Metric Regularity Properties of Monotone Mappings

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Abstract

The theory of metric regularity deals with properties of set-valued mappings that provide estimates useful in solving inverse problems and generalized equations. Maximal monotone mappings, which dominate applications related to convex optimization, have valuable special features in this respect that have not previously been recorded. Here it is shown that the property of strong metric subregularity is generic in an almost everywhere sense. Metric regularity not only coincides with strong metric regularity but also implies local single-valuedness of the inverse, rather than just of a graphical localization of the inverse. Consequences are given for the solution mapping associated with a monotone generalized equation.

Keywords: *maximal monotone mappings, metric regularity, generic subregularity, generalized equations, inverse problems, variational inequalities, convex optimization.*

Version of 16 Jun 2022

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1 Introduction

For a Hilbert space $\mathcal{H}$ and a set-valued mapping $F : \mathcal{H} \rightrightarrows \mathcal{H}$, the graph of F is the set $\operatorname{gph} F = \{\,(x,y)\,|\, y \in F(x)\}$ and the inverse F^{-1} has $\operatorname{gph} F^{-1} = \{\,(y,x)\,|\,(x,y) \in \operatorname{gph} F\}$. The *generalized equation* problem associated with such a mapping F and a continuous single-valued mapping $f : \mathcal{H}' \times \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}'$ is another Hilbert space, has the form

$$\text{for given } \bar{p} \text{ and } \bar{y} \text{ find } \bar{x} \text{ such that } \bar{y} - f(\bar{p}, \bar{x}) \in F(\bar{x}). \tag{1}$$

With y and p in the role of parameters in the problem, attention turns to the *solution mapping* $S : \mathcal{H}' \times \mathcal{H} \rightrightarrows \mathcal{H}$ defined by

$$S(p,y) = \{\,x\,|\, y - f(p,x) \in F(x)\}. \tag{2}$$

The classical implicit function problem of solving $f(p,x) = y$ for x as a function of p or (p,y) is the case where $F(x)$ is just the zero mapping, assigning 0 to every x, but set-valuedness of F allows coverage of much more, for instance conditions for optimality that go beyond equations. An important question is whether S has a *single-valued graphical localization* around the elements in (1), i.e., whether there exist neighborhoods $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{P}$ of $\bar{x}$, $\bar{y}$, $\bar{p}$, such that, for $(p,y) \in \mathcal{P} \times \mathcal{Y}$, there is one and only one $x \in S(p,y) \cap \mathcal{X}$, and if so, whether that function from (p,y) to x is continuous, or Lipschitz continuous, or perhaps even differentiable. But even if the answer to that localization question is no, the solution mapping may have generalized continuity or differentiability behaviors of a set-valued variety that furnish helpful information.

The book [1] by Asen Dontchev and the author of this paper, which is dedicated to his memory,[2] presents that subject in great detail. Properties of *metric regularity* of set-valued mappings, which will be recalled below, are a vital tool in the analysis there. The book [1] does not, however, adequately address the situation where F is maximal monotone, which is tied to applications in convex optimization, in particular. Here we fill that gap, describing the very special features of metric regularity on its different levels when monotonicity is at hand and the spaces $\mathcal{H}$ and $\mathcal{H}'$ are *finite-dimensional.*

The basic theory of monotone mappings is laid out in [3, Chapter 12] for $\mathcal{H}$ being $\mathbb{R}^n$ with its canonical inner product, which obviously carries over to our finite-dimensional Hilbert space $\mathcal{H}$ through the option of introducing coordinates with respect to an orthonormal basis. That theory presumes global monotonicity, but here we really only need it locally, in the sense that F is *monotone in* $\mathcal{X} \times \mathcal{Y}$, a product of subsets of $\mathcal{H}$, when

$$\langle x_1 - x_0, y_1 - y_0 \rangle \geq 0 \text{ for all } (x_0,y_0),\, (x_1,y_1) \in (\mathcal{X} \times \mathcal{Y}) \cap \operatorname{gph} F. \tag{3}$$

Such monotonicity is *maximal* if there is no mapping F' that shares the same property (3) and has $(\mathcal{X} \times \mathcal{Y}) \cap \operatorname{gph} F'$ properly larger than $(\mathcal{X} \times \mathcal{Y}) \cap \operatorname{gph} F$. The global version has $\mathcal{X} \times \mathcal{Y} = \mathcal{H} \times \mathcal{H}$. For any mapping F that satisfies (3), there always exists a globally maximal monotone mapping $\bar{F}$ such that $(\mathcal{X} \times \mathcal{Y}) \cap \operatorname{gph} \bar{F} = (\mathcal{X} \times \mathcal{Y}) \cap F$ [3, 12.6]. The maximal monotonicity of F, assumed here from now on, entails the closedness of its graph and the closed convexity of the sets $F(x)$ and $F^{-1}(y)$.

A prime example of a globally maximal monotone mapping is the subgradient mapping $\partial\varphi$ associated with a closed proper convex function φ on $\mathcal{H}$ [3, 12.17]. A key case of that is the one where φ is the indicator δ_C of a nonempty closed convex set $C \subset \mathcal{H}$. Then $\partial\varphi$ is the normal cone mapping $N_C : \mathcal{H} \rightrightarrows \mathcal{H}$ with

$$y \in N_C(x) \iff x \in C \text{ and } \langle y, x' - x \rangle \geq 0 \text{ for all } x' \in C. \tag{4}$$

[2] He died 16 September 2021 at age 73.

The generalized equation problem (1) is a so-called *variational inequality problem of geometric type* when $F = N_C$, and one of *function type* more broadly when $F = \partial\varphi$. However much of what distinguishes those cases holds as well simply when F is maximal monotone. An example of that still in the domain of convex optimization comes up when F is the monotone mapping derived from subgradients of a convex-concave saddle function, such as a Lagrangian function in congugate duality theory [3, 12.27].

The version of (1) in which not only F but also the functions $f(p,\cdot)$ are monotone is a *monotone generalized equation*. Because f is single-valued and continuous, the monotonicity of $f(p,x)$ with respect to x, in the sense of (3), is automatically maximal [3, 12.7].

Of course the parameterization in (1) is merely an option. Without it, we are looking for $\bar{x}$ such that $\bar{y} - f(\bar{x}) \in F(\bar{x})$, which is equivalent to finding $\bar{x} \in (f+F)^{-1}(\bar{y})$. Here $f+F$ is a mapping $F' : \mathcal{H} \rightrightarrows \mathcal{H}$, so we are solving an inverse problem for F', which for general purposes we can just as well pose as an *inverse problem for F*:

$$\text{given } \bar{y}, \text{ find } \bar{x} \text{ such that } F(\bar{x}) \ni \bar{y}, \text{ or in other words } \bar{x} \in F^{-1}(\bar{y}). \tag{5}$$

The solution mapping associated with this problem is just F^{-1} itself.

The theory of metric regularity deals with properties of F and F^{-1} that directly concern the inverse problem (5), although they ultimately are important also in understanding the broader problem (1). First there is basic *metric regularity* of F at $\bar{x}$ for $\bar{y} \in F(\bar{x})$, according to which

$$\exists\, \kappa > 0 \text{ with } d(x, F^{-1}(y)) \leq \kappa d(y, F(x)) \text{ for all } (x,y) \in \operatorname{gph} F \text{ close enough to } (\bar{x},\bar{y}). \tag{6}$$

Here $d(x,C)$ gives the distance of a point x from a closed set C, and "close enough" means for a small enough neighborhood. The *modulus* of such metric regularity is the liminf of the possible κ values as the neighborhood gets smaller and smaller, and is denoted by $\operatorname{reg}(F;\bar{x}\,|\,\bar{y})$. Metric *subregularity* asks instead only for

$$\exists\, \kappa > 0 \text{ with } d(x, F^{-1}(\bar{y})) \leq \kappa ||y - \bar{y}|| \text{ for all } (x,y) \in \operatorname{gph} F \text{ close enough to } (\bar{x},\bar{y}) \tag{7}$$

and has a similarly defined modulus, denoted by $\operatorname{subreg}(F;\bar{x}\,|\,\bar{y})$. It is known in fact to be equivalent to the seemingly more demanding condition that

$$\exists\, \kappa > 0 \text{ with } d(x, F^{-1}(\bar{y})) \leq \kappa d(\bar{y}, F(x)) \text{ for all } (x,y) \in \operatorname{gph} F \text{ having } x \text{ close enough to } \bar{x}, \tag{8}$$

[1, 3H.4]. The metric regularity in (6) is *strong* if, in addition, the graphical localization in (6) makes F^{-1} be single-valued. In parallel, *strong metric subregularity* refers to (7) holding with $\bar{x}$ an isolated point of $F^{-1}(y)$, i.e., having a neighborhood in which there is no other $x \in F^{-1}(\bar{y})$.

At each of these four levels of metric regularity there is an equivalent *inverse* property with respect to F^{-1}. For strong metric regularity, it is the existence of a single-valued graphical localization of F^{-1} around $\bar{y}$ that is Lipschitz continuous; the Lipshitz modulus at $\bar{y}$ equals $\operatorname{reg}(F;\bar{x}\,|\,\bar{y})$. For metric regularity, it is a local Lipschitz-like behavior called the *Aubin property* [1, p. 172]. The inverse property paired with subregularity is the *calmness* of F^{-1} at $\bar{y}$ for $\bar{x}$, which is a one-sided reduction of the Aubin property [1, p. 197]. The inverse of strong metric subregularity combines that calmness with $\bar{x}$ being an isolated point of $F^{-1}(\bar{y})$.

Maximal monotonicity of F has a powerful effect on these properties, as we will now make clear.

2 Results

A deep connection between monotonicity and convexity lies behind the results that are about to be presented. For F maximal monotone, F^{-1} is maximal monotone as well, and the effective domain and effective range sets

$$\begin{array}{c} \operatorname{dom} F = \{\, x \,|\, F(x) \neq \emptyset\}, \quad \operatorname{rge} F = \{\, y \,|\, \exists\, x,\ y \in F(x)\}, \\ \text{with} \;\; \operatorname{dom} F^{-1} = \operatorname{rge} F, \quad \operatorname{rge} F^{-1} = \operatorname{dom} F, \end{array} \tag{9}$$

are *nearly convex*: their closures are convex, and they include the relative interiors of those closures [3, 12.41]. The recession cone of $F(x)$ when $x \in \operatorname{dom} F$ is moreover the normal cone to the convex set $\operatorname{cl} \operatorname{dom} F$ at x [3, 12.37]. Thus, $F(x)$ is nonempty and bounded if and only if $x \in \operatorname{int} \operatorname{dom} F$ [1, 12.67]. That way, the nonemptiness of $\operatorname{int} \operatorname{dom} F$ corresponds to no set $F(x)$ having with in it an entire line, or equivalently none having a nonzero subspace in its recession cone. (The normal cones to a convex set with empty interior include the subspace orthogonal to the affine hull of the set.)

Generalized differentiation will offer assistance here along with convexity. The *graphical derivative* of F at $\bar{x}$ for $\bar{y} \in F(\bar{x})$ is the set-valued mapping $DF(\bar{x}\,|\,\bar{y}) : \mathcal{H} \rightrightarrows \mathcal{H}$ having as its graph the tangent cone to $\operatorname{gph} F$ at $(\bar{x}, \bar{y})$ in the sense of variational analysis. This means that $\eta \in DF(\bar{x}\,|\,\bar{y})(\xi)$ if and only if there are sequences $\tau^\nu \searrow 0$ and $(\xi^\nu, \eta^\nu) \to (\xi, \nu)$ such that $(\bar{x}, \bar{y}) + \tau^\nu(\xi^\nu, \eta^\nu) \in \operatorname{gph} F$. The definition is symmetric in x and y, so $DF^{-1}(\bar{y}\,|\,\bar{x}) = DF(\bar{x}\,|\,\bar{y})^{-1}$. We have from [1, 4E.1] that

$$\begin{array}{l} F \text{ is strongly metrically subregular at } \bar{x} \text{ for } \bar{y} \in F(\bar{x}) \quad \Longleftrightarrow \\ DF(\bar{x}\,|\,\bar{y})(\xi) \ni 0 \;\text{ only for } \xi = 0, \text{ or equivalently, } DF^{-1}(\bar{y}\,|\,\bar{x})(0) = \{0\}. \end{array} \tag{10}$$

This criterion is fulfilled in particular when $DF^{-1}(\bar{x}\,|\,\bar{y})$ is (generalized) *differentiable* at $\bar{x}$, which means that F^{-1} is single-valued at $\bar{y}$ with the mapping $DF^{-1}(\bar{x}\,|\,\bar{y}) : \mathcal{H} \to \mathcal{H}$ is single-valued and *linear*. These facts and definitions don't draw on F being monotone, but in the case of differentiability such monotonicity carries over to monotonicity of the derivative mapping.

Theorem 1 (strong metric subregularity is generic). *Suppose the set $D = \operatorname{rge} F = \operatorname{dom} F^{-1}$ has nonempty interior (without which $F^{-1}(y)$, whenever nonempty, would be unbounded with a nonzero subspace in its recession cone). Then for almost every $\bar{y}$ in that convex interior of D (having the rest of D in its closure), there is a unique solution $\bar{x}$ to the inverse problem (5) and F is strongly metrically subregular at $\bar{x}$ for $\bar{y}$. In fact, F^{-1} will be differentiable at almost every such $\bar{y}$.*

Proof. This falls into place right away from the facts about domains and interiors reviewed at the beginning of this section, because F^{-1}, like any maximal monotone mapping, is differentiable almost everywhere on the interior of its effective domain [3, 12.66(b)]. □

Turning from subregularity to regularity, we look next at a property already established in [1, 3G.5] but with a complicated proof. We furnish an alternative proof based on a fundamental fact about when a maximal monotone mapping can be continuous. We furthermore augment the statement by a new observation about single-valuedness of the inverse.

Theorem 2 (metric regularity is automatically strong). *If F is metrically regular at $\bar{x}$ for $\bar{y} \in F(\bar{x})$, then it is strongly metrically regular there. The equivalent property of F^{-1} is that it is single-valued and Lipschitz continuous on a neighborhood of $\bar{y}$ with its Lipschitz modulus at $\bar{y}$ being the same as* $\operatorname{reg}(F; \bar{x}\,|\,\bar{y})$. *No localization also in the x argument is needed for this single-valuedness.*

Proof. Metric regularity of F at $\bar{x}$ for $\bar{y} \in F(\bar{x})$ is equivalent F^{-1} having the Aubin property at $\bar{y}$ for $\bar{x} \in F^{-1}(\bar{y})$ [1, 3E.7]. This was mentioned earlier without reviewing the definition of the Aubin property (on page 172 of [1]), and the details won't be necessary. All that's needed are two immediate implications of that definition. The first is that when the Aubin property holds at $\bar{y}$ for $\bar{x} \in F^{-1}(\bar{y})$, then $\bar{y} \in \operatorname{int} F^{-1}(\bar{x})$ and the Aubin property persists for all $(y, x) \in \operatorname{gph} F^{-1}$ near $(\bar{y}, \bar{x})$. That way, to show that F^{-1} has a single-valued graphical localization around $\bar{y}$, it suffices to show that F^{-1} is single-valued at $\bar{y}$. The second consequence is the existence, on some neighborhood of $\bar{y}$, of a single-valued function s with $s(y) \in F^{-1}(y)$ and $s(\bar{y}) = \bar{x}$, which is continuous at $\bar{x}$. That's incompatible with $F^{-1}(\bar{y})$ being more than a singleton, because from the property of maximal monotonicity in [3, 12.63(a)], if a sequence of points y^ν approaches $\bar{y}$ from a particular direction, the elements $s(y^\nu)$ of $F^{-1}(y^\nu)$, in converging to $s(\bar{y})$, would have $s(\bar{y})$ in the face of $F^{-1}(\bar{y})$ in that direction. But without single-valuedness, $F^{-1}(\bar{x})$ would have more than one face.

The single-valudeness seen here requires no localization in the x argument, since a convex set can't have an isolated point unless it is just a singleton; there is only one element of $F^{-1}(y)$ in all of $\mathcal{H}$. That, confirms the final assertion in the theorem. (It's not covered by Theorem 1 because it also addresses points of single-valuedness where F^{-1} might not also be differentiable.) □

Theorems 1 and 2 capture the major differences in working with metric regularity properties of a maximal monotone mapping F, but what does monotonicity contribute to the original picture of the solution mapping (2) for (1)? The problem there involves, along with F, a continuous single-valued mapping $f : \mathcal{H}' \times \mathcal{H} \to \mathcal{H}$, with H' likewise a finite-dimensional Hilbert space.[3] It poses a *monotone* generalized equation when $f(p, x)$ is monotone with respect to x.

For insight into this, we recall, for $\sigma > 0$, the concept of *σ-strong monotonicity* of a mapping $G : \mathcal{H} \rightrightarrows \mathcal{H}$ around $\bar{x}$ for $\bar{y} \in G(\bar{x})$. It is the existence of neighborhoods $\mathcal{X}$ of $\bar{x}$ and $\mathcal{Y}$ of $\bar{y}$ such that

$$\langle x_1 - x_0, y_1 - y_0 \rangle \geq \sigma ||x_1 - x_0||^2 \text{ for all } (x_0, y_0), (x_1, y_1) \in (\mathcal{X} \times \mathcal{Y}) \cap \operatorname{gph} G. \tag{11}$$

This will provide a criterion beyond the usual kind in [1] for a well-behaved solution mapping S when the generalized equation is monotone.

Theorem 3 (implicit function theorem with monotonicity). *In problem (1) with solution $\bar{x}$ and $f(p, x)$ monotone in x, let f be differentiable around $(\bar{p}, \bar{x})$ with its linear derivative mapping $Df(\bar{p}, \bar{x})$ having components $D_p(\bar{p}, \bar{x})$ and $D_x(\bar{p}, \bar{x})$ in the separate arguments. Consider the corresponding linearization of $f(\bar{p}, x)$ at $\bar{x}$, namely*

$$g(x) = f(\bar{p}, \bar{x}) + D_x f(\bar{p}, \bar{x})(x - \bar{x}). \tag{12}$$

Suppose the mapping $G = g + F : \mathcal{H} \rightrightarrows \mathcal{H}$, having $\bar{y} \in G(\bar{x})$, is σ-strongly monotone around $\bar{x}$ for $\bar{y}$ as in (11). Then the solution mapping S in (2), which is monotone in the y argument, is single-valued and Lipschitz continuous on a neighborhood of $(\bar{p}, \bar{y})$ with Lipschitz modulus

$$\operatorname{lip} S(\bar{p}, \bar{y}) \leq \sigma^{-1} \max\{1, ||D_p f(\bar{p}, \bar{x})||\}. \tag{13}$$

Proof. This will be derived from [1, 3F.9] and Theorem 2. The monotonicity of the mapping $y \mapsto S(p, y)$ will also enter into this. That monotonicity follows from that mapping being the inverse of the mapping $x \mapsto f(p, x) + F(x)$, which is monotone through the monotonicity of $x \mapsto f(p, x)$. The impact of the σ-strong monotonicity of G in (11) will come from its implication that the graphical localization of G^{-1} in $\mathcal{Y} \times \mathcal{X}$ is Lipschitz continuous with constant σ^{-1}, inasmuch as $\langle x_1 - x_0, y_1 - y_0 \rangle \leq ||x_1 - x_0|| \cdot ||y_1 - y_0||$ in (11).

[3] Not all of $\mathcal{H}'$ need be brought in here; only p in some subset of $\mathcal{H}'$ need to be involved. But we proceed this way to to avoid extra notation.

The result in [1, 3F.9] keeps $\bar{y}$ fixed at 0 (no y as parameter), so to apply it we have to shift the perspective slightly by introducing $\bar{f}(p,y,x) = f(p,x) - y$ and passing to the representation

$$S(p,y) = \{\, x \mid \bar{f}(p,y,x) + F(x) \ni 0\}. \tag{14}$$

The linearization of $\bar{f}(\bar{p},\bar{y},\cdot)$ at $\bar{x}$ is the same as that of $f(\bar{p},\cdot)$, namely the function g in (12). In this setting, we get from [1, 3F.9] that, if the mapping $G = g + F$ is metrically regular with modulus $\operatorname{reg}(G;\bar{x}\,|\,\bar{y})$, then the solution mapping in (13) will have the Lipschitz-like Aubin property at $(\bar{p},\bar{y})$ for $\bar{x}$ with modulus

$$\operatorname{lip}(S : \bar{p},\bar{y}\,|\,\bar{x}) \leq \operatorname{reg}(G;\bar{x}\,|\,\bar{y})\cdot\|D_{p,y}\bar{f}(\bar{p},\bar{y},\bar{x})\|. \tag{15}$$

However, since $S(p,y)$ is monotone in y, that Aubin property, being the inverse of metric regularity, demands single-valuedness of S by Theorem 2. It thereby reduces to ordinary Lipschitz continuity with the same modulus estimate (15).

As the sum of a continuous monotone mapping g and the maximal monotone mapping F, G is maximal monotone [3, 12.44]. Its metric regularity is therefore by Theorem 2 the same as strong metric regularity and corresponds to G^{-1} being single-valued and Lipschitz continuous locally with its modulus at $\bar{y}$ equal to $\operatorname{reg}(G;\bar{x}\,|\,\bar{y})$. Here we have from our assumption of strong metric regularity that G^{-1} has σ^{-1} as a local Lipschitz constant, so $\operatorname{reg}(G;\bar{x}\,|\,\bar{y}) \leq \sigma^{-1}$ in (14).

On the other hand, the mapping $D_{p,y}\bar{f}(\bar{p},\bar{y},\bar{x})$ takes a pair (π,η) to the pair $(D_p f(\bar{p},\bar{x})(\pi),\eta)$. The norm of a linear mapping $(\pi,\eta) \mapsto (A\pi, B\eta)$ is the max of $\|A\|$ and $\|B\|$, so the norm of $D_{p,y}\bar{f}(\bar{p},\bar{y})$ in (14) is the max of $\|D_p f(\bar{p},\bar{x})\|$ and 1. That validates the Lipschitz constant in the theorem. □

For an illustration of Theorem 3, suppose $F = N_C$ for a closed convex set C as in (4) and let $f(p,x) = \nabla_x \varphi(p,x)$ for a continuously differentiable function φ that is convex in the x argument. Then

$$S(p,y) = \operatorname*{argmin}_{x\in C} \{\, \varphi(p,x) - \langle x,y\rangle\},$$

because having $y - \nabla_x\varphi(x,p) \in N_C(x)$ is the necessary and sufficient condition in convex analysis for x to give a global solution in that problem of minimization. The strong monotonicity assumption in Theorem 3 corresponds, in function terms, to a sort of quadratic growth property that enhances optimality behavior at $\bar{x}$ when $(p,y) = (\bar{p},\bar{y})$ through tilt stability [2]. According to Theorem 3, that property induces the argmin mapping to be locally single-valued and Lipschitz continuous.

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