3.1 (improvement guarantee). *Consider an equilibrium with holdings $\bar{x}_i$ and price vector p. If the agents i have out-of-equilibrium holdings x_i^0 leading to that equilibrium, there must exist a pair of agents i_1 and i_2 along with a goods vector $d \in \mathbb{R}^n$ having $p \cdot d = 0$ and such that*

$$\text{(3.6)} \quad x_{i_1}^0 + d \in \mathbb{R}^n_{++}, \; u_{i_1}(x_{i_1}^0 + d) > u_{i_1}(x_{i_1}^0), \quad \textit{while } x_{i_2}^0 - d \in \mathbb{R}^n_{++}, \; u_{i_2}(x_{i_2}^0 - d) > u_{i_2}(x_{i_2}^0).$$

Then, by trading d from i_1 to i_2, budgets are kept in balance and both agents see an improvement in their levels of utility.

Proof. To check whether (3.6) can be satisfied for a given pair of agents, it suffices to look for a vector d for which

$$\text{(3.7)} \qquad \nabla u_{i_1}(x_{i_1}^0) \cdot d > 0, \qquad \nabla u_{i_2}(x_{i_2}^0) \cdot d < 0, \qquad p \cdot d = 0,$$

because then (3.6) will hold for $d' = td$ and small enough $t > 0$. We can exclude from this any consideration of agents i having $\nabla g_i(x_i^0) = \lambda_i p$ for some $\lambda_i > 0$, because they already have solved their optimization problems with no need for adjustment, as noted in (1.5). Under our out-of-equilibrium assumption, there must exist more than a single agent not in that finalized category, for otherwise the condition in (1.7) would fail to hold.

In the simplifying notation (3.2), let

$$L = \{ (g_{i_1}{\cdot}d,\ g_{i_2}{\cdot}d,\ p{\cdot}d) \in \mathbb{R}^3. \,|\, d \in \mathbb{R}^n \}. \tag{3.8}$$

This is a subspace of $\mathbb{R}^3$ with orthogonal complement

$$L^\perp = \{ (a_1, a_2, a_3) \,|\, a_1 g_{i_1} + a_2 g_{i_2} + a_3 d = 0 \}. \tag{3.9}$$

The question is whether L meets the interval product $(0, \infty) \times (-\infty, 0) \times [0, 0]$. If not, we can invoke a separation rule developed in [9, 27.6]:

$$\exists (a_1, a_2, a_3) \in L^\perp \ \text{ such that } \ 0 < a_1(0, \infty) + a_2(-\infty, 0) + a_3[0, 0],$$

which says through (3.9) that

$$\exists (a_1, a_2, a_3) \ \text{ with } a_1 \geq 0 \text{ and } a_2 \leq 0, \text{ not both } 0, \text{ such that} \quad a_1 g_{1_1} + a_2 g_{i_2} + a_3 p = 0. \tag{3.10}$$

We can analyze the meaning of this, case by case relative to the two inequalities, while in taking into account that the vectors g_{1_1}, g_{i_2} and p all lie in $\mathbb{R}^n_{++}$.

Case 1: $a_1 = 0$, $a_2 < 0$. Then necessarily $a_3 > 0$, and $g_{i_2} = \lambda p$ for $\lambda = a_3/|a_2| > 0$. That situation has been excluded.

Case 2: $a_1 > 0$, $a_2 = 0$. Then necessarily $a_3 < 0$, and $g_{i_1} = \lambda p$ for $\lambda = |a_3|/a_1 > 0$. Again, that situation has been excluded.

Case 3: $a_1 > 0$, $a_2 < 0$. Then we obtain from the equation in (3.10) that

$$\exists\, \alpha > 0,\ \beta \in \mathbb{R}, \ \text{ such that } \ \alpha g_{i_1} = g_{i_2} + \beta p. \tag{3.11}$$

Thus, if there were no pair of agents for which (3.7) is satisfied by some vector d, the condition in (3.11) would have to hold for all pairs of agents i_1 and i_2. Putting this another way, in fixing on a reference agent i_0,

$$\forall i \neq i_0, \exists\, \alpha_i > 0,\ \beta_i \in \mathbb{R}, \ \text{ such that } \ \alpha_i g_i = g_{i_0} + \beta_i p. \tag{3.12}$$

With such coefficients we calculate that

$$\begin{aligned}\sum_{i=1}^{m} \alpha_i g_i \cdot [\bar{x}_i - x_i^0] &= g_{i_0} \cdot [\bar{x}_{i_0} - x_{i_0}^0] \\ &\quad + \sum_{i \neq i_0} [g_{i_0} + \beta_i p] \cdot [\bar{x}_i - x_i^0] \\ &= g_{i_0} \cdot \sum_{i=1}^{m} [\bar{x}_i - x_i^0] \\ &\quad + \sum_{i \neq i_0} \beta_i p \cdot [\bar{x}_i - x_i^0] \\ &= 0,\end{aligned} \tag{3.13}$$

with the conclusion coming from the equilibrium conditions that $p \cdot [\bar{x}_i - x_i^0] = 0$ for all agents i and $\sum_{i=1}^{m} x_i^0 = \sum_{i=1}^{m} \bar{x}_i$. But this stands in contradiction to the property in (1.6) that is associated with the holdings x_i^0 not already being in equilibrium with p, in accordance with which the initial sum in (3.13) must be positive. □

The important thing to recognize about the opportunity for improvement guaranteed in Theorem 3.1 is that, after the shifts in (3.5) are carried out, the resultant new holdings of the agents are *still holdings leading to the same equilibrium.* This is a consequence of the budgets and quantities of goods both being maintained. At that stage, unless the equilibrium has just been reached, there must by Theorem 3.1 exist a next possible improvement. That suggests the following economic mechanism for progressing towards an equilibrium.

Iterative improvements toward an equilibrium. *Initially, the agents have out-of-equalibrium holdings associated with an equilibrium having price vector p and holdings $\bar{x}_i$. Some pair of agents i_1 and i_2 is sure to be able to make a budget-balenced trade in which a goods vector d passes from from i_1 to i_2 and both utilities increase. Let the holdings of all the agents, after that takes place, be denoted by x_i^1. These are still associated with the same equilibrium. If they all coincide with the holdings $\bar{x}_i$, equilibrium has been achieved. If not, the improvement step can be repeated to get the next stage of holdings x_i^2, and so forth.*

Many details could be elaborated here in exploring how the improvements at each stage might be made to best advantage. The most interesting point, however, is that this scheme could furnish an *economically motivated* mechanism for realizing a Walrasian exchange. The transfer of goods takes place through transactions of agents who can come together in *pairs* and act in their mutual self-interest *without the need for any guidance or enforcement by a central authority.*

Much remains to be understood, nevertheless. From the mathematical perspective there is the issue of convergence. Although utility values only improve, so that no configuration of holdings can ever be repeated, and the utility levels for the agents must therefore tend to specific limits, assurance is required that those limits are the utility values at equilibrium. Without some precautions, the limits might fall short, because the procedure mires down. Also unclear, of course, is how the pairs of agents would be able to locate each other and recognize the correct trade to make. The proof of Theorem 3.1 is only an existence argument; it is not constructive.

In the background, moreover, lies a more fundamental shortcoming. The proposed actions depend on the agents already knowing the equilibrium price vector p, but what would make that knowledge possible? An impediment from the start in the Walras approach is the perpetual absence of a convincing economic story behind the determination of equilibrium prices, when the initial holdings x_i^0 are the only information. New basic theory will be needed in order to understand satisfactorily how agents themselves might, even then, bring an equilibrium into being. Some efforts in that direction can be seen in [4], [5], [3].

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