ZEROS

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ABSTRACT

The purpose of this book is two fold.

(1) To give a systematic account of classical "zero theory" as developed by Jensen, Pólya, Titchmarsh, Cartwright, Levinson and others.

(2) To set forth developments of a more recent nature with a view toward their possible application to the Riemann Hypothesis.

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CONTENTS

§1. *INFINITE PRODUCTS*

Let $\{z_n : n = 1,2,\ldots\}$ be a sequence of complex numbers.

1.1 DEFINITION The infinite product

$$\prod_{n=1}^{\infty} (1 + z_n)$$

is <u>convergent</u> if the following conditions are satisfied.

- The partial products

$$\prod_{n=1}^{N} (1 + z_n)$$

approach a finite limit as $N \to \infty$.

- From some point on, say $n > N_0$, $z_n \neq -1$, and then

$$\lim_{N \to \infty} \prod_{N_0+1}^{N} (1 + z_n) \neq 0.$$

[Note: The infinite product

$$\prod_{n=1}^{\infty} (1 + z_n)$$

is <u>divergent</u> if it is not convergent.]

<u>N.B.</u> The convergence of

$$\prod_{n=1}^{\infty} (1 + z_n)$$

implies that $1 + z_n \to 1$, hence that $z_n \to 0$.

1.2 REMARK It can happen that

$$\prod_{n=1}^{\infty} (1 + z_n) = 0$$

but only when at least one factor is zero.

1.3 EXAMPLE On the one hand,

$$\prod_{n=2}^{\infty} (1 - \frac{1}{n^2}) = \frac{1}{2},$$

while on the other,

$$\prod_{n=1}^{\infty} (1 - \frac{1}{n^2}) = 0.$$

1.4 EXAMPLE For all $N_0 > 1$,

$$\lim_{N \to \infty} \prod_{N_0+1}^{N} (1 - \frac{1}{n}) = 0.$$

Therefore the infinite product

$$\prod_{n=2}^{\infty} (1 - \frac{1}{n})$$

is divergent.

Turning to the theory, we shall first consider the case of real numbers.

1.5 LEMMA If $\{a_n : n = 1,2,\ldots\}$ is a sequence of nonnegative real numbers, then $\prod_{n=1}^{\infty} (1 + a_n)$ is convergent iff $\sum_{n=1}^{\infty} a_n$ is convergent.

PROOF In fact, $\forall$ N,

$$a_1 + a_2 + \cdots + a_N \leq \prod_{n=1}^{N} (1 + a_n) \leq \exp(a_1 + a_2 + \cdots + a_N).$$

1.6 EXAMPLE The infinite product

$$\prod_{n=1}^{\infty} (1 + \frac{1}{n^p})$$

is convergent for $p > 1$ and divergent for $p \leq 1$.

1.7 LEMMA If $\{a_n : n = 1,2,\ldots\}$ is a sequence of nonnegative real numbers, then $\prod_{n=1}^{\infty} (1 - a_n)$ is convergent iff $\sum_{n=1}^{\infty} a_n$ is convergent.

PROOF If a_n does not tend to 0, then both the product and the series are divergent, so there is no loss of generality in assuming from the beginning that $a_n < \frac{1}{2}$ ($\Rightarrow 1 - a_n > \frac{1}{2}$).

- Suppose that $\prod_{n=1}^{\infty} (1 - a_n)$ is convergent -- then the partial products

$$\prod_{n=1}^{N} (1 - a_n)$$

constitute a monotone decreasing sequence with a positive limit L: $\forall$ N,

$$\prod_{n=1}^{N} (1 - a_n) \geq L > 0.$$

But

$$1 + a_n \leq \frac{1}{1 - a_n},$$

thus

$$\prod_{n=1}^{N} (1 + a_n) \leq \prod_{n=1}^{N} \frac{1}{1 - a_n} \leq \frac{1}{L}.$$

Since the partial products

$$\prod_{n=1}^{N} (1 + a_n)$$

constitute a monotone increasing sequence, it follows that $\prod_{n=1}^{\infty} (1 + a_n)$ is convergent, hence the same is true of $\sum_{n=1}^{\infty} a_n$ (cf. 1.5).

• Suppose that $\sum_{n=1}^{\infty} a_n$ is convergent -- then $\sum_{n=1}^{\infty} 2a_n$ is convergent, thus $\prod_{n=1}^{\infty} (1 + 2a_n)$ is convergent (cf. 1.5), so there exists $K > 0$ such that $\forall$ N,

$$\prod_{n=1}^{N} (1 + 2a_n) \leq K.$$

But

$$0 \leq a_n < \frac{1}{2} \Rightarrow 1 - a_n \geq \frac{1}{1 + 2a_n}$$

=>

$$\prod_{n=1}^{N} (1 - a_n) \geq \prod_{n=1}^{N} \frac{1}{1 + 2a_n} \geq \frac{1}{K} > 0.$$

And

$$\prod_{n=1}^{\infty} (1 - a_n)$$

is monotone increasing.

1.8 EXAMPLE The infinite product

$$\prod_{n=1}^{\infty} (1 - \frac{1}{n^p})$$

is convergent for $p > 1$ and divergent for $p \leq 1$.

1.9 LEMMA Let $\{a_n : n = 1,2,\ldots\}$ be a sequence of real numbers. Assume: $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_n^2$ are convergent -- then $\prod_{n=1}^{\infty} (1 + a_n)$ is convergent.

PROOF Supposing as we may that $\forall\, n,\ |a_n| < \frac{1}{2}$, note that

$$\log(1 + a_n) = a_n + O(a_n^2).$$

Therefore the series

$$\sum_{n=1}^{\infty} \log(1 + a_n)$$

is convergent to L, say, hence

$$\prod_{n=1}^{N} (1 + a_n) = \exp(\log \prod_{n=1}^{N} (1 + a_n))$$

$$= \exp(\sum_{n=1}^{N} \log(1 + a_n))$$

$$\xrightarrow[N \to \infty]{} e^L \neq 0.$$

1.10 EXAMPLE The infinite product

$$\prod_{n=1}^{\infty} (1 + \frac{(-1)^{n-1}}{n})$$

is convergent.

1.11 LEMMA Let $\{a_n : n = 1,2,\ldots\}$ be a sequence of real numbers. Assume: $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} a_n^2$ is divergent -- then $\prod_{n=1}^{\infty} (1 + a_n)$ is divergent.

[Use the inequality

$$x - \log(1 + x) > \begin{cases} \frac{x^2}{2}/(1 + x) & (x > 0) \\ \frac{x^2}{2} & (0 > x > -1).] \end{cases}$$

1.12 EXAMPLE The infinite product

$$\prod_{n=1}^{\infty} \left(1 + \frac{(-1)^{n-1}}{\sqrt{n}}\right)$$

is divergent.

1.13 REMARK It can happen that both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_n^2$ are divergent, yet $\prod_{n=1}^{\infty} (1 + a_n)$ is convergent.

[Consider

$$\left(1 - \frac{1}{\sqrt{2}}\right)\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{2}\right)\left(1 - \frac{1}{\sqrt{3}}\right)\left(1 + \frac{1}{\sqrt{3}} + \frac{1}{3}\right) \cdots .]$$

Let $\{z_n : n = 1,2,\ldots\}$ be a sequence of complex numbers.

1.14 CRITERION The infinite product

$$\prod_{n=1}^{\infty} (1 + z_n)$$

is convergent iff $\forall\ \varepsilon > 0$, $\exists\ N(\varepsilon)$ such that $\forall\ N > N(\varepsilon)$ and every $k \geq 1$,

$$|(1 + z_{N+1}) \cdots (1 + z_{N+k}) - 1| < \varepsilon.$$

PROOF

- <u>Necessity</u> Choose N_0 per 1.1, put

$$P_N = \prod_{N_0+1}^{N} (1 + z_n)$$

and fix $C > 0$:

$$\forall\ N > N_0,\ |P_N| > C.$$

Since $\{P_N\}$ is a Cauchy sequence, by taking N_0 large enough, one can arrange that

$\forall\ N > N_0$ and every $k \geq 1$,

$$|P_{N+k} - P_N| < C\varepsilon.$$

Therefore

$$\left| \frac{P_{N+k}}{P_N} - 1 \right| < \frac{C}{P_N} \varepsilon < \varepsilon$$

or still,

$$|(1 + z_{N+1}) \cdots (1 + z_{N+k}) - 1| < \varepsilon.$$

- <u>Sufficiency</u> First take $\varepsilon = \frac{1}{2}$, hence $\forall\ N > N(\frac{1}{2})$ and every $k \geq 1$,

$$|(1 + z_{N+1}) \cdots (1 + z_{N+k}) - 1| < \frac{1}{2}.$$

So, for all $n > N_0 \equiv N(\frac{1}{2}) + 1$, $z_n \neq -1$, and if

$$\lim_{N \to \infty} \prod_{N_0+1}^{N} (1 + z_n)$$

exists, it cannot be zero since

$$\frac{1}{2} < \left| \prod_{N_0+1}^{N} (1 + z_n) \right| < \frac{3}{2}.$$

Take now $\varepsilon > 0$ and choose $N(\frac{\varepsilon}{2}) > N(\frac{1}{2})$ -- then $\forall\ N > N(\frac{\varepsilon}{2})$ and every $k \geq 1$,

$$|(1 + z_{N+1}) \cdots (1 + z_{N+k}) - 1| < \frac{\varepsilon}{2},$$

from which

$$\left| \frac{P_{N+k}}{P_N} - 1 \right| < \frac{\varepsilon}{2}$$

or still,

$$|P_{N+k} - P_N| < |P_N| \frac{\varepsilon}{2} < (\frac{3}{2}) \frac{\varepsilon}{2}$$

$$= \frac{3}{4} \varepsilon < \varepsilon.$$

Therefore

$$\{ \prod_{N_0+1}^{N} (1 + z_n) \}$$

is a Cauchy sequence, thus is convergent.

1.15 DEFINITION The infinite product

$$\prod_{n=1}^{\infty} (1 + z_n)$$

is absolutely convergent if the infinite product

$$\prod_{n=1}^{\infty} (1 + |z_n|)$$

is convergent.

1.16 LEMMA An absolutely convergent infinite product

$$\prod_{n=1}^{\infty} (1 + z_n)$$

is convergent.

PROOF One has only to note that

$$|(1 + z_{N+1}) \cdots (1 + z_{N+k}) - 1|$$

$$\leq (1 + |z_{N+1}|) \cdots (1 + |z_{N+k}|) - 1$$

and then apply 1.14.

1.17 REMARK In view of 1.5, $\prod_{n=1}^{\infty} (1 + |z_n|)$ is convergent iff $\sum_{n=1}^{\infty} |z_n|$ is convergent.

1.18 EXAMPLE The infinite product

$$\prod_{n=1}^{\infty} \sin(z/n)/(z/n)$$

is absolutely convergent for all finite z (with the usual convention at z = 0).

[Observe that

$$\sin(z/n)/(z/n) -1 = O_z(\frac{1}{n^2}) \quad (n \to \infty).]$$

It is initially tempting to think that absolute convergence should be the demand that $\prod_{n=1}^{\infty} |1 + z_n|$ is convergent but this will not do since then it is no longer true that "absolute convergence" implies convergence.

1.19 EXAMPLE The infinite product

$$\prod_{n=1}^{\infty} (1 + \frac{\sqrt{-1}}{n})$$

is divergent but the infinite product

$$\prod_{n=1}^{\infty} |1 + \frac{\sqrt{-1}}{n}|$$

is convergent.

1.20 LEMMA If the infinite product

$$\prod_{n=1}^{\infty} (1 + z_n)$$

is absolutely convergent, then it can be rearranged at will without changing its value, which is thus independent of the order of the factors.

1.21 EXAMPLE The infinite product

$$P = (1 - \frac{1}{2})(1 + \frac{1}{3})(1 - \frac{1}{4})(1 + \frac{1}{5})(1 - \frac{1}{6}) \cdots$$

is convergent (cf. 1.10) but not absolutely convergent and has value 1/2, while the rearrangement

$$Q = (1 - \frac{1}{2})(1 - \frac{1}{4})(1 + \frac{1}{3})(1 - \frac{1}{6})(1 - \frac{1}{8})(1 + \frac{1}{5}) \cdots$$

has value $1/2\sqrt{2}$.

1.22 EXAMPLE Fix a complex number $q: |q| < 1$. Introduce the absolutely convergent infinite products

$$q_0 = \prod_{n=1}^{\infty} (1 - q^{2n}) \quad , \quad q_1 = \prod_{n=1}^{\infty} (1 + q^{2n}) ,$$

$$q_2 = \prod_{n=1}^{\infty} (1 + q^{2n-1}) \quad , \quad q_3 = \prod_{n=1}^{\infty} (1 - q^{2n-1}) .$$

Then

$$q_0 q_3 = \prod_{n=1}^{\infty} (1 - q^n) \quad , \quad q_1 q_2 = \prod_{n=1}^{\infty} (1 + q^n) .$$

In addition,

$$q_0 = \prod_{n=1}^{\infty} (1 - q^{2n})$$

$$= \prod_{m=1}^{\infty} (1 - q^{4m}) \prod_{m=1}^{\infty} (1 - q^{4m-2})$$

$$= \prod_{m=1}^{\infty} (1 - q^{2m}) \prod_{m=1}^{\infty} (1 + q^{2m}) \prod_{m=1}^{\infty} (1 + q^{2m-1}) \prod_{m=1}^{\infty} (1 - q^{2m-1})$$

$$= q_0 q_1 q_2 q_3,$$

so

$$q_1 q_2 q_3 = 1.$$

1.23 EXAMPLE The infinite product

$$\prod_{n=1}^{\infty} (1 - \frac{z^2}{n^2})$$

is absolutely convergent and has value

$$\frac{\sin \pi z}{\pi z}.$$

Consider now the infinite product

$$(1 - z)(1 + z)(1 - \frac{z}{2})(1 + \frac{z}{2}) \cdots .$$

Officially, therefore

$$z_1 = - z,\ z_2 = z,\ z_3 = - \frac{z}{2},\ z_4 = \frac{z}{2},\ \ldots,$$

and the associated series of absolute values is

$$|z| + |z| + \frac{|z|}{2} + \frac{|z|}{2} + \cdots,$$

which is not convergent if $z \neq 0$. Nevertheless, our infinite product is convergent and has value

$$\frac{\sin \pi z}{\pi z},$$

as can be seen by looking at the sequence of partial products. To correct for the failure of absolute convergence, form instead the infinite product

$$\{(1 - z)e^{z}\}\{(1 + z)e^{-z}\}\{(1 - \tfrac{z}{2})e^{z/2}\}\{(1 + \tfrac{z}{2})e^{-z/2}\} \cdots .$$

To place it into the $\prod_{n=1}^{\infty} (1 + z_n)$ format, note that the $(2n-1)^{\text{th}}$ term is

$$(1 - \tfrac{z}{n})e^{z/n} - 1$$

and the $(2n)^{\text{th}}$ term is

$$(1 + \tfrac{z}{n})e^{-z/n} - 1.$$

But

$$(1 \mp \tfrac{z}{n})e^{\pm z/n} = 1 + O_z(\tfrac{1}{n^2}) \quad (n \to \infty).$$

Since

$$1 + 1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \cdots$$

is convergent, it follows that the foregoing infinite product is absolutely convergent and it too has value

$$\frac{\sin \pi z}{\pi z} .$$

1.24 EXAMPLE The infinite product

$$(1 - z)(1 - \tfrac{z}{2})(1 + z)(1 - \tfrac{z}{3})(1 - \tfrac{z}{4})(1 + \tfrac{z}{2}) \cdots$$

is convergent and has value

$$\exp(- z \log 2) \frac{\sin \pi z}{\pi z} .$$

[Judiciously insert the appropriate exponential correction factors.]

Let $\{f_n(z) : n = 1,2,\ldots\}$ be a sequence of complex valued functions defined on some nonempty subset S of the complex plane.

1.25 DEFINITION The infinite product

$$\prod_{n=1}^{\infty} (1 + f_n(z))$$

is <u>uniformly convergent in S</u> if $\forall\ \varepsilon > 0$, $\exists\ N(\varepsilon)$ such that $\forall\ N > N(\varepsilon)$ and every $k \geq 1$ and every $z \in S$,

$$|(1 + f_{N+1}(z)) \cdots (1 + f_{N+k}(z)) - 1| < \varepsilon.$$

1.26 LEMMA Suppose that $\forall\ n > 0$, $\exists\ M_n > 0$ such that $\forall\ z \in S$, $|f_n(z)| \leq M_n$. Assume: $\sum_{n=1}^{\infty} M_n$ is convergent -- then the infinite product

$$\prod_{n=1}^{\infty} (1 + f_n(z))$$

is absolutely and uniformly convergent in S.

PROOF Absolute convergence is immediate (cf. 1.17):

$$\sum_{n=1}^{\infty} |f_n(z)| \leq \sum_{n=1}^{\infty} M_n < \infty.$$

As for uniform convergence, the assumption on the M_n implies that $\prod_{n=1}^{\infty} (1 + M_n)$ is convergent (cf. 1.5). On the other hand,

$$|(1 + f_{N+1}(z)) \cdots (1 + f_{N+k}(z)) - 1|$$

$$\leq (1 + |f_{N+1}(z)|) \cdots (1 + |f_{N+k}(z)|) - 1$$

$$\leq (1 + M_{N+1}) \cdots (1 + M_{N+k}) - 1,$$

thus it remains only to quote 1.14.

1.27 REMARK It suffices to assume that $\sum_{n=1}^{\infty} |f_n(z)|$ is uniformly convergent in S with a bounded sum.

1.28 EXAMPLE Take for S a compact subset of $\{z:|z| < 1\}$ -- then S is contained in $\{z:|z| \leq \delta\}$ for some $\delta < 1$, so $\forall\, z \in S$,

$$\sum_{n=1}^{\infty} |z^n| \leq \sum_{n=1}^{\infty} \delta^n = \frac{\delta}{1-\delta} .$$

Therefore the infinite product

$$\prod_{n=1}^{\infty} (1 + z^n)$$

is absolutely and uniformly convergent in S.

1.29 THEOREM Let $f_n(z)$ $(n = 1,2,\ldots)$ be continuous (holomorphic) in a region† D and suppose that the infinite product

$$\prod_{n=1}^{\infty} (1 + f_n(z))$$

is uniformly convergent on compact subsets of D -- then the function defined by

$$\prod_{n=1}^{\infty} (1 + f_n(z))$$

is continuous (holomorphic) in D.

1.30 EXAMPLE The infinite product

$$\prod_{n=1}^{\infty} (1 + \frac{z}{n})\exp(- \frac{z}{n})$$

is uniformly convergent on compact subsets of $\mathbf{C}$ and if as usual, $\Gamma(z)$ stands for

---

† a.k.a.: nonempty open connected subset of $\mathbf{C}$

the gamma function, then

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} (1 + \frac{z}{n})\exp(- \frac{z}{n}),$$

where

$$\gamma = \lim_{n \to \infty} (H_n - \log n)$$

is Euler's constant.

[Note:

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} (1 + \frac{1}{n})^z (1 + \frac{z}{n})^{-1}$$

is meromorphic with simple poles at 0 (residue 1) and the negative integers

$- n = - 1, - 2, \ldots$ (residue $\frac{(-1)^n}{n!}$).]

APPENDIX

Given a complex number τ whose imaginary part is positive, let $q = \exp(\pi \sqrt{-1}\, \tau)$, thus $|q| < 1$.

LEMMA The <u>theta functions</u>

$$\begin{bmatrix} \Theta_1(z|\tau) \\ \Theta_2(z|\tau) \\ \Theta_3(z|\tau) \\ \Theta_4(z|\tau) \end{bmatrix}$$

defined by the series

$$\begin{cases} \Theta_1(z|\tau) = 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} \sin(2n+1)z \\ \Theta_2(z|\tau) = 2 \sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} \cos(2n+1)z \\ \Theta_3(z|\tau) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2nz \\ \Theta_4(z|\tau) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nz \end{cases}$$

are entire functions of z.

[The defining series are uniformly convergent on compact subsets of $\mathbf{C}$.]

RELATIONS

- $\Theta_1(z|\tau) = -\sqrt{-1}\exp(\sqrt{-1}\,z + \frac{1}{4}\pi\sqrt{-1}\,\tau)\Theta_4(z + \frac{\pi\tau}{2}|\tau)$
- $\Theta_2(z|\tau) = \Theta_1(z + \frac{\pi}{2}|\tau)$
- $\Theta_3(z|\tau) = \Theta_4(z + \frac{\pi}{2}|\tau)$.

ZEROS Let m,n be integers.

- $\Theta_1(m\pi + n\pi\tau|\tau) = 0$
- $\Theta_2(\frac{\pi}{2} + m\pi + n\pi\tau|\tau) = 0$
- $\Theta_3(\frac{\pi}{2} + \frac{\pi\tau}{2} + m\pi + n\pi\tau|\tau) = 0$
- $\Theta_4(\frac{\pi\tau}{2} + m\pi + n\pi\tau|\tau) = 0$.

These formulas give all the zeros of the respective theta functions and each zero is simple.

PRODUCTS Let

$$q_0 = \prod_{n=1}^{\infty} (1 - q^{2n}) \qquad \text{(cf. 1.22)}.$$

- $\Theta_1(z|\tau) = 2q_0 q^{1/4} \sin z \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2z + q^{4n})$
- $\Theta_2(z|\tau) = 2q_0 q^{1/4} \cos z \prod_{n=1}^{\infty} (1 + 2q^{2n} \cos 2z + q^{4n})$
- $\Theta_3(z|\tau) = q_0 \prod_{n=1}^{\infty} (1 + 2q^{2n-1} \cos 2z + q^{4n-2})$
- $\Theta_4(z|\tau) = q_0 \prod_{n=1}^{\infty} (1 - 2q^{2n-1} \cos 2z + q^{4n-2})$.

TRANSFORMATIONS

- $\Theta_1(z|\tau) = \sqrt{-1}\,(-\sqrt{-1}\,\tau)^{-\frac{1}{2}} \exp\left(\frac{z^2}{\pi\sqrt{-1}\,\tau}\right) \Theta_1\left(\frac{z}{\tau}\middle| -\tau^{-1}\right)$
- $\Theta_2(z|\tau) = (-\sqrt{-1}\,\tau)^{-\frac{1}{2}} \exp\left(\frac{z^2}{\pi\sqrt{-1}\,\tau}\right) \Theta_4\left(\frac{z}{\tau}\middle| -\tau^{-1}\right)$
- $\Theta_3(z|\tau) = (-\sqrt{-1}\,\tau)^{-\frac{1}{2}} \exp\left(\frac{z^2}{\pi\sqrt{-1}\,\tau}\right) \Theta_3\left(\frac{z}{\tau} - \tau^{-1}\right)$
- $\Theta_4(z|\tau) = (-\sqrt{-1}\,\tau)^{-\frac{1}{2}} \exp\left(\frac{z^2}{\pi\sqrt{-1}\,\tau}\right) \Theta_2\left(\frac{z}{\tau}\middle| -\tau^{-1}\right)$.

[Note: The square root is real and positive when τ is purely imaginary.]

EXAMPLE Take z = x real and $\tau = \sqrt{-1}\, t$ $(t > 0)$ -- then

$$\Theta_3(x|\sqrt{-1}\, t) = \frac{1}{\sqrt{t}} \exp(-\frac{x^2}{\pi t})\Theta_3(\frac{x}{\sqrt{-1}\, t}|\frac{\sqrt{-1}}{t}).$$

Specializing still further, let x = 0, and put

$$\Theta(t) = \sum_{n=1}^{\infty} e^{-n^2 \pi t},$$

thus

$$\begin{aligned} 1 + 2\Theta(t) &= \Theta_3(0|\sqrt{-1}\, t) \\ &= \frac{1}{\sqrt{t}} \Theta_3(0|\frac{\sqrt{-1}}{t}) \\ &= \frac{1}{\sqrt{t}} (1 + 2\Theta(\frac{1}{t})). \end{aligned}$$

§2. *ORDER*

Given an entire function

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \quad (\Rightarrow \lim_{n \to \infty} |c_n|^{1/n} = 0),$$

put

$$M(r;f) = \max_{|z| = r} |f(z)|.$$

2.1 LEMMA $M(r;f)$ is a continuous increasing function of r.

2.2 LEMMA If f is not a constant, then

$$M(r;f) \to \infty \quad (r \to \infty).$$

2.3 LEMMA If for some $\lambda > 0$,

$$\varliminf_{r \to \infty} \frac{M(r;f)}{r^{\lambda}} = 0,$$

then f is a polynomial of degree $\leq \lambda$.

PROOF In general,

$$|c_n| \leq \frac{M(r;f)}{r^n},$$

so for $n > \lambda$,

$$|c_n| \leq \varliminf_{r \to \infty} \frac{M(r;f)}{r^{\lambda}} = 0.$$

2.4 EXAMPLE We have

$$\begin{cases} M(r;\exp z^n) = \exp r^n \ (n = 1,2,\ldots) \\ M(r;\exp e^z) = \exp e^r. \end{cases}$$

2.5 EXAMPLE We have

$$\begin{cases} M(r;\sin z) = \dfrac{e^r - e^{-r}}{2} \\ M(r;\cos z) = \dfrac{e^r + e^{-r}}{2} \end{cases}.$$

2.6 LEMMA Let

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n \quad (a_n \neq 0,\ n \geq 1)$$

be a polynomial of degree n -- then

$$M(r;p(z)) \sim |a_n| r^n \quad (r \to \infty).$$

2.7 DEFINITION An entire function is said to be <u>transcendental</u> if it is not a polynomial.

2.8 LEMMA If f is transcendental, then for any polynomial p,

$$\lim_{r \to \infty} \frac{M(r;p)}{M(r;f)} = 0.$$

2.9 DEFINITION If $f \not\equiv C$ is an entire function, then its <u>order</u> $\rho (= \rho(f))$ is given by

$$\overline{\lim_{r \to \infty}} \frac{\log \log M(r;f)}{\log r}.$$

[Note: Conventionally, the order of $f \equiv C$ is 0.]

2.10 REMARK The reason that one works with log log M(r;f) rather than log M(r;f) is that if f is transcendental, then

$$\lim_{r \to \infty} \frac{\log M(r;f)}{\log r} = \infty.$$

2.11 EXAMPLE Every polynomial is an entire function of order 0 (cf. 2.6) but there are transcendental entire functions of order 0, e.g., $\sum_{n=0}^{\infty} e^{-n^2} z^n$ (cf. 2.27).

2.12 EXAMPLE The entire function $\exp z^n$ $(n = 1,2,\ldots)$ is of order n. On the other hand, the entire function $\exp e^z$ is of order ∞.

2.13 DEFINITION f is of <u>finite order</u> if ρ is finite; otherwise, f is of <u>infinite order</u>.

2.14 LEMMA An entire function f is of finite order iff there exists a positive constant K such that

$$M(r;f) < \exp r^K \quad (r >> 0),$$

the greatest lower bound of the set of all such K then being the order of f.

2.15 LEMMA An entire function f is of finite order iff there exist positive constants B, C, and K such that

$$M(r;f) < B \exp Cr^K \quad (r >> 0),$$

the greatest lower bound of the set of all such K then being the order of f.

[Note: In general, the constants B and C depend on K.]

2.16 APPLICATION Suppose that f is an entire function of finite order. Given a complex constant A, let $f_A(z) = f(z + A)$ -- then $\rho(f) = \rho(f_A)$.

[For $\exists$ K > 0:

$$M(r;f) < \exp r^K \quad (r >> 0).$$

But

$$|z| < |A| \Rightarrow |z + A| < 2|z|$$

=>

$$M(r;f_A) < \exp 2^K r^K \quad (r >> 0).]$$

2.17 APPLICATION Suppose that f is an entire function of finite order. Given a nonzero complex constant A, let $f_A(z) = f(Az)$ -- then $\rho(f) = \rho(f_A)$.

[For $\exists\ K > 0$:

$$M(r;f) < \exp r^K \quad (r >> 0).$$

But

$$|Az| \leq |A||z|$$

=>

$$M(r;f_A) < \exp |A|^K r^K \quad (r >> 0).]$$

2.18 LEMMA If $M(r;f) \sim h(r) \quad (r \to \infty)$, then

$$\overline{\lim_{r \to \infty}} \frac{\log \log M(r;f)}{\log r} = \overline{\lim_{r \to \infty}} \frac{\log \log h(r)}{\log r}.$$

PROOF Assuming that $r >> 0$, write

$$\log M(r;f) = \log\left(\frac{M(r;f)}{h(r)} h(r)\right)$$

$$= \log h(r) + \log \frac{M(r;f)}{h(r)}$$

$$= \log h(r) \left[1 + \frac{1}{\log h(r)} \log \frac{M(r;f)}{h(r)}\right]$$

=>

$$\frac{\log \log M(r;f)}{\log r} = \frac{\log \log h(r)}{\log r}$$

$$+ \frac{\log[1 + \frac{1}{\log h(r)} \log \frac{M(r;f)}{h(r)}]}{\log r},$$

from which the assertion.

2.19 EXAMPLE If C is a positive constant, then

$$\overline{\lim_{r \to \infty}} \frac{\log \log Ce^r}{\log r} = 1.$$

This said, take now in 2.18

$$h(r) = \frac{e^r}{2}$$

to conclude that the entire functions sin z and cos z are both of order 1 (cf. 2.5).

[Note: Define entire functions

$$\frac{\sin \sqrt{z}}{\sqrt{z}}, \cos \sqrt{z}$$

by the appropriate power series -- then each is of order $\frac{1}{2}$.]

2.20 EXAMPLE Put

$$\Gamma_1(z) = \int_1^\infty t^z e^{-t} dt.$$

Then Γ_1 is entire and

$$M(r;\Gamma_1) = \sqrt{2\pi r}\ (\frac{r}{e})^r (1 + O(\frac{1}{r})).$$

Therefore

$$\log M(r;\Gamma_1) \sim r \log r \quad (r \to \infty),$$

so $\rho(\Gamma_1) = 1$.

Sometimes it is simpler to work directly with log M(r;f).

2.21 EXAMPLE Fix $\alpha > 0$ and let

$$f_\alpha(z) = \prod_{n=1}^{\infty} (1 + \frac{z^n}{n^{\alpha n}}).$$

Then

$$\log M(r;f_\alpha) = \sum_{n=1}^{\infty} \log(1 + \frac{r^n}{n^{\alpha n}})$$

$$= \int_0^\infty \log(1 + \frac{r^u}{u^{\alpha u}})du + O(r^{\frac{1}{\alpha}})$$

$$\sim r^{\frac{2}{\alpha}} \frac{1}{\alpha} \int_1^\infty t^{-\frac{2}{\alpha} - 1} \log t \, dt \quad (r \to \infty),$$

where we made the change of variable $t = \frac{r}{u^\alpha}$. In the integral

$$\int_1^\infty t^{-\frac{2}{\alpha} - 1} \log t \, dt,$$

let $x = t^{\frac{2}{\alpha}}$, hence

$$\frac{\alpha}{2} \int_1^\infty \frac{\log x^{\frac{\alpha}{2}}}{x^2} dx$$

$$= \frac{\alpha^2}{4} \int_1^\infty \frac{\log x}{x^2} dx = \frac{\alpha^2}{4} \Gamma(2) = \frac{\alpha^2}{4}.$$

Therefore

$$\log M(r;f_\alpha) \sim \frac{\alpha}{4} r^{\frac{2}{\alpha}} \quad (r \to \infty),$$

so

$$\rho(f_\alpha) = \frac{2}{\alpha}.$$

As will now be seen, the order ρ of an entire function f can be computed from the coefficients of its power series expansion at the origin.

2.22 SUBLEMMA If there exist positive constants A and K such that

$$M(r;f) < \exp Ar^k \quad (r >> 0),$$

then

$$|c_n| < (\frac{eAK}{n})^{n/K} \quad (n >> 0).$$

PROOF For $r >> 0$, say $r \geq r_0$,

$$|c_n| \leq \frac{M(r;f)}{r^n} < \exp(Ar^K - n \log r).$$

As a function of r,

$$Ar^K - n \log r$$

achieves its minimum at r_n, where $r_n^K = n/(AK)$. But for $n >> 0$, $r_n \geq r_0$. And

$$\begin{aligned} &\exp(Ar_n^K - n \log r_n) \\ &= \exp(A \frac{n}{AK})\exp(- n \log(\frac{n}{AK})^{1/K}) \\ &= \exp(\frac{n}{K})\exp(\log(\frac{n}{AK})^{- n/K}) \\ &= (\frac{eAK}{n})^{n/K}. \end{aligned}$$

2.23 LEMMA If there exist positive constants A and K such that

$$|c_n| < (\frac{eAK}{n})^{n/K} \quad (n >> 0),$$

then $\forall\ \varepsilon > 0$,

$$M(r;f) < \exp(A + \varepsilon)r^K \quad (r >> 0),$$

hence

$$M(r;f) < \exp r^{K + \varepsilon} \quad (r >> 0).$$

PROOF We can and will assume that $c_0 = 0$ and

$$|c_n| < \left(\frac{eAK}{n}\right)^{n/K} \quad \forall\, n \geq 1.$$

Accordingly,

$$M(r;f) \leq \sum_{n=1}^{\infty} |c_n| r^n$$

$$\leq \sum_{n=1}^{\infty} \left(\frac{eAK}{n}\right)^{n/K} r^n$$

$$= \sum_{n=1}^{\infty} \left(\frac{eAr^K}{n/K}\right)^{n/K}.$$

Put $m = [n/K]$:

$$\left[\begin{array}{l} m! \sim \left(\frac{m}{e}\right)^m \sqrt{2\pi m} \\ \\ \sqrt{2\pi m} < C_1 \left(\frac{A + \varepsilon/2}{A}\right)^{m+1}. \end{array}\right.$$

Therefore

$$\left(\frac{eAr^K}{m}\right)^{m+1} = \left(\frac{e}{m}\right)\left(\frac{e}{m}\right)^m (Ar^K)^{m+1}$$

$$= \left(\frac{e}{m}\right) \frac{\left(\frac{e}{m}\right)^m}{\frac{\sqrt{2\pi m}}{m!}} \frac{\sqrt{2\pi m}}{m!} (Ar^K)^{m+1}$$

$$< C_2 \frac{\sqrt{2\pi m}}{m!} (Ar^K)^{m+1}$$

$$< C_3 \frac{1}{m!} \left(\frac{A + \varepsilon/2}{A}\right)^{m+1} (Ar^K)^{m+1}$$

$$= C_3 \frac{(A + \varepsilon/2)^{m+1} r^{K(m+1)}}{m!}$$

=>

$$\sum_{m=1}^{\infty} \frac{(A + \varepsilon/2)^{m+1} r^{K(m+1)}}{m!}$$

$$= (A + \varepsilon/2)(r^K)(\exp (A + \varepsilon/2)r^K - 1)$$

$$< (A + \varepsilon/2)(r^K)\exp (A + \varepsilon/2)r^K$$

$$< \exp(A + \varepsilon)r^K \quad (r >> 0).$$

2.24 THEOREM The order of the entire function

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

is given by

$$\rho = \overline{\lim_{r \to \infty}} \frac{n \log n}{\log(1/|c_n|)}$$

or, equivalently, is given by

$$\rho = \overline{\lim_{r \to \infty}} \frac{\log n}{\log \frac{1}{|c_n|^{1/n}}}.$$

[Note: The terms for which $c_n = 0$ are taken to be 0.]

PROOF Suppose first that ρ is finite -- then for any $K > \rho$,

$$M(r;f) < \exp r^K \quad (r >> 0),$$

thus by 2.22,

$$|c_n| < \left(\frac{eK}{n}\right)^{n/K} \quad (n >> 0).$$

Therefore

$$K > \frac{\log n}{\log \frac{1}{|c_n|^{1/n}}} + \frac{\log \frac{1}{eK}}{\log \frac{1}{|c_n|^{1/n}}} \quad (n >> 0).$$

But

$$\lim_{n \to \infty} \log \frac{1}{|c_n|^{1/n}} = \infty,$$

so

$$K \geq \overline{\lim_{n \to \infty}} \frac{\log n}{\log \frac{1}{|c_n|^{1/n}}}$$

=>

$$\rho \geq \overline{\lim_{n \to \infty}} \frac{\log n}{\log \frac{1}{|c_n|^{1/n}}} .$$

To reverse this, let

$$K' > \overline{\lim_{n \to \infty}} \frac{\log n}{\log \frac{1}{|c_n|^{1/n}}} .$$

Choose a positive integer $N(K')$:

$$\frac{\log n}{\log \frac{1}{|c_n|^{1/n}}} < K' \quad (n > N(K'))$$

or still,

$$|c_n| < \left(\frac{1}{n}\right)^{n/K'} \quad (n > N(K')).$$

Then, thanks to 2.23 (with $A = \frac{1}{eK'}$), given $\varepsilon > 0$, there is an $R(\varepsilon)$:

$$M(r;f) < \exp\left(\frac{1}{eK'} + \varepsilon\right)r^{K'} < \exp r^{K'+\varepsilon} \quad (r > R(\varepsilon)),$$

hence

$$\rho \leq K' + \varepsilon \Rightarrow \rho \leq K' \Rightarrow \rho \leq \overline{\lim_{n \to \infty}} \frac{\log n}{\log \frac{1}{|c_n|^{1/n}}}.$$

In summary: For ρ finite,

$$\rho = \overline{\lim_{n \to \infty}} \frac{\log n}{\log \frac{1}{|c_n|^{1/n}}}.$$

Turning to the case of an infinite ρ, on the basis of what has been said above, it is clear that if

$$\overline{\lim_{n \to \infty}} \frac{\log n}{\log \frac{1}{|c_n|^{1/n}}}$$

is finite, then ρ is finite, i.e., if ρ is infinite, then

$$\overline{\lim_{n \to \infty}} \frac{\log n}{\log \frac{1}{|c_n|^{1/n}}}$$

is infinite.

2.25 APPLICATION The order of an entire function is unchanged by differentiation: $\rho(f) = \rho(f')$.

2.26 EXAMPLE Let $0 < \rho < \infty$ — then the entire function

$$f(z) = \sum_{n=1}^{\infty} \left(\frac{\rho e}{n}\right)^{n/\rho} z^n$$

is of order ρ.

2.27 EXAMPLE The entire function

$$f(z) = \sum_{n=2}^{\infty} \left(\frac{1}{\log n}\right)^n z^n$$

is of infinite order and the entire function

$$f(z) = \sum_{n=0}^{\infty} e^{-n^2} z^n$$

is of zero order.

2.28 EXAMPLE Fix $\alpha > 0$ -- then the entire function

$$ML_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}$$

is of order $\frac{1}{\alpha}$.

[Note: Obviously,

$$\begin{cases} ML_1(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n + 1)} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z \\ ML_2(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(2n + 1)} = \sum_{n=0}^{\infty} \frac{z^n}{(2n)!} = \cosh \sqrt{z}.] \end{cases}$$

2.29 EXAMPLE The <u>Bessel function</u> $J_\nu(z)$ of the first kind of real index $\nu > -1$

is defined by the series

$$\left(\frac{z}{2}\right)^{\nu} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n}}{n!\Gamma(\nu + n + 1)},$$

where $\left(\frac{z}{2}\right)^{\nu} = \exp(\nu \log \frac{z}{2})$, the logarithm having its principal value. Multiplying up,

$$\left(\frac{z}{2}\right)^{-\nu} J_{\nu}(z)$$

is therefore entire and, moreover, it is of order 1.

2.30 EXAMPLE Fix $\alpha > 1$ -- then the entire function

$$\Phi_{\alpha}(z) = \int_0^{\infty} \exp(-t^{\alpha})\cos zt\, dt$$

is of order $\frac{\alpha}{\alpha-1}$.

[One first has to check that $\Phi_{\alpha}(z)$ really is entire, which can be seen by noting that it is uniformly convergent on compact subsets of $\mathbf{C}$:

$$|\cos zt| \leq e^{t|z|}$$

$\Rightarrow$

$$|\exp(-t^{\alpha})\cos zt| \leq \exp(t|z| - t^{\alpha}) \leq \exp(-t)$$

for all t such that $t^{\alpha-1} > 1 + |z|$. This settled, to compute the order, write

$$\Phi_{\alpha}(z) = \int_0^{\infty} \exp(-t^{\alpha}) \left[\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n} t^{2n}}{(2n)!}\right] dt$$

$$= \sum_{n=0}^{\infty} \left[\int_0^{\infty} \exp(-t^{\alpha}) t^{2n} dt\right] \frac{(-1)^n z^{2n}}{(2n)!}$$

$$= \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \Gamma(\frac{2n+1}{\alpha}) z^{2n},$$

and then proceed... .]

[Note: As a special case,

$$\Phi_2(z) = \frac{1}{2}\sqrt{\pi}\exp(-\frac{z^2}{4}),$$

an entire function of order 2 (by direct inspection).]

2.31 LEMMA If f_1, f_2 are entire functions of respective orders ρ_1, ρ_2 and if $\rho_1 \le \rho_2$ $(\rho_1 < \rho_2)$, then the order of $f_1 + f_2$ is $\le \rho_2$ $(= \rho_2)$.

2.32 EXAMPLE Take $f_1 = e^z$, $f_2 = -e^z$ -- then $\rho_1 = \rho_2 = 1$ but the order of $f_1 + f_2$ is 0.

2.33 EXAMPLE If f is an entire function of order ρ, then for any polynomial p, the order of f + p is equal to ρ.

2.34 LEMMA If f_1, f_2 are entire functions of respective orders ρ_1, ρ_2 and if $\rho_1 \le \rho_2$ $(\rho_1 < \rho_2)$, then the order of $f_1 f_2$ is $\le \rho_2$ $(= \rho_2)$.

2.35 EXAMPLE Take $f_1 = e^z$, $f_2 = e^{-z}$ -- then $\rho_1 = \rho_2 = 1$ but the order of $f_1 f_2$ is 0.

2.36 EXAMPLE If f is an entire function of order ρ, then for any nonzero polynomial p, the order of pf is equal to ρ.

[Note: If the quotient $\frac{f}{p}$ is an entire function, then it too is of order ρ.

Proof: $\rho(\frac{f}{p}) = \rho(p \cdot \frac{f}{p}) = \rho(f)$.]

2.37 LEMMA If f,g are entire functions and if $\frac{f}{g}$ is an entire function, then

$$\rho(\frac{f}{g}) \leq \max(\rho(f),\rho(g)).$$

PROOF Since $g \cdot \frac{f}{g} = f$, in the event that $\rho(\frac{f}{g}) > \rho(g)$, we have

$$\rho(\frac{f}{g}) = \rho(g \cdot \frac{f}{g}) = \rho(f) \qquad \text{(cf. 2.34)},$$

leaving the case $\rho(\frac{f}{g}) \leq \rho(g)$.

2.38 EXAMPLE Consider the theta functions

$$\begin{cases} \Theta_1(z|\tau) \\ \Theta_2(z|\tau) \\ \Theta_3(z|\tau) \\ \Theta_4(z|\tau) \end{cases}$$

of the Appendix to §1 -- then each is of order 2. First

$$\begin{cases} \Theta_2(z|\tau) = \Theta_1(z + \frac{\pi}{2}|\tau) \\ \Theta_3(z|\tau) = \Theta_4(z + \frac{\pi}{2}|\tau). \end{cases}$$

Therefore

$$\begin{cases} \rho(\Theta_2) = \rho(\Theta_1) \\ \rho(\Theta_3) = \rho(\Theta_4), \end{cases}$$

provided that Θ_1 and Θ_4 are of finite order (cf. 2.16). Next, recall the relation

$$\Theta_1(z|\tau) = -\sqrt{-1}\exp(\sqrt{-1}\,z + \frac{1}{4}\pi\sqrt{-1}\,\tau)\Theta_4(z + \frac{\pi\tau}{2}|\tau).$$

Granting for the moment that $\rho(\Theta_1) = 2$, the fact that $\exp(\sqrt{-1}\,z)$ is of order 1 in conjunction with 2.34 forces

$$\rho(\Theta_4(z + \frac{\pi\tau}{2}|\tau)) = 2$$

from which $\rho(\Theta_4) = 2$ (cf. 2.16). To deal with Θ_1, given z, let

$$\lambda = (2|z| + \log 2)/\log|1/q| - \frac{1}{2}.$$

Then

$$|\Theta_1(z|\tau)| \le 2\sum_{n=0}^{\infty} |q|^{(n+\frac{1}{2})^2} e^{(2n+1)|z|}$$

$$\le 2\sum_{n\le\lambda} |q|^{(n+\frac{1}{2})^2} e^{(2n+1)|z|} + 2\sum_{n>\lambda} (\frac{1}{2})^{n+\frac{1}{2}}$$

$$= O(e^{(2\lambda+1)|z|}) = O(e^{C|z|^2}).$$

Therefore $\rho(\Theta_1) \le 2$. That $\rho(\Theta_1) = 2$ is established in 4.27.

2.39 EXAMPLE The entire function

$$1 + \sum_{n=1}^{\infty} (\frac{1}{2})^{n^2} e^{nz}$$

is of order 2.

2.40 NOTATION Given an entire function f, let

$$A(r;f) = \max_{|z|=r} \operatorname{Re} f(z).$$

2.41 RAPPEL If for some $C > 0$, $d > 0$,

$$A(r;f) < Cr^{d} \quad (r >> 0),$$

then f is a polynomial of degree $\leq [d]$.

2.42 LEMMA If f is entire and if the order of $F = e^{f}$ is finite, then f is a polynomial (and the order of F is equal to the degree of f).

PROOF From the definitions,

$$\log |F(z)| = \text{Re } f(z),$$

hence

$$\log M(r;f) = A(r;f).$$

But $\forall\ \varepsilon > 0$,

$$\frac{\log \log M(r;F)}{\log r} < \rho(F) + \varepsilon \quad (r >> 0),$$

thus

$$\log M(r;F) < r^{\rho(F) + \varepsilon} \quad (r >> 0)$$

and so

$$A(r;f) < r^{\rho(F) + \varepsilon} \quad (r >> 0).$$

Therefore f is a polynomial of degree $\leq [\rho(F) + \varepsilon]$ or still, f is a polynomial of degree $\leq [\rho(F)]$.

§3. *TYPE*

Let f be an entire function of order ρ, where $0 < \rho < \infty$.

3.1 DEFINITION The <u>type</u> τ (= $\tau(f)$) of f is given by

$$\overline{\lim_{r \to \infty}} \frac{\log M(r;f)}{r^{\rho}}.$$

3.2 EXAMPLE The entire function

$$\exp(a_0 + a_1 z + \cdots + a_n z^n) \quad (a_n \neq 0,\ n \geq 1)$$

is of order n and type $|a_n|$.

3.3 EXAMPLE The entire functions

$$\begin{cases} \sin Az \\ \cos Az \end{cases} \quad (A \neq 0)$$

are of order 1 and type $|A|$.

3.4 DEFINITION f is of <u>maximal type</u> if $\tau = \infty$, of <u>minimal type</u> if $\tau = 0$, and of <u>intermediate type</u> if $0 < \tau < \infty$.

3.5 REMARK f is of <u>finite type</u> if $0 \leq \tau < \infty$, which will be the case iff there exists a positive constant C such that

$$M(r;f) < \exp Cr^{\rho} \quad (r >> 0),$$

the greatest lower bound of the set of all such C then being the type of f.

Here is a formula for the type parallel to that of 2.24 for the order.

3.6 THEOREM The type of the entire function

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

is given by

$$\tau = \frac{1}{\rho e} \overline{\lim_{n \to \infty}} (n|c_n|^{\rho/n}).$$

PROOF Suppose first that τ is finite -- then for any $A > \tau$,

$$M(r;f) < \exp Ar^{\rho} \quad (r >> 0),$$

thus by 2.22,

$$|c_n| < (\frac{\rho e A}{n})^{n/\rho} \quad (n >> 0),$$

so

$$A > \frac{1}{\rho e} n|c_n|^{\rho/n} \quad (n >> 0).$$

Therefore

$$A \geq \frac{1}{\rho e} \overline{\lim_{n \to \infty}} (n|c_n|^{\rho/n})$$

$$\Rightarrow$$

$$\tau \geq \frac{1}{\rho e} \overline{\lim_{n \to \infty}} (n|c_n|^{\rho/n}).$$

To go the other way, let

$$K' > \frac{1}{\rho e} \overline{\lim_{n \to \infty}} (n|c_n|^{\rho/n}).$$

Choose a positive integer $N(K')$:

$$\frac{1}{\rho e} n|c_n|^{\rho/n} < K' \quad (n > N(K'))$$

or still,

$$|c_n| < \left(\frac{\rho e K'}{n}\right)^{n/\rho} \quad (n > N(K')).$$

Then, thanks to 2.23 (with $A = K'$, $K = \rho$), given any $\varepsilon > 0$, there is an $R(\varepsilon)$:

$$M(r;f) < \exp(K' + \varepsilon)r^{\rho} \quad (r > R(\varepsilon)),$$

hence

$$\tau \leq K' + \varepsilon \Rightarrow \tau \leq K' \Rightarrow \tau \leq \frac{1}{\rho e} \overline{\lim_{n \to \infty}} (n|c_n|^{\rho/n}).$$

In summary: For τ finite,

$$\tau = \frac{1}{\rho e} \overline{\lim_{n \to \infty}} (n|c_n|^{\rho/n}).$$

Turning to the case of an infinite τ, on the basis of what has been said above, it is clear that if

$$\frac{1}{\rho e} \overline{\lim_{n \to \infty}} (n|c_n|^{\rho/n})$$

is finite, then τ is finite, i.e., if τ is infinite, then

$$\frac{1}{\rho e} \overline{\lim_{n \to \infty}} (n|c_n|^{\rho/n})$$

is infinite.

3.7 APPLICATION The type of an entire function is unchanged by differentiation: $\tau(f) = \tau(f')$.

3.8 EXAMPLE Let $0 < \rho < \infty$ -- then the entire function

$$f(z) = \sum_{n=2}^{\infty} \left(\frac{\rho e}{n \log n}\right)^{n/\rho} z^n$$

is of order ρ and of minimal type.

3.9 EXAMPLE Let $0 < \rho < \infty$ -- then the entire function

$$f(z) = \sum_{n=2}^{\infty} \left(\rho e \frac{\log n}{n}\right)^{n/\rho} z^n$$

is of order ρ and of maximal type.

3.10 EXAMPLE The entire function

$$z \longrightarrow \int_0^1 e^{zt^2}\, dt$$

is of order 1 and of type 1.

3.11 EXAMPLE Let $0 < \rho < \infty$, $0 < \tau < \infty$ -- then the entire function

$$f(z) = \sum_{n=1}^{\infty} \left(\frac{\rho e \tau}{n}\right)^{n/\rho} z^n$$

is of order ρ and of type τ (cf. 2.26).

3.12 EXAMPLE Fix $\alpha > 0$, $A > 0$ -- then the entire function

$$ML_{\alpha,A}(z) = \sum_{n=0}^{\infty} \frac{(Az)^n}{\Gamma(\alpha n + 1)}$$

is of order $\frac{1}{\alpha}$ and of type A (cf. 2.28).

3.13 EXAMPLE Fix $t > 0$ and let

$$\theta_t(z) = 1 + \sum_{n=1}^{\infty} (e^{-\pi t})^{n^2} e^{nz}.$$

Then θ_t is of order 2 and of type $\frac{1}{4\pi t}$.

[Note: As a special case,

$$\theta_{\frac{\log 2}{\pi}} = 1 + \sum_{n=1}^{\infty} (\tfrac{1}{2})^{n^2} e^{nz},$$

an entire function of order 2 and of type $\frac{1}{4 \log 2}$ (cf. 2.39).]

3.14 LEMMA Let f_1, f_2 be entire functions of respective orders ρ_1, ρ_2, where $0 < \rho_1 < \infty$, $0 < \rho_2 < \infty$, and respective types τ_1, τ_2.

- If $\rho_1 < \rho_2$, then $\rho(f_1 f_2) = \rho(f_2)$ and $\tau(f_1 f_2) = \tau_2$.

- If $\rho_1 = \rho_2$, if $0 < \tau_1 < \infty$, if $\tau_2 = 0$, then $\rho(f_1 f_2) = \rho_1 = \rho_2$ and $\tau(f_1 f_2) = \tau_1$.

- If $\rho_1 = \rho_2$, if $\tau_1 = \infty$, if $0 \leq \tau_2 < \infty$, then $\rho(f_1 f_2) = \rho_1 = \rho_2$ and $\tau(f_1 f_2) = \infty$.

§4. *CONVERGENCE EXPONENT*

Let $\{r_n : n = 1,2,\ldots\}$ be a sequence of positive real numbers with

$$0 < r_1 \le r_2 \le \cdots \quad (r_n \to \infty),$$

finite repetitions being permitted.

4.1 DEFINITION The greatest lower bound κ of the positive p for which the series

$$\sum_{n=1}^{\infty} \frac{1}{r_n^p}$$

is convergent is called the <u>convergence exponent</u> of the sequence $\{r_n : n = 1,2,\ldots\}$.

<u>N.B.</u> If $\forall\, p$,

$$\sum_{n=1}^{\infty} \frac{1}{r_n^p} = \infty,$$

then take $\kappa = \infty$.

4.2 EXAMPLE The sequence $\{e^n\}$ has convergence exponent 0.

4.3 EXAMPLE The sequence $\{\log n\}$ has convergence exponent ∞.

4.4 REMARK Take $\kappa < \infty$ -- then the series

$$\sum_{n=1}^{\infty} \frac{1}{r_n^{\kappa}}$$

may or may not converge.

[The sequence $\{n\}$ has convergence exponent 1 and $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is divergent while the sequence $\{n(\log n)^2\}$ also has convergence exponent 1 but $\sum\limits_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ is convergent.]

4.5 LEMMA We have

$$\kappa = \overline{\lim_{n \to \infty}} \frac{\log n}{\log r_n} .$$

4.6 DEFINITION The <u>counting function</u> $n(r)$ $(r \geq 0)$ of the sequence $\{r_n : n = 1,2,\ldots\}$ is the number of r_n such that $r_n \leq r$, i.e.,

$$n(r) = \sum_{r_n \leq r} 1.$$

[Note: $n(r) = 0$ for $0 \leq r < r_1$. In addition, $n(r)$ is right continuous, increasing, integer valued, and piecewise constant.]

4.7 EXAMPLE Take $r_n = n \ \forall \ n$ -- then $n(r) = [r]$.

4.8 EXAMPLE Let $\{r_n : n = 1,2,\ldots\}$ be the sequence derived from the lattice points in the plane (excluding $(0,0)$) -- then

$$\sum_{n=1}^{\infty} \frac{1}{r_n^p} = \sum_{(m,n) \neq (0,0)} \frac{1}{(m^2 + n^2)^{p/2}} ,$$

the series on the right being convergent if $p > 2$ and divergent if $p \leq 2$, hence $\kappa = 2$. And here

$$n(r) \sim \pi r^2 \quad (r \to \infty).$$

4.9 LEMMA We have

$$\overline{\lim_{r \to \infty}} \frac{\log n(r)}{\log r} = \overline{\lim_{n \to \infty}} \frac{\log n}{\log r_n} .$$

4.10 APPLICATION The convergence exponent κ is given by

$$\overline{\lim_{r \to \infty}} \frac{\log n(r)}{\log r} \quad (\text{cf. } 4.5).$$

4.11 DEFINITION Take $\kappa < \infty$ -- then the density of the sequence $\{r_n : n = 1,2,\ldots\}$ is

$$\Delta = \overline{\lim_{n \to \infty}} \frac{n}{r_n^\kappa} .$$

4.12 EXAMPLE Fix $p > 1$ and let $r_n = n^p$ -- then $\kappa = 1/p$ and $\Delta = 1$.

4.13 LEMMA We have

$$\Delta = \overline{\lim_{r \to \infty}} \frac{n(r)}{r^\kappa} .$$

4.14 DEFINITION Take $\kappa < \infty$ -- then the genus of the sequence $\{r_n : n = 1,2,\ldots\}$ is the smallest nonnegative integer g such that

$$\sum_{n=1}^{\infty} \frac{1}{r_n^{g+1}}$$

is convergent.

4.15 LEMMA Assume that κ is finite.

- If κ is not an integer, then $g = [\kappa]$.

- If κ is an integer, then $g = \kappa - 1$ if $\sum_{n=1}^{\infty} \frac{1}{r_n^\kappa}$ is convergent while $g = \kappa$ if $\sum_{n=1}^{\infty} \frac{1}{r_n^\kappa}$ is divergent.

Having dispensed with the formalities, we shall now come back to complex variable theory. So suppose that f is a transcendental entire function of finite order ρ. Arrange the nonzero zeros of f in a sequence $z_1, z_2, \ldots$ such that

$$0 < |z_1| \leq |z_2| \leq \ldots$$

with multiple zeros counted according to their multiplicities and let $r_n = |z_n|$.

4.16 THEOREM Given $\varepsilon > 0$,

$$\overline{\lim_{r \to \infty}} \frac{n(r)}{r^{\rho + \varepsilon}} \leq e(\rho + \varepsilon).$$

Before detailing the proof, it will be best to make some initial reductions.

• If the number of zeros of f is finite, then $n(r)$ is eventually constant and the result is trivial. It will therefore be assumed that $r_n = |z_n| \to \infty$.

• If $f(0) = 0$, write $f(z) = z^m g(z)$ $(g(0) \neq 0)$ -- then the order of f equals the order of g (cf. 2.36) so we can just as well assume from the beginning that $f(0) \neq 0$.

• Since multiplication by a nonzero constant does not affect the order of the zeros, there is no loss of generality in assuming that $|f(0)| = 1$.

4.17 JENSEN INEQUALITY If $|f(0)| = 1$, then $\forall\ r > 0$,

$$\int_0^r \frac{n(t)}{t}\, dt \leq \log M(r;f).$$

Proceeding to the proof of 4.16, fix a parameter $\lambda \in]0,1[$ -- then

$$\begin{aligned}\int_0^r \frac{n(t)}{t}\, dt &\geq \int_{\lambda r}^r \frac{n(t)}{t}\, dt \\ &\geq n(\lambda r) \int_{\lambda r}^r \frac{dt}{t} \\ &= n(\lambda r) \log \frac{1}{\lambda}\end{aligned}$$

or still,

$$n(\lambda r) \leq \frac{1}{\log \frac{1}{\lambda}} \log M(r;f)$$

or still,

$$\frac{n(\lambda r)}{\log M(r;f)} \leq \frac{1}{\log \frac{1}{\lambda}}.$$

Therefore

$$\overline{\lim_{r \to \infty}} \frac{n(\lambda r)}{\log M(r;f)} \leq \frac{1}{\log \frac{1}{\lambda}}.$$

But

$$\log M(r;f) < r^{\rho + \varepsilon} \quad (r >> 0),$$

thus

$$\overline{\lim_{r \to \infty}} \frac{n(\lambda r)}{r^{\rho + \varepsilon}} \leq \frac{1}{\log \frac{1}{\lambda}}$$

or still,

$$\overline{\lim_{r \to \infty}} \frac{n(r)}{r^{\rho + \varepsilon}} \leq \frac{1}{\lambda^{\rho + \varepsilon}} \frac{1}{\log \frac{1}{\lambda}}.$$

To finish up, simply take

$$\lambda = e^{-1/(\rho + \varepsilon)}.$$

4.18 APPLICATION If f is a transcendental entire function of finite order ρ, then $\forall\ \varepsilon > 0$,

$$n(r) = O(r^{\rho + \varepsilon}).$$

4.19 LEMMA If $|f(0)| = 1$, then

$$n(r) \leq \log M(er;f).$$

PROOF In fact,

$$n(r) = n(r) \int_r^{er} \frac{dt}{t}$$

$$\leq \int_r^{er} \frac{n(t)}{t}\,dt$$

$$\leq \int_0^{er} \frac{n(t)}{t}\,dt$$

$$\leq \log M(er;f).$$

4.20 THEOREM If f is a transcendental entire function of finite order ρ, then the convergence exponent κ of the sequence $\{r_n = |z_n|\}$ is $\leq \rho$.

PROOF This, of course, is trivial if f has a finite number of zeros (for then $\kappa = 0$), so as above it will be assumed that f has an infinite number of zeros (hence that $r_n = |z_n| \to \infty$), matters reducing to the case when $|f(0)| = 1$:

$$\kappa = \overline{\lim_{r \to \infty}} \frac{\log n(r)}{\log r} \qquad \text{(cf. 4.10)}$$

$$\leq \overline{\lim_{r \to \infty}} \frac{\log \log M(er;f)}{\log r} \qquad \text{(cf. 4.19)}$$

$$\leq \overline{\lim_{r \to \infty}} \frac{\log \log M(er;f)}{\text{lor } er} \cdot \frac{\log er}{\log r}$$

$$= \overline{\lim_{r \to \infty}} \frac{\log \log M(r;f)}{\log r}$$

$$= \rho.$$

4.21 COROLLARY If $p > \rho$, then

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^p} < \infty.$$

4.22 EXAMPLE It can happen that $\kappa < \rho$. E.g.: If $f(z) = e^z$, then $\rho = 1$ but there are no zeros, thus $\kappa = 0$. Another "for instance" is given by $e^{z^2} \sin z$, where $\kappa = 1 < 2 = \rho$.

[Note: The so-called canonical products constitute a class of entire functions of finite order for which $\kappa = \rho$ (cf. 5.10).]

4.23 REMARK If κ is positive, then f has an infinite number of zeros.

4.24 DEFINITION Let f be a transcendental entire function of finite order ρ -- then f is said to be of convergence class or divergence class according to whether

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\kappa}}$$

is convergent or divergent.

4.25 EXAMPLE The transcendental entire function

$$f(z) = \prod_{n=2}^{\infty} \left(1 - \frac{z}{n(\log n)^2}\right)$$

is of order 1. Here $\kappa = 1$ and f(z) is of convergence class (cf. 4.4).

4.26 EXAMPLE The transcendental entire functions

$$\begin{cases} \sin z \\ \cos z \end{cases}$$

are of order 1 and of divergence class.

4.27 EXAMPLE Consider the theta functions

$$\begin{cases} \Theta_1(z|\tau) \\ \Theta_2(z|\tau) \\ \Theta_3(z|\tau) \\ \Theta_4(z|\tau) \end{cases}$$

of the Appendix to §1 — then the zeros of each of them are enumerated there and in all four cases,

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^p}$$

is convergent if $p > 2$ and divergent if $p \le 2$ (cf. 4.8), hence $\kappa = 2$. On the other hand, it was shown in 2.38 that $\rho(\Theta_1) \le 2$, so $\rho(\Theta_1) = 2$ ($\Rightarrow \rho(\Theta_2) = \rho(\Theta_3) = \rho(\Theta_4) = 2$). Therefore the theta functions are of divergence class.

4.28 LEMMA If $|f(0)| = 1$ and if $0 < \rho = \kappa < \infty$, then

$$\Delta \le e^{\rho}\tau.$$

PROOF In fact,

$$\begin{aligned}
\Delta &= \overline{\lim_{r \to \infty}} \frac{n(r)}{r^{\kappa}} \quad \text{(cf. 4.13)} \\
&\le \overline{\lim_{r \to \infty}} \, e^{\kappa} \frac{\log M(er;f)}{(er)^{\kappa}} \quad \text{(cf. 4.19)} \\
&= \overline{\lim_{r \to \infty}} \, e^{\rho} \frac{\log M(er;f)}{(er)^{\rho}} \\
&= \overline{\lim_{r \to \infty}} \, e^{\rho} \frac{\log M(r;f)}{r^{\rho}} \\
&= e^{\rho}\tau \quad \text{(cf. 3.1)}.
\end{aligned}$$

Maintaining the assumption that f is a transcendental entire function of finite order ρ, suppose further that f is of finite type τ (cf. 3.5), so $\rho > 0$.

4.29 THEOREM We have

$$\overline{\lim_{r \to \infty}} \frac{n(r)}{r^{\rho}} \le \rho e \tau.$$

The technical key to proving this is to employ a generalization of 4.17.

4.30 JENSEN INEQUALITY If f has a zero of order m at the origin, then

$$\int_0^r \frac{n(t)}{t}\,dt \le \log M(r;f) - \log \left|\frac{f^{(m)}(0)}{m!}\right| r^m.$$

[Note: When m = 0, the correction term becomes

$$-\log |f(0)|$$

which disappears if in addition $|f(0)| = 1$.]

To establish 4.29, start by fixing a parameter $\lambda \in]0,1[$ and then proceed as in the proof of 4.16:

$$\int_0^r \frac{n(t)}{t}\,dt \ge n(\lambda r) \log \frac{1}{\lambda}$$

or still,

$$n(\lambda r) \le \frac{1}{\log \frac{1}{\lambda}} (\log M(r;f) - \log \left|\frac{f^{(m)}(0)}{m!}\right| r^m)$$

or still,

$$\frac{n(\lambda r)}{\log M(r;f)} \le \frac{1}{\log \frac{1}{\lambda}} (1 - \frac{\log \left|\frac{f^{(m)}(0)}{m!}\right| r^m}{\log M(r;f)}).$$

But

$$\lim_{r \to \infty} \frac{\log r}{\log M(r;f)} = 0 \qquad \text{(cf. 2.10)}.$$

Therefore

$$\overline{\lim_{r \to \infty}} \frac{n(\lambda r)}{\log M(r;f)} \le \frac{1}{\log \frac{1}{\lambda}}.$$

Since f is of finite type, $\forall\ \varepsilon > 0$,

$$\log M(r;f) < (\tau + \varepsilon)r^{\rho} \quad (r >> 0).$$

And this implies that

$$\overline{\lim_{r \to \infty}} \frac{n(\lambda r)}{(\tau + \varepsilon)r^{\rho}} \leq \frac{1}{\log \frac{1}{\lambda}}$$

or still,

$$\overline{\lim_{r \to \infty}} \frac{n(r)}{r^{\rho}} \leq \frac{\tau + \varepsilon}{\lambda^{\rho} \log \frac{1}{\lambda}}.$$

Setting $\lambda = e^{-1/\rho}$ then gives

$$\overline{\lim_{r \to \infty}} \frac{n(r)}{r^{\rho}} \leq \rho e(\tau + \varepsilon),$$

so in the limit $(\varepsilon \to 0)$

$$\overline{\lim_{r \to \infty}} \frac{n(r)}{r^{\rho}} \leq \rho e \tau.$$

4.31 REMARK It follows that if f has finite order and finite type, then 4.18 can be sharpened to

$$n(r) = O(r^{\rho}).$$

§5. *CANONICAL PRODUCTS*

Given a nonnegative integer p, let

$$E(z,0) = 1 - z \quad (p = 0)$$

and

$$E(z,p) = (1 - z)\exp(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}) \quad (p > 0).$$

[Note: The polynomial

$$z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}$$

is the p^{th} partial sum of the expansion

$$\log \frac{1}{1 - z} = \sum_{k=1}^{\infty} \frac{z^k}{k}.]$$

5.1 DEFINITION The functions E(z,p) are called <u>primary factors</u>.

5.2 LEMMA If $|z| \leq 1$, then

$$|E(z,p) - 1| \leq |z|^{p+1}.$$

PROOF Assuming that p is positive, write

$$E(z,p) = 1 + \sum_{n=1}^{\infty} A_n z^n.$$

Then

$$E'(z,p) = \sum_{n=1}^{\infty} nA_n z^{n-1}.$$

Meanwhile,

$$E'(z,p) = - z^p \exp(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}).$$

Therefore

$$A_1 = A_2 = \ldots = A_p = 0 \text{ and } A_n < 0 \quad (n > p).$$

On the other hand, $E(1,p) = 0$, so

$$\sum_{n=p+1}^{\infty} |A_n| = 1.$$

Accordingly,

$$\begin{aligned} |z| \leq 1 \Rightarrow |E(z,p) - 1| &\leq \sum_{n=p+1}^{\infty} |A_n|\ |z|^n \\ &= |z|^{p+1} \sum_{n=p+1}^{\infty} |A_n|\ |z|^{n-p-1} \\ &\leq |z|^{p+1} \sum_{n=p+1}^{\infty} |A_n| \\ &= |z|^{p+1}. \end{aligned}$$

Let $\{z_n : n = 1,2,\ldots\}$ be a sequence of nonzero complex numbers with

$$0 < |z_1| \leq |z_2| \leq \ldots \quad (|z_n| \to \infty),$$

finite repetitions being permitted. Put $r_n = |z_n|$ and assume that the convergence exponent κ of the sequence $\{r_n : n = 1,2,\ldots\}$ is finite.

Fix a nonnegative integer p such that the series

$$\sum_{n=1}^{\infty} \frac{1}{r_n^{p+1}}$$

is convergent.

5.3 NOTATION Let

$$P(z,p) = \prod_{n=1}^{\infty} E(\frac{z}{z_n},p).$$

N.B. At the origin,

$$P(0,p) = 1.$$

5.4 THEOREM $P(z,p)$ is an entire function whose zeros are the z_n.

PROOF Taking into account 5.2, it is a question of applying 1.26 and 1.29. So consider the series

$$\sum_{n=1}^{\infty} (E(\frac{z}{z_n},p) - 1).$$

Given $R > 0$, choose $N >> 0 : n > N \Rightarrow |z_n| > R$ -- then for $|z| \leq R$,

$$|E(\frac{z}{z_n},p) - 1| \leq \left|\frac{z}{z_n}\right|^{p+1} \leq \frac{R^{p+1}}{|z_n|^{p+1}}$$

and by assumption

$$\sum_{n>N} \frac{1}{|z_n|^{p+1}} < \infty.$$

5.5 LEMMA For all complex z, if $p = 0$,

$$\log|E(z,0)| \leq \log(1 + |z|),$$

and if $p > 0$,

$$\log|E(z,p)| \leq C_p \frac{|z|^{p+1}}{1 + |z|},$$

where $C_p = 3e(2 + \log p)$.

PROOF The first inequality is trivial. To establish the second inequality, consider two cases.

- $|z| \le \frac{p}{p+1}$ -- then

$$\begin{aligned}\log|E(z,p)| &= \log|(E(z,p) - 1) + 1| \\ &\le \log(|E(z,p) - 1| + 1) \\ &\le |E(z,p) - 1| \\ &\le |z|^{p+1} \quad (\text{cf. } 5.2),\end{aligned}$$

since $\log(x+1) \le x$ for $x \ge 0$.

- $|z| > \frac{p}{p+1}$ -- then

$$\begin{aligned}\log|E(z,p)| &\le 2|z| + \frac{|z|^2}{2} + \cdots + \frac{|z|^p}{p} \\ &= |z|^p\left(\frac{1}{p} + \frac{1}{p-1}\frac{1}{|z|} + \cdots + \frac{1}{2}\frac{1}{|z|^{p-2}} + 2\frac{1}{|z|^{p-1}}\right) \\ &\le |z|^p\left(\frac{p+1}{p}\right)^{p-1}\left(2 + \frac{1}{2} + \cdots + \frac{1}{p}\right) \\ &\le |z|^p\left(1 + \frac{1}{p}\right)^p\left(2 + \int_1^p \frac{dt}{t}\right) \\ &\le |z|^p\, e(2 + \log p) \\ &= e(2 + \log p)|z|^p \frac{1+|z|}{1+|z|} \\ &= e(2 + \log p)\left(1 + \frac{1}{|z|}\right)\frac{|z|^{p+1}}{1+|z|} \\ &\le 3e(2 + \log p)\frac{|z|^{p+1}}{1+|z|} \\ &= C_p \frac{|z|^{p+1}}{1+|z|},\end{aligned}$$

since

$$1 + \frac{1}{|z|} < 1 + \frac{p+1}{p} = 1 + 1 + \frac{1}{p} \le 3.$$

5.6 SUBLEMMA We have

$$\lim_{r \to \infty} \frac{n(r)}{r^{p+1}} = 0.$$

PROOF In fact,

$$\sum_{n=1}^{\infty} \frac{1}{r_n^{p+1}} = \int_0^{\infty} \frac{dn(t)}{t^{p+1}}$$

$$= \lim_{r \to \infty} \frac{n(r)}{r^{p+1}} + (p+1) \int_0^{\infty} \frac{n(t)}{t^{p+2}}\, dt.$$

And

$$\frac{n(r)}{r^{p+1}} = (p+1)n(r) \int_r^{\infty} \frac{dt}{t^{p+2}}$$

$$\le (p+1) \int_r^{\infty} \frac{n(t)}{t^{p+2}}\, dt \to 0 \quad (r \to \infty).$$

5.7 LEMMA Put $r = |z|$ -- then for $p = 0$,

$$\log|P(z,0)| \le \int_0^r \frac{n(t)}{t}\, dt + r \int_r^{\infty} \frac{n(t)}{t^2}\, dt,$$

and for $p > 0$,

$$\log|P(z,p)| \le (p+1)C_p r^p \left(\int_0^r \frac{n(t)}{t^{p+1}}\, dt + r\int_r^{\infty} \frac{n(t)}{t^{p+2}}\, dt\right).$$

PROOF If $p = 0$,

$$\log|P(z,0)| \le \sum_{n=1}^{\infty} \log\left(1 + \frac{r}{r_n}\right) \quad (\text{cf. } 5.5)$$

$$= \int_0^\infty \log(1 + \frac{r}{t}) dn(t)$$

$$= \log(1 + \frac{r}{t}) n(t) \Big|_0^\infty + r\int_0^\infty \frac{n(t)}{t(t+r)}\, dt$$

$$= \log(1 + \frac{r}{t}) t \frac{n(t)}{t} \Big|_0^\infty + r\int_0^\infty \frac{n(t)}{t(t+r)}\, dt$$

$$= r\int_0^\infty \frac{n(t)}{t(t+r)}\, dt$$

$$\le \int_0^r \frac{n(t)}{t}\, dt + r\int_r^\infty \frac{n(t)}{t^2}\, dt$$

and if $p > 0$,

$$\log|P(z,p)| \le C_p \sum_{n=1}^{\infty} \frac{r^{p+1}}{r_n^p(r+r_n)} \quad \text{(cf. 5.5)}$$

$$= C_p\, r^{p+1} \int_0^\infty \frac{dn(t)}{t^p(t+r)}$$

$$= C_p\, r^{p+1} \frac{n(t)}{t^p(t+r)} \Big|_0^\infty$$

$$+ C_p\, r^{p+1} \int_0^\infty (\frac{p}{t^{p+1}(t+r)} + \frac{1}{t^p(t+r)^2}) n(t)dt$$

$$= C_p\, r^{p+1} \frac{n(t)}{t^{p+1}(1 + r/t)} \Big|_0^\infty$$

$$+ C_p\, r^{p+1} \int_0^\infty (\frac{p}{t^{p+1}(t+r)} + \frac{1}{t^p(t+r)^2}) n(t)dt$$

$$= C_p\, r^{p+1} \int_0^\infty (\frac{p}{t^{p+1}(t+r)} + \frac{1}{t^p(t+r)^2}) n(t)dt$$

$$= C_p\, r^{p+1}\left(\int_0^r + \int_r^\infty\right)\left(\frac{p}{t^{p+1}(t+r)} + \frac{1}{t^p(t+r)^2}\right)n(t)\,dt$$

$$\leq (p+1)C_p\, r^p\left(\int_0^r \frac{n(t)}{t^{p+1}}\,dt + r\int_r^\infty \frac{n(t)}{t^{p+2}}\,dt\right).$$

5.8 REMARK For use below, note that these inequalities involve z only through its modulus r, hence provide estimates for

$$\log M(r;P(z,p)).$$

It has been assumed from the outset that the convergence exponent κ of the sequence $\{r_n : n = 1,2,\ldots\}$ is finite, thus it makes sense to take $p = \mathfrak{g}$, the genus of the sequence $\{r_n : n = 1,2,\ldots\}$ (cf. 4.14).

5.9 DEFINITION

$$P(z,\mathfrak{g}) = \prod_{n=1}^{\infty} E\left(\frac{z}{z_n},\mathfrak{g}\right)$$

is called the <u>canonical product</u> formed from the z_n.

[Note: $P(z,\mathfrak{g})$ is a transcendental entire function and the infinite product defining $P(z,\mathfrak{g})$ is absolutely convergent (cf. 5.4).]

5.10 THEOREM The order ρ of $P(z,\mathfrak{g})$ is equal to κ.

PROOF It suffices to show that $\rho \leq \kappa$, hence is finite (for then, on general grounds, $\kappa \leq \rho$ (cf. 4.20)). In any event,

$$\mathfrak{g} \leq \kappa \leq \mathfrak{g} + 1 \quad \text{(cf. 4.15)}$$

and it will be assumed that $\mathfrak{g}$ is positive.

<u>Case 1:</u> $\kappa < \mathfrak{g} + 1$. Choose $\varepsilon > 0 : \kappa + \varepsilon < \mathfrak{g} + 1$ -- then

$$n(t) < t^{\kappa+\varepsilon} \quad (t >> 0) \quad \text{(cf. 4.10)},$$

so

$$\log M(r;P(z,g))$$

$$\leq (g+1)C_g r^g (O(1) + \int_0^r t^{\kappa+\varepsilon-g-1}dt + r\int_r^\infty t^{\kappa+\varepsilon-g-2}dt)$$

$$\leq (g+1)C_g r^g (O(1) + \frac{r^{\kappa+\varepsilon-g}}{\kappa+\varepsilon-g} + \frac{r^{\kappa+\varepsilon-g}}{g+1-\kappa-\varepsilon})$$

$$< r^{\kappa+2\varepsilon} \quad (r >> 0).$$

Therefore $\rho \leq \kappa$.

<u>Case 2</u>: $\kappa = g+1$. Owing to 5.6,

$$\lim_{r \to \infty} \frac{n(r)}{r^{g+1}} = 0.$$

Fix $\varepsilon > 0$ and choose r_0:

$$r > r_0 \Rightarrow \frac{n(r)}{r^{g+1}} < \varepsilon, \int_r^\infty \frac{n(t)}{t^{g+2}}\, dt < \varepsilon.$$

Then

$$\log M(r;P(z,g))$$

$$\leq (g+1)C_g r^g (r\frac{n(r)}{r^{g+1}} + r\varepsilon)$$

$$\leq (g+1)C_g r^g (r\varepsilon + r\varepsilon)$$

$$= 2(g+1)C_g \varepsilon r^{g+1}$$

$$= 2(g+1)C_g \varepsilon r^{\kappa}.$$

Restated: $\forall\ C > 0$,

$$\log M(r;P(z,g)) \leq Cr^{\kappa} \quad (r >> 0).$$

Therefore $\rho \le \kappa$ (and more (cf. 5.16)).

[Note: The discussion when $\mathfrak{g} = 0$ is similar but simpler.]

5.11 LEMMA Let Q be a polynomial of degree q and put

$$f(z) = e^{Q(z)} P(z,\mathfrak{g}).$$

Then

$$\rho(f) = \max(q,\kappa).$$

PROOF Since q equals the order of e^Q and since κ equals the order of $P(z,\mathfrak{g})$, it follows from 2.34 that

$$\rho(f) \le \max(q,\kappa).$$

On the other hand, $\kappa \le \rho(f)$ (cf. 4.20). And

$$\frac{f}{P} = e^Q \Rightarrow q = \rho(e^Q) \le \max(\rho(f),\kappa) \quad \text{(cf. 2.37)}$$
$$= \rho(f).$$

Therefore

$$\max(q,\kappa) \le \rho(f).$$

[Note: It is a corollary that if $\rho(f)$ is not an integer, then $\rho(f) = \kappa$.]

5.12 EXAMPLE The canonical product

$$\{(1-z)e^z\}\{(1+z)e^{-z}\}\{(1-\frac{z}{2})e^{z/2}\}\{(1+\frac{z}{2})e^{-z/2} \ldots\}$$

represents

$$\frac{\sin \pi z}{\pi z} \quad \text{(cf. 1.23)}.$$

5.13 EXAMPLE The reciprocal

$$\frac{1}{z\Gamma(z)} = e^{\gamma z} \prod_{n=1}^{\infty} (1 + \frac{z}{n})\exp(- \frac{z}{n})$$

is a transcendental entire function of order 1. To see this, take $z_n = -n$ $(n = 1,2,\ldots)$ -- then $\kappa = 1$ and $\mathfrak{g} = 1$ (cf. 4.15). In view of 5.10, the order of the canonical product

$$\prod_{n=1}^{\infty} (1 + \frac{z}{n})\exp(-\frac{z}{n})$$

is 1, as is the order of $e^{\gamma z}$. Therefore the order of $\frac{1}{z\Gamma(z)}$ equals

$$\max(1,1) = 1 \quad (\text{cf. } 5.11).$$

5.14 EXAMPLE Let ω_1, ω_2 be two nonzero complex constants whose ratio is not purely real. Put

$$\Omega_{m,n} = m\omega_1 + n\omega_2 \quad ((m,n) \neq (0,0))$$

and consider

$$\prod_{m,n} (1 - \frac{z}{\Omega_{m,n}})\exp(\frac{z}{\Omega_{m,n}} + \frac{1}{2}(\frac{z}{\Omega_{m,n}})^2).$$

Then here, $\kappa = 2$ and $\mathfrak{g} = 2$ (cf. 4.15). Setting

$$\sigma(z|\omega_1,\omega_2) = \prod_{m,n} \cdots ,$$

it follows that $\sigma(z|\omega_1,\omega_2)$ is a transcendental entire function of order 2.

The proof of 5.10 fell into two cases:

$$\kappa < \mathfrak{g} + 1 \text{ or } \kappa = \mathfrak{g} + 1.$$

5.15 RAPPEL (cf. 4.15)

- If κ is not an integer, then $\mathfrak{g} = [\kappa]$.
- If κ is an integer, then $\mathfrak{g} = \kappa - 1$ if $\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\kappa}}$ is convergent, while

$\mathfrak{g} = \kappa$ if $\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\kappa}}$ is divergent.

[Note: Employing the terminology of 4.24, in this situation

$$\begin{cases} P(z,\mathfrak{g}) \text{ of convergence class} \Rightarrow \mathfrak{g} = \kappa - 1 \\ P(z,\mathfrak{g}) \text{ of divergence class} \Rightarrow \mathfrak{g} = \kappa.] \end{cases}$$

So, if κ is not an integer, then $\kappa < \mathfrak{g} + 1$ and if κ is an integer, then $\kappa < \mathfrak{g} + 1$ if $\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\kappa}}$ is divergent but $\kappa = \mathfrak{g} + 1$ if $\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\kappa}}$ is convergent.

With these points in mind, we shall now proceed to the determination of the type τ of $P(z,\mathfrak{g})$.

[Note: The very definition of type requires that $0 < \rho < \infty$. It is automatic that ρ is finite and it is also automatic that ρ is positive if κ is not an integer or if κ is an integer and $\mathfrak{g} = \kappa - 1$ but if κ is an integer and $\mathfrak{g} = \kappa$, then it will be assumed that κ $(= \rho)$ is positive.]

5.16 THEOREM If κ is an integer and if $\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\kappa}}$ is convergent, then $P(z,\mathfrak{g})$ is of minimal type.

[Here $\kappa = \mathfrak{g} + 1$, thus the assertion is implied by the "Case 2" analysis in 5.10.]

5.17 LEMMA Take $\rho > 0$ -- then

$$\Delta \leq e^{\rho}\tau.$$

PROOF Since $P(0,\mathfrak{g}) = 1$, in view of 4.19,

$$n(r) \leq \log M(er;P(z,\mathfrak{g})),$$

thus

$$\frac{n(r)}{r^{\kappa}} \leq \frac{\log M(er;P(z,g))}{r^{\kappa}}$$

=>

$$\Delta = \overline{\lim_{r \to \infty}} \frac{n(r)}{r^{\kappa}} \text{ (cf. 4.13) } \leq \overline{\lim_{r \to \infty}} e^{\kappa} \frac{\log M(er;P(z,g))}{(er)^{\kappa}}$$

$$= \overline{\lim_{r \to \infty}} e^{\rho} \frac{\log M(er;P(z,g))}{(er)^{\rho}}$$

$$= e^{\rho} \overline{\lim_{r \to \infty}} \frac{\log M(er;P(z,g))}{(er)^{\rho}}$$

$$= e^{\rho}\tau.$$

Suppose that κ is not an integer (hence $\rho > 0$ and $g < \kappa < g + 1$).

5.18 LEMMA Put

$$K_{0,\kappa} = \frac{1}{\kappa} + \frac{1}{1-\kappa}$$

and

$$K_{g,\kappa} = (g + 1)C_g \left[\frac{1}{\kappa-g} + \frac{1}{g+1-\kappa}\right] \quad (g > 0).$$

Then

$$\tau \leq 2K_{g,\kappa}\Delta.$$

PROOF Given $\varepsilon > 0$, we have

$$n(t) < (\Delta + \varepsilon)t^{\kappa} \quad (t >> 0).$$

Therefore, taking $g > 0$,

$$\log M(r;P(z,g))$$

$$\leq (g + 1)C_g r^g (\int_0^r \frac{n(t)}{t^{g+1}} dt + r\int_r^\infty \frac{n(t)}{t^{g+2}} dt) \qquad \text{(cf. 5.7)}$$

$$\leq (g + 1)C_g r^g (O(1) + (\Delta + \varepsilon)\int_0^r t^{\kappa-g-1} dt + (\Delta + \varepsilon)r\int_r^\infty t^{\kappa-g-2} dt)$$

$$\leq (g + 1)C_g r^g (O(1) + (\Delta + \varepsilon) \frac{r^{\kappa-g}}{\kappa-g} + (\Delta + \varepsilon) \frac{r^{\kappa-g}}{g+1-\kappa})$$

$$< 2K_{g,\kappa}(\Delta + \varepsilon)r^\kappa \quad (r >> 0).$$

Since $\rho = \kappa$, it follows that

$$\overline{\lim_{r \to \infty}} \frac{\log M(r;P(z,g))}{r^\rho} \leq K_{g,\kappa}(\Delta + \varepsilon),$$

i.e.,

$$\tau \leq 2K_{g,\kappa}\Delta.$$

[Note: The discussion when $g = 0$ is similar but simpler.]

5.19 THEOREM If κ is not an integer, then $P(z,g)$ is of maximal, minimal, or intermediate type according to whether $\Delta = \infty$, $\Delta = 0$, or $0 < \Delta < \infty$ and conversely.

[This is implied by 5.17 and 5.18.]

There remains the case when κ is an integer > 0 and $\sum_{n=1}^\infty \frac{1}{|z_n|^\kappa}$ is divergent (hence $g = \kappa$). To this end, let

$$\delta(r) = \left| \frac{1}{\kappa} \sum_{|z_n|<r} z_n^{-\kappa} \right|,$$

put

$$\delta = \overline{\lim_{r \to \infty}} \delta(r),$$

and set

$$\Gamma = \max(\delta,\Delta).$$

5.20 THEOREM Under the preceding conditions, $P(z,\mathfrak{g})$ is of maximal, minimal, or intermediate type according to whether $\Gamma = \infty$, $\Gamma = 0$, or $0 < \Gamma < \infty$ and conversely.

The proof can be divided into two parts.

- $\exists\ C > 1$:

$$\Gamma \leq Ce^{\rho}\tau.$$

[First, it can be shown that for some $C > 1$,

$$\delta(r) < C\,\frac{\log M(er;P(z,\mathfrak{g}))}{r^{\kappa}} \qquad (r >> 0).$$

Thus

$$\delta(r) < (Ce^{\rho})\,\frac{\log M(er;P(z,\mathfrak{g}))}{(er)^{\rho}} \qquad (r >> 0)$$

and so

$$\delta \leq Ce^{\rho}\tau.$$

Meanwhile,

$$\Delta \leq e^{\rho}\tau \qquad (\text{cf. } 5.17).$$

Therefore

$$\Gamma \leq Ce^{\rho}\tau.$$

- $\exists\ K > 0$:

$$\tau \leq K\Gamma.$$

[Write

$$P(z,\mathfrak{g}) = \exp\Big(\big(\frac{1}{\kappa}\sum_{|z_n|<r} z_n^{-\kappa}\big)z^{\kappa}\Big)$$

$$\times \prod_{|z_n|<r} E(\frac{z}{z_n}, \mathfrak{g} - 1) \prod_{|z_n|\geq r} E(\frac{z}{z_n}, \mathfrak{g}),$$

where $r = |z|$ and take $\kappa > 1$ -- then

$$\log M(r;P(z,\mathfrak{g}))$$

$$\leq \delta(r)r^{\kappa}$$

$$+ C_{\mathfrak{g}}(r^{\mathfrak{g}}\int_0^r \frac{dn(t)}{t^{\mathfrak{g}-1}(t+r)} + r^{\mathfrak{g}+1} \int_r^{\infty} \frac{dn(t)}{t^{\mathfrak{g}}(t+r)})$$

$$\leq \delta(r)r^{\kappa}$$

$$+ (\mathfrak{g} + 1)C_{\mathfrak{g}}(r^{\mathfrak{g}-1} \int_0^r \frac{n(t)}{t^{\mathfrak{g}}} dt + r^{\mathfrak{g}+1} \int_r^{\infty} \frac{n(t)}{t^{\mathfrak{g}+2}} dt).$$

But $\forall \varepsilon > 0$,

$$n(t) < (\Delta + \varepsilon)t^{\kappa} \quad (t >> 0).$$

Therefore

$$\log M(r;P(z,\mathfrak{g}))$$

$$\leq \delta(r)r^{\kappa} + 2(\mathfrak{g} + 1)C_{\mathfrak{g}}(\Delta + \varepsilon)r^{\kappa} \quad (r >> 0).$$

And finally

$$\tau = \overline{\lim_{r \to \infty}} \frac{\log M(r;P(z,\mathfrak{g}))}{r^{\kappa}} \leq \delta + 2(\mathfrak{g} + 1)C_{\mathfrak{g}}\Delta$$

$$\leq \Gamma + 2(\mathfrak{g} + 1)C_{\mathfrak{g}}\Gamma$$

$$= (1 + 2(\mathfrak{g} + 1)C_{\mathfrak{g}})\Gamma$$

$$\equiv K\Gamma.$$

[Note: Minor modifications in the argument are needed if $\kappa = 1$.]

5.21 EXAMPLE In the setup of 5.12, the zeros are $\pm n$ $(n = 1,2,\ldots)$, say $z_1 = 1$, $z_2 = -1$, $z_3 = 2$, $z_4 = -2, \ldots$, hence $r_1 = 1$, $r_2 = 1$, $r_3 = 2$, $r_4 = 2, \ldots$. Here $\kappa = 1$ and $\frac{\sin \pi z}{\pi z}$ is of divergence class. Moreover,

$$\delta(r) = 0 \ (r > 0) \Rightarrow \delta = 0.$$

On the other hand,

$$\Delta = \overline{\lim_{n \to \infty}} \frac{n}{r_n} \quad \text{(cf. 4.11)}.$$

But

$$\frac{1}{r_1} = \frac{1}{1}, \frac{2}{r_2} = \frac{2}{1}, \frac{3}{r_3} = \frac{3}{2}, \frac{4}{r_4} = \frac{4}{2}, \ldots .$$

Therefore $\Delta = 2$ and

$$\Gamma = \max(\delta,\Delta) = \max(0,2) = 2.$$

I.e.: $\frac{\sin \pi z}{\pi z}$ is of intermediate type.

5.22 EXAMPLE In the setup of 5.13, the zeros are $-n$ $(n = 1,2,\ldots)$, say $z_n = -n$. Here $\kappa = 1$ and $\frac{1}{z\Gamma(z)}$ is of divergence class. However, in contrast with 5.21,

$$\delta = \overline{\lim_{n \to \infty}} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) = \infty.$$

Since it is clear that $\Delta = 1$, we thus have

$$\Gamma = \max(\delta,\Delta) = \max(\infty,1) = \infty.$$

Consequently,

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)\exp\left(-\frac{z}{n}\right)$$

is of maximal type. But the order of $e^{\gamma z}$ is 1 and the type of $e^{\gamma z}$ is γ. An appeal to 3.14 then implies that

$$\frac{1}{z\Gamma(z)} = \prod_{n=1}^{\infty} (1 + \frac{z}{n})\exp(- \frac{z}{n})$$

is of maximal type.

§6. *EXPONENTIAL FACTORS*

Take a canonical product $P(z,\mathfrak{g})$ per §5, let Q be a polynomial of degree $q \geq 1$ and put

$$f(z) = e^{Q(z)}P(z,\mathfrak{g}).$$

Then

$$\rho(= \rho(f)) = \max(q,\kappa) \quad \text{(cf. 5.11)}.$$

[Note: Recall that it is always true that $\kappa \leq \rho$ (cf. 4.20).]

6.1 DEFINITION The genus of f is the nonnegative integer

$$\underline{\text{gen}}\ f = \max(q,\mathfrak{g}).$$

6.2 LEMMA We have

$$\underline{\text{gen}}\ f \leq \rho.$$

[This is because $\mathfrak{g} \leq \kappa$ (cf. 5.15).]

6.3 LEMMA If ρ is not an integer, then the genus of f is $[\rho]$.

PROOF For here $\rho = \kappa$ (and $\rho > q$). But in general,

$$\mathfrak{g} \leq \kappa \leq \mathfrak{g} + 1,$$

so in this case

$$\mathfrak{g} < \rho < \mathfrak{g} + 1,$$

thus

$$\underline{\text{gen}}\ f = \max(q,\mathfrak{g}) = \max(q,[\rho]) = [\rho].$$

6.4 LEMMA If ρ is an integer, then the genus of f is either equal to ρ or to $\rho - 1$.

PROOF The genus of f is necessarily less than or equal to ρ (cf. 6.2). If it is less than ρ, then $q < \rho$ ($\Rightarrow q \leq \rho - 1$) and $\rho = \kappa$, hence

$$\mathfrak{g} \leq \rho \leq \mathfrak{g} + 1.$$

But by assumption, $\mathfrak{g} < \rho$. Therefore $\mathfrak{g} = \rho - 1$ and

$$\underline{\text{gen}}\ f = \max(q,\mathfrak{g}) = \max(q,\rho - 1) = \rho - 1.$$

6.5 REMARK When ρ is an integer, there are five possibilities.

(i) $\kappa < \rho$, $\mathfrak{g} \leq \kappa$, $q = \rho$, $\underline{\text{gen}}\ f = \rho$

(ii) $\kappa = \rho$, $\mathfrak{g} = \rho$, $q = \rho$, $\underline{\text{gen}}\ f = \rho$

(iii) $\kappa = \rho$, $\mathfrak{g} = \rho$, $q < \rho$, $\underline{\text{gen}}\ f = \rho$

(iv) $\kappa = \rho$, $\mathfrak{g} = \rho - 1$, $q = \rho$, $\underline{\text{gen}}\ f = \rho$

(v) $\kappa = \rho$, $\mathfrak{g} = \rho - 1$, $q < \rho$, $\underline{\text{gen}}\ f = \rho - 1$.

And examples illustrating the various possibilities can be constructed.

6.6 THEOREM Suppose that ρ is nonintegral -- then f is of maximal, minimal, or intermediate type according to whether $\Delta = \infty$, $\Delta = 0$, or $0 < \Delta < \infty$ and conversely.

PROOF In this situation, $\rho = \kappa$ (the order of P (cf. 5.10)), while $\rho > q$ (q the order of e^Q). Therefore the type of f equals the type of P (cf. 3.14), so we can quote 5.19.

6.7 THEOREM Suppose that ρ is integral. Assume: $\mathfrak{g} < \rho$ -- then f is either of minimal type or of intermediate type.

PROOF The assumption that $\mathfrak{g}$ is less than ρ puts us in cases (i), (iv), or (v) above. Since the series $\sum\limits_{n=1}^{\infty} \frac{1}{|z_n|^\rho}$ is convergent, one can replace κ by ρ in 5.16 and conclude that $P(z,\mathfrak{g})$ is of minimal type.

• In case (i), the order of e^Q is strictly greater than the order of $P: q > \kappa$. Therefore

$$\tau(f) = \tau(e^Q) = |a_q| \neq 0 \quad \text{(cf. 3.14)},$$

so f is of intermediate type.

• In case (iv), the order of e^Q and the order of P are one and the same: $q = \kappa$. Since $0 < \tau(e^Q) = |a_q| < \infty$, $0 = \tau(P)$, the conclusion is that $\tau(f) = |a_q|$ (cf. 3.14), thus f is of intermediate type.

• In case (v), the order of e^Q is strictly smaller than the order of $P: q > \kappa$. Therefore

$$\tau(f) = \tau(P) = 0 \quad \text{(cf. 3.14)},$$

i.e., f is of minimal type.

Assuming still that ρ is integral, it remains to deal with cases (ii) and (iii) ($\Rightarrow g = \rho$). Agreeing to write

$$\begin{cases} a_\rho = a_q \text{ if } q = \rho \\ a_\rho = 0 \text{ if } q < \rho, \end{cases}$$

let

$$\delta(r) = \left| a_\rho + \frac{1}{\rho} \sum_{|z_n|<r} z_n^{-\rho} \right|,$$

put

$$\delta = \overline{\lim_{r \to \infty}}\, \delta(r),$$

and set

$$\Gamma = \max(\delta, \Delta).$$

6.8 THEOREM Suppose that ρ is integral. Assume: $\mathfrak{g} = \rho$ -- then f is of maximal, minimal, or intermediate type according to whether $\Gamma = \infty$, $\Gamma = 0$, or $0 < \Gamma < \infty$ and conversely.

PROOF The case (iii) scenario is straightforward: $q < \kappa = \rho$, hence $\tau(f) = \tau(P)$, the latter being controlled by 5.20 ($a_\rho = 0$, so the Γ there is the Γ here). As for what happens in case (ii), simply repeat the proof of 5.20 subject to the complication resulting from the presence of $a_q \neq 0$ in the definition of δ, the trick being to write

$$f(z) = \exp((a_\rho + \frac{1}{\rho} \sum_{|z_n|<r} z_n^{-\rho}) z^\rho) \exp(Q(z) - a_\rho z^\rho)$$

$$\times \prod_{|z_n|<r} E(\frac{z}{z_n}, \mathfrak{g} - 1) \prod_{|z_n|\geq r} E(\frac{z}{z_n}, \mathfrak{g}).$$

6.9 REMARK Under the preceding assumptions, if f is of minimal type, then

$$\frac{1}{\rho} \sum_{n=1}^{\infty} \frac{1}{z_n^\rho} = - a_\rho.$$

§7. *REPRESENTATION THEORY*

Let f be an entire function -- then as regards its zeros, there are three possibilities.

1. f has no zeros.

2. f has a finite number of zeros.

3. f has an infinite number of zeros.

7.1 THEOREM If f has no zeros, then there is an entire function g such that $f = e^g$.

PROOF Since f has no zeros, $\frac{1}{f}$ is entire, as is $\frac{f'}{f}$. Define g by the prescription

$$g(z) = \int_0^z \frac{f'(t)}{f(t)}\,dt,$$

the path of integration being immaterial -- then $g' = \frac{f'}{f}$. And

$$\begin{aligned}(fe^{-g})' &= f'e^{-g} - fg'e^{-g}\\ &= e^{-g}(f' - f\,\frac{f'}{f})\\ &= 0.\end{aligned}$$

Therefore

$$f(z)e^{-g(z)} = f(0)e^{-g(0)} = f(0)$$

$$\Rightarrow$$

$$f(z) = f(0)e^{g(z)}.$$

Conclude by absorbing f(0) into the exponential.

7.2 REMARK If f has no zeros, if $f = e^g$, and if f is of finite order, then g is a polynomial (cf. 2.42).

Suppose now that f is an entire function with finitely many zeros $z_1 \neq 0, \ldots, z_n \neq 0$ (each counted with multiplicity), as well as a zero of order $m \geq 0$ at the origin -- then the entire function

$$f(z)/z^m \prod_{k=1}^{n} (1 - \frac{z}{z_k})$$

has no zeros, hence equals

$$e^{g(z)},$$

where $g(z)$ is entire, so

$$f(z) = z^m e^{g(z)} \prod_{k=1}^{n} (1 - \frac{z}{z_k}).$$

N.B. If f is of finite order, then g is a polynomial (cf. 7.2).

Assume henceforth that f is a transcendental entire function of finite order ρ with an infinite number of nonzero zeros $\{z_n : n \geq 1\}$ and a zero of order $m \geq 0$ at the origin. Set $\Pi(z) = P(z,g)$.

7.3 HADAMARD FACTORIZATION We have

$$f(z) = z^m e^{Q(z)} \Pi(z),$$

where $Q(z)$ is a polynomial of degree $q \leq \rho$.

PROOF The quotient

$$\frac{f(z)}{z^m \Pi(z)}$$

is entire and has no zeros, thus can be written as $e^{Q(z)}$, where $Q(z)$ is entire. Owing to 2.37, the order of

$$\frac{f(z)}{z^m \Pi(z)}$$

is $\leq$ the maximum of ρ and the order of $z^m \Pi(z)$, the order of the latter being that of $\Pi(z)$ (cf. 2.36), which in turn is equal to κ (cf. 5.10). But κ is $\leq \rho$ (cf. 4.20). Therefore the order of $e^{Q(z)}$ is $\leq \rho$, so $Q(z)$ is a polynomial of degree $q \leq \rho$ (cf. 2.42).

7.4 REMARK If f is a transcendental entire function of finite nonintegral order ρ, then it is automatic that f has an infinity of zeros.

[In fact,

$$\rho = \max(q,\kappa) \quad \text{(cf. 5.11)} \Rightarrow \rho = \kappa.$$

But if f had finitely many zeros, then of necessity, $\kappa = 0\ldots$.]

By definition (cf. 6.1),

$$\underline{\text{gen}}\ f = \max(q,g)$$

and the simplest cases

$$\underline{\text{gen}}\ f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

are of special interest.

7.5 LEMMA If $\underline{\text{gen}}$ f = 0 or 1, then $\rho \leq 2$.

PROOF If ρ is not an integer, then $\underline{\text{gen}}\ f = [\rho]$ (cf. 6.3), hence $\rho < 2$. On the other hand, if ρ is an integer, then $\underline{\text{gen}}\ f = \rho$ or $\rho - 1$ (cf. 6.4), hence $\rho \leq 2$.

- $\underline{\text{gen}}\ f = 0$. Here $q = 0$, so $Q(z) = C$, and

$$f(z) = z^m e^C \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right),$$

where

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|} < \infty.$$

- <u>gen</u> f = 1.

$$\left.\begin{array}{l} q = 1 \\ g = 1 \end{array}\right] \Rightarrow f(z) = z^m e^{az+b} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{z/z_n},$$

where a ≠ 0 and

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^2} < \infty$$

but

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|} = \infty.$$

$$\left.\begin{array}{l} q = 0 \\ g = 1 \end{array}\right] \Rightarrow f(z) = z^m e^C \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{z/z_n},$$

where

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^2} < \infty$$

but

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|} = \infty.$$

$$\left.\begin{array}{l} q = 1 \\ g = 0 \end{array}\right] \Rightarrow f(z) = z^m e^{az+b} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right),$$

where a ≠ 0 and

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|} < \infty.$$

§8. *ZEROS*

Let f be an entire function.

8.1 DEFINITION A critical point of f is a zero of f'.

Suppose that

$$f(z) = \prod_{i=1}^{k} (z - z_i)^{m_i}$$

is a polynomial of degree n, thus $\sum_{i=1}^{k} m_i = n$ and the z_i are distinct. There are then two kinds of critical points.

- A zero z_i of multiplicity $m_i > 1$ is said to be of the first kind. Counting it $m_i - 1$ times (its multiplicity as a zero of f'), it follows that there are n - k critical points of the first kind.

- Since the degree of f' is n - 1, there are k - 1 additional critical points, these being termed of the second kind. They are not zeros of f but are zeros of $\frac{f'}{f}$ (defined on $C - \{z_1,\ldots,z_k\}$), i.e., are zeros of

$$\sum_{i=1}^{k} \frac{m_i}{z - z_i}.$$

8.2 REMARK There is no simple relation between the number of distinct zeros of a polynomial and its derivative.

(1) The polynomial $\prod_{i=1}^{k} (z - i)^2$ has k distinct zeros while its derivative has 2k - 1 distinct zeros.

(2) The polynomial $z^n - 1$ has n distinct zeros but its derivative has just one.

(3) The polynomial $z^{n-1}(z - 1)$ has two distinct zeros as does its derivative.

8.3 THEOREM The zeros of f' belong to the convex hull of the zeros of f.

PROOF It suffices to consider a zero z_0 of the second kind:

$$\sum_{i=1}^{k} \frac{m_i}{z_0 - z_i} = 0 \Rightarrow \sum_{i=1}^{k} \frac{m_i}{\bar{z}_0 - \bar{z}_i} = 0$$

=>

$$\sum_{i=1}^{k} m_i \frac{z_0 - z_i}{|z_0 - z_i|^2} = 0$$

=>

$$z_0 \sum_{i=1}^{k} \frac{m_i}{|z_0 - z_i|^2} = \sum_{i=1}^{k} m_i \frac{z_i}{|z_0 - z_i|^2}$$

=>

$$z_0 = \sum_{i=1}^{k} \lambda_i z_i,$$

where

$$\lambda_i = \frac{\dfrac{m_i}{|z_0 - z_i|^2}}{\displaystyle\sum_{j=1}^{k} \frac{m_j}{|z_0 - z_j|^2}} > 0$$

and

$$\sum_{i=1}^{k} \lambda_i = 1.$$

8.4 EXAMPLE There are transcendental entire functions for which this result is false.

[Take

$$f(z) = z \exp \frac{z^2}{2}.$$

It has one zero, viz. $z = 0$, but its derivative

$$f'(z) = (1 + z^2)\exp \frac{z^2}{2}$$

has two zeros, viz. $\pm \sqrt{-1}$.]

8.5 NOTATION Given a nonempty closed subset T of $\mathbf{C}$, let $< T >$ stand for its closed convex hull.

8.6 LEMMA Let f be a transcendental entire function of finite order ρ with <u>gen</u> $f = 0$. Assume: The zeros of f lie in T — then the zeros of f' lie in $< T >$.

PROOF Decompose f per 7.3:

$$f(z) = Cz^m \prod_{n=1}^{\infty} (1 - \frac{z}{z_n}),$$

and put

$$f_N(z) = Cz^m \prod_{n=1}^{N} (1 - \frac{z}{z_n}).$$

Then

$$f_N \to f \quad (N \to \infty)$$

uniformly on compact subsets of $\mathbf{C}$, so

$$f_N' \to f' \quad (N \to \infty)$$

uniformly on compact subsets of $\mathbf{C}$. But the zeros of f' are limits of zeros of the

f'_N, these in turn being elements of < T > (cf. 8.3).

[Note: In terms of ρ,

$$0 \le \rho < 1 \Rightarrow \underline{\text{gen}}\ f = [\rho] = 0 \quad \text{(cf. 6.3)}$$

or

$$\rho = 1 \text{ and } \underline{\text{gen}}\ f = \rho - 1 = 1 - 1 = 0 \quad \text{(cf. 6.4).]}$$

8.7 EXAMPLE The transcendental entire function

$$f(z) = \prod_{k=0}^{K} \cos(z - k\sqrt{-1})^{1/2}$$

is of order 1/2 and its zeros lie in the set

$$T: \text{Re } z \ge 0 \ \& \ 0 \le \text{Im } z \le K.$$

Since here T = < T >, the zeros of its derivative also lie in T.

8.8 REMARK Take $\rho = 1$ and suppose that the conditions of 6.8 are in force with f of minimal type, hence $\Gamma = 0$ and

$$\sum_{n=1}^{\infty} \frac{1}{t_n} = - a_1 \quad \text{(cf. 6.9)}$$

$$\equiv - a.$$

Then 8.6 still goes through. Thus write

$$f(z) = Cz^m e^{az} \prod_{n=1}^{\infty} (1 - \frac{z}{z_n}) e^{-z/z_n} \quad \text{(cf. 7.3)}$$

and let

$$f_N(z) = Cz^m e^{az} \prod_{n=1}^{N} (1 - \frac{z}{z_n}) e^{-z/z_n}.$$

Since

$$\sum_{n=1}^{N} \frac{1}{z_n} - a \to 0 \quad (N \to \infty),$$

it follows that

$$f_N \to f \quad (N \to \infty)$$

uniformly on compact subsets of $\mathbf{C}$.

8.9 EXAMPLE Fix $\tau > 0$ -- then

$$f(z) = (z^2 - 1)^m e^{\tau z}$$

is a transcendental entire function of order 1 and type τ and its zeros lie in the convex set $[-1,1]$. On the other hand, f has a critical point at

$$-\frac{1}{\tau} \ (m + \sqrt{m^2 + \tau^2}) \notin [-1,1].$$

Therefore the assumption of minimal type cannot be dropped in 8.8.

Before proceeding further, it will be best to recall some standard generalities.

8.10 LEMMA Suppose that f is a real analytic function -- then in any finite interval I, f has at most a finite number of distinct zeros.

[Note: This is false if f is merely C^∞: Take $I = [0,1]$ and consider $f(x) = x \sin(\frac{1}{x})$.]

8.11 ROLLE'S THEOREM Suppose that f is a real analytic function -- then between any two consecutive zeros of f, say $f(a) = 0$, $f(b) = 0$ $(a < b)$, f' has an odd number of zeros in $]a,b[$ counted according to multiplicity.

8.12 LEMMA Suppose that f is a real analytic function and let I be a finite interval. Assume: f' has Z' zeros in I counted according to multiplicity -- then f has at most Z' + 1 zeros in I counted according to multiplicity.

PROOF Let d denote the number of distinct zeros of f in I and let D denote the number of zeros of f in I counted according to multiplicity. At a zero of f of multiplicity m_k, f' has a zero of multiplicity $m_k - 1$. In addition, by Rolle's theorem, f' has at least one zero between two consecutive zeros of f. Therefore

$$Z' \geq \sum_{k=1}^{d} (m_k - 1) + d - 1$$

$$= D - d + d - 1 = D - 1$$

$$\Rightarrow$$

$$D \leq Z' + 1.$$

[Note: It is thus a corollary that if f has Z zeros in I counted according to multiplicity, then f' has at least Z - 1 zeros in I counted according to multiplicity.]

8.13 DEFINITION An entire function is said to be <u>real</u> if it assumes real values on the real axis.

[Note: The restriction of a real entire function to the real axis is a real analytic function.]

<u>N.B.</u> If

$$f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

then f is real iff $\forall$ n, c_n is real.

8.14 EXAMPLE If f is a polynomial and if the zeros of f are real, then f is real (to within a multiplicative constant) but not conversely.

8.15 REMARK If f is a transcendental entire function of finite order and if <u>gen</u> f = 0, then the reality of its zeros forces the reality of f (up to a constant factor) but this need not be true if <u>gen</u> f > 0 (although it will be if f is a canonical product with real zeros).

8.16 THEOREM If f is a polynomial and if the zeros of f are real, then the zeros of f' are real.

[In view of 8.3, this is immediate.]

[Note: Suppose that $z_1 < \cdots < z_k$ are the distinct zeros of f -- then by Rolle's theorem, f has at least one critical point in each of the intervals $]z_i, z_{i+1}[$ $(i = 1, \ldots, k - 1)$ and these critical points are of the second kind. Since there are k - 1 critical points of the second kind, there is but one critical point in $]z_i, z_{i+1}[$ and it is simple. Finally, all critical points of f are to be found in $[z_1, z_k]$.]

8.17 EXAMPLE The zeros of the following polynomials are real and simple.

- The <u>Legendre polynomials</u>:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

- The <u>Laguerre polynomials</u>:

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} e^{-x} x^n.$$

- The Hermite polynomials:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

A polynomial

$$f(z) = \prod_{n=1}^{N} E(\frac{z}{z_n}, 0) = \prod_{n=1}^{N} (1 - \frac{z}{z_n})$$

of degree N is, in particular, a canonical product, so 8.16 is a special case of the next result (compare too 8.6).

8.18 THEOREM Let

$$f(z) = \prod_{n=1}^{\infty} E(\frac{z}{z_n}, g)$$

be a canonical product whose zeros are real -- then the zeros of f' are real.

PROOF Working with the zeros of f' that are not zeros of f, pass to

$$\frac{f'(z)}{f(z)} = z^g \sum_{n=1}^{\infty} \frac{1}{z_n^g (z-z_n)},$$

which shows that the origin is a zero of multiplicity g of $f'(z)$. Let

$$F(z) = z^{-g} \frac{f'(z)}{f(z)}$$

and write $z_n = x_n + \sqrt{-1}\, 0$, hence

$$F(z) = \sum_{n=1}^{\infty} \frac{1}{x_n^g (z-x_n)}.$$

Suppose now that

$$f'(c) = f'(a + \sqrt{-1}\, b) = 0,$$

the claim being that b = 0. To see this, separate the real and imaginary parts in $F(c) = 0$ to get

$$a \sum_{n=1}^{\infty} \frac{1}{x_n^{g}|c-x_n|^2} - \sum_{n=1}^{\infty} \frac{1}{x_n^{g-1}|c-x_n|^2} = 0$$

and

$$b \sum_{n=1}^{\infty} \frac{1}{x_n^{g}|c-x_n|^2} = 0.$$

- If g is even or if $\forall$ n, $x_n > 0$ ($x_n < 0$), then b = 0.

- If g is odd and there are positive as well as negative x_n, then

$$b \neq 0 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{x_n^{g}|c-x_n|^2} = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{x_n^{g-1}|c-x_n|^2} = 0.$$

But this is impossible since $g - 1$ is even.

8.19 ADDENDUM Let $\zeta' < \zeta''$ be consecutive zeros of f of the same sign -- then there is exactly one distinct zero of f' in $]\zeta',\zeta''[$.

[By Rolle's theorem, there is at least one ζ in $]\zeta',\zeta''[$ such that $f'(\zeta) = 0$ (bear in mind that f is real). As for its uniqueness, if g is even or if $\forall$ n, $x_n > 0$ ($x_n < 0$), then the sign of

$$F'(x) = - \sum_{n=1}^{\infty} \frac{1}{x_n^{g}(x-x_n)^2}$$

is constant, thus F(x) is monotonic between ζ' and ζ'', thus cannot vanish more than once in $]\zeta',\zeta''[$. So, if $\alpha \neq \beta$ were distinct zeros of f' in $]\zeta',\zeta''[$, then g would have to be odd and there would have to be both positive and negative x_n. But

$$\begin{cases} 0 = F(\alpha) + F(\beta) = (\alpha + \beta)X - 2Y \\ 0 = F(\alpha) - F(\beta) = (\beta - \alpha)X \end{cases}$$

$$\Rightarrow X = 0 \quad (\alpha \neq \beta)$$

$\Rightarrow$

$$- 2 \sum_{n=1}^{\infty} \frac{1}{x_n^{g-1}(\alpha-x_n)(\beta-x_n)} = 0.$$

This, however, is impossible: $g - 1$ is even and $\forall\, n$, $(\alpha - x_n)(\beta - x_n) > 0$.]

8.20 REMARK It can be shown that the genus of f' is equal to the genus of f. [This is obvious if the order ρ of f is not an integer (for $\rho = \rho'$ (the order of f') (cf. 2.25) and $\underline{\text{gen}}\ f = [\rho] = [\rho'] = \underline{\text{gen}}\ f'$ (cf. 6.3)) but not so obvious otherwise.]

8.21 EXAMPLE Let

$$f_\alpha(z) = \prod_{n=2}^{\infty} \left(1 + \frac{1}{n(\log n)^\alpha}\right) \qquad (1 < \alpha < 2).$$

Then $\rho(f_\alpha) = 1$, $\underline{\text{gen}}\ f_\alpha = 0$, and $\underline{\text{gen}}\ f_\alpha' = 0$. On the other hand,

$$A \neq 0 \Rightarrow \underline{\text{gen}}(f_\alpha - A) = 1$$

$$\Rightarrow$$

$$\underline{\text{gen}}(f_\alpha - A)' = \underline{\text{gen}}\ f_\alpha' = 0.$$

If f is a nonconstant real entire function, then the zeros of f are either real or, if nonreal, occur in conjugate pairs $(z_0, \bar{z}_0)$.

N.B. The multiplicity of z_0 is the same as the multiplicity of $\bar{z}_0$.

8.22 LEMMA If f is a nonconstant real polynomial, then the number of nonreal zeros of f' counted according to multiplicity is $\leq$ the number of nonreal zeros of f counted according to multiplicity.

PROOF Suppose that the degree of f is n, the number of real zeros of f counted according to multiplicity is r, and the number of nonreal zeros of f counted according to multiplicity is $n - r$, then for f' they are $= n - 1$, $\geq r - 1$ (cf. 8.12), and $\leq n - 1 - (r - 1) = n - r$.

Let f be a nonconstant real entire function of finite order ρ and suppose that f has $0 \leq C = 2D < \infty$ nonreal zeros counted according to multiplicity -- then f' has $0 \leq C' = 2D' \leq C = 2D < \infty$ nonreal zeros counted according to multiplicity (see 8.24 below).

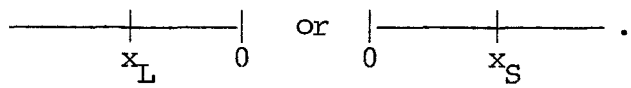
Extra Zeros This refers to f' and there are two kinds.

- If $\zeta' < \zeta''$ are consecutive real zeros of f, then by Rolle's theorem, f' has an odd number of zeros in $]\zeta',\zeta''[$ counted according to multiplicity, say $2k + 1$. One then says that f' has 2k extra zeros between ζ' and ζ''.

- If f has a largest real zero x_L or a smallest real zero x_S, then any zero of f' in $]x_L,\infty[$ or $]-\infty,x_S[$ is called extra and will be counted according to multiplicity.

Let E' denote the total number of extra zeros of f'.

8.23 EXAMPLE Take for f a canonical product whose zeros are real (cf. 8.18) -- then it might be that 0 is extra as in



8.24 THEOREM[†] Under the preceding assumptions on f,

$$E' + C' \leq C + \underline{\text{gen}}\ f,$$

and

$$\underline{\text{gen}}\ f = \underline{\text{gen}}\ f'.$$

8.25 SCHOLIUM If f is a canonical product whose zeros are real, then $E' \leq \mathfrak{g}$ (cf. 8.18).

[Note: As a special case, if f is a polynomial and if the zeros of f are real, then $E' = 0$ (the critical points guaranteed by Rolle's theorem are simple (cf. 8.16)).]

8.26 EXAMPLE Take

$$f(z) = (z + 1)\exp \frac{z^2}{2}.$$

It has one real zero, viz. $z = -1$, and its derivative

$$f'(z) = (1 + z + z^2)\exp \frac{z^2}{2}$$

has two nonreal zeros, viz.

$$z = \frac{-1 \pm \sqrt{-3}}{2}.$$

[†] E. Borel, *Lecons sur les Fonctions Entières*, Gauthier-Villars, 1900, pp. 37-47.

Here

$$\begin{bmatrix} E' = 0 \\ C' = 2 \end{bmatrix}, \quad \begin{bmatrix} C = 0 \\ \underline{\text{gen}}\ f = 2. \end{bmatrix}$$

8.27 EXAMPLE Take

$$f(z) = (z^2 - 4)\exp \frac{z^2}{3}.$$

It has two real zeros, viz. $z = \pm 2$, and its derivative

$$f'(z) = \frac{2}{3} z(z^2 - 1)\exp \frac{z^2}{3}$$

has three real zeros, viz. $z = -1, 0, 1$. Here

$$\begin{bmatrix} E' = 2 \\ C' = 0 \end{bmatrix}, \quad \begin{bmatrix} C = 0 \\ \underline{\text{gen}}\ f = 2. \end{bmatrix}$$

[Note: The three zeros between -2 and 2 are per Rolle and $3 = 2 + 1$, so $E' = 2$.]

8.28 EXAMPLE Take

$$f(z) = (z^2 - 1)e^z.$$

It has two real zeros, viz. $z = \pm 1$, and its derivative

$$f'(z) = (z^2 + 2z - 1)e^z$$

has two real zeros, viz. $z = -1 \pm \sqrt{2}$. Here

$$\begin{bmatrix} E' = 1 \\ C' = 0 \end{bmatrix}, \quad \begin{bmatrix} C = 0 \\ \underline{\text{gen}}\ f = 1. \end{bmatrix}$$

[Note: The zero $-1 + \sqrt{2}$ lies between -1 and 1 and is per Rolle but the zero $-1 - \sqrt{2}$ lies to the left of -1, hence is extra.]

8.29 REMARK If f is a nonconstant real polynomial, then

$$E' + C' = \begin{cases} C & \text{if } \deg f > C \\ C - 1 & \text{if } \deg f = C. \end{cases}$$

[Note: In particular, $C' \leq C$ (cf. 8.22).]

8.30 THEOREM Let f be a nonconstant real entire function of finite order ρ. Assume: The zeros of f are real and $\underline{\text{gen}}\ f = 0$ or 1 -- then the zeros of f' are real and

$$\underline{\text{gen}}\ f = \underline{\text{gen}}\ f'.$$

PROOF In this situation,

$$E' + C' \leq \underline{\text{gen}}\ f \quad \text{(cf. 8.24)},$$

so

$$\underline{\text{gen}}\ f = 0 \Rightarrow C' = 0.$$

And

$$\underline{\text{gen}}\ f = 1 \Rightarrow E' + C' \leq 1$$
$$\Rightarrow C' \leq 1.$$

But C' is even. Therefore $C' = 0$ (although E' might be 1 (cf. 8.28)).

[Note: It follows that f' satisfies the same general conditions as f.]

§9. *JENSEN CIRCLES*

We begin with a computation.

9.1 LEMMA Let $c = a + \sqrt{-1}\, b$ -- then $\forall\, z = x + \sqrt{-1}\, y$,

$$\text{Im}\left[\frac{1}{z - c} + \frac{1}{z - \bar{c}}\right]$$

$$= -\text{Im}\left[\frac{\bar{z} - \bar{c}}{|z - c|^2} + \frac{\bar{z} - c}{|z - \bar{c}|^2}\right]$$

$$= -\text{Im}\left[\frac{(z - c)(z - \bar{c})(\bar{z} - c) + (z - \bar{c})(z - c)(\bar{z} - \bar{c})}{|z - c|^2\, |z - \bar{c}|^2}\right]$$

$$= -2\text{Im}\left[\frac{(z - c)(z - \bar{c})(\bar{z} - a)}{|z - c|^2\, |z - \bar{c}|^2}\right]$$

$$= -2\text{Im}\left[\frac{(z - a - \sqrt{-1}\, b)(z - a + \sqrt{-1}\, b)(\bar{z} - a)}{|z - c|^2\, |z - \bar{c}|^2}\right]$$

$$= -2y\, \frac{|z - a|^2 - b^2}{|z - c|^2\, |z - \bar{c}|^2}$$

$$= -2y\, \frac{(x - a)^2 + y^2 - b^2}{|z - c|^2\, |z - \bar{c}|^2}.$$

Given a real polynomial f, denote by $z_1,\ldots,z_\ell$ those zeros of f which lie in the open upper half-plane.

9.2 DEFINITION Put

$$\mathfrak{C}_j = \{z \in \mathbb{C}: |z - \operatorname{Re} z_j| \leq \operatorname{Im} z_j \ (j = 1,\ldots,\ell)\}.$$

Then the $\mathfrak{C}_j$ are called the Jensen circles of f.

[Note: The line segment joining the pair z_j, $\bar{z}_j$ is the vertical diameter of $\mathfrak{C}_j$.]

9.3 THEOREM Let f be a real polynomial -- then the nonreal critical points of f lie in the union

$$\bigcup_{j=1}^{\ell} \mathfrak{C}_j$$

of the Jensen circles of f.

PROOF Take f monic of degree n, so

$$f(z) = \prod_{i=1}^{k} (z - z_i)^{m_i}$$

$$= \prod_{\operatorname{Im} z_i = 0} (z - z_i)^{m_i} \cdot \prod_{\operatorname{Im} z_i > 0} (z - z_i)^{m_i} (z - \bar{z}_i)^{m_i}$$

$$= \prod_{\operatorname{Im} z_i = 0} (z - z_i)^{m_i} \cdot \prod_{j=1}^{\ell} (z - z_j)^{m_j} (z - \bar{z}_j)^{m_j}.$$

Since the only issue is the position of the critical points of the second kind, pass to

$$\frac{f'(z)}{f(z)} = \sum_{\operatorname{Im} z_i = 0} \frac{m_i}{z - z_i} + \sum_{j=1}^{\ell} m_j \left[\frac{1}{z - z_j} + \frac{1}{z - \bar{z}_j} \right].$$

Write

$$z = x + \sqrt{-1}\, y \text{ and } z_j = x_j + \sqrt{-1}\, y_j \quad (j = 1,\ldots,\ell).$$

Then

$$\operatorname{Im}\frac{f'(z)}{f(z)} = -y\left[\sum_{\operatorname{Im} z_i=0}\frac{m_i}{|z-z_i|^2}\right.$$

$$\left.+\,2\sum_{j=1}^{\ell} m_j\,\frac{(x-x_j)^2+y^2-y_j^2}{|z-z_j|^2\,|z-\bar{z}_j|^2}\qquad(\text{cf. } 9.1)\right].$$

To say that $z \in \mathfrak{C}_j$ means that

$$|x+\sqrt{-1}\,y-x_j| \le y_j$$

or still, that

$$(x-x_j)^2+y^2 \le y_j^2.$$

Therefore

$$z \notin \mathfrak{C}_j \Rightarrow (x-x_j)^2+y^2-y_j^2 > 0.$$

Accordingly, outside the union of the $\mathfrak{C}_j$, at a z with $y \neq 0$, we have

$$\operatorname{syn}\operatorname{Im}\frac{f'(z)}{f(z)} = -\operatorname{sgn} y \neq 0$$

$$\Rightarrow$$

$$f'(z) \neq 0.$$

Inspection of the preceding proof then leads to the following conclusion.

9.4 SCHOLIUM A nonreal critical point of the second kind lies in the interior of at least one of the Jensen circles of f unless it is a boundary point of each of them (in which case f has no real zeros).

9.5 LEMMA Let x_0 be a point on the real line lying outside all the Jensen circles of f. Assume: $f(x_0) = 0$ -- then in each of the half-planes

$$\begin{cases} \{z \in C: \mathrm{Re}\, z < x_0\} \\ \{z \in C: \mathrm{Re}\, z > x_0\}, \end{cases}$$

the number of zeros is the same as the number of critical points.

9.6 LEMMA Ket x_0 be a point on the real line lying outside all the Jensen circles of f. Assume: $f(x_0) \neq 0$ -- then in each of the half-planes

$$\begin{cases} \{z \in C: \mathrm{Re}\, z < x_0\} \\ \{z \in C: \mathrm{Re}\, z > x_0\}, \end{cases}$$

the number of zeros is at least as large as the number of critical points (but can exceed it by at most one).

9.7 THEOREM Let $a < b$ be two real numbers lying outside all the Jensen circles of f. Denote by M the number of zeros and by M' the number of critical points in the strip

$$\{z \in C: a < \mathrm{Re}\, z < b\}.$$

Then

- $f(a) = 0$ and $f(b) = 0 \Rightarrow M' = M + 1$.
- $f(a) = 0$ or $f(b) = 0 \Rightarrow M \leq M' \leq M + 1$.
- $f(a) \neq 0$ and $f(b) \neq 0 \Rightarrow M - 1 \leq M' \leq M + 1$.

9.8 EXAMPLE The assumption that a and b lie outside all the Jensen circles of f cannot be dropped.

[Take

$$f(z) = z^4 + 4$$

and let

$$\begin{cases} a = -1 \\ b = 1, \end{cases} \quad \text{so} \quad \begin{cases} f(a) \neq 0 \\ f(b) \neq 0. \end{cases}$$

Then $M = 0$ but $M' = 3$.]

§10. *CLASSES OF ENTIRE FUNCTIONS*

Let T be a nonempty closed subset of C.

10.1 DEFINITION A <u>T-polynomial</u> is a polynomial whose zeros are in T.

10.2 DEFINITION A <u>T-function</u> is an entire function $\not\equiv 0$ which is the uniform limit on compact subsets of C of a sequence of T-polynomials.

10.3 NOTATION Let

$$\text{ent}(T)$$

stand for the class of T-functions.

<u>N.B.</u> The product of two T-functions is a T-function.

10.4 LEMMA If $f \in \text{ent}(T)$, then all its zeros lie in T.

[Note: As will be seen below (cf. 10.14), the converse to this assertion is false: An entire function whose zeros are in T need not belong to ent(T).]

10.5 LEMMA If T is bounded, then ent(T) is the set of T-polynomials.

PROOF Let $f \in \text{ent}(T)$ and suppose that $f_n \to f$ uniformly on compact subsets of C, where $\{f_n\}$ is a sequence of T-polynomials. Since all the zeros of f lie in T and since T is bounded, their number is finite, call if N. By Rouché's theorem, the number of zeros of f_n is also N provided $n >> 0$, thus the f_n are of degree N provided $n >> 0$. But the Taylor coefficients of f are the limits of the Taylor coefficients of the f_n, hence f is a polynomial of degree N.

Abstractly, the problem then is to characterize ent(T) in terms of the properties of T. This can be done (more or less) but instead of delving into the general theory, we shall consider only those special cases that will be needed later on, namely:

$$\left[\begin{array}{l} T =]-\infty,0] \text{ or } [0,+\infty[\\ \\ T =]-\infty,+\infty[\end{array}\right. \quad \text{subject to the restriction that here}$$

"T-polynomials" and "T-functions" are real (so, e.g., $\sqrt{-1}\,(z^2 - 1)$ is not a T-polynomial even though its zeros are real).

10.6 LEMMA We have

$$\left[\begin{array}{l} \text{ent}(]-\infty,0]) \\ \\ \text{ent}([0,+\infty[) \end{array}\right. \subset \text{ent}(]-\infty,+\infty[).$$

[This is obvious.]

10.7 EXAMPLE If $f = C$ $(C \neq 0)$, then $f \in \text{ent}([0,+\infty[)$.

[Consider

$$C(1 - \frac{z}{k}) \qquad (k = 1,2,\ldots).]$$

10.8 EXAMPLE Since

$$e^{-z} = \lim_{n\to\infty} (1 - \frac{z}{n})^n,$$

it follows that

$$e^{-z} \in \text{ent}([0,+\infty[).$$

10.9 EXAMPLE The zeros of

$$(1 - \frac{z^2}{n^2})$$

are $z = \pm n$, so

$$\prod_{n=1}^{N} (1 - \frac{z^2}{n^2}) \in \text{ent}(]-\infty, +\infty[),$$

which implies that

$$\frac{\sin \pi z}{\pi z} \in \text{ent}(]-\infty, +\infty[) \quad \text{(cf. 1.23).}$$

10.10 EXAMPLE The zeros of the Laguerre polynomials (cf. 8.17) are real and positive, hence $\forall$ n,

$$L_n \in \text{ent}([0, +\infty[).$$

Consider now the Bessel function of index 0:

$$J_0(z) = 1 - \frac{1}{1!1!} (\frac{z}{2})^2 + \frac{1}{2!2!} (\frac{z}{2})^4 - \frac{1}{3!3!} (\frac{z}{2})^6 + \cdots .$$

Then

$$J_0(z) = \lim_{n \to \infty} L_n(\frac{z^2}{4n})$$

uniformly on compact subsets of $\mathbf{C}$, thus

$$J_0(z) \in \text{ent}([0, +\infty[).$$

[In fact,

$$L_n(\frac{z^2}{4n}) = 1 - \frac{z^2}{2\cdot 2} + \frac{z^4}{2\cdot 4\cdot 2\cdot 4} (1 - \frac{1}{n}) - \frac{z^6}{2\cdot 4\cdot 6\cdot 2\cdot 4\cdot 6} (1 - \frac{1}{n})(1 - \frac{2}{n}) + \cdots .]$$

10.11 THEOREM Let $f \not\equiv 0$ be a real entire function -- then $f \in \text{ent}([0, +\infty[)$ iff f has a representation of the form

$$f(z) = Cz^m e^{az} \prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n}),$$

where $C \neq 0$ is real, m is a nonnegative integer, a is real and ≤ 0, the λ_n are real and > 0 with $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$.

[Note: Functions having finitely many zeros are accommodated by the convention that $\lambda_n = \infty$ and $0 = \frac{1}{\lambda_n}$ $(n \geq n_0)$ and an empty product is taken to be 1.]

10.12 REMARK ent([0, + ∞[) is closed under differentiation (cf. 8.16).

10.13 REMARK Let $f \in$ ent([0, + ∞[) -- then $\mathfrak{g} = 0$, so

$$\underline{\text{gen}}\ f = \begin{cases} 0 \text{ if } a = 0 \\ 1 \text{ if } a \neq 0 \end{cases}$$

and $\rho \leq 1$.

10.14 EXAMPLE The real entire function

$$e^{-z^2} \prod_{n=1}^{\infty} (1 - \frac{z}{n^2})$$

has its zeros in [0, + ∞[but does not belong to ent([0, + ∞[).

That the conditions of 10.11 are necessary is straightforward: Consider

$$p_k(z) = C(1 - \frac{z}{k})(z - \frac{1}{k})^m (1 + \frac{az}{k})^k \prod_{n=1}^{k} (1 - \frac{z}{\lambda_n}).$$

This said, suppose now that $f \in$ ent([0, + ∞[) and write

$$f(z) = a_0 - a_1 z + a_2 z^2 - \cdots .$$

Let

$$p_k(z) = a_{k0} - a_{k1} z + a_{k2} z^2 - \cdots + (-1)^k a_{kk} z^k$$

be a sequence of polynomials whose zeros are real and positive such that $p_k \to f$

uniformly on compact subsets of C -- then

$$\lim_{k \to \infty} a_{k\ell} = a_\ell .$$

10.15 REDUCTION There is no loss of generality in assuming that $a_0 \neq 0$.

[Fix a positive real number α which is smaller than the smallest positive zero of f (cf. 10.4), pass to $f(z + \alpha)$, and note that $f(\alpha) \neq 0$.]

Therefore one can work instead with

$$\frac{f(z)}{a_0}, \frac{p_k(z)}{a_{k0}} \quad (\text{since } \lim_{k \to \infty} a_{k0} = a_0 \neq 0).$$

So, recast,

$$f(z) = 1 - a_1 z + a_2 z^2 - \cdots$$

and

$$p_k(z) = 1 - a_{k1} z + a_{k2} z^2 - \cdots + (-1)^k a_{kk} z^k$$

$$\equiv (1 - \frac{z}{\lambda_{k1}})(1 - \frac{z}{\lambda_{k2}}) \cdots (1 - \frac{z}{\lambda_{kk}}),$$

where the zeros $\lambda_{k\ell} \neq 0$ are positive and

$$0 < \lambda_{k1} \leq \lambda_{k2} \leq \cdots \leq \lambda_{kk}.$$

<u>N.B.</u> The a_k and the $a_{k\ell}$ are nonnegative.

10.16 LEMMA[†] Let

$$\Phi(z) = 1 - c_1 z + c_2 z^2 - \cdots + (-1)^n c_n z^n$$

[†] O. Schlömilch, *Zeitschr. f. Math. und Physik* 3 (1858), pp. 301-308 (see page 308, formula 15).

be a real polynomial whose zeros are real and positive -- then

$$\frac{c_1}{n} \geq \left[\frac{c_2}{\binom{n}{2}}\right]^{1/2} \geq \cdots \geq \left[\frac{c_p}{\binom{n}{p}}\right]^{1/p} \geq \cdots \geq (c_n)^{1/n}.$$

Take $\Phi = p_k$, thus

$$\frac{a_{k1}}{k} \geq \left[\frac{a_{k\ell}}{\binom{k}{\ell}}\right]^{1/\ell}$$

$$\Rightarrow (a_{k1})^{\ell} \frac{k(k-1)\cdots(k-\ell+1)}{k^{\ell}} \frac{1}{\ell!} \geq a_{k\ell},$$

so in the limit as $k \to \infty$,

$$\frac{(a_1)^{\ell}}{\ell!} \geq a_{\ell}.$$

10.17 LEMMA f is of finite order $\rho \leq 1$.

PROOF In fact,

$$\begin{aligned} |f(z)| &\leq \sum_{\ell=0}^{\infty} a_{\ell}|z|^{\ell} \\ &\leq \sum_{\ell=0}^{\infty} \frac{(a_1)^{\ell}}{\ell!} |z|^{\ell} \\ &= \exp(a_1|z|) \end{aligned}$$

$$\Rightarrow$$

$$M(r;f) \leq \exp a_1 r,$$

from which the assertion (cf. 2.15).

Enumerate the zeros of f in the usual way:

$$0 < \lambda_1 \le \lambda_2 \le \cdots.$$

Then

$$\lim_{k \to \infty} \lambda_{k\ell} = \lambda_\ell.$$

But

$$a_{k1} = \frac{1}{\lambda_{k1}} + \frac{1}{\lambda_{k2}} + \cdots + \frac{1}{\lambda_{kk}}$$

$$\ge \frac{1}{\lambda_{k1}} + \frac{1}{\lambda_{k2}} + \cdots + \frac{1}{\lambda_{k\ell}}$$

=>

$$a_1 = \lim_{k \to \infty} a_{k1}$$

$$\ge \lim_{k \to \infty} \left(\frac{1}{\lambda_{k1}} + \frac{1}{\lambda_{k2}} + \cdots + \frac{1}{\lambda_{k\ell}}\right)$$

$$= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_\ell}.$$

Therefore the series $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots$ converges and

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} \le a_1.$$

Proceeding, write

$$f(z) = e^{Q(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) \qquad \text{(cf. 7.3)},$$

where $q \le \rho \le 1$ and $\mathfrak{g} = 0$, hence

$$\underline{\text{gen}}\ f = \max(q,\mathfrak{g}) = q.$$

And

$$Q(z) = az + b,$$

the final claim being that a is real and ≤ 0.

[Note: $1 = f(0) = e^b \prod_{n=1}^{\infty} 1 = e^b$.]

However

$$1 - a_1 z + \cdots = (1 + az + \cdots)(1 - (\sum_{n=1}^{\infty} \frac{1}{\lambda_n})z + \cdots)$$

=>

$$- a_1 = a - \sum_{n=1}^{\infty} \frac{1}{\lambda_n}$$

=>

$$a = - a_1 + \sum_{n=1}^{\infty} \frac{1}{\lambda_n}$$

$$\leq 0,$$

thereby completing the proof of 10.11.

10.18 REMARK The fact that f is of finite order $\rho \leq 1$ was established by appealing to 10.16. This can be avoided. Indeed, $\{a_{k1} : k = 1,2,\ldots\}$ converges to a_1, hence is bounded, say $0 \leq a_{k1} \leq M$, hence

$$| p_k(z) | \leq \sum_{\ell=1}^{k} |1 - \frac{z}{\lambda_{k\ell}}|$$

$$\leq \prod_{\ell=1}^{k} (1 + \frac{|z|}{\lambda_{k\ell}})$$

$$\leq \exp(|z| \sum_{\ell=1}^{k} \frac{1}{\lambda_{k\ell}})$$

$$\leq \exp(|z|\, a_{k1})$$

$$\leq \exp(M|z|).$$

And then

$$|f(z)| = \lim_{k \to \infty} |p_k(z)| \leq \exp(M|z|).$$

10.19 THEOREM Let $f \not\equiv 0$ be a real entire function -- then $f \in \text{ent}(]-\infty, +\infty[)$ iff f has a representation of the form

$$f(z) = Cz^m e^{az^2+bz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) e^{z/\lambda_n},$$

where $C \neq 0$ is real, m is a nonnegative integer, a is real and ≤ 0, b is real, the λ_n are real with $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty$.

[Note: Functions having finitely many zeros are accommodated by the convention that $\lambda_n = \infty$ and $0 = \frac{1}{\lambda_n}$ $(n \geq n_0)$ and an empty product is taken to be 1.]

10.20 REMARK $\text{ent}(]-\infty, +\infty[)$ is closed under differentiation (cf. 8.16).

10.21 REMARK Let $f \in \text{ent}(]-\infty, +\infty[)$.

- $\mathfrak{g} = 0 \Rightarrow \underline{\text{gen}}\ f = 0,1,2$
- $\mathfrak{g} = 1 \Rightarrow \underline{\text{gen}}\ f = 1,2.$

To see that the conditions of 10.19 are necessary, introduce

$$\Lambda_k = b + \sum_{n=1}^{k} \frac{1}{\lambda_n}$$

and let

$$p_k(z) = C(1 - \frac{z}{k})(z - \frac{1}{k})^m (1 + \frac{az^2}{k})^k (1 + \frac{\Lambda_k z}{n_k})^{n_k} \prod_{n=1}^{k} (1 - \frac{z}{\lambda_n}),$$

where the $n_k \to \infty$ $(k \to \infty)$ are chosen subject to

$$|z| \le k \Rightarrow \left| (1 + \frac{\Lambda_k z}{n_k})^{n_k} - e^{\Lambda_k z} \right|$$

$$< \frac{1}{k} \exp(-k \sum_{n=1}^{k} \frac{1}{|\lambda_n|}).$$

Turning to the sufficiency, let $f \in \text{ent}(]-\infty, +\infty[)$ and normalize the situation so that as before

$$f(z) = 1 - a_1 z + a_2 z^2 - \cdots$$

and

$$p_k(z) = 1 - a_{k1} z + a_{k2} z^2 - \cdots + (-1)^k a_{kk} z^k$$

$$\equiv (1 - \frac{z}{\lambda_{k1}})(1 - \frac{z}{\lambda_{k2}}) \cdots (1 - \frac{z}{\lambda_{kk}}),$$

where the zeros $\lambda_{k\ell} \neq 0$ are real and

$$0 < |\lambda_{k1}| \le |\lambda_{k2}| \le \cdots \le |\lambda_{kk}|.$$

10.22 SUBLEMMA $\forall$ complex z,

$$|(1 + z)e^{-z}| \le e^{4|z|^2}.$$

PROOF If $|z| \le \frac{1}{2}$, then

$$|(1 + z)e^{-z}| \le e^{|z|^2} \le e^{4|z|^2}.$$

On the other hand, if $|z| \geq \frac{1}{2}$, then

$$|(1 + z)e^{-z}| \leq (1 + |z|)e^{|z|}$$

$$\leq e^{2|z|} \leq e^{4|z|^2}.$$

From the definitions,

$$a_1 = \lim_{k \to \infty} a_{k1} = \lim_{k \to \infty} \sum_{\ell=1}^{k} \frac{1}{\lambda_{k\ell}} .$$

Next

$$a_{k2} = \sum_{i<j} \frac{1}{\lambda_{ki}} \frac{1}{\lambda_{kj}}$$

$$= \frac{1}{2} \sum_{i \neq j} \frac{1}{\lambda_{ki}} \frac{1}{\lambda_{kj}} .$$

But

$$\left(\sum_{i=1}^{k} \frac{1}{\lambda_{ki}}\right)\left(\sum_{j=1}^{k} \frac{1}{\lambda_{kj}}\right)$$

$$= \sum_{\ell=1}^{k} \frac{1}{\lambda_{k\ell}^2} + \sum_{i \neq j} \frac{1}{\lambda_{ki}} \frac{1}{\lambda_{kj}}$$

$$= \sum_{\ell=1}^{k} \frac{1}{\lambda_{k\ell}^2} + 2 \sum_{i<j} \frac{1}{\lambda_{ki}} \frac{1}{\lambda_{kj}} .$$

So, upon letting $k \to \infty$, we get

$$a_1^2 = \lim_{k \to \infty} \sum_{\ell=1}^{k} \frac{1}{\lambda_{k\ell}^2} + 2a_2$$

or still,

$$a_1^2 - 2a_2 = \lim_{k \to \infty} \sum_{\ell=1}^{k} \frac{1}{\lambda_{k\ell}^2} .$$

Fix constants $\begin{cases} U > 0 \\ V > 0 \end{cases}$ such that $\forall$ k,

$$\begin{cases} \left| \sum_{\ell=1}^{k} \frac{1}{\lambda_{k\ell}} \right| \leq U \\ \\ \sum_{\ell=1}^{k} \frac{1}{\lambda_{k\ell}^2} \leq V. \end{cases}$$

10.23 LEMMA We have

$$|p_k(z)| \leq \exp(U|z| + 4V|z|^2).$$

PROOF Write

$$|p_k(z)e^{a_{k1}z}| = |p_k(z)\exp(\sum_{\ell=1}^{k} \frac{z}{\lambda_{k\ell}})|$$

$$= |\prod_{\ell=1}^{k} (1 - \frac{z}{\lambda_{k\ell}})\exp(\frac{z}{\lambda_{k\ell}})|$$

$$\leq \prod_{\ell=1}^{k} |(1 - \frac{z}{\lambda_{k\ell}})\exp(\frac{z}{\lambda_{k\ell}})|$$

$$\leq \prod_{\ell=1}^{k} \exp(4|\frac{z}{\lambda_{k\ell}}|^2) \qquad \text{(cf. 10.22)}$$

$$\leq \exp(4(\sum_{\ell=1}^{k} \frac{1}{\lambda_{k\ell}^2})|z|^2)$$

$$\leq \exp(4V|z|^2).$$

Therefore

$$|p_k(z)| = |p_k(z)e^{a_{k1}z}\, e^{-a_{k1}z}|$$

$$\leq |p_k(z)e^{a_{k1}z}|\,|e^{-a_{k1}z}|$$

$$\leq \exp(4V|z|^2)\exp(|a_{k1}|\,|z|)$$

$$\leq \exp(U|z| + 4V|z|^2).$$

Consequently, f is of finite order $\rho \leq 2$ (cf. 10.18).

10.24 LEMMA If $\lambda_1, \lambda_2, \ldots$ are the zeros of f and if

$$0 \leq |\lambda_1| \leq |\lambda_2| \leq \cdots,$$

then

$$\lim_{k \to \infty} \lambda_{k\ell} = \lambda_\ell$$

and

$$a_1^2 - 2a_2 \geq \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2}.$$

PROOF Start by writing

$$\frac{1}{\lambda_{k1}^2} + \frac{1}{\lambda_{k2}^2} + \cdots + \frac{1}{\lambda_{kk}^2}$$

$$\geq \frac{1}{\lambda_{k1}^2} + \frac{1}{\lambda_{k2}^2} + \cdots + \frac{1}{\lambda_{k\ell}^2}$$

and then let $k \to \infty$, hence

$$a_1^2 - 2a_2 = \lim_{k \to \infty} \left(\sum_{\ell=1}^{k} \frac{1}{\lambda_{k\ell}^2} \right)$$

$$\geq \lim_{k \to \infty} \left(\frac{1}{\lambda_{k1}^2} + \frac{1}{\lambda_{k2}^2} + \cdots + \frac{1}{\lambda_{k\ell}^2} \right)$$

$$= \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \cdots + \frac{1}{\lambda_\ell^2},$$

which implies that

$$a_1^2 - 2a_2 \geq \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2}.$$

Accordingly,

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty \quad (\Rightarrow g = 0 \text{ or } 1)$$

and the product

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) e^{z/\lambda_n}$$

is an entire function whose zeros are the λ_n (cf. 5.4). To see that its order is also ≤ 2, write

$$\left| \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) e^{z/\lambda_n} \right|$$

$$\leq \prod_{n=1}^{\infty} \left| \left(1 - \frac{z}{\lambda_n}\right) e^{z/\lambda_n} \right|$$

$$\leq \prod_{n=1}^{\infty} \exp\left(4 \frac{|z|^2}{\lambda_n^2}\right) \quad \text{(cf. 10.22)}$$

$$\leq \exp\left(4\left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2}\right)|z|^2\right).$$

Thanks to 2.37, the order of

$$\frac{f(z)}{\prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n})e^{z/\lambda_n}}$$

is $\leq$ the maximum of ρ and the order of

$$\prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n})e^{z/\lambda_n},$$

thus is ≤ 2, so

$$\frac{f(z)}{\prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n})e^{z/\lambda_n}} = e^{Q(z)},$$

where

$$Q(z) = az^2 + bz + c$$

is a polynomial of degree ≤ 2 (cf. 2.42).

[Note: $1 = f(0) = e^c \prod_{n=1}^{\infty} 1 = e^c$.]

There remain the claims that (1) b is real and (2) a is real and ≤ 0. To this end, compare coefficients:

(1) $b = -a_1 = \lim_{k \to \infty} a_{k1}$, which is real.

(2) $a = -\frac{1}{2}\left(a_1^2 - 2a_2 - \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2}\right)$

and

$$a_1^2 - 2a_2 - \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \geq 0 \quad \text{(cf. 10.24)}.$$

The proof of 10.19 is therefore complete.

N.B. Take an $f \in \text{ent}([0, +\infty[)$ and write

$$f(z) = Cz^m e^{az} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) \qquad \text{(cf. 10.11)}.$$

Then since the λ_n are real and > 0 with $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$, we have

$$f(z) = Cz^m \exp\left(\left(a - \sum_{n=1}^{\infty} \frac{1}{\lambda_n}\right) z\right) \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) e^{z/\lambda_n}$$

and $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty$.

10.25 DEFINITION The Laguerre-Polya class of entire functions is comprised of the elements of $\text{ent}(]-\infty, +\infty[)$.

10.26 DEFINITION The type I Laguerre-Polya class of entire functions is comprised of the elements of

$$\text{ent}(]-\infty, 0]) \cup \text{ent}([0, +\infty[).$$

10.27 DEFINITION The type II Laguerre-Polya class of entire functions is comprised of the elements of $\text{ent}(]-\infty, +\infty[)$ which are not type I.

10.28 NOTATION $\mathcal{L} - \mathcal{P}$, $\text{I} - \mathcal{L} - \mathcal{P}$, $\text{II} - \mathcal{L} - \mathcal{P}$.

10.29 EXAMPLE Let p be a real polynomial with real zeros only.

- If all the nonzero zeros of p are either positive or negative, then $p \in \text{I} - \mathcal{L} - \mathcal{P}$.
- If p has both positive and negative zeros, then $p \in \text{II} - \mathcal{L} - \mathcal{P}$.

10.30 EXAMPLE The function

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} (1 + \frac{z}{n}) \exp(- \frac{z}{n})$$

is in II - L - P (cf. 1.30).

Given $A \geq 0$ $(A < \infty)$, put

$$S(A) = \{z: |\text{Im } z| \leq A\}.$$

10.31 NOTATION A - L - P stands for the class of real entire functions $f \not\equiv 0$ that have a representation of the form

$$f(z) = Cz^m e^{az^2+bz} \prod_{n=1}^{\infty} (1 - \frac{z}{z_n}) e^{z/z_n},$$

where $C \neq 0$ is real, m is a nonnegative integer, a is real and ≤ 0, b is real, the $z_n \in S(A) - \{0\}$ with $\sum_{n=1}^{\infty} \frac{1}{|z_n|^2} < \infty$.

[Note: Therefore

$$0 - L - P = L - P.]$$

10.32 THEOREM $f \in A - L - P$ iff f is the uniform limit on compact subsets of $\mathbf{C}$ of a sequence of real polynomials whose only zeros are in S(A).

10.33 REMARK Take T = S(A) -- then

$$A - L - P \subset \text{ent}(S(A)),$$

the containment being proper if $A > 0$.

[Note: It is possible to characterize ent(S(A)) but we shall omit the details as they will not be needed.]

10.34 EXAMPLE The real polynomial $z(z^2 + 1)$ belongs to $1 - L - P$.

10.35 LEMMA $A - L - P$ is closed under differentiation.

[This is because S(A) is convex, so 8.3 is applicable.]

10.36 NOTATION Denote by

$$* - L - P$$

the class of real entire functions of the form

$$\varphi(z) = p(z)f(z),$$

where p is a real polynomial and $f \in L - P$.

10.37 LEMMA $\varphi \in * - L - P$ iff $\varphi \in A - L - P$ for some A and φ has at most a finite number of nonreal zeros.

10.38 LEMMA $* - L - P$ is closed under differentiation.

PROOF Take a $\varphi \in * - L - P$ and fix an $A:\varphi \in A - L - P$ -- then $\varphi' \in A - L - P$ (cf. 10.35) and has at most a finite number of nonreal zeros (cf. 8.24).

Let $\varphi \in * - L - P$ and suppose that $a \pm \sqrt{-1}\, b$ is a pair of conjugate nonreal zeros of φ.

10.39 DEFINITION Given $k \geq 1$, the ellipse whose minor axis has $a + \sqrt{-1}\, b$ and $a - \sqrt{-1}\, b$ as endpoints and whose major axis has length $2b\sqrt{k}$ is called the <u>Jensen ellipse</u> of order k of φ.

The notion of "Jensen ellipse" generalizes that of "Jensen circle" (in the context of a real polynomial) and the proof of the following result is a computation similar to that used in 9.3.

10.40 THEOREM Let $\varphi \in * - L - P$ -- then every nonreal zero of $\varphi^{(k)}$ lies in the union of the Jensen ellipses of order k of φ.

[Note: Restated, if $a_j \pm \sqrt{-1}\, b_j$ $(j = 1,\ldots,d)$ are the nonreal zeros of φ and if $z = x + \sqrt{-1}\, y$ is a nonreal zero of $\varphi^{(k)}$, then for some j,

$$\frac{(x - a_j)^2}{k} + y^2 \leq b_j^2.]$$

The symbols C, C', E' employed in 8.24 make sense in the present setting (replace the "f" there by the "φ" here). Therefore

$$E' + C' \leq C + \underline{\text{gen}}\ \varphi$$

and

$$\underline{\text{gen}}\ \varphi = \underline{\text{gen}}\ \varphi'.$$

10.41 LEMMA Let $\varphi \in * - L - P$ -- then $C' \leq C$ (cf. 8.22).

§11. *DERIVATIVES*

11.1 DEFINITION An entire function φ is said to be of growth (2,A) $(0 \leq A < \infty)$ if its order is < 2 or is of order 2 with type not exceeding A.

Denote by

$$\text{ent}(2,A)$$

the class of entire functions of growth (2,A) -- then

$$A < A' \Rightarrow \text{ent}(2,A) \subset \text{ent}(A,A').$$

In particular:

$$\text{ent}(2,0) \subset \text{ent}(2,A).$$

11.2 LEMMA The class ent(2,A) is closed under differentiation (cf. 2.25 and 3.7).

N.B. If $\varphi \in \text{ent}(2,A)$, then for every $a > A$,

$$M(r;\varphi) < e^{ar^2} \quad (r >> 0).$$

We shall now establish some technicalities that will be needed for the proof of the main result (viz. 11.9 infra).

11.3 NOTATION Given positive real numbers $A > 0$, $B > 0$, let

$$C = (B + \sqrt{B^2 + 2A^{-1}})/2,$$

thus

$$2AC(C - B) = 1.$$

11.4 LEMMA If $\varphi \in \text{ent}(2,A)$, then

$$\overline{\lim_{n \to \infty}} \sqrt{n} \left[\frac{M(B\sqrt{n};\varphi^{(n)})}{n!} \right]^{1/n} \leq 2ACe^{AC^2}.$$

PROOF Take $a > A$ and let

$$c = (B + \sqrt{B^2 + 2a^{-1}})/2,$$

so that

$$2ac(c - B) = 1.$$

Determine r_0:

$$r \geq r_0 \Rightarrow M(r;\varphi) < e^{ar^2}.$$

Then for $n = 1,2,\ldots,$

$$\log \left[\frac{M(B\sqrt{n};\varphi^{(n)})}{n!} \right]^{1/n} \leq \frac{ar^2}{n} - \log(r - B\sqrt{n})$$

if $r > \max(r_0,B\sqrt{n})$. Since the RHS attains its minimum

$$\log \frac{2ace^{ac^2}}{\sqrt{n}}$$

at $r = c\sqrt{n}$, it follows that

$$\overline{\lim_{n \to \infty}} \sqrt{n} \left[\frac{M(B\sqrt{n};\varphi^{(n)})}{n!} \right]^{1/n} \leq 2ace^{ac^2}.$$

To finish, let $a \downarrow A$.

Let f be an entire function and suppose that $z_0, z_1, \ldots$ is a sequence of complex numbers such that $\forall\, n \geq 0$, $f^{(n)}(z_n) = 0$ -- then $\forall\, n > 0$,

$$f(z) = \int_{z_0}^{z} \int_{z_1}^{\zeta_1} \cdots \int_{z_{n-1}}^{\zeta_{n-1}} f^{(n)}(\zeta_n)\,d\zeta_n \cdots d\zeta_2 d\zeta_1.$$

11.5 SUBLEMMA We have

$$|f(z)| \le \frac{1}{n!} \sup_{w \in H_n} |f^{(n)}(w)| \ (|z - z_0| + |z_0 - z_1| + \cdots + |z_{n-2} - z_{n-1}|)^n,$$

where H_n is the convex hull of the set $\{z, z_0, z_1, \ldots, z_{n-1}\}$.

11.6 SUBLEMMA If $w \in H_n$, then

$$|w| \le |z| + |z - z_0| + |z_0 - z_1| + \cdots + |z_{n-2} - z_{n-1}|.$$

PROOF Let D_n be the closed disk of radius the RHS centered at the origin: $z \in D_n$. Next,

$$|z_0| \le |z| + |z - z_0| \Rightarrow z_0 \in D_n$$

$$|z_1| \le |z| + |z - z_0| + |z_0 - z_1| \Rightarrow z_1 \in D_n$$

$$\vdots$$

Therefore D_n contains $z, z_0, z_1, \ldots, z_{n-1}$, hence being convex, D_n contains w.

Accordingly, $H_n \subset D_n$, and

$$|f(z)| \le \frac{1}{n!} \sup_{w \in D_n} |f^{(n)}(w)| (|z - z_0| + |z_0 - z_1| + \cdots + |z_{n-2} - z_{n-1}|)^n.$$

11.7 LEMMA Maintaining the notation and assumptions of 11.4, suppose further that

$$2ABCe^{AC^2} < 1.$$

Impose the following conditions: $\exists$ a sequence $z_0, z_1, \ldots$ of complex numbers such that $\forall\ n \ge 0$, $\varphi^{(n)}(z_n) = 0$ and

$$\overline{\lim_{n \to \infty}} \ (|z_0 - z_1| + |z_1 - z_2| + \cdots + |z_{n-1} - z_n|)/\sqrt{n} < B.$$

Then

$$\varphi \equiv 0.$$

PROOF In fact,

$$\overline{\lim_{n \to \infty}} \ B\sqrt{n} \left[\frac{M(B\sqrt{n};\varphi^{(n)})}{n!} \right]^{1/n} \leq 2ABCe^{AC^2} \qquad \text{(cf. 11.4)}$$

$$< 1$$

=>

$$\lim_{n \to \infty} \frac{M(B\sqrt{n};\varphi^{(n)})}{n!} (B\sqrt{n})^n = 0.$$

Fix z and determine n_0:

$$n \geq n_0 \Rightarrow |z| + |z - z_0| + |z_0 - z_1| + \cdots + |z_{n-2} - z_{n-1}| \leq B\sqrt{n},$$

so $n \geq n_0$,

$$\Rightarrow |\varphi(z)| \leq \frac{M(B\sqrt{n};\varphi^{(n)})}{n!} (B\sqrt{n})^n \qquad \text{(cf. 11.5 and 11.6)}$$

$$\Rightarrow |\varphi(z)| = 0 \Rightarrow \varphi(z) = 0.$$

11.8 SUBLEMMA Let $\gamma_k = \alpha_k + \sqrt{-1}\,\beta_k$ $(\beta_k > 0)$ $(k = 0,1,\ldots,n)$ be complex numbers such that

$$|\gamma_{k+1} - \alpha_k| \leq \beta_k \qquad (k = 0,1,\ldots,n-1).$$

Then

$$0 \leq \beta_n \leq \beta_{n-1} \leq \cdots \leq \beta_0$$

and

$$|\gamma_0 - \gamma_1| + |\gamma_1 - \gamma_2| + \cdots + |\gamma_{n-1} - \gamma_n|$$

$$\le \beta_0 - \beta_n + \sqrt{n}\,(\beta_0^2 - \beta_n^2)^{1/2}.$$

PROOF The decrease of the β_k is immediate and induction on n leads to the inequality

$$|\alpha_0 - \alpha_1| + |\alpha_1 - \alpha_2| + \cdots + |\alpha_{n-1} - \alpha_n| \le \sqrt{n}\,(\beta_0^2 - \beta_n^2)^{1/2},$$

from which

$$|\gamma_0 - \gamma_1| + |\gamma_1 - \gamma_2| + \cdots + |\gamma_{n-1} - \gamma_n|$$

$$\le |\alpha_0 - \alpha_1| + |\alpha_1 - \alpha_2| + \cdots + |\alpha_{n-1} - \alpha_n|$$

$$+ (\beta_0 - \beta_1)+(\beta_1 - \beta_2) + \cdots + (\beta_{n-1} - \beta_n)$$

$$\le \sqrt{n}\,(\beta_0^2 - \beta_n^2)^{1/2} + \beta_0 - \beta_n.$$

[Note: Extending the setup to infinity, let $\beta = \lim\limits_{n \to \infty} \beta_n$, hence

$$\overline{\lim_{n \to \infty}}\, (|\gamma_0 - \gamma_1|+|\gamma_1 - \gamma_2| + \cdots + |\gamma_{n-1} - \gamma_n|)/\sqrt{n}$$

$$\le (\beta_0^2 - \beta^2)^{1/2}.]$$

To see how data of this type is going to arise, take a $\varphi \in * - L - P$ -- then $\forall\, n \ge 0$, $\varphi^{(n)} \in * - L - P$ (cf. 10.38) and given a nonreal zero z_{n+1} of $\varphi^{(n+1)}$ in the open upper half-plane, there is a nonreal zero z_n of $\varphi^{(n)}$ in the open upper half-plane such that

$$|z_{n+1} - \operatorname{Re} z_n| \le \operatorname{Im} z_n.$$

[Note: This is a consequence of 10.40 (use Jensen circles, replacing the φ there by $\varphi^{(n)}$ and then applying the theory to the pair $(\varphi^{(n)}, \varphi^{(n+1)})$).]

11.9 THEOREM Let $\varphi \in * - L - P$ -- then there is a positive integer N_0 such that $\forall\ N \geq N_0$, $\varphi^{(N)}$ has only real zeros, thus is in $L - P$.

In order to utilize the machinery developed above, there is one crucial preliminary to be dealt with.

Let $\varphi \in * - L - P$ and let $c_1, \bar{c}_1, \ldots, c_J, \bar{c}_J$ denote the nonreal zeros of φ -- then φ has a representation of the form

$$C \prod_{j=1}^{J} (z - c_j)(z - \bar{c}_j) z^m e^{az^2+bz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) e^{z/\lambda_n},$$

where the various parameters are subject to the conditions enumerated in 10.19.

11.10 LEMMA A given $\varphi \in * - L - P$ is of growth $(2, |a|)$.

PROOF It is simply a matter of examining the various possibilities.

[Note: The polynomial

$$C \prod_{j=1}^{J} (z - c_j)(z - \bar{c}_j) z^m$$

can be safely ignored.]

1. If $a = 0$, $b = 0$, and if the product $\prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) e^{z/\lambda_n}$ is finite (recall the conventions set forth in 10.19), then the order of φ is 0.

2. If $a = 0$, $b \neq 0$, and if the product $\prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) e^{z/\lambda_n}$ is finite, then the order of φ is 1 (cf. 2.36).

3. If $a \neq 0$, $b = 0$ or $\neq 0$, and if the product $\prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) e^{z/\lambda_n}$ is finite, then the order of φ is 2 and its type is $|a|$ (cf. 3.2).

4. If $a = 0$, $b = 0$, and if the product $\prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n})e^{z/\lambda_n}$ is infinite, then there are two possibilities.

- $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|} = \infty$ -- then $g = 1$ is the genus of the sequence $\{|\lambda_n|:n = 1,2,\ldots\}$ (cf. 4.14), hence $\prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n})e^{z/\lambda_n}$ is the associated canonical product (cf. 5.9). As such, its order is κ (the convergence exponent of the sequence $\{|\lambda_n|:n = 1,2,\ldots\}$) (cf. 5.10). But $1 \leq \kappa \leq 1 + 1$ (cf. 4.15), so the order of the product $\prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n})e^{z/\lambda_n}$ is ≤ 2. It remains to analyze the situation when $\kappa = 2$. This, however, is immediate: $\prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n})e^{z/\lambda_n}$ is of minimal type (cf. 5.16), thus is of growth $(2,0)$ or still, is of growth $(2,|a|)$ (since here $a = 0$).

- $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|} < \infty$ -- then $g = 0$ is the genus of the sequence $\{|\lambda_n :n = 1,2,\ldots\}$ (cf. 4.14) and we can write

$$\prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n})e^{z/\lambda_n} = \exp((\sum_{n=1}^{\infty} \frac{1}{\lambda_n})z) \prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n}).$$

Thanks to 5.11, the order of the RHS is $\max(1,\kappa) \leq \max(1,1) = 1$ if $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} \neq 0$ or $\kappa \leq 1$ if $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = 0$.

5. If $a = 0$, $b \neq 0$, and if the product $\prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n})e^{z/\lambda_n}$ is infinite, then there are two possibilities.

- $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|} = \infty$. Suppose first that κ is < 2 -- then the

order of

$$e^{bz} \prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n}) e^{z/\lambda_n}$$

is $\max(1,\kappa) < 2$ (cf. 5.11). On the other hand, if $\kappa = 2$, then the order of

$$e^{bz} \prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n}) e^{z/\lambda_n}$$

is $\max(1,2) = 2$ (cf. 5.11). As for its type, use 3.14 in the "$\rho_1 < \rho_2$" scenario to see that it is minimal, thus

$$e^{bz} \prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n}) e^{z/\lambda_n}$$

is of growth $(2,0)$ or still, is of growth $(2,|a|)$ (since here $a = 0$).

- $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|} < \infty$ -- then the order of the product

$\prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n})$ is ≤ 1, hence the order of

$$e^{bz} \prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n}) e^{z/\lambda_n}$$

$$= \exp((b + \sum_{n=1}^{\infty} \frac{1}{\lambda_n}) z) \prod_{n=1}^{\infty} (1 - \frac{z}{z_n})$$

is ≤ 1 (cf. 5.11).

6. If $a \neq 0$, $b = 0$ or $\neq 0$, and if the product $\prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n}) e^{z/\lambda_n}$ is infinite, then there are two possibilities.

- $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|} = \infty$. Suppose first that κ is < 2 -- then the

order of

$$e^{az^2+bz} \prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n})e^{z/\lambda_n}$$

is $\max(2,\kappa) = 2$ (cf. 5.11) and its type is $|a|$ (apply 3.14 (first bullet point)). As for what happens when $\kappa = 2$, the product $\prod_{n=1}^{\infty} (1 - z/\lambda_n)e^{z/\lambda_n}$ is of minimal type (see above), so another appeal to 3.14 (second bullet point) allows one to conclude that the type of

$$e^{az^2+bz} \prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n})e^{z/\lambda_n}$$

is again $|a|$.

- $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|} < \infty$ -- then the order of the product $\prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n})$

is ≤ 1, hence the order of

$$e^{az^2+bz} \prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n})e^{z/\lambda_n}$$

$$= \exp(az^2 + (b + \sum_{n=1}^{\infty} \frac{1}{\lambda_n})z) \prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n})$$

is 2 (cf. 5.11) and its type is $|a|$ (use 3.14 in the "$\rho_1 < \rho_2$" scenario).

Passing now to the proof of 11.9, it suffices to show that there is a positive N_0 such that $\varphi^{(N_0)}$ has only real zeros (cf. 10.38 and 10.41). Proceeding by contradiction, suppose that $\forall\, n \geq 0$, $\varphi^{(n)}$ has a nonreal zero and let X_n denote the set of nonreal zeros of $\varphi^{(n)}$ in the open upper half-plane $\text{Im}\, z > 0$ -- then each X_n is finite and the product $X = \prod_{n=0}^{\infty} X_n$ is a nonempty compact set. Given $n = 1,2,\ldots$, put

$$E_n = \{(\zeta_0,\zeta_1,\ldots) \in X: |\zeta_{j+1} - \text{Re}\, \zeta_j| \leq \text{Im}\, \zeta_j,\ j=0,1,\ldots,n\}.$$

Then E_n is a closed subset of X and $E_1 \supset E_2 \supset \cdots$. Furthermore, E_n is nonempty, so $\bigcap_{n=1}^{\infty} E_n \neq \emptyset$, thus one can find a sequence $z_0,z_1,\ldots$ of complex numbers such that

$$\text{Im}\, z_n > 0,\ \varphi^{(n)}(z_n) = 0,\ |z_{n+1} - \text{Re}\, z_n| \leq \text{Im}\, z_n.$$

Write $z_n = a_n + \sqrt{-1}\, b_n$ $(b_n > 0)$ -- then $\{b_n\}$ is a decreasing sequence and

$$\begin{aligned} &|z_m - z_{m+1}| + |z_{m+1} - z_{m+2}| + \cdots + |z_{m+n-1} - z_{m+n}| \\ &\leq b_m - b_{m+n} + \sqrt{n}\, (b_m^2 - b_{m+n}^2)^{1/2}. \end{aligned}$$

Here $m = 0,1,\ldots$ and $n = 1,2,\ldots$. Therefore

$$\begin{aligned} &\overline{\lim_{n \to \infty}}\, (|z_m - z_{m+1}| + |z_{m+1} - z_{m+2}| + \cdots + |z_{m+n-1} - z_{m+n}|)/\sqrt{n} \\ &\leq (b_m^2 - b^2)^{1/2}, \end{aligned}$$

where we have set $b = \lim_{n \to \infty} b_n$. Fix $A > |a|$, hence

$$\varphi \in \text{ent}(2,A) \qquad (\text{cf. } 11.10).$$

Choose $B > 0$:

$$2ABCe^{Ac^2} < 1$$

and choose m:

$$(b_m^2 - b^2)^{1/2} < B.$$

Then

$$\overline{\lim_{n \to \infty}} \, (|z_m - z_{m+1}| + |z_{m+1} - z_{m+2}| + \cdots + |z_{m+n-1} - z_{m+n}|)/\sqrt{n} < B.$$

But

$$\varphi \in \text{ent}(2,A) \Rightarrow \varphi^{(m)} \in \text{ent}(2,A) \quad \text{(cf. 11.2)}.$$

And this means that 11.7 is applicable to $\varphi^{(m)}$:

$$\Rightarrow \varphi^{(m)} \equiv 0.$$

Contradiction... .

11.11 EXAMPLE The real entire function e^{z^2} belongs to ent(2,1). However, it is not in $* - \mathcal{L} - \mathcal{P}$ and 11.9 does not obtain.

§12. *JENSEN POLYNOMIALS*

Given a real entire function

$$f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

put $\gamma_n = f^{(n)}(0)$, thus

$$f(z) = \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} z^n.$$

12.1 DEFINITION The $\underline{n^{th}\text{ Jensen polynomial}}$ J_n associated with f is defined by

$$J_n(f;z) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k z^k.$$

12.2 LEMMA The sequence $\{J_n(f;t)\}$ is generated by $e^x f(xt)$, i.e.,

$$e^x f(xt) = \sum_{n=0}^{\infty} J_n(f;t) \frac{x^n}{n!} \quad (x,t \in R).$$

12.3 LEMMA We have

$$zJ_n'(f;z) = nJ_n(f;z) - nJ_{n-1}(f;z) \; (n \geq 1).$$

12.4 DEFINITION The $\underline{n^{th}\text{ Appell polynomial}}$ J_n^* associated with f is defined by

$$J_n^*(f;z) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k z^{n-k}.$$

12.5 LEMMA The sequence $\{J_n^*(f;t)\}$ is generated by $e^{xt} f(x)$, i.e.,

$$e^{xt} f(x) = \sum_{n=0}^{\infty} J_n^*(f;t) \frac{x^n}{n!} \quad (x,t \in R).$$

12.6 LEMMA We have

$$\frac{d}{dz} J_n^*(f;z) = nJ_{n-1}^*(f;z) \quad (n \geq 1).$$

N.B. Obviously,

$$\begin{cases} J_n(f;z) = z^n J_n^*(f;\frac{1}{z}) \\ J_n^*(f;z) = z^n J_n(f;\frac{1}{z}). \end{cases}$$

Therefore the zeros of J_n are real iff the zeros of J_n^* are real.

12.7 DEFINITION The $(n,m)^{th}$ Jensen polynomial associated with f is defined by

$$J_{n,m}(f;z) = \sum_{k=0}^{n} \binom{n}{k} \gamma_{k+m} z^k.$$

N.B. Therefore

$$J_{n,m}(f;z) = J_n(f^{(m)};z).$$

12.8 LEMMA We have

$$J_n^{(m)}(f;z) = \frac{n!}{(n-m)!} J_{n-m,m}(f;z)$$

$$= \frac{n!}{(n-m)!} J_{n-m}(f^{(m)};z).$$

12.9 THEOREM On compact subsets of $\mathbb{C}$,

$$J_n(f;\frac{z}{n}) \rightarrow f(z)$$

uniformly.

PROOF Fix a compact set $K \subset \mathbb{C}$. Given $\varepsilon > 0$, choose $N > 2$:

$$\sum_{n=N+1}^{\infty} \left|\frac{\gamma_n}{n!} z^n\right| < \frac{\varepsilon}{4} \quad (z \in K).$$

Next, choose $N' > N$:

$$n \geq N' \Rightarrow \left|\sum_{k=2}^{N} \left(\frac{\gamma_k}{k!} - \left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right)\frac{\gamma_k}{k!}\right) z^k\right| < \frac{\varepsilon}{2} \quad (z \in K).$$

Then $\forall\, z \in K$ and $\forall\, n \geq N'$:

$$\left|f(z) - J_n\left(f;\frac{z}{n}\right)\right|$$

$$= \left|\sum_{n=N+1}^{\infty} \frac{\gamma_n}{n!} z^n + \sum_{k=0}^{N} \frac{\gamma_k}{k!} z^k - \left(\gamma_0 + \gamma_1 z + \sum_{k=2}^{n} \left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right)\frac{\gamma_k}{k!} z^k\right)\right|$$

$$= \left|\sum_{n=N+1}^{\infty} \frac{\gamma_n}{n!} z^n + \sum_{k=2}^{N} \frac{\gamma_k}{k!} z^k - \sum_{k=2}^{n} \left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right)\frac{\gamma_k}{k!} z^k\right|$$

$$= \left|\sum_{n=N+1}^{\infty} \frac{\gamma_n}{n!} z^n + \sum_{k=2}^{N} \left(\frac{\gamma_k}{k!} - \left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right)\frac{\gamma_k}{k!}\right) z^k\right.$$

$$\left. - \sum_{k=N+1}^{n} \left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right)\frac{\gamma_k}{k!} z^k\right|$$

$$\leq \sum_{n=N+1}^{\infty} \left|\frac{\gamma_n}{n!} z^n\right| + \left|\sum_{k=2}^{N} \left(\frac{\gamma_k}{k!} - \left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right)\frac{\gamma_k}{k!}\right) z^k\right|$$

$$+ \sum_{k=N+1}^{n} \left|\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right)\frac{\gamma_k}{k!} z^k\right|$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.$$

In what follows, certain classical facts from the theory of equations will be admitted without proof. To begin with:

12.10 HERMITE-POULAIN CRITERION Suppose that the real polynomial

$$a_0 + a_1 z + \cdots + a_n z^n$$

has real zeros only. Let $p(z)$ be a real polynomial -- then the polynomial

$$P(z) = a_0 p(z) + a_1 p'(z) + \cdots + a_n p^{(n)}(z)$$

has at least as many real zeros as $p(z)$ does.

[Note: By taking limits, one can extend 12.10, viz. replace the real polynomial

$$a_0 + a_1 z + \cdots + a_n z^n$$

by an element $f \in L - P$ -- then for any real polynomial $p(z)$, the polynomial

$$\sum_{k=0}^{d} \frac{f^{(k)}(0)}{k!} p^{(k)}(z) \qquad (d = \deg p)$$

has at least as many real zeros as $p(z)$ does.]

12.11 APPLICATION A real polynomial has real zeros only iff its Jensen polynomials have real zeros only.

[Suppose that

$$f(z) = \gamma_0 + \frac{\gamma_1}{1!} z + \cdots + \frac{\gamma_d}{d!} z^d$$

is a real polynomial of degree d.

- If $f(z)$ has real zeros only, take $p(z) = z^n$ in 12.10 to see that $\forall\ n = 1,2,\ldots,$

$$J_n^*(f;z) = \gamma_0 z^n + \binom{n}{1}\gamma_1 z^{n-1} + \cdots$$

has real zeros only, so the same is true of $J_n(f;z)$.

- If $\forall\ n = 1,2,\ldots,\ J_n(f;z)$ has real zeros only, then

$$f(z) = \lim_{n \to \infty} J_n(f;\frac{z}{n})$$

has real zeros only (cf. 12.9).]

12.12 MALO-SCHUR CRITERION Suppose that the zeros of

$$a_0 + a_1 z + \cdots + a_n z^n$$

are real and the zeros of

$$b_0 + b_1 z + \cdots + b_m z^m$$

are real and of the same sign. Put $k = \min(n,m)$ -- then the zeros of

$$a_0 b_0 + 1! a_1 b_1 z + \cdots + k! a_k b_k z^k$$

are real.

12.13 EXAMPLE Suppose that the zeros of

$$a_0 + a_1 z + \cdots + a_n z^n$$

are real -- then the zeros of

$$a_n + a_{n-1} z + \cdots + a_0 z^n$$

are real. Working now with

$$(1 + z)^n = 1 + \binom{n}{1} z + \cdots + z^n,$$

it follows that the zeros of

$$a_n + n a_{n-1} z + \cdots + n! a_0 z^n$$

are real, or still, that the zeros of

$$\frac{a_n}{n!} + \frac{a_{n-1}}{(n-1)!} z + \cdots + a_0 z^n$$

are real, or still, that the zeros of

$$a_0 + \frac{a_1}{1!} z + \cdots + \frac{a_n}{n!} z^n$$

are real. Consequently, if the zeros of

$$b_0 + b_1 z + \cdots + b_m z^m$$

are real and of the same sign, then the zeros of

$$a_0 b_0 + a_1 b_1 z + \cdots + a_k b_k z^k \qquad (k = \min(n,m))$$

are real.

12.14 THEOREM Let $f \not\equiv 0$ be a real entire function -- then $f \in L - P$ iff its Jensen polynomials have real zeros only.

PROOF In view of 12.9, it is clear that the condition is sufficient. Turning to the necessity, given that $f \in L - P$, choose a sequence $\{p_k : k = 1,2,\ldots\}$ of real polynomials having real zeros only such that $p_k \to f$ uniformly on compact subsets of $\mathbf{C}$, say

$$p_k(z) = \gamma_{k0} + \frac{\gamma_{k1}}{1!} + \cdots .$$

Then the Jensen polynomials $J_n(p_k;z)$ have real zeros only (cf. 12.11). But for fixed n,

$$\lim_{k \to \infty} J_n(p_k;z) = J_n(f;z)$$

uniformly on compact subsets of $\mathbf{C}$.

12.15 REMARK If $f \in L - P$, then

$$J_n(f;\frac{z}{n}) \to f(z)$$

uniformly on compact subsets of $\mathbb{C}$ and the zeros of $J_n(f;\frac{z}{n})$ are real. By comparison, the partial sums

$$\sum_{k=0}^{n} \frac{\gamma_k}{k!} z^k,$$

while uniformly convergent on compact subsets of $\mathbb{C}$, may very well have nonreal zeros. E.g.: Take $f(z) = e^z$ -- then

$$\sum_{k=0}^{n} \frac{z^k}{k!}$$

has no real zeros if n is even and has one real zero if n is odd.

12.16 DEFINITION A sequence $\gamma_0,\gamma_1,\ldots$ of real numbers is said to be a <u>multiplier sequence</u> if $\forall$ n = 1,2,..., the real polynomial

$$\sum_{k=0}^{n} \binom{n}{k}\gamma_k z^k$$

has real zeros only or, equivalently, if $\forall$ n = 1,2,..., the real polynomial

$$\sum_{k=0}^{n} \binom{n}{k}\gamma_k z^{n-k}$$

has real zeros only.

If $f \in L - P$, then the associated sequence $\gamma_0,\gamma_1,\ldots$ is a multiplier sequence (cf. 12.14).

12.17 EXAMPLE Take

$$f(z) = \begin{cases} e^z \\ \\ e^{-z} \end{cases}$$

to see that

$$\begin{cases} 1, 1, 1, \ldots \\ 1, -1, 1, \ldots \end{cases}$$

are multiplier sequences.

12.18 EXAMPLE Let p be a positive integer and take $f(z) = z^p e^z$ -- then

$$z^p e^z = p! \frac{z^p}{p!} + \frac{(p+1)!}{1!} \frac{z^{p+1}}{(p+1)!} + \cdots .$$

Therefore the sequence

$$0, 0, \ldots, 0, p!, \frac{(p+1)!}{1!}, \ldots$$

is a multiplier sequence.

[Note: Specialize and let p = 1, thus 0,1,2,... is a multiplier sequence.]

12.19 EXAMPLE Take $f(z) = e^{-z^2/2}$ -- then

$$e^{-z^2/2} = 1 - \frac{z^2}{2!} + 1 \cdot 3 \frac{z^4}{4!} - 1 \cdot 3 \cdot 5 \frac{z^6}{6!} + \cdots .$$

Therefore the sequence

$$1, 0, -1, 0, 1 \cdot 3, 0, - 1 \cdot 3 \cdot 5, 0, \ldots$$

is a multiplier sequence.

12.20 EXAMPLE Take

$$f(z) = \begin{cases} \cos z \\ \sin z \end{cases} \text{--}$$

then

$$\begin{cases} 1, 0, -1, 0, 1, 0, -1, \ldots \\ 0, 1, 0, -1, 0, 1, 0, \ldots \end{cases}$$

are multiplier sequences.

12.21 THEOREM Let $\gamma_0,\gamma_1,\ldots$ be a multiplier sequence and put $c_n = \frac{\gamma_n}{n!}$ -- then

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

is a real entire function and, as such, is in $L - P$.

PROOF The objective is to find an estimate for $|c_n|$ that suffices to ensure the convergence of the series at every z. This said, let γ_r be the first nonzero entry in the sequence $\gamma_0,\gamma_1,\ldots$. Take $n > r$:

$$\sum_{k=0}^{n} \binom{n}{k} \gamma_k z^{n-k}$$

$$= \sum_{k=0}^{n} \frac{n!}{(n-k)!} \frac{\gamma_k}{k!} z^{n-k}$$

$$= \sum_{k=0}^{n} \frac{n!}{(n-k)!} c_k z^{n-k}$$

$$= c_0 z^n + nc_1 z^{n-1} + \cdots + n!c_n$$

$$= n(n-1)\cdots(n-r+1)c_r z^{n-r} + \cdots + n!c_n$$

and denote by $\lambda_1,\lambda_2,\ldots,\lambda_{n-r}$ its (necessarily real) zeros -- then

$$\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_{n-r}^2$$

$$= (n-r)^2 \left(\frac{c_{r+1}}{c_r}\right)^2 - 2(n-r)(n-r-1) \frac{c_{r+2}}{c_r}$$

and

$$\lambda_1\lambda_2\cdots\lambda_{n-r} = (-1)^{n-r}(n-r)!\,\frac{c_n}{c_r}\,.$$

But

$$\frac{\lambda_1^2+\lambda_2^2+\cdots+\lambda_{n-r}^2}{n-r} \geq ((\lambda_1\lambda_2\cdots\lambda_{n-r})^2)^{\frac{1}{n-r}}.$$

Therefore

$$|c_n| < C\,\frac{(Mn)^{(n-r)/2}}{(n-r)!}\,,$$

where C and M are positive constants independent of n. And this estimate will do the trick.

12.22 LEMMA Let $\gamma_0,\gamma_1,\ldots$ be a multiplier sequence. Suppose that

$$c_0 + c_1z + \cdots + c_dz^d$$

is a real polynomial whose zeros are real and of the same sign -- then the zeros of the real polynomial

$$\gamma_0c_0 + \gamma_1c_1z + \cdots + \gamma_dc_dz^d$$

are real.

PROOF Thanks to 12.12, the zeros of the real polynomial

$$\gamma_0c_0 + 1!\ \binom{n}{1}\gamma_1c_1z + \cdots + d!\ \binom{n}{d}\gamma_dc_dz^d \qquad (n > d)$$

are real. Replacing z by $\frac{z}{n}$, it follows that the zeros of the real polynomial

$$\gamma_0c_0 + \gamma_1c_1z + \cdots + (1-\tfrac{1}{n})(1-\tfrac{2}{n})\cdots(1-\tfrac{d-1}{n})\gamma_dc_dz^d$$

are real so, upon letting $n \to \infty$, we conclude that the zeros of the real polynomial

$$\gamma_0 c_0 + \gamma_1 c_1 z + \cdots + \gamma_d c_d z^d$$

are real.

[Note: The stated property is characteristic. Proof: The zeros of the real polynomial

$$(1 + z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k$$

are real and of the same sign.]

12.23 APPLICATION Let $\gamma_0, \gamma_1, \ldots$ be a multiplier sequence -- then the Turan inequalities obtain:

$$\gamma_n^2 - \gamma_{n-1}\gamma_{n+1} \geq 0 \quad (n = 1,2,\ldots).$$

[The zeros of the real polynomial

$$z^{n-1} + 2z^n + z^{n+1}$$

are real and ≤ 0. Therefore the zeros of the real polynomial

$$\gamma_{n-1} z^{n-1} + 2\gamma_n z^n + \gamma_{n+1} z^{n+1}$$

are real, from which the assertion.]

12.24 LAGUERRE CRITERION Let $Q(x)$ be a real polynomial whose zeros are real and lie outside the interval $[0,d]$ -- then for any real sequence $c_0, c_1, \ldots, c_d$, the number of nonreal zeros of the real polynomial

$$Q(0)c_0 + Q(1)c_1 z + \cdots + Q(d)c_d z^d$$

is $\leq$ the number of nonreal zeros of the real polynomial

$$c_0 + c_1 z + \cdots + c_d z^d.$$

[Note: Accordingly, if the zeros of

$$c_0 + c_1 z + \cdots + c_d z^d$$

are real, then the zeros of

$$Q(0)c_0 + Q(1)c_1 z + \cdots + Q(d)c_d z^d$$

are also real.]

12.25 THEOREM Let $f \in \mathcal{L} - \mathcal{P}$ and assume that the zeros of f are negative. Suppose that

$$c_0 + c_1 z + \cdots + c_d z^d$$

is a real polynomial whose zeros are real -- then the zeros of the real polynomial

$$f(0)c_0 + f(1)c_1 z + \cdots + f(d)c_d z^d$$

are real.

PROOF Take $f(0) = 1$ and write

$$f(z) = e^{az^2+bz} \prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n})e^{z/\lambda_n} \quad \text{(cf. 10.19)}.$$

Choose $k > 0$: $\sqrt{k} > d\sqrt{-a}$ $(a \le 0)$ and put

$$Q_k(z) = (1 + \frac{az^2}{k})^k (1 - \frac{z}{\lambda_1}) \cdots (1 - \frac{z}{\lambda_k}),$$

the interval of exclusion thus being $[0,d]$. Let

$$B_k = b + \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_k}.$$

Then the zeros of the real polynomial

$$c_0 + c_1 e^{B_k} z + \cdots + c_d e^{dB_k} z^d$$

are real, hence the zeros of the real polynomial

$$c_0 Q_k(0) + c_1 Q_k(1) e^{B_k z} + \cdots + c_d Q_k(d) e^{dB_k z^d}$$

are also real. Now let $k \to \infty$.

N.B. An additional assumption to the effect that the zeros of

$$c_0 + c_1 z + \cdots + c_d z^d$$

are of the same sign is inutile.

12.26 SCHOLIUM If $f \in L - P$ and if the zeros of f are negative, then the sequence $f(0), f(1), \ldots$ is a multiplier sequence.

12.27 EXAMPLE Take $f(z) = e^{z^2 \log q}$ $(0 < q \leq 1)$ -- then $f(n) = q^{n^2}$, so $\{q^{n^2}\}$ is a multiplier sequence.

12.28 EXAMPLE Take $f(z) = \frac{1}{\Gamma(z+1)}$ (cf. 10.30) -- then $f(n) = \frac{1}{n!}$, so $\{\frac{1}{n!}:n = 0,1,\ldots\}$ is a multiplier sequence.

[Note: Given $\alpha > 0$, put $(\alpha)_0 = 1$ and

$$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha + n-1) \qquad (n \geq 1).$$

Take now

$$f(z) = \frac{\Gamma(\alpha)}{\Gamma(z+\alpha)}.$$

Then

$$f(n) = \frac{\Gamma(\alpha)}{\Gamma(n+\alpha)} = \frac{1}{(\alpha)_n},$$

so $\{\frac{1}{(\alpha)_n} : n = 0,1,\ldots\}$ is a multiplier sequence.]

12.29 THEOREM Let $f \in L - P$ and assume that the zeros of f are negative. Suppose that

$$F(z) = C_0 + C_1 z + \cdots$$

is in $L - P$ -- then the series

$$f(0)C_0 + f(1)C_1 z + \cdots$$

is a real entire function and, as such, is in $L - P$.

PROOF The initial claim is that the series

$$f(0)C_0 + f(1)C_1 z + \cdots$$

is convergent for every z. Thus decompose f per 10.19:

$$f(z) = Ce^{az^2+bz} \prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n})e^{z/\lambda_n}.$$

Then

$$(1 + t)e^{-t} \leq 1 \quad (t \geq 0)$$

=>

$$(1 - \frac{t}{\lambda_n})e^{t/\lambda_n} = (1 + (\frac{t}{-\lambda_n}))e^{-(t/-\lambda_n)} \leq 1 \quad (\lambda_n < 0).$$

So, for k a nonnegative integer,

$$|f(k)| \leq |C|e^{ak^2} e^{bk} \leq |C|e^{bk} \quad (a \leq 0).$$

Therefore

$$\overline{\lim_{k \to \infty}} |f(k)|^{1/k} |C_k|^{1/k} = 0,$$

which settles the convergence issue. To verify the $L - P$ contention, note first that the zeros of

$$J_n(F;z) = C_0 + nC_1z + n(n-1)C_2z^2 + \cdots$$

are real (cf. 12.14). Therefore the zeros of the real polynomial

$$f(0)C_0 + nf(1)C_1z + n(n-1)f(2)C_2z^2 + \cdots$$

are real (cf. 12.25). But this polynomial is the n^{th} Jensen polynomial of the series

$$f(0)C_0 + f(1)C_1z + \cdots,$$

so another application of 12.14 finishes the argument.

12.30 EXAMPLE Take $F(z) = e^z$ -- then

$$\sum_{n=0}^{\infty} \frac{f(n)}{n!} z^n$$

is in $L - P$.

12.31 EXAMPLE Take $F(z) = e^{-z^2}$ -- then

$$\sum_{n=0}^{\infty} (-1)^n \frac{f(2n)}{n!} z^{2n}$$

is in $L - P$.

12.32 EXAMPLE Fix a positive integer m and take

$$f(z) = \frac{\Gamma(z+1)}{\Gamma(mz+1)} .$$

Then

$$f(n) = \frac{n!}{(mn)!} ,$$

hence

$$\sum_{n=0}^{\infty} \frac{z^n}{(mn)!} \equiv ML_m(z) \qquad \text{(cf. 2.28)}$$

is in $L - P$.

[Note: The poles of the numerator, viz. -1, -2, ..., are absorbed by the poles of the denominator, viz. $-\frac{1}{m}, -\frac{2}{m}, \ldots, -\frac{m}{n}, \ldots$.]

12.33 EXAMPLE Recall that the Bessel function $J_\nu(z)$ of the first kind of real index $\nu > -1$ is defined by the series

$$(\tfrac{z}{2})^\nu \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n!\Gamma(\nu + n + 1)} \qquad \text{(cf. 2.29)}.$$

To apply the foregoing machinery, rewrite this as

$$J_\nu(z) = (\tfrac{z}{2})^\nu \, \Psi_\nu(\tfrac{z}{2}),$$

where

$$\Psi_\nu(z) = \sum_{n=0}^{\infty} (-1)^n \frac{f_\nu(2n)}{n!} z^{2n}.$$

Here

$$f_\nu(z) = \frac{1}{\Gamma(\nu + \frac{z}{2} + 1)}$$

is in $L - P$ and its zeros are negative (since $\nu > -1$). Therefore the zeros of $J_\nu(z)$ are real[†].

[†] E. Lommel, *Studien über die Bessel'schen Functionen*, Teubner, Leipzig, 1868, §19.

12.34 EXAMPLE Given p = 1, 2, ...,

$$\Phi_{2p}(z) = \int_0^\infty \exp(-t^{2p}) \cos zt \, dt \qquad \text{(cf. 2.30)}$$

is in $L - P$.

[In fact,

$$2p\ \Phi_{2p}(z) = \sum_{n=0}^{\infty} (-1)^n \frac{f_p(2n)}{n!} z^{2n},$$

where

$$f_p(z) = \frac{\Gamma(\frac{z}{2} + 1)\Gamma(\frac{z+1}{2p})}{\Gamma(z+1)},$$

the poles of the numerator, viz.

$$-2, -4, -6, \ldots, -1, -(1+2p), -(1+4p), \ldots,$$

being absorbed by the poles of the denominator, viz. -1, -2, -3,... .

[Note: $\Phi_2(z)$ has no zeros but $\Phi_4(z)$, $\Phi_6(z)$, ..., have an infinity of zeros.

Proof: The order of $\Phi_{2p}(z)$ is $\frac{2p}{2p-1}$, which lies strictly between 1 and 2 if $p > 1$, so one can cite 7.4.]

If $f \in L - P$, then $f' \in L - P$ (cf. 10.20 and 10.25).

[Note: Letting $\gamma_0, \gamma_1, \ldots$ be the multiplier sequence associated with f, it follows that $\gamma_0' = \gamma_1$, $\gamma_1' = \gamma_2, \ldots$ is a multiplier sequence (namely the one associated with f').]

12.35 EXAMPLE The n^{th} Hermite polynomial is, by definition,

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2} \qquad \text{(cf. 8.17)},$$

so

$$\frac{d^n}{dz^n} e^{-z^2} = (-1)^n H_n(z)e^{-z^2}.$$

The fact that e^{-z^2} is in $L - P$ then implies that $\frac{d^n}{dz^n} e^{-z^2}$ is in $L - P$, thus the zeros of $H_n(z)$ must be real.

While $L - P$ is not a vector space, there are circumstances in which it is closed under addition.

12.36 LEMMA If $f \in L - P$, then $\forall\, a \in R$,

$$af + f' \in L - P \qquad \text{(cf. 12.10)}.$$

PROOF The product $f(z)e^{az}$ is in $L - P$, as is the derivative $\frac{d}{dz}(f(z)e^{az})$, as is the product $e^{-az} \frac{d}{dt}(f(z)e^{az})$, thus

$$af(z) + f'(z)$$

is in $L - P$.

12.37 EXAMPLE Let p be a real polynomial with real zeros only. Take $\alpha > 0$, $\beta \in R$, and define F by

$$F(z) = \int_{-\infty}^{\infty} p(\sqrt{-1}\, t)\exp(-\alpha t^2 + \sqrt{-1}\, \beta t + \sqrt{-1}\, zt)dt.$$

Then $F \in L - P$.

[Supposing that p is monic, write

$$p(z) = (z + a_1)\dots(z + a_n)\ (a_1,\dots,a_n \in R).$$

Put

$$F_0(z) = \int_{-\infty}^{\infty} \exp(-\alpha t^2 + \sqrt{-1}\, \beta t + \sqrt{-1}\, zt)dt.$$

Then

$$F_0(z) = (\frac{\pi}{\alpha})^{1/2} \exp(\frac{-(z + \beta)^2}{4\alpha}),$$

so $F_0 \in L - P$. Now define F_k $(k = 1,\ldots,n)$ by

$$F_k(z) = \int_{-\infty}^{\infty} p_k(\sqrt{-1}\, t)\exp(-\alpha t^2 + \sqrt{-1}\, \beta t + \sqrt{-1}\, zt)dt,$$

where

$$p_k(z) = (z + a_1)\ldots(z + a_k).$$

Then

$$\begin{aligned} F_1 &= a_1 F_0 + F_0' \\ &\vdots \\ F = F_n &= a_n F_{n-1} + F_{n-1}', \end{aligned}$$

so $F \in L - P$.]

APPENDIX

A multiplier sequence $\gamma_0,\gamma_1,\ldots$ is said to be <u>strict</u> if it has the following property: Given any real polynomial

$$c_0 + c_1 z + \cdots + c_d z^d$$

whose zeros are real, the zeros of the real polynomial

$$\gamma_0 c_0 + \gamma_1 c_1 z + \cdots + \gamma_d c_d z^d$$

are also real (cf. 12.22).

EXAMPLE Let $f \in L - P$ and assume that the zeros of f are negative -- then the sequence f(0), f(1),... is a strict multiplier sequence (cf. 12.25). In particular: $\{\frac{1}{n!}: n = 0,1,\ldots\}$ is a strict multiplier sequence (cf. 12.28 (or 12.13)).

LEMMA A strict multiplier sequence acting on a polynomial whose zeros are real and of the same sign preserves the reality and the sign of the zeros.

EXAMPLE Take $f(z) = (z^2 + 2z - 1)e^z$ and consider the corresponding multiplier sequence $\{-1 + n + n^2 : n = 0,1,\ldots\}$ -- then its action on $(z + 1)^2$ is

$$-1(1) + 1(2)z + 5(2)z^2.$$

The zeros of this polynomial are $\frac{-1 \pm \sqrt{11}}{10}$, hence are real but of opposite sign. Therefore the multiplier sequence $\{-1 + n + n^2 : n = 0,1,\ldots\}$ is not strict.

DEFINITION Given two sequences

$$\begin{bmatrix} a_0, a_1, \ldots \\ b_0, b_1, \ldots \end{bmatrix}$$

of real numbers, their <u>component wise product</u> is the sequence $a_0 b_0, a_1 b_1, \ldots$.

LEMMA If

$$\begin{bmatrix} \alpha_0, \alpha_1, \ldots \\ \beta_0, \beta_1, \ldots \end{bmatrix}$$

are strict multiplier sequences, then so is their component wise product.

LEMMA If

$$\begin{bmatrix} \alpha_0, \alpha_1, \ldots \\ \beta_0, \beta_1, \ldots \end{bmatrix}$$

are multiplier sequences and if $\alpha_0, \alpha_1, \ldots$ is strict, then their component wise product is a multiplier sequence.

PROOF Let

$$c_0 + c_1 z + \cdots + c_d z^d$$

be a real polynomial whose zeros are real and of the same sign -- then

$$\alpha_0 c_0 + \alpha_1 c_1 z + \cdots + \alpha_d c_d z^d$$

is a real polynomial whose zeros are real and of the same sign, thus the zeros of the real polynomial

$$\alpha_0 \beta_0 c_0 + \alpha_1 \beta_1 c_1 z + \cdots + \alpha_d \beta_d c_d z^d$$

are real (cf. 12.22), which implies that $\alpha_0\beta_0$, $\alpha_1\beta_1, \ldots$ is a multiplier sequence (see the comment appended to 12.22).

APPLICATION Let $f \in L - P$, say

$$f(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Then $c_0, c_1, \ldots$ is a multiplier sequence.

[For

$$c_n = \frac{\gamma_n}{n!}$$

and $\{\frac{1}{n!}: n = 0,1,\ldots\}$ is a strict multiplier sequence while $\gamma_0, \gamma_1, \ldots$ is a multiplier sequence (cf. 12.14).]

[Note: A priori,

$$c_n^2 - c_{n-1}\, c_{n+1} \geq 0 \quad (n = 1,2,\ldots) \qquad \text{(cf. 12.23)}$$

but this can be sharpened:

$$\gamma_n^2 - \gamma_{n-1}\, \gamma_{n+1} \geq 0$$

=>

$$(n!)^2 c_n^2 - (n-1)!(n+1)! c_{n-1} c_{n+1} \geq 0$$

=>

$$n c_n^2 - (n+1) c_{n-1} c_{n+1} \geq 0$$

=>

$$c_n^2 - (1 + \tfrac{1}{n}) c_{n-1} c_{n+1} \geq 0$$

=>

$$c_n^2 - c_{n-1} c_{n+1} \geq 0.]$$

§13. *CHARACTERIZATIONS*

Let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

be in $L - P$ -- then

$$c_n = \frac{\gamma_n}{n!} \quad (\gamma_n = f^{(n)}(0))$$

and $\gamma_0, \gamma_1, \ldots$ is a multiplier sequence (cf. 12.14). Therefore (cf. 12.23)

$$\gamma_n^2 - \gamma_{n-1}\gamma_{n+1} \geq 0 \qquad (n = 1,2,\ldots).$$

13.1 EXAMPLE Consider the Hermite polynomials $\{H_n : n = 0,1,\ldots\}$ (cf. 12.35) -- then for real t and complex z,

$$\exp(2tz - z^2) = \sum_{n=0}^{\infty} \frac{H_n(t)}{n!} z^n.$$

Since $\forall$ t, the function

$$z \to \exp(2tz - z^2)$$

is in $L - P$, it follows that

$$H_n^2(t) - H_{n-1}(t)H_{n+1}(t) \geq 0 \qquad (n = 1,2,\ldots).$$

13.2 EXAMPLE Consider the Laguerre polynomials $\{L_n^{(\alpha)} : n = 0,1,\ldots\}$ of index $\alpha > -1$ and degree n, thus

$$L_n^{(\alpha)}(t) = \frac{t^{-\alpha}e^t}{n!} \frac{d^n}{dt^n} e^{-t} t^{n+\alpha} \qquad (\text{cf. } 8.17 \ (L_n^{(0)} \equiv L_n)),$$

where

$$L_n^{(\alpha)}(0) = \frac{(1+\alpha)_n}{n!}.$$

In terms of the Bessel function J_α, for real $t > 0$ and complex z,

$$\Gamma(1+\alpha)(tz)^{-\alpha/2} J_\alpha(2\sqrt{tz})$$

$$= \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(t)}{(1+\alpha)_n} z^n$$

$$= \sum_{n=0}^{\infty} \frac{n!}{(1+\alpha)_n} L_n^{(\alpha)}(t) \frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(t)}{L_n^{(\alpha)}(0)} \frac{z^n}{n!}.$$

Since $\forall\, t > 0$, the function

$$z \rightarrow (tz)^{-\alpha/2} J_\alpha(2\sqrt{tz})$$

is in $L - P$ (cf. 12.33), it follows that

$$\left[\frac{L_n^{(\alpha)}(t)}{L_n^{(\alpha)}(0)}\right]^2 - \frac{L_{n-1}^{(\alpha)}(t)}{L_{n-1}^{(\alpha)}(0)} \frac{L_{n+1}^{(\alpha)}(t)}{L_{n+1}^{(\alpha)}(0)} \geq 0 \qquad (n = 1,2,\ldots).$$

[Note: As we know,

$$\left(\frac{z}{2}\right)^{-\alpha} J_\alpha(z) \in L - P,$$

so by evenness,

$$\left(\frac{\sqrt{z}}{2}\right)^{-\alpha} J_\alpha(\sqrt{z}) \in L - P$$

=>

$$2^{\alpha} z^{-\alpha/2} J_{\alpha}(\sqrt{z}) \in L - P$$

=>

$$2^{\alpha}(4z)^{-\alpha/2} J_{\alpha}(2\sqrt{z}) \in L - P$$

=>

$$z^{-\alpha/2} J_{\alpha}(2\sqrt{z}) \in L - P.]$$

13.3 LEMMA If $f \in L - P$, then for all real t,

$$(f^{(n)}(t))^2 - f^{(n-1)}(t)f^{(n+1)}(t) \geq 0 \quad (n \geq 1),$$

with equality iff $f^{(n-1)}(z)$ is of the form Ce^{bz} or t is a multiple zero of $f^{(n-1)}(z)$.

PROOF Decompose f per 10.19:

$$f(z) = Cz^m e^{az^2+bz} \prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n})e^{z/\lambda_n}.$$

Then

$$\frac{f'(t)}{f(t)} = \frac{m}{t} + 2at + b + \sum_{n=1}^{\infty} (\frac{1}{t-\lambda_n} + \frac{1}{\lambda_n})$$

=>

$$\frac{d}{dt}(\frac{f'(t)}{f(t)}) = \frac{f(t)f''(t) - (f'(t))^2}{(f(t))^2}$$

$$= - \frac{m}{t^2} + 2a - \sum_{n=1}^{\infty} \frac{1}{(t-\lambda_n)^2}.$$

If $f(z) = Ce^{bz}$ or if t is a multiple zero of $f(z)$, then

$$f(t)f''(t) - (f'(t))^2 = 0.$$

On the other hand, if $f(z) \neq Ce^{bz}$ and if c is not a zero of f(z), then

$$-\frac{m}{c^2} + 2a - \sum_{n=1}^{\infty} \frac{1}{(c-\lambda_n)^2} < 0$$

=>

$$f(c)f''(c) - (f'(c))^2 < 0,$$

so by continuity,

$$f(t)f''(t) - (f'(t))^2 \leq 0$$

for all real t. If equality obtains and if $f(z) \neq Ce^{bz}$, then t must be a zero of f(z) (cf. supra), hence t must be a multiple zero of f(z):

$$(f'(t))^2 = 0 \Rightarrow f'(t) = 0.$$

Proceed from here by iteration (bear in mind that $L - P$ is closed under differentiation (cf. 10.20 and 10.25)).

[Note: In particular,

$$(f^{(n)}(0))^2 - f^{(n-1)}(0)f^{(n+1)}(0) \geq 0,$$

i.e.,

$$\gamma_n^2 - \gamma_{n-1}\gamma_{n+1} \geq 0 \quad (n = 1,2,\ldots).]$$

13.4 EXAMPLE Take

$$f(z) = z(z^2 + 1).$$

Then

$$f'(t)^2 - f(t)f''(t) = 3t^4 + 1 > 0.$$

Still, $f \notin L - P$ (because it has the nonreal zeros $\pm\sqrt{-1}$).

13.5 EXAMPLE Take

$$f(z) = e^z - e^{2z}.$$

Then

$$(f^{(n)}(t))^2 - f^{(n-1)}(t)f^{(n+1)}(t) = 2^{n-1}e^{3t} > 0 \quad (n \geq 1).$$

Still, $f \notin L - P$ (because it has the nonreal zeros $2\pi\sqrt{-1}\,k$ $(k = \pm 1, \pm 2, \ldots)$).

Therefore the inequalities

$$(f^{(n)}(t))^2 - f^{(n-1)}(t)f^{(n+1)}(t) \geq 0 \quad (n \geq 1)$$

do not serve to characterize the elements of $L - P$ (even if they are strict).

13.6 NOTATION Given a real entire function f, let $L_0(f)(t) = f(t)^2$ and for $n = 1,2,\ldots,$ let

$$L_n(f)(t) = \sum_{k=0}^{2n} \frac{(-1)^{k+n}}{(2n)!} \binom{2n}{k} f^{(k)}(t)f^{(2n-k)}(t) \qquad (t \in R).$$

<u>N.B.</u> For the record,

$$L_1(f)(t) = \sum_{k=0}^{2} \frac{(-1)^{k+1}}{2} \binom{2}{k} f^{(k)}(t)f^{(2-k)}(t)$$

$$= -\frac{f(t)f''(t)}{2} + (f'(t))^2 - \frac{f''(t)f(t)}{2}$$

$$= (f'(t))^2 - f(t)f''(t).$$

13.7 THEOREM Let $f \in A - L - P$ (cf. 10.31) -- then $f \in 0 - L - P$ $(= L - P)$ iff $\forall\, n \geq 0$ and $\forall\, t \in R$,

$$L_n(f)(t) \geq 0.$$

Some preparation will help ease the way.

13.8 NOTATION Given a real entire function f, for fixed $x \in R$, let

$$f_x(y) = |f(x + \sqrt{-1}\, y)|^2$$

$$\equiv f(x + \sqrt{-1}\, y) f(x - \sqrt{-1}\, y).$$

Then f_x is an even function of y and

$$f_x(y) = \sum_{n=0}^{\infty} \Lambda_n(f)(x) y^{2n},$$

where

$$\Lambda_n(f)(x) = \frac{f_x^{(2n)}(0)}{(2n)!}.$$

13.9 LEMMA We have

$$\Lambda_n(f)(x) = L_n(f)(x).$$

PROOF In fact,

$$(2n)!\Lambda_n(f)(x) = f_x^{(2n)}(0)$$

$$= \frac{d}{dy} |f(x + \sqrt{-1}\, y)|^2 \Big|_{y=0}$$

$$= \frac{d}{dy} (f(x + \sqrt{-1}\, y) f(x - \sqrt{-1}\, y)) \Big|_{y=0}$$

$$= \sum_{k=0}^{n} \binom{2n}{k} \frac{d^k}{dy^k} f(x + \sqrt{-1}\, y) \Big|_{y=0} \cdot \frac{d^{2n-k}}{dy^{2n-k}} f(x - \sqrt{-1}\, y) \Big|_{y=0}$$

$$= \sum_{k=0}^{n} (-1)^{k+n} \binom{2n}{k} f^{(k)}(x) f^{(2n-k)}(x)$$

$$= (2n)!L_n(f)(x).$$

When convenient to do so, write

$$\begin{cases} L_n(f)(t) = L_n(f(t)) \\ \Lambda_n(f)(t) = \Lambda_n(f(t)). \end{cases}$$

13.10 LEMMA For every real a,

$$L_n((x + a)f(x)) = (x + a)^2\, L_n(f(x)) + L_{n-1}(f(x)) \quad (n = 1,2,\ldots).$$

PROOF From the definitions,

$$\sum_{n=0}^{\infty} L_n((x + a)f(x))y^{2n}$$

$$= \sum_{n=0}^{\infty} \Lambda_n((x + a)(f(x))y^{2n}$$

$$= |(x + a + \sqrt{-1}\, y)f(x + \sqrt{-1}\, y)|^2$$

$$= ((x + a)^2 + y^2) \sum_{n=0}^{\infty} \Lambda_n(f(x))y^{2n}$$

$$= (x + a)^2 \sum_{n=0}^{\infty} \Lambda_n(f(x))y^{2n} + \sum_{n=0}^{\infty} \Lambda_n(f(x))y^{2n+2}$$

$$= (x + a)^2 \sum_{n=0}^{\infty} \Lambda_n(f(x))y^{2n} + \sum_{n=1}^{\infty} \Lambda_{n-1}(f(x))y^{2n}$$

$$= (x + a)^2\Lambda_0(f(x)) + \sum_{n=1}^{\infty} [(x + a)^2\Lambda_n(f(x)) + \Lambda_{n-1}(f(x))]y^{2n}$$

$$= (x + a)^2 L_0(f(x)) + \sum_{n=1}^{\infty} [(x + a)^2 L_n(f(x)) + L_{n-1}(f(x))]y^{2n}.$$

To establish the necessity in 13.7, it can be assumed that f is a real polynomial with real zeros only. For this purpose, proceed by induction on the degree of f, the assertion being clear when deg f = 0. If deg f > 0, write f(x) = (x + a)g(x), where $a \in R$ and g(x) is a real polynomial with real zeros only. By the induction hypothesis, $L_n(g(x)) \geq 0$ for all $n \geq 0$. Now apply 13.10 to see that the same is true of f.

Turning to the sufficiency in 13.7, if $f \not\equiv 0$ is not in $L - P$, then f has a nonreal zero $z_0 = x_0 + \sqrt{-1}\, y_0$, so

$$0 = |f(z_0)|^2 = \sum_{n=0}^{\infty} L_n(f)(x_0) y_0^{2n} \quad (y_0 \neq 0).$$

Since each term in the sum on the right is nonnegative, it follows that $L_n(f)(x_0) = 0 \ \forall\, n \geq 0$, hence $\forall\, y \in R$,

$$0 = |f(x_0 + \sqrt{-1}\, y)|^2 = \sum_{n=0}^{\infty} L_n(f)(x_0) y^{2n},$$

implying thereby that $f \equiv 0$.

[Note: The assumption that $f \in A - L - P$ serves to ensure that if $f \notin 0 - L - P$ $(= L - P)$, then f has a nonreal zero.]

13.11 EXAMPLE Take $f(z) = (z^2+1)e^z$ -- then

$$\begin{cases} L_1(f)(t) = 2(t^2 - 1)e^{2t} \\ L_2(f)(t) = e^{2t} \end{cases}$$

and $L_n(f)(t) = 0$ $(n > 2)$. Here

$$t^2 < 1 \Rightarrow L_1(f)(t) < 0$$

and, of course, $f \notin L - P$ (but $f \in * - L - P$).

13.12 THEOREM Let $f \in A - L - P$ (cf. 10.31) -- then $f \in 0 - L - P$ $(= L - P)$ iff $\forall\, z$,

$$|f'(z)|^2 \geq \text{Re}(f(z)\overline{f''(z)}).$$

PROOF Suppose first that $f \in L - P$:

$$|f(x + \sqrt{-1}\, y)|^2 = \sum_{n=0}^{\infty} L_n(f)(x)y^{2n}$$

=>

$$\frac{\partial^2}{\partial y^2} |f(x + \sqrt{-1}\, y)|^2$$

$$= \sum_{n=0}^{\infty} (2n + 2)(2n + 1)L_{n+1}(f)(x)y^{2n}$$

$$\geq 0 \text{ (cf. 13.7)}.$$

On the other hand,

$$\frac{\partial^2}{\partial y^2} |f(x + \sqrt{-1}\, y)|^2 = 2|f'(z)|^2 - 2\text{Re}(f(z)\overline{f''(z)}).$$

As for the converse, let $z_0 = x_0 + \sqrt{-1}\, y_0$ be a zero of f and consider

$$f_0(y) \equiv f_{x_0}(y) = |f(x_0 + \sqrt{-1}\, y)|^2.$$

Then

$$\frac{d^2}{dy^2} f_0(y) \geq 0,$$

so $f_0(y)$ is a convex even function of y, thus has a unique minimum, which must be taken on at $y = 0$. But

$$0 = f(z_0) = f(x_0 + \sqrt{-1}\, y_0) \Rightarrow y_0 = 0.$$

Therefore the zeros of f are real, hence $f \in 0 - \mathcal{L} - \mathcal{P}$ $(= \mathcal{L} - \mathcal{P})$.

13.13 THEOREM Let $f \in A - \mathcal{L} - \mathcal{P}$ (cf. 10.31) -- then $f \in 0 - \mathcal{L} - \mathcal{P}$ $(= \mathcal{L} - \mathcal{P})$ iff $\forall\ z = x + \sqrt{-1}\, y$ $(y \neq 0)$,

$$\frac{1}{y} \operatorname{Im}(- f'(z)\overline{f(z)}) \geq 0.$$

[This is a simple consequence of the canonical computation... .]

APPENDIX

Let $f \in \mathcal{L} - \mathcal{P}$ be transcendental. If $f(t_0) \neq 0$ and $f'(t_0) = 0$, then $f(t_0)f''(t_0) < 0$ (cf. 13.3), so t_0 is a simple zero of $f' \in \mathcal{L} - \mathcal{P}$.

LEMMA Let $f \in \mathcal{L} - \mathcal{P}$ be transcendental. Suppose that $f^{(n)}$ has a multiple zero at t_0 -- then

$$f(t_0) = f'(t_0) = \cdots = f^{(n)}(t_0) = 0.$$

SCHOLIUM If the zeros of f are simple, then the zeros of all of its derivatives are simple.

THEOREM Let $f \in \mathcal{L} - \mathcal{P}$ be transcendental. Assume: f satisfies the differential equation

$$f^{(n)}(z) = A(z)f(z),$$

where $A|R$ is real analytic -- then the zeros of f are simple.

PROOF Proceeding by contradiction, suppose that at some t_0, $f(t_0) = f'(t_0) = 0$, thus $f^{(n)}(t_0) = 0$. Since

$$f^{(n+1)}(z) = A'(z)f(z) + A(z)f'(z),$$

it follows that $f^{(n+1)}(t_0) = 0$. Owing now to the lemma,

$$f(t_0) = f'(t_0) = \cdots = f^{(n)}(t_0) = f^{(n+1)}(t_0) = 0.$$

But

$$f^{(n+k)}(z) = \sum_{\ell=0}^{k} \binom{k}{\ell} A^{(k-\ell)}(z) f^{(\ell)}(z).$$

Therefore f and all its derivatives vanish at t_0, a non sequitur.

§14. SHIFTED SUMS

Let $f \not\equiv 0$ be a real entire function.

14.1 NOTATION Given a real number λ, put

$$f_\lambda(z) = f(z + \sqrt{-1}\,\lambda) + f(z - \sqrt{-1}\,\lambda).$$

[Note: f_λ is again a real entire function.]

Obviously,

$$f_\lambda = f_{-\lambda}.$$

14.2 EXAMPLE Take $f(z) = z^n$ -- then

$$f_\lambda(z) = 2 \prod_{k=0}^{n-1} \left(z - \lambda \cot\left[\frac{(2k+1)\pi}{2n}\right]\right).$$

14.3 EXAMPLE Take $f(z) = \begin{cases} \sin z \\ \cos z \end{cases}$ -- then

$$f_\lambda(z) = 2 \cosh \lambda \begin{cases} \sin z \\ \cos z. \end{cases}$$

Let EX_f denote the set of λ such that $f_\lambda \equiv 0$ or for which f_λ has the form $C_\lambda \exp(b_\lambda z)$, where $C_\lambda \neq 0$ and b_λ are real constants.

14.4 LEMMA Suppose that f is not of the form Ce^{bz}, where $C \neq 0$ and b are real constants -- then EX_f is a discrete subset of R (if not empty).

[In fact,

$$EX_f = \{\lambda : L_1(f_\lambda) \equiv 0\}.]$$

14.5 EXAMPLE Take $f(z) = e^z$ -- then

$$f_\lambda(z) = 2(\cos \lambda)e^z,$$

so $EX_f = R$.

[Note: f is in $L - P$ but technically the zero function (e.g., $f_{\frac{\pi}{2}}$) is not in $L - P$.]

14.6 EXAMPLE Take $f(z) = e^z(a_0 + a_1 z)$, where a_0 and $a_1 \neq 0$ are real -- then

$$f_\lambda(z) = e^z(A_1 z + A_0),$$

where

$$A_1 = 2a_1 \cos \lambda$$

and

$$A_0 = 2a_0 \cos \lambda - 2a_1 \lambda \sin \lambda.$$

Therefore

$$EX_f = \{(2k + 1) \frac{\pi}{2}: k = 0, \pm 1, \ldots\}.$$

And

$$\lambda \in EX_f \ (\lambda \neq 0) \Rightarrow A_0 = -2a_1 \lambda \sin \lambda \neq 0$$

$$\Rightarrow f_\lambda \not\equiv 0.$$

14.7 EXAMPLE Take

$$f(z) = e^{bz}p(z) \qquad (b \text{ real}),$$

where

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n \qquad (a_n \neq 0)$$

is a real polynomial of degree $n \geq 2$ with real zeros only -- then

$$f_\lambda(z) = e^{bz}(A_n z^n + A_{n-1}z^{n-1} + \cdots + A_0).$$

Here

$$A_n = 2a_n \cos \lambda b$$

and

$$A_{n-1} = 2a_{n-1} \cos \lambda b - 2\lambda n\, a_n \sin \lambda b.$$

- If $\cos \lambda b \neq 0$, then $A_n \neq 0$ and f_λ has n zeros.

- If $\cos \lambda b = 0$, then $A_n = 0$ but if in addition $\lambda \neq 0$, then $A_{n-1} \neq 0$,

thus f_λ has n-1 zeros.

Since $n \geq 2$, the conclusion is that $EX_f = \emptyset$.

14.8 REMARK It is clear that if $\forall\ \lambda$, $f_\lambda \not\equiv 0$ has a zero, then $EX_f = \emptyset$.

[For instance, if $f \in L - P$ and if

$$f(z) = Cz^m e^{bz} \prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n})e^{z/\lambda_n} \quad \text{(cf. 10.19)}$$

has an infinite number of zeros, then $\forall\ \lambda$, $f_\lambda \not\equiv 0$ has an infinite number of zeros, hence $EX_f = \emptyset$.]

14.9 LEMMA If $f \in L - P$, then $\forall\ \lambda \in R$, either $f_\lambda \in L - P$ or $f_\lambda \equiv 0$.

PROOF By the usual approximation argument, it will be enough to consider the case when f is a real polynomial with real zeros only, say

$$f(z) = Cz^m \prod_{n=1}^{N} (1 - \frac{z}{\lambda_n}) \quad (C \neq 0).$$

So take $\lambda > 0$ and suppose that $f_\lambda(z) = 0$ $(z = x + \sqrt{-1}\, y)$ -- then

$$|f(z + \sqrt{-1}\,\lambda)| = |f(z - \sqrt{-1}\,\lambda)|$$

=>

$$1 = \frac{|f(z + \sqrt{-1}\,\lambda)|^2}{|f(z - \sqrt{-1}\,\lambda)|^2}$$

$$= \frac{|(z + \sqrt{-1}\,\lambda)^2|^m}{|(z - \sqrt{-1}\,\lambda)^2|^m} \cdot \frac{\prod_{n=1}^{N} |\lambda_n - (z + \sqrt{-1}\,\lambda)|^2}{\prod_{n=1}^{N} |\lambda_n - (z - \sqrt{-1}\,\lambda)|^2}$$

$$= \left[\frac{x^2 + (y + \lambda)^2}{x^2 + (y - \lambda)^2}\right]^m \cdot \prod_{n=1}^{N} \frac{(x - \lambda_n)^2 + (y + \lambda)^2}{(x - \lambda_n)^2 + (y - \lambda)^2}.$$

If $y > 0$, then all factors on the RHS are > 1, while if $y < 0$, then all factors on the RHS are < 1. As this is impossible, it follows that $y = 0$.

[Note: More generally, the same argument can be used to show that the polynomial

$$f(z + \sqrt{-1}\,\lambda) - \gamma f(z - \sqrt{-1}\,\lambda) \qquad (\gamma \in \mathbf{C},\ |\gamma| = 1)$$

has real zeros only.]

N.B. Consequently, $\forall\ \lambda \in \mathbf{R}$,

$$f \in L - P \Rightarrow L_1(f_\lambda)(t) \geq 0 \qquad (t \in \mathbf{R}) \qquad (\text{cf. } 13.3).$$

14.10 EXAMPLE Take $f(z) = z(1 + z^2)$ -- then

$$L_1(f_\lambda)(t) = 12t^4 + (6\lambda^2 - 2)^2 \geq 0,$$

yet $f \notin L - P$.

[Note:

$$L_1(f_\lambda)(0) = (6\lambda^2 - 2)^2$$

and the expression on the right vanishes at $\lambda = \pm \frac{1}{\sqrt{3}}$.]

14.11 LEMMA If $f \in \mathcal{L} - \mathcal{P}$ and if $EX_f = \emptyset$, then $\forall\ \lambda \neq 0$, the zeros of f_λ are simple.

PROOF Take $\lambda > 0$ and suppose that t_0 is a multiple zero of f_λ:

$$\begin{cases} f_\lambda(t_0) = 0 \Rightarrow f(t_0 + \sqrt{-1}\,\lambda) = -f(t_0 - \sqrt{-1}\,\lambda) \\ f_\lambda'(t_0) = 0 \Rightarrow f'(t_0 - \sqrt{-1}\,\lambda) = -f'(t_0 + \sqrt{-1}\,\lambda). \end{cases}$$

Now

$$f(t_0 - \sqrt{-1}\,\lambda)f'(t_0 + \sqrt{-1}\,\lambda)$$

is real iff

$$f(t_0 - \sqrt{-1}\,\lambda)f'(t_0 + \sqrt{-1}\,\lambda) = \overline{f(t_0 - \sqrt{-1}\,\lambda)f'(t_0 + \sqrt{-1}\,\lambda)}.$$

But

$$\overline{f(t_0 - \sqrt{-1}\,\lambda)f'(t_0 + \sqrt{-1}\,\lambda)}$$

$$= f(t_0 + \sqrt{-1}\,\lambda)f'(t_0 - \sqrt{-1}\,\lambda)$$

$$= (-f(t_0 - \sqrt{-1}\,\lambda))(-f'(t_0 + \sqrt{-1}\,\lambda))$$

$$= f(t_0 - \sqrt{-1}\,\lambda)f'(t_0 + \sqrt{-1}\,\lambda).$$

On the other hand, for Im z > 0,

$$\operatorname{Im} \frac{f'(z)}{f(z)} = \operatorname{Im}\Big(\frac{m}{z} + 2az + b + \sum_{n=1}^{\infty} \Big(\frac{1}{z - \lambda_n} + \frac{1}{\lambda_n}\Big)\Big)$$

$$< 0.$$

Setting $z = t_0 + \sqrt{-1}\,\lambda$ then leads to a contradiction:

$$\operatorname{Im} \frac{f'(t_0 + \sqrt{-1}\,\lambda)}{f(t_0 + \sqrt{-1}\,\lambda)} = \operatorname{Im} \frac{f'(t_0 + \sqrt{-1}\,\lambda)\overline{f(t_0 + \sqrt{-1}\,\lambda)}}{|f(t_0 + \sqrt{-1}\,\lambda)|^2}$$

$$= \frac{1}{|f(t_0 + \sqrt{-1}\,\lambda)|^2} \operatorname{Im}(f'(t_0 + \sqrt{-1}\,\lambda) f(t_0 - \sqrt{-1}\,\lambda))$$

$$= 0.$$

[Note: This point is illustrated by 14.2 and 14.3.]

14.12 THEOREM If $f \in L - P$ and if $EX_f = \emptyset$, then $\forall\, \lambda \neq 0$,

$$L_1(f_\lambda)(t) > 0 \quad (t \in R) \quad \text{(cf. 13.3)}.$$

14.13 REMARK Suppose that $f \in A - L - P$ has the property that $\forall\, \lambda \neq 0$,

$$L_1(f_\lambda)(t) > 0 \quad (t \in R) \quad \text{(cf. 13.3)}.$$

Then $EX_f = \emptyset$ and it is an open question as to whether $f \in L - P$.

[Note: If specialized to the case when $f \in * - L - P$, the stated condition does indeed imply that $f \in L - P$. In passing, observe that the strict inequality $L_1(f_\lambda)(t) > 0$ is necessary (cf. 14.10).]

§15. *JENSEN CIRCLES* [BIS]

Given a real polynomial f, denote by $z_1,\ldots,z_\ell$ those zeros of f which lie in the open upper half-plane.

15.1 NOTATION Given a real polynomial f and a real number λ, for $j = 1,\ldots,\ell$, put

$$\mathbb{C}_j(\lambda) = \{z \in \mathbf{C}: |z - \operatorname{Re} z_j|^2 \leq (\operatorname{Im} z_j)^2 - \lambda^2\}.$$

[Note: Take $\mathbb{C}_j(\lambda) = \emptyset$ if $|\lambda| > |\operatorname{Im} z_j|$.]

<u>N.B.</u> In particular:

$$\mathbb{C}_j(0) = \mathbb{C}_j \quad \text{(cf. 9.2)}.$$

15.2 THEOREM For any $\lambda \neq 0$, the nonreal zeros of the polynomial

$$f(z + \sqrt{-1}\,\lambda) - \gamma f(z - \sqrt{-1}\,\lambda) \quad (\gamma \in \mathbf{C},\ |\gamma| = 1)$$

lie in the union of the $\mathbb{C}_j(\lambda)$.

PROOF Take f monic of degree n, so

$$f(z) = \prod_{\operatorname{Im} z_i=0} (z - z_i)^{m_i} \cdot \prod_{j=1}^{\ell} (z - z_j)^{m_j} (z - \bar{z}_j)^{m_j} \quad \text{(cf. 9.3)}.$$

Write

$$z = x + \sqrt{-1}\,y \text{ and } z_j = x_j + \sqrt{-1}\,y_j \quad (j = 1,\ldots,\ell).$$

Then

- $|z + \sqrt{-1}\,\lambda - z_i|^2 - |z - \sqrt{-1}\,\lambda - z_i|^2$

$$= 4\lambda y \quad (\operatorname{Im} z_i = 0).$$

- $$|z + \sqrt{-1}\,\lambda - z_j|^2 |z + \sqrt{-1}\,\lambda - \bar{z}_j|^2$$

$$- |z - \sqrt{-1}\,\lambda - z_j|^2 |z - \sqrt{-1}\,\lambda - \bar{z}_j|^2$$

$$= 8\lambda y[(x - x_j)^2 + y^2 + \lambda^2 - y_j^2].$$

If now z is nonreal and lies outside all the $\mathbb{C}_j(\lambda)$, then

$$(x - x_j)^2 + y^2 + \lambda^2 - y_j^2 > 0.$$

Therefore every factor in the product representation of $|f(z + \sqrt{-1}\,\lambda)|^2$ is larger than the corresponding factor in the product representation of $|f(z - \sqrt{-1}\,\lambda)|^2$ if $\lambda y > 0$ and vice-versa if $\lambda y < 0$. To recapitulate:

$$\left[\begin{array}{l} \lambda y > 0 \Rightarrow |f(z + \sqrt{-1}\,\lambda)| > |f(z - \sqrt{-1}\,\lambda)| \\ \lambda y < 0 \Rightarrow |f(z + \sqrt{-1}\,\lambda)| < |f(z - \sqrt{-1}\,\lambda)|. \end{array}\right.$$

Accordingly, at such a z, the polynomial

$$f(z + \sqrt{-1}\,\lambda) - \gamma f(z - \sqrt{-1}\,\lambda)$$

cannot vanish.

<u>N.B.</u> If $|\lambda| = |\text{Im } z_j| = |y_j|$, then

$$\mathbb{C}_j(\lambda) = \{z \in \mathbb{C}: (x - x_j)^2 + y^2 \le y_j^2 - \lambda^2 = 0\},$$

so in this situation, $x = x_j$ and $y = 0$, thus

$$\mathbb{C}_j(\lambda) = \{(x_j, 0)\}.$$

15.3 COROLLARY For any $\lambda \neq 0$, the nonreal zeros of the polynomial

$$f_\lambda(z) = f(z + \sqrt{-1}\,\lambda) + f(z - \sqrt{-1}\,z)$$

lie in the union of the $\mathbb{C}_j(\lambda)$.

[Simply take $\gamma = -1$.]

15.4 COROLLARY For any $\lambda \neq 0$ and any $\xi \in C$ $(\xi \neq 0)$, the nonreal zeros of the polynomial

$$\xi f(z + \sqrt{-1}\,\lambda) + \bar{\xi} f(z - \sqrt{-1}\,\lambda)$$

lie in the union of the $\mathbb{C}_j(\lambda)$.

[Simply take $\gamma = -\frac{\bar{\xi}}{\xi}$.]

15.5 REMARK One can recover 9.3 from 15.2. Thus let $\lambda_n = \frac{1}{n}$ and consider

$$f_n(z) = \frac{f(z + \sqrt{-1}\,\lambda_n) - f(z - \sqrt{-1}\,\lambda_n)}{2\lambda_n}.$$

Then

$$\lim_{n \to \infty} f_n(z) = f'(z)$$

uniformly on compact subsets of C. Moreover, the zeros of $f_n(z)$ are contained in the union of the $\mathbb{C}_j(\lambda_n)$ and the real line which is a subset of the union of the Jensen circles of f and the real line.

15.6 LEMMA Let f be a real polynomial whose zeros lie in the strip

$$S(A) = \{z: |\mathrm{Im}\, z| \leq A\} \quad (A > 0).$$

Then $\forall\, \lambda \neq 0$, the zeros of the polynomial

$$f(z + \sqrt{-1}\,\lambda) - \gamma f(z - \sqrt{-1}\,\lambda) \quad (\gamma \in C,\ |\gamma| = 1)$$

lie in $S(\sqrt{A^2 - \lambda^2})$ if $|\lambda| < A$ and lie in $S(0) = R$ if $A \leq |\lambda|$.

PROOF If $z = x + \sqrt{-1}\, y \in \mathfrak{C}_j(\lambda)$ is a nonreal zero and if $|\lambda| < A$, then

$$y^2 \leq (x - x_j)^2 + y^2 \leq y_j^2 - \lambda^2 \leq A^2 - \lambda^2,$$

hence $z \in S(\sqrt{A^2 - \lambda^2})$. Meanwhile, at the transition point $A = |\lambda|$, there is no nonreal zero in any of the $\mathfrak{C}_j(\lambda)$ and on the other side $A < |\lambda|$, all the $\mathfrak{C}_j(\lambda)$ are empty.

15.7 REMARK If $A = 0$, hence if $f \in L - P$, then $\forall\ \lambda \neq 0$, the zeros of the polynomial

$$f(z + \sqrt{-1}\,\lambda) - \gamma f(z - \sqrt{-1}\,\lambda) \quad (\gamma \in \mathbb{C},\ |\gamma| = 1)$$

are real (cf. 14.9) and this persists to $\lambda = 0$:

$$f(z) - \gamma f(z) = (1 - \gamma) f(z).$$

15.8 THEOREM Let $f \in A - L - P$ (cf. 10.31) -- then the zeros of f_λ lie in $S(\sqrt{A^2 - \lambda^2})$ if $|\lambda| < A$ and lie in $S(0) = R$ if $A \leq |\lambda|$.

[Taking into account 15.6 and 15.7, apply 10.32.]

[Note: It is a corollary that

$$f_\lambda \in A_\lambda - L - P,$$

where

$$A_\lambda = (\max(A^2 - \lambda^2, 0))^{1/2}.]$$

§16. *STURM CHAINS*

Given nonconstant real polynomials P and Q, put

$$F(z) = P(z) + \sqrt{-1}\, Q(z).$$

16.1 LEMMA Suppose that F(z) has all its zeros in either the open upper half-plane or the open lower half-plane -- then P and Q have real zeros only.

PROOF Working under the open lower half-plane supposition, write

$$F(z) = C_n(z - z_1)\ldots(z - z_n) \qquad (C_n \neq 0).$$

Then for Im z > 0,

$$|z - z_k| > |\bar{z} - z_k| \ (\text{Im } z_k < 0,\ k = 1,\ldots,n)$$

$$\Rightarrow$$

$$|F(z)| > |F(\bar{z})|$$

$$\Rightarrow$$

$$2\sqrt{-1}\ (P(\bar{z})Q(z) - P(z)Q(\bar{z}))$$

$$= F(z)\overline{F(z)} - F(\bar{z})\overline{F(\bar{z})}$$

$$> 0.$$

Therefore P and Q have real zeros only (nonreal zeros of either P or Q would occur in conjugate pairs).

[Note: P and Q have no common zero (otherwise F would have a real zero: $|F(x)|^2 = P(x)^2 + Q(x)^2$).]

Here is an application. Let f be a nonconstant real polynomial with real zeros only, so $f \in L - P$, thus taking $\lambda > 0$, the zeros of $f(z + \sqrt{-1}\,\lambda)$ lie in the open lower half-plane. Define nonconstant real polynomials P and Q by writing

$$f(z + \sqrt{-1}\,\lambda) = P(z) + \sqrt{-1}\,Q(z).$$

Then $P,Q \in L - P$ and $\forall\, x \in R$,

$$f_\lambda(x) = f(x + \sqrt{-1}\,\lambda) + \overline{f(x + \sqrt{-1}\,\lambda)} = 2P(x)$$

$$\Rightarrow f_\lambda \in L - P \text{ (cf. 14.9)}.$$

16.2 REMARK If μ and ν are real and if $\mu^2 + \nu^2 > 0$, then the zeros of F and

$$(\mu - \sqrt{-1}\,\nu)F = (\mu P + \nu Q) + \sqrt{-1}\,(\mu Q - \nu P)$$

are the same. Therefore

$$\begin{cases} \mu P + \nu Q \\ \mu Q - \nu P \end{cases}$$

have real zeros only.

16.3 SUBLEMMA The zeros of

$$\left(1 + \frac{\sqrt{-1}\,\lambda z}{n}\right)^n \qquad (\lambda > 0)$$

lie in the open upper half-plane, hence the zeros of

$$1 - \binom{n}{2}\frac{\lambda^2 z^2}{n^2} + \binom{n}{4}\frac{\lambda^4 z^4}{n^4} - \cdots$$

are real (cf. 16.1).

16.4 LEMMA Let f be a real polynomial -- then f_λ has at least as many real zeros as f does.

PROOF Take $\lambda > 0$ -- then the polynomial

$$f(z) - \binom{n}{2} \frac{\lambda^2}{n^2} f''(z) + \binom{n}{4} \frac{\lambda^4}{n^4} f''''(z) - \cdots$$

has at least as many real zeros as $f(z)$ does (cf. 12.10). But there is an expansion

$$\frac{f_\lambda(z)}{2} = f(z) - \frac{\lambda^2}{2!} f''(z) + \frac{\lambda^4}{4!} f''''(z) - \cdots,$$

so it remains only to let $n \to \infty$.

16.5 LEMMA Assume:

- $F(z)$ has n zeros in the closed lower half-plane

or

- $F(z)$ has n zeros in the closed upper half-plane.

Then P and Q have n pairs of nonreal zeros at most.

[Note: The case $n = 0$ is 16.1.]

There is more to be said about (P,Q) and F but for this it will be best to first introduce some machinery.

Let

$$P_n(x), P_{n-1}(x), \ldots, P_1(x), P_0(x)$$

be a sequence of real polynomials such that $\deg P_k = k$ and $P_k^{(k)}(0) > 0$ $(k = 0, \ldots, n)$.

[Note: Therefore $P_0(x)$ is a positive constant.]

16.6 DEFINITION The P_k are a Sturm chain if the following conditions are satisfied.

- Two consecutive terms P_k, P_{k+1} cannot vanish simultaneously.

- Whenever one of the $P_{n-1},\ldots,P_1$ vanishes, the neighboring terms have opposite signs.

16.7 EXAMPLE Consider the Legendre polynomials

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \quad \text{(cf. 8.17).}$$

Then

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{3}{2} x^2 - \frac{1}{2},$$

and for $k > 2$,

$$P_k(x) = \frac{2^k (\frac{1}{2})_k}{k!} x^k + \pi_{k-2}(x),$$

where π_{k-2} is a polynomial of degree $(k-2)$ in x. Furthermore, there is a recurrence relation

$$(k + 1)P_{k+1}(x) = (2k + 1)xP_k(x) - kP_{k-1}(x).$$

Thus, in consequence, the sequence

$$P_n(x),\ P_{n-1}(x),\ldots,\ P_1(x),\ P_0(x)$$

is a Sturm chain.

[Note: This setup is the tip of the iceberg: Consider a weight function $w(x) > 0$ $(a < x < b)$ (a or b potentially infinite) and an associated sequence $\{P_n(x)\}$ of orthogonal real polynomials.]

16.8 EXAMPLE Fix $\lambda > -1$ and let

$$P_{\lambda,n}(x) = \int_{-1}^{1} (1 - t^2)^\lambda (x + \sqrt{-1}\ t)^n dt \quad (n = 0,1,\ldots).$$

Then the sequence

$$P_{\lambda,n}(x),\ P_{\lambda,n-1}(x),\ldots,\ P_{\lambda,1}(x),\ P_{\lambda,0}(x)$$

is a Sturm chain.

16.9 STURM CRITERION Suppose that

$$P_n(x),\ P_{n-1}(x),\ldots,\ P_1(x),\ P_0(x)$$

is a Sturm chain -- then the zeros of the P_k $(k = 1,\ldots,n)$ are real and simple.

Return now to

$$F(z) = P(z) + \sqrt{-1}\,Q(z).$$

16.10 LEMMA Under the assumptions of 16.1, P and Q have real zeros only and, in addition, these zeros are simple.

[Note: The new information is the assertion of simplicity.]

It suffices to work with P (since $-\sqrt{-1}\,F = Q - \sqrt{-1}\,P$), the idea being to exhibit a Sturm chain

$$P(x) = P_n(x),\ P_{n-1}(x),\ldots,\ P_1(x),\ P_0(x),$$

thereby enabling one to quote 16.9.

As before, write

$$F(z) = C_n(z - z_1)\ldots(z - z_n) \quad (C_n \neq 0),$$

take $C_n = 1$, and let

$$z_1 = a_1 + \sqrt{-1}\,b_1 \ (b_1 < 0),\ldots,\ z_n = a_n + \sqrt{-1}\,b_n \ (b_n < 0).$$

Put

$$F_k(x) = (x - a_1 - \sqrt{-1}\,b_1)\ldots(x - a_k - \sqrt{-1}\,b_k)$$

$$\equiv P_k(x) + \sqrt{-1}\,Q_k(x).$$

Then

$$\begin{cases} P_k(x) = (x - a_k)P_{k-1}(x) + b_kQ_{k-1}(x) \\ Q_k(x) = - b_kP_{k-1}(x) + (x - a_k)Q_{k-1}(x). \end{cases}$$

Replacing k by k + 1 gives

$$P_{k+1}(x) = (x - a_{k+1})P_k(x) + b_{k+1}Q_k(x)$$

from which (by elimination of $Q_k(x)$)

$$b_kP_{k+1}(x) = (b_k(x - a_{k+1}) + b_{k+1}(x - a_k))P_k(x)$$

$$- b_{k+1}(b_k^2 + (x - a_k)^2)P_{k-1}(x).$$

Setting $P_0(x) = 1$ and noting that by construction, the P_k are monic, it thus follows that

$$P(x) = P_n(x), P_{n-1}(x), \ldots, P_1(x), P_0(x)$$

is a Sturm chain, as desired.

At this juncture, return to the inequality

$$2\sqrt{-1}\ (P(\bar{z})Q(z) - P(z)Q(\bar{z})) > 0 \quad (\text{Im } z > 0)$$

and divide it by $- 2\sqrt{-1}\ (z - \bar{z})$ to get

$$- \frac{P(\bar{z})(Q(z) - Q(\bar{z})) - Q(\bar{z})(P(z) - P(\bar{z}))}{z - \bar{z}} > 0 \quad (\text{Im } z > 0).$$

Letting z approach the real axis, we conclude that

$$Q(x)P'(x) - P(x)Q'(x) \geq 0.$$

16.11 REMARK Recall that P and Q have no common zeros, so if $P(x_0) = 0$, then $Q(x_0) \neq 0$. On the other hand, x_0 is simple (cf. 16.10), hence $P'(x_0) \neq 0$. Therefore

$$Q(x_0)P'(x_0) - P(x_0)Q'(x_0) = Q(x_0)P'(x_0) > 0.$$

Accordingly,

$$Q(x)P'(x) - P(x)Q'(x) > 0$$

whenever $P(x) = 0$ (and, analogously, whenever $Q(x) = 0$).

16.12 LEMMA Between any two consecutive zeros of Q there is one and only one zero of P and between any two consecutive zeros of P there is one and only one zero of Q, i.e., P and Q have <u>interlacing zeros</u>.

PROOF The rational function

$$R(x) = \frac{P(x)}{Q(x)}$$

has a nonnegative derivative at all x except at the zeros of $Q(x)$. Moreover, between any two consecutive zeros of $Q(x)$, $R(x)$ climbs from $-\infty$ to $+\infty$ and, in so doing, determines a unique zero of $P(x)$.

16.13 REMARK This property of the data forces an after the fact restriction on the degrees of P and Q, viz.

$$\deg P = \deg Q \text{ or } \left[\begin{array}{l} \deg P = \deg Q + 1 \\ \\ \deg Q = \deg P + 1. \end{array}\right.$$

The preceding considerations can be turned around. Spelled out, make the following assumptions.

- The zeros of P and Q are real and simple.
- The zeros of P and Q are interlacing.

- There exists an x_0 such that

$$Q(x_0)P'(x_0) - P(x_0)Q'(x_0) > 0.$$

Then

$$F(z) = P(z) + \sqrt{-1}\, Q(z)$$

has all its zeros in the open lower half-plane.

To begin with, it is clear that P and Q do not have a common zero (their zeros being interlacing), thus F cannot have a real zero. Suppose, therefore, that $F(z_0) = 0$, where $z_0 = x_0 + \sqrt{-1}\, y_0$ $(y_0 \neq 0)$ -- then

$$\frac{P(z_0)}{Q(z_0)} + \sqrt{-1} = 0.$$

Denoting by $a_1 < a_2 < \cdots < a_n$ the zeros of Q, pass to the decomposition

$$\frac{P(z)}{Q(z)} = A + \frac{A_1}{z - a_1} + \frac{A_2}{z - a_2} + \cdots + \frac{A_n}{z - a_n},$$

where A is a real constant and

$$A_k = \frac{P(a_k)}{Q'(a_k)} \quad (k = 1,2,\ldots,n).$$

Here

$$\begin{cases} P(a_k)P(a_{k+1}) < 0 \\ Q'(a_k)Q'(a_{k+1}) < 0, \end{cases}$$

so

$$A_1, A_2, \ldots, A_n$$

have one and the same sign. But

$$-\sqrt{-1} = A + \frac{A_1}{z_0 - a_1} + \frac{A_2}{z_0 - a_2} + \cdots + \frac{A_n}{z_0 - a_n}$$

=>

$$-1 = -y_0 \sum_{k=1}^{n} \frac{A_k}{(x_0-a_k)^2 + y_0^2}$$

=>

$$1 = y_0 \sum_{k=1}^{n} \frac{A_k}{(x_0-a_k)^2 + y_0^2} .$$

There are then two possibilities: All the A_k are > 0, in which case y_0 is positive, or all the A_k are negative, in which case y_0 is negative. And this means that $F(z)$ has all its zeros either in the open upper half-plane or the open lower half-plane.

It remains to eliminate the first contingency. However, it it held, then, arguing as before, we would have

$$Q(x)P'(x) - P(x)Q'(x) \leq 0,$$

contradicting the assumption that there exists an x_0 such that

$$Q(x_0)P'(x_0) - P(x_0)Q'(x_0) > 0.$$

[Note:

$$\forall\ k,\ A_k < 0 \Rightarrow \left(\frac{P(x)}{Q(x)}\right)' > 0 \qquad (x \neq a_k)$$

$$\Rightarrow Q(x)P'(x) - P(x)Q'(x) > 0.]$$

In summary:

$$F(z) = P(z) + \sqrt{-1}\, Q(z)$$

has all its zeros in the open lower half-plane.

16.14 REMARK The developments in this § are known collectively as <u>Hermite-Bieler theory</u>.

§17. *EXPONENTIAL TYPE*

Given an entire function

$$f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

put

$$T(f) = \overline{\lim_{r \to \infty}} \frac{\log M(r;f)}{r}.$$

17.1 DEFINITION f is of <u>exponential type</u> if $T(f) < \infty$, in which case $T(f)$ is called the <u>exponential type</u> of f.

<u>N.B.</u> f is of exponential type iff there exists a positive constant K:

$$f(z) = O(e^{K|z|}),$$

the greatest lower bound of the set of K for which such a relation holds then being the exponential type of f.

17.2 LEMMA If f is of exponential type, then its order $\rho(f)$ is ≤ 1.

17.3 LEMMA If f is of exponential type and if $T(f) > 0$, then its order $\rho(f)$ is $= 1$ and $T(f) = \tau(f)$.

17.4 LEMMA If f is of exponential type and if $T(f) = 0$, then there are two possibilities: $\rho(f) < 1$ or $\rho(f) = 1$ and $\tau(f) = 0$.

17.5 SCHOLIUM The set of entire functions of exponential type is comprised of the entire functions of order < 1 and the entire functions of order 1 and of finite type.

17.6 EXAMPLE The entire function

$$\frac{\sin \sqrt{z}}{\sqrt{z}}$$

is of order $\frac{1}{2}$. It is of type 1 but of exponential type 0.

17.7 EXAMPLE The entire function

$$\frac{1}{z\Gamma(z)}$$

is of order 1 (cf. 5.13). However, it is of maximal type (cf. 5.22), hence is not of exponential type.

17.8 LEMMA If f is of exponential type, then f' is of exponential type and $T(f) = T(f')$ (cf. 2.25 and 3.7).

17.9 LEMMA If f,g are of exponential type and if $\frac{f}{g}$ is entire, then $\frac{f}{g}$ is of exponential type.

PROOF On general grounds,

$$\begin{aligned} \rho\left(\frac{f}{g}\right) &\leq \max(\rho(f),\rho(g)) \quad \text{(cf. 2.37)} \\ &\leq \max(1,1) = 1. \end{aligned}$$

There is nothing to prove if $\rho(\frac{f}{g}) < 1$, so assume that $\rho(\frac{f}{g}) = 1$ and distinguish two cases.

<u>Case 1</u>: $\rho(g) < 1$ -- then $\rho(f) = 1$

and

$$\tau(f) = \tau(g \cdot \frac{f}{g}) = \tau(\frac{f}{g}) \quad \text{(cf. 3.14)},$$

thus $\frac{f}{g}$ is of finite type.

<u>Case 2</u>: $\rho(g) = 1$ -- then $0 \le \tau(g) < \infty$ and if $\tau(\frac{f}{g}) = \infty$, it would follow that

$$\tau(f) = \tau(g \cdot \frac{f}{g}) = \infty \quad \text{(cf. 3.14)},$$

contradicting $0 \le \tau(f) < \infty$.

17.10 THEOREM Suppose that f is an entire function -- then

$$T(f) = \frac{1}{e} \overline{\lim_{n \to \infty}} n|a_n|^{1/n} \quad \text{(cf. 3.6)}.$$

[Note: In terms of the γ_n,

$$T(f) = \overline{\lim_{n \to \infty}} |\gamma_n|^{1/n}.$$

Proof:

$$\frac{1}{e} \overline{\lim_{n \to \infty}} n|a_n|^{1/n}$$

$$= \frac{1}{e} \overline{\lim_{n \to \infty}} n\left|\frac{\gamma_n}{n!}\right|^{1/n}$$

$$= \lim_{n \to \infty} \left(\frac{n^n e^{-n}\sqrt{2\pi n}}{n!}\right)^{1/n} \frac{n}{e(n^n e^{-n}\sqrt{2\pi n})^{1/n}} |\gamma_n|^{1/n}$$

$$= \overline{\lim_{n \to \infty}} |\gamma_n|^{1/n}.]$$

17.11 APPLICATION An entire function f is of exponential type iff

$$\overline{\lim_{n \to \infty}} n|a_n|^{1/n} < \infty.$$

17.12 NOTATION E_0 is the set of entire functions of exponential type.

17.13 LEMMA E_0 is a vector space.

PROOF Let

$$\begin{cases} f(z) = \sum_{n=0}^{\infty} a_n z^n \\ g(z) = \sum_{n=0}^{\infty} b_n z^n \end{cases}$$

be elements of E_0 -- then

$$|a_n + b_n|^{1/n} \leq (2\max(|a_n|,|b_n|))^{1/n}$$

$$\leq 2^{1/n}(|a_n|^{1/n} + |b_n|^{1/n})$$

=>

$$\overline{\lim_{n \to \infty}}\ n|a_n + b_n|^{1/n}$$

$$\leq \overline{\lim_{n \to \infty}}\ 2^{1/n} n(|a_n|^{1/n} + |b_n|^{1/n})$$

$$\leq \lim_{n \to \infty} 2^{1/n} \cdot \overline{\lim_{n \to \infty}}\ (n|a_n|^{1/n} + n|b_n|)^{1/n}$$

$$\leq \overline{\lim_{n \to \infty}}\ n|a_n|^{1/n} + \overline{\lim_{n \to \infty}}\ n|b_n|^{1/n}$$

$$< \infty.$$

17.14 EXAMPLE A trigonometric polynomial

$$\sum_{k=-n}^{n} c_k e^{\sqrt{-1}\, kz}$$

is an entire function of exponential type n.

17.15 LEMMA E_0 is an algebra.

PROOF Given

$$\begin{cases} f \in E_0 \\ g \in E_0, \end{cases}$$

choose positive constants

$$\begin{cases} (K,M) \\ (L,N) \end{cases} : \begin{cases} |f(z)| \leq Me^{K|z|} \\ |g(z)| \leq Ne^{L|z|}. \end{cases}$$

Then

$$|f(z)g(z)| \leq MNe^{(K+L)|z|}.$$

17.16 LEMMA E_0 is closed under translation: If $f(z)$ is of exponential type $T(f)$ and if A,B are complex constants, then $f(AZ + B)$ is of exponential type $|A|T(f)$.

Embedded in the theory are a variety of estimates, a sampling of the simplest of these being given below.

17.17 LEMMA Let $f \in E_0$, say

$$|f(z)| \leq C_K e^{K|z|}.$$

Assume: $\forall$ real x,

$$|f(x)| \leq M.$$

Then $\forall$ real y,

$$|f(x + \sqrt{-1}\, y)| \leq Me^{K|y|}.$$

[This is a standard application of Phragmén-Lindelöf... .]

17.18 THEOREM Let $f \in E_0$. Assume: $\forall$ real x,

$$|f(x)| \leq M.$$

Then $\forall$ real y,

$$|f(x + \sqrt{-1}\, y)| \leq Me^{T(f)|y|}.$$

PROOF Given $\varepsilon > 0$, $\exists\, C_\varepsilon > 0$:

$$|f(z)| \leq C_\varepsilon \exp((T(f) + \varepsilon)|z|).$$

So, $\forall$ real y,

$$|f(x + \sqrt{-1}\, y)| \leq M \exp((T(f) + \varepsilon)|y|).$$

Now let $\varepsilon \to 0$:

$\Rightarrow$

$$|f(x + \sqrt{-1}\, y)| \leq Me^{T(f)|y|}.$$

[Note: Accordingly, if $T(f) = 0$, then f is a constant. In particular: Every entire function of order less than one which is bounded on the real axis must be a constant.]

17.19 EXAMPLE Given $\phi \in L^1[-A,A]$ $(0 < A < \infty)$, put

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \phi(t) e^{\sqrt{-1}\, zt}\, dt.$$

Then f(z) is entire and

$$|f(z)| \leq \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} |\phi(t)| e^{-yt}\, dt \quad (z = x + \sqrt{-1}\, y)$$

$$\leq \frac{1}{\sqrt{2\pi}} e^{A|y|} \int_{-A}^{A} |\phi(t)|\, dt$$

$$\Rightarrow T(f) \le A,$$

thus f(z) is of exponential type. And:

$$|f(x)| \le \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} |\phi(t)| dt$$

$$\equiv M,$$

thereby realizing the assumption of 17.18.

17.20 LEMMA Let $f \in E_0$. Suppose that

$$f(x) \to 0 \text{ as } |x| \to \infty.$$

Then

$$f(x + \sqrt{-1}\, y) \to 0 \text{ as } |x| \to \infty$$

uniformly in every horizontal strip.

[On the basis of the foregoing, this follows from Montel's theorem.]

17.21 EXAMPLE Take the data as in 17.19 -- then by the Riemann-Lebesgue lemma (cf. 21.6),

$$f(x) \to 0 \text{ as } |x| \to \infty.$$

17.22 LEMMA Let $f \in E_0$ with $T(f) > 0$. Assume: $\forall$ real x,

$$|f(x)| \le M.$$

Then

$$f'(x) = \frac{4T(f)}{\pi^2} \sum_{k=-\infty}^{\infty} (-1)^k \frac{1}{(2k+1)^2} f\left(x + \frac{2k+1}{2T(f)}\pi\right),$$

the convergence being uniform on compact subsets of R.

PROOF Suppose initially that $T(f) = 1$ and consider the meromorphic function

$$F(z) = \frac{f(z)}{z^2 \cos z}.$$

Let Γ_n be the square contour with corners at $(1 + \sqrt{-1})\pi n$, $(-1 + \sqrt{-1})\pi n$, $(-1 - \sqrt{-1})\pi n$, $(1 - \sqrt{-1})\pi n$ -- then F has no singularities on Γ_n but inside Γ_n it might have a pole at the origin or at the points $\frac{2k+1}{2}\pi$ $(-n \le k \le n-1)$. So, from residue theory,

$$\frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_n} F(z)dz$$

$$= f'(0) - \sum_{k=-n}^{n-1} (-1)^k \frac{4}{\pi^2 (2k+1)^2} f(\frac{2k+1}{2}\pi).$$

Next

$$z \in \Gamma_n \Rightarrow |\cos z| > \frac{e^{|y|}}{4} \quad (y = \text{Im } z).$$

Meanwhile (cf. 17.18),

$$|f(x + \sqrt{-1}\, y)| \le Me^{|y|} \quad (T(f) = 1).$$

Therefore

$$z \in \Gamma_n \Rightarrow |F(z)| = \frac{|f(z)|}{|z^2 \cos z|}$$

$$< 4M|z|^{-2}$$

$\Rightarrow$

$$\int_{\Gamma_n} F(z)dz \to 0 \quad (n \to \infty)$$

$\Rightarrow$

$$f'(0) = \frac{4}{\pi^2} \sum_{k=-\infty}^{\infty} (-1)^k \frac{1}{(2k+1)^2} f(\frac{2k+1}{2}\pi).$$

Working now with $f(z + x_0)$ at a fixed $x_0 \in R$ (the exponential type of this function is still 1 (cf. 17.16)), we conclude that

$$f'(x_0) = \frac{4}{\pi^2} \sum_{k=-\infty}^{\infty} (-1)^k \frac{1}{(2k+1)^2} f(x_0 + \frac{2k+1}{2} \pi).$$

Finally, to eliminate the restriction that $T(f) = 1$, consider the function $f(\frac{z}{T(f)})$ of exponential type 1 (cf. 17.16) -- then

$$f'(\frac{x}{T(f)}) \frac{1}{T(f)} = \frac{4}{\pi^2} \sum_{k=-\infty}^{\infty} (-1)^k \frac{1}{(2k+1)^2} f(\frac{x}{T(f)} + \frac{2k+1}{2T(f)} \pi),$$

i.e., $\forall$ real x,

$$f'(x) = \frac{4T(f)}{\pi^2} \sum_{k=-\infty}^{\infty} (-1)^k \frac{1}{(2k+1)^2} f(x + \frac{2k+1}{2T(f)} \pi).$$

17.23 APPLICATION Take $f(z) = \sin z$ and evaluate at $x = 0$:

$$\Rightarrow 1 = \frac{4}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{1}{(2k+1)^2}.$$

17.24 THEOREM Let $f \in E_0$ with $T(f) > 0$. Assume: $\forall$ real x,

$$|f(x)| \leq M.$$

Then

$$|f'(x)| \leq MT(f).$$

PROOF In fact,

$$\begin{aligned} |f'(x)| &\leq T(f) \frac{4}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{1}{(2k+1)^2} |f(x + \frac{2k+1}{2T(f)} \pi)| \\ &\leq MT(f) \frac{4}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{1}{(2k+1)^2} \\ &= MT(f). \end{aligned}$$

17.25 COROLLARY Let $f \in E_0$ with $T(f) > 0$. Assume: $\forall$ real x,

$$|f(x)| \leq M.$$

Then (cf. 17.8)

$$|f^{(n)}(x)| \leq MT(f)^n \quad (n = 1,2,\ldots).$$

17.26 EXAMPLE Take

$$f(z) = \sum_{k=-n}^{n} c_k e^{\sqrt{-1}\,kz} \qquad \text{(cf. 17.14)}$$

and let M be the maximum of $|f(x)|$ -- then

$$|f'(x)| \leq Mn.$$

17.27 REMARK Here is a suggestive way to write the assumption and the conclusion of 17.24:

$$|f(x)| \leq |Me^{\sqrt{-1}\,T(f)x}| \Rightarrow |f'(x)| \leq |(Me^{\sqrt{-1}\,T(f)x})'|.$$

Working on the real axis, let $||.||_p$ be the L^p-norm:

$$||f||_p = \left[\int_{-\infty}^{\infty} |f(x)|^p dx\right]^{1/p} \quad (p \geq 1).$$

[Note: $||.||_p$ is translation invariant: $\forall$ f, $\forall$ t, $||f_t||_p = ||f||_p$, where $f_t(x) = f(x + t)$.]

17.28 THEOREM Let $f \in E_0$. Assume:

$$||f||_p < \infty.$$

Then $\forall$ real y,

$$\int_{-\infty}^{\infty} |f(x + \sqrt{-1}\,y)|^p dx \leq ||f||_p^p \, e^{pT(f)|y|}.$$

PROOF It suffices to consider the case when $y > 0$. To this end, let

$$F_A(z) = \int_{-A}^{A} |f(z + t)|^p dt.$$

Then

$$|F_A(x)| \leq \int_{-\infty}^{\infty} |f(x + t)|^p dt$$

$$= ||f||_p^p < \infty.$$

In addition, $|f(z)|^p$ is subharmonic, thus $F_A(z)$ is subharmonic. Using Phragmén-Lindelöf in its subharmonic formulation, it follows that

$$|F_A(x + \sqrt{-1}\, y)| \leq ||f||_p^p \, e^{pT(f)|y|}.$$

Finish by sending A to infinity.

17.29 LEMMA Let $f \in E_0$. Assume:

$$||f||_p < \infty.$$

Then f is bounded on the real axis: $\forall$ real x,

$$|f(x)| \leq M.$$

PROOF Because $|f(z)|^p$ is subharmonic, we have

$$|f(x)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} |f(x + re^{\sqrt{-1}\,\theta})|^p d\theta$$

=>

$$|f(x)|^p \int_0^1 r dr \leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 |f(x + re^{\sqrt{-1}\,\theta})|^p r dr d\theta$$

$$\leq \frac{1}{2\pi} \iint_{s^2+t^2 \leq 1} |f(x + s + \sqrt{-1}\, t)|^p ds dt$$

$$\leq \frac{1}{2\pi} \int_{-1}^{1} dt \int_{-1}^{1} |f(x + s + \sqrt{-1}\, t)|^p ds$$

=>

$$|f(x)|^p \le \frac{1}{\pi}\int_{-1}^{1} dt \int_{-\infty}^{\infty} |f(x + s + \sqrt{-1}\, t)|^p ds$$

$$= \frac{1}{\pi}\int_{-1}^{1} dt \int_{-\infty}^{\infty} |f(s + \sqrt{-1}\, t)|^p ds$$

$$\le \frac{1}{\pi}\int_{-1}^{1} ||f||_p^p \, e^{pT(f)|t|} dt$$

$$= \frac{2}{\pi} ||f||_p^p \int_0^1 e^{pT(f)t} dt$$

$$\equiv M^p.$$

17.30 REMARK If $||f||_p < \infty$ and if $T(f) = 0$, then arguing as above,

$$|f(x + \sqrt{-1}\, y)|^p \le \frac{1}{\pi}\int_{y-1}^{y+1} dt \int_{-\infty}^{\infty} |f(s + \sqrt{-1}\, t)|^p ds$$

$$\le \frac{1}{\pi}\int_{y-1}^{y+1} ||f||_p^p \, dt \qquad \text{(cf. 17.28)}$$

$$= \frac{2}{\pi} ||f||_p^p < \infty.$$

Therefore f is a constant, hence f is identically zero (cf. 17.34).

17.31 THEOREM Let $f \in E_0$ with $T(f) > 0$. Assume:

$$f \in L^p(-\infty,\infty).$$

Then $f' \in L^p(-\infty,\infty)$ and

$$||f'||_p \le ||f||_p T(f).$$

PROOF Apply 17.22 in the obvious way (legal in view of 17.29).

17.32 SUBLEMMA If $f \in L^1(-\infty,\infty)$ and if f is uniformly continuous, then the limit of f(x) as x approaches plus or minus infinity is zero.

PROOF Given $\varepsilon > 0$, choose $\delta > 0$:

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}.$$

Choose $R > 0$:

$$\int_R^\infty |f| + \int_{-\infty}^{-R} |f| < \varepsilon\delta.$$

Claim:

$$\begin{cases} x > R + \delta \Rightarrow |f(x)| < \varepsilon \\ x < -R - \delta \Rightarrow |f(x)| < \varepsilon. \end{cases}$$

Consider the first of these assertions and to get a contradiction, assume instead that $|f(x)| \geq \varepsilon$ -- then

$$\begin{aligned} & x - \delta < y < x + \delta \\ & \Rightarrow |f(y)| = |f(x) + f(y) - f(x)| \\ & \qquad \geq |f(x)| - |f(y) - f(x)| \\ & \qquad = |f(x)| - |f(x) - f(y)| \\ & \qquad > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2} \end{aligned}$$

$\Rightarrow$

$$\int_{x-\delta}^{x+\delta} |f| > \frac{\varepsilon}{2}(2\delta) = \varepsilon\delta.$$

But

$$\int_{x-\delta}^{x+\delta} |f| < \int_R^\infty |f| < \varepsilon\delta.$$

17.33 LEMMA Let

$$\Phi = \phi * \chi_{-1,1},$$

where $\phi \in L^1(-\infty,\infty)$ and $\chi_{-1,1}$ is the characteristic function of [-1,1] -- then $\Phi \in L^1(-\infty,\infty)$ is uniformly continuous and

$$\begin{cases} \lim\limits_{x \to +\infty} \Phi(x) = 0 \\ \lim\limits_{x \to -\infty} \Phi(x) = 0. \end{cases}$$

[Note: The * stands, of course, for convolution.]

17.34 THEOREM Let $f \in E_0$. Assume:

$$||f||_p < \infty.$$

Then

$$f(x) \to 0 \text{ as } |x| \to \infty.$$

PROOF Proceeding as in 17.29,

$$\pi|f(x)|^p \le \int_{-1}^{1} dt \int_{-1}^{1} |f(x + s + \sqrt{-1}\, t)|^p ds.$$

Let

$$\phi(s) = \int_{-1}^{1} |f(s + \sqrt{-1}\, t)|^p dt.$$

Then

$$\begin{aligned} \int_{-\infty}^{\infty} |\phi(s)| ds &= \int_{-\infty}^{\infty} \left(\int_{-1}^{1} |f(s + \sqrt{-1}\, t)|^p dt\right) ds \\ &= \int_{-1}^{1} dt \int_{-\infty}^{\infty} |f(s + \sqrt{-1}\, t)|^p ds \\ &< \infty. \end{aligned}$$

I.e.: $\phi \in L^1(-\infty,\infty)$. And

$$\phi * \chi_{-1,1}(x) = \int_{-\infty}^{\infty} \phi(x - s)\chi_{-1,1}(s)ds$$

$$= \int_{-1}^{1} \phi(x - s)ds$$

$$= \int_{-1}^{1} \phi(x + s)ds$$

$$= \int_{-1}^{1} (\int_{-1}^{1} |f(x + s + \sqrt{-1}\, t)|^p dt)ds$$

$$= \int_{-1}^{1} dt \int_{-1}^{1} |f(x + s + \sqrt{-1}\, t)|^p ds.$$

Now quote 17.33.

Let $\{\lambda_n\}$ be a real increasing sequence such that $\lambda_{n+1} - \lambda_n \geq 2\delta > 0$.

[Note: The intervals $]\lambda_n - \delta, \lambda_n + \delta[$ are then pairwise disjoint:

$$\left[\begin{array}{l} x < \lambda_n + \delta \\ \\ x > \lambda_{n+1} - \delta \end{array}\right. \Rightarrow \lambda_n + \delta > \lambda_{n+1} - \delta \Rightarrow 2\delta > \lambda_{n+1} - \lambda_n.]$$

17.35 THEOREM Let $f \in E_0$. Assume:

$$||f||_p < \infty.$$

Then

$$\sum_n |f(\lambda_n)|^p \leq 2 \frac{e^{\delta p T(f)}}{\delta\pi} ||f||_p^p.$$

PROOF We have

$$\sum_n |f(\lambda_n)|^p \leq \frac{1}{\delta^2\pi} \sum_n \iint_{|z|\leq\delta} |f(\lambda_n + z)|^p dxdy$$

$$\leq \frac{1}{\delta^2\pi} \sum_n \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} |f(\lambda_n + x + \sqrt{-1}\, y)|^p dxdy$$

$$= \frac{1}{\delta^2\pi} \sum_n \int_{-\delta}^{\delta} \int_{\lambda_n-\delta}^{\lambda_n+\delta} |f(x + \sqrt{-1}\, y)|^p dxdy$$

$$\leq \frac{1}{\delta^2\pi} \int_{-\delta}^{\delta} \int_{-\infty}^{\infty} |f(x + \sqrt{-1}\, y)|^p dxdy$$

$$\leq \frac{1}{\delta^2\pi} \int_{-\delta}^{\delta} ||f||_p^p \, e^{pT(f)\,|y|} dy \qquad \text{(cf. 17.28)}$$

$$\leq \frac{2}{\delta^2\pi} \left(\int_0^{\delta} e^{pT(f)y} dy\right) ||f||_p^p$$

$$\leq 2\, \frac{e^{\delta pT(f)}}{\delta\pi}\, ||f||_p^p.$$

§18. *THE BOREL TRANSFORM*

Let K be a nonempty convex compact subset of C.

18.1 DEFINITION Put

$$H_K(z) = \sup_{w \in K} \operatorname{Re}(wz).$$

Then

$$H_K : C \to C$$

is called the <u>support function</u> of K.

<u>N.B.</u> H_K is homogeneous of degree 1:

$$H_K(tz) = tH_K(z) \qquad (t > 0).$$

Therefore

$$H_K(z) = H_K(|z|e^{\sqrt{-1}\,\theta}) = |z|H_K(e^{\sqrt{-1}\,\theta}).$$

[Note: Of course, $H_K(0) = 0$.]

<u>N.B.</u> H_K is convex:

$$H_K(\lambda z_1 + (1 - \lambda)z_2) \le \lambda H_K(z_1) + (1 - \lambda)H(z_2) \qquad (0 < \lambda < 1).$$

[Note: It thus follows that H_K is continuous.]

18.2 EXAMPLE Take $K = \{x_0 + \sqrt{-1}\, y_0\}$ (a singleton) -- then

$$H_K(z) = |z|(x_0 \cos \theta - y_0 \sin \theta).$$

18.3 EXAMPLE Take $K = \{z : |z| \le R\}$ -- then

$$H_K(z) = R|z|.$$

18.4 EXAMPLE Take $K = [-a,a]$ $(a > 0)$ -- then

$$H_K(z) = a|z||\cos\theta|.$$

18.5 EXAMPLE Take $K = [-\sqrt{-1}\,a, \sqrt{-1}\,a]$ $(a > 0)$ -- then

$$H_K(z) = a|z||\sin\theta|.$$

18.6 LEMMA $\forall\ w \in K$,

$$(\text{Re } w)\cos\theta - (\text{Im } w)\sin\theta$$

$$= \text{Re}(we^{\sqrt{-1}\,\theta}) \leq H_K(e^{\sqrt{-1}\,\theta}).$$

18.7 APPLICATION

- Take $\theta = 0$ to get

$$\text{Re } w \leq H_K(1).$$

- Take $\theta = \pi$ to get

$$-\text{Re } w \leq H_K(-1).$$

Therefore

$$-H_K(-1) \leq \text{Re } w \leq H_K(1).$$

18.8 APPLICATION

- Take $\theta = \frac{\pi}{2}$ to get

$$-\text{Im } w \leq H_K(\sqrt{-1}).$$

- Take $\theta = \frac{3\pi}{2}$ to get

$$-\text{Im } w(-1) \leq H_K(-\sqrt{-1}).$$

Therefore

$$-H_K(\sqrt{-1}) \leq \text{Im } w \leq H_K(-\sqrt{-1}).$$

18.9 EXAMPLE Suppose that

$$\begin{cases} H_K(1) \le 0 \\ H_K(-1) \le 0. \end{cases}$$

Then

$$0 \le -H_K(-1) \le \operatorname{Re} w \le H_K(1) = 0$$

$$\Rightarrow \operatorname{Re} w = 0.$$

Therefore K is contained in the imaginary axis.

18.10 DEFINITION Suppose that

$$f(z) = \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} z^n$$

is of exponential type -- then its <u>Borel transform</u> $\mathcal{B}_f$ is defined by the prescription

$$\mathcal{B}_f(w) = \sum_{n=0}^{\infty} \frac{\gamma_n}{w^{n+1}} .$$

[Note: The series converges if $|w| > T(f)$ and diverges if $|w| < T(f)$.]

18.11 EXAMPLE Take $f(z) = e^z$ -- then

$$\mathcal{B}_f(w) = \frac{1}{w-1} .$$

18.12 EXAMPLE Take $f(z) = e^{\sqrt{-1}\, z}$ -- then

$$\mathcal{B}_f(w) = \frac{1}{w-\sqrt{-1}} .$$

18.13 LEMMA Fix $T' > T(f)$ and suppose that $\operatorname{Re} w > 2T'$ -- then

$$\mathcal{B}_f(w) = \int_0^{\infty} f(t) e^{-wt} dt.$$

PROOF First of all,

$$\left|f(z) - \sum_{k=0}^{n} c_k z^k\right| \le \sum_{k=n+1}^{\infty} |c_k||r|^k$$

$$= \sum_{k=n+1}^{\infty} |c_k| R^k \left(\frac{r}{R}\right)^k \quad (R > r)$$

$$\le M(R;f) \sum_{k=n+1}^{\infty} \left(\frac{r}{R}\right)^k$$

$$= \left(\frac{r}{R}\right)^{n+1} M(R;f) \frac{1}{1 - \frac{r}{R}}$$

$$\le \left(\frac{r}{R}\right)^{n+1} e^{RT'} \frac{R}{R-r} .$$

Now take $R = 2r$ to get

$$\left|f(z) - \sum_{k=0}^{n} c_k z^k\right| \le \left(\frac{1}{2}\right)^n e^{2rT'}.$$

Since

$$|e^{-wt}| = \exp(-\ (\text{Re } w)t),$$

it then follows that

$$\left|\int_0^\infty f(t)e^{-wt}dt - \int_0^\infty \left(\sum_{k=0}^{n} c_k t^k\right)e^{-wt}dt\right|$$

$$\le \int_0^\infty \left|f(t) - \sum_{k=0}^{n} c_k t^k\right| \exp(-\ (\text{Re } w)t)dt$$

$$\le \left(\frac{1}{2}\right)^n \int_0^\infty \exp((2T' - \text{Re } w)t)dt.$$

But

$$\text{Re } w > 2T' \Rightarrow (2T' - \text{Re } w) < 0$$

=>

$$\int_0^\infty \exp((2T' - \mathrm{Re}\, w)t)dt < \infty.$$

Therefore the infinite series

$$\sum_{n=0}^{\infty} c_n \int_0^\infty t^n e^{-wt} dt$$

is convergent and has sum $\int_0^\infty f(t)e^{-wt}dt$. And finally

$$\sum_{n=0}^{\infty} c_n \int_0^\infty t^n e^{-wt} dt$$

$$= \sum_{n=0}^{\infty} \gamma_n \int_0^\infty \frac{t^n}{n!} e^{-wt} dt$$

$$= \sum_{n=0}^{\infty} \frac{\gamma_n}{w^{n+1}} = \mathcal{B}_f(w).$$

[Note: The constant implicit in the asymptotics has been set equal to 1. To proceed in general, break $\int_0^\infty \ldots dt$ into $\int_0^{t_0} \ldots dt + \int_{t_0}^\infty \ldots dt$.]

Keeping still to the assumption that f is of exponential type, let K_f denote the intersection of all the convex compact subsets of $\mathbf{C}$ outside of which $\mathcal{B}_f$ is holomorphic.

N.B. Therefore K_f is the smallest convex compact subset of $\mathbf{C}$ outside of which $\mathcal{B}_f$ is holomorphic.

18.14 DEFINITION K_f is the indicator diagram of f.

18.15 LEMMA The extreme points of K_f are singular points of $\mathcal{B}_f$.

PROOF If $p \in K_f$ were an extreme point of K_f which was not a singular point of $\mathcal{B}_f$, then upon removing a certain neighborhood of p from K_f one would be led to a smaller convex compact subset of $\mathbb{C}$ outside of which $\mathcal{B}_f$ is holomorphic.

18.16 EXAMPLE Let

$$f(z) = \sum_{k=1}^{n} P_k(z) e^{c_k z}$$

be an <u>exponential polynomial</u> (meaning that the P_k are polynomials and the c_k are complex numbers). Since the Borel transform of a monomial $z^p e^{c_k z}$ equals $p!(w - c_k)^{-p-1}$, the poles at the c_k are the only singularities of the Borel transform of f, so the indicator diagram of f is the convex hull of the set $\{c_1, \ldots, c_n\}$.

18.17 NOTATION Write H_f in place of H_{K_f}.

18.18 EXAMPLE Take $f(z) = \sin \pi z$ -- then

$$\mathcal{B}_f(w) = \frac{1}{2\sqrt{-1}} \left[\frac{1}{w - \sqrt{-1}\,\pi} - \frac{1}{w + \sqrt{-1}\,\pi} \right]$$

and

$$K_f = [-\sqrt{-1}\,\pi, \sqrt{-1}\,\pi].$$

Here

$$H_f(z) = \pi |z| |\sin \theta| \quad \text{(cf. 18.5)},$$

so

$$H_f(\pm \sqrt{-1}) = \pi = \tau(f).$$

Let Γ be a rectifiable Jordan curve containing K_f in its interior.

18.19 THEOREM We have

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \mathcal{B}_f(w) e^{zw} dw.$$

PROOF Take for Γ the circle $|w| = T(f) + \varepsilon$ $(\varepsilon > 0)$ -- then

$$\frac{1}{2\pi\sqrt{-1}} \int_\Gamma \mathcal{B}_f(w) e^{zw} dw$$

$$= \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \left(\sum_{n=0}^{\infty} \frac{n!c_n}{w^{n+1}}\right) e^{zw} dw$$

$$= \sum_{n=0}^{\infty} n!c_n \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \frac{e^{zw}}{w^{n+1}} dw$$

$$= \sum_{n=0}^{\infty} c_n z^n = f(z).$$

18.20 LEMMA $K_f = \emptyset$ iff $f \equiv 0$.

PROOF If $K_f = \emptyset$, then $\mathcal{B}_f$ is everywhere holomorphic (including ∞), thus $\mathcal{B}_f$ is a constant. But $\mathcal{B}_f(\infty) = 0$, so $\mathcal{B}_f \equiv 0 \Rightarrow f \equiv 0$ (cf. 18.19). Conversely, if $f \equiv 0$, then $\forall\, n$, $\gamma_n = 0$, hence $\mathcal{B}_f \equiv 0$.

18.21 EXAMPLE Suppose that

$$\begin{cases} H_f(\sqrt{-1}) < 0 \\ H_f(-\sqrt{-1}) < 0. \end{cases}$$

Then $K_f = \emptyset$, implying thereby that $f \equiv 0$.

[From 18.8,

$$\begin{cases} -H_f(\sqrt{-1}) > 0 \Rightarrow \operatorname{Im} w > 0 \\ H_f(-\sqrt{-1}) < 0 \Rightarrow \operatorname{Im} w < 0.] \end{cases}$$

18.22 NOTATION $H_0(\infty)$ is the set of functions that are holomorphic near ∞ and vanish at ∞.

[Note: If $\Phi \in H_0(\infty)$, then there is an expansion

$$\Phi(z) = \sum_{n=0}^{\infty} \frac{A_n}{z^{n+1}},$$

where

$$A_n = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \Phi(w) w^n dw \quad (n = 0,1,\ldots),$$

Γ a suitable contour.]

E.g.:

$$f \in E_0 \Rightarrow \mathcal{B}_f \in H_0(\infty).$$

18.23 LEMMA The arrow

$$\mathcal{B}: E_0 \to H_0(\infty)$$

that sends f to $\mathcal{B}_f$ is a linear injection.

PROOF Using the inversion formula for the Laplace transform, if $\mathcal{B}_f = \mathcal{B}_g$, then for $u = \operatorname{Re} w >> 0$ (cf. 18.13),

$$f(t) = \frac{1}{2\pi\sqrt{-1}} \int_{u-\sqrt{-1}\infty}^{u+\sqrt{-1}\infty} e^{tw} \mathcal{B}_f(w) dw$$

$$= \frac{1}{2\pi\sqrt{-1}} \int_{u-\sqrt{-1}\infty}^{u+\sqrt{-1}\infty} e^{tw} B_g(w)\,dw = g(t).$$

<u>N.B.</u> The inverse

$$\mathcal{B}^{-1}:\mathcal{B}E_0 \to E_0$$

is constructed via 18.19:

$$\mathcal{B}^{-1}(\mathcal{B}_f)(z) = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \mathcal{B}_f(w)e^{zw}dw.$$

18.24 LEMMA The arrow

$$\mathcal{B}:E_0 \to H_0(\infty)$$

that sends f to $\mathcal{B}_f$ is a linear surjection.

PROOF Fix $\Phi \in H_0(\infty)$ and let $S(\Phi)$ be the smallest convex compact subset of $\mathbf{C}$ in whose complement Φ is holomorphic. Put

$$N(S(\Phi),r) = \{w \in \mathbf{C}:d(w,S(\Phi)) < r\}$$

and let Γ be a rectifiable Jordan curve containing $S(\Phi)$ in its interior:

$$S(\Phi) \subset \text{int } \Gamma \subset N(S(\Phi),r).$$

Consider now the holomorphic function

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \Phi(w)e^{zw}dw.$$

Then

$$\sup_{w\in\Gamma} \text{Re }(zw) \leq \sup_{w\in S(\Phi)} (\text{Re}(zw) + r|z|)$$

$$= H_{S(\Phi)}(z) + r|z|$$

=>

$$|f(z)| \le C \exp(H_{S(\Phi)}(z) + r|z|),$$

where

$$C = \frac{\text{len } \Gamma}{2\pi} \sup_{w \in \Gamma} |\Phi(w)|.$$

Choose R > > 0:

$$S(\Phi) \subset \{z: |z| \le R\}$$

=>

$$|f(z)| \le C \exp(R|z| + r|z|) \qquad \text{(cf. 18.3)}.$$

Therefore $f \in E_0$. And $\mathcal{B}_f = \Phi$ (details below).

[Let T be the analytic functional defined by the rule

$$\langle F, T\rangle = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \Phi(w) F(w)\, dw.$$

Then by definition its FL-transform $\hat{T}$ is the function

$$\langle e^{zw}, \hat{T}\rangle = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \Phi(w) e^{zw} dw,$$

thus here

$$\langle e^{zw}, \hat{T}\rangle = f(z).$$

On the other hand, the prescription

$$F \to \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \mathcal{B}_f(w) F(w)\, dw$$

defines an analytic functional S whose FL-transform is also f(z) (cf. 18.19). But

$$f(z) = \begin{cases} \langle e^{zw}, \hat{T}\rangle = \displaystyle\sum_{n=0}^{\infty} \frac{\langle w^n, T\rangle}{n!} z^n \\ \\ \langle e^{zw}, \hat{S}\rangle = \displaystyle\sum_{n=0}^{\infty} \frac{\langle w^n, S\rangle}{n!} z^n \end{cases}$$

=>

$$\langle w^n, T\rangle = \langle w^n, S\rangle \quad (n = 0,1,\ldots)$$

=>

$$\Phi = B_f.]$$

[Note: See 20.2 for the definition of "analytic functional".]

§19. *THE INDICATOR FUNCTION*

Let f be an entire function of exponential type.

19.1 DEFINITION The <u>indicator function</u>

$$h_f : C^\times \to C$$

of f is defined by

$$h_f(z) = \overline{\lim_{r \to \infty}} \frac{\log|f(rz)|}{r}$$

[Note: Sometimes

$$h_f(e^{\sqrt{-1}\,\theta}) = \overline{\lim_{r \to \infty}} \frac{\log|f(re^{\sqrt{-1}\,\theta})|}{r}$$

is referred to as the exponential type of f in the direction θ. Obviously,

$$h_f(e^{\sqrt{-1}\,\theta}) \leq T(f).]$$

19.2 EXAMPLE Take $f(z) = \exp(a + \sqrt{-1}\,b)z$ $(a,b \in R)$ -- then

$$h_f(z) = |z|(a \cos\theta - b \sin\theta) \quad (z = |z|e^{\sqrt{-1}\,\theta}).$$

19.3 LEMMA If $f \equiv 0$, then $h_f \equiv -\infty$ and if $h_f \equiv -\infty$, then $f \equiv 0$.

19.4 LEMMA If $f \not\equiv 0$, then $h_f(e^{\sqrt{-1}\,\theta}) > -\infty$ everywhere.

19.5 LEMMA If $f \not\equiv 0$, then $h_f(z)$ is a continuous function of $z \in C$ if $h_f(0)$ is defined to be 0.

<u>N.B.</u> h_f $(f \not\equiv 0)$ is homogeneous of degree 1:

$$h_f(tz) = th_f(z) \quad (t > 0).$$

Therefore

$$h_f(z) = h_f(|z|e^{\sqrt{-1}\,\theta}) = |z|h_f(e^{\sqrt{-1}\,\theta}).$$

19.6 REMARK It can be shown that h_f $(f \not\equiv 0)$ is subharmonic.

19.7 THEOREM If $f \not\equiv 0$, then $H_f = h_f$.

PROOF It will be enough to prove that $\forall\ \theta$,

$$H_f(e^{\sqrt{-1}\,\theta}) = h_f(e^{\sqrt{-1}\,\theta}).$$

To this end, we shall first show that

$$h_f(e^{\sqrt{-1}\,\theta}) \leq H_f(e^{\sqrt{-1}\,\theta}).$$

Thus write

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_\varepsilon} \mathcal{B}_f(w)e^{zw}dw \quad \text{(cf. 18.19)},$$

choosing Γ_ε so as to remain within the ε-neighborhood of K_f subject to $K_f \subset \text{int}\ \Gamma_\varepsilon$ -- then

$$|f(re^{\sqrt{-1}\,\theta})| \leq \frac{\text{len}\ \Gamma_\varepsilon}{2\pi} \cdot \sup_{w\in\Gamma_\varepsilon} |\mathcal{B}_f(w)| \cdot \sup_{w\in\Gamma_\varepsilon} \exp(r\text{Re}(we^{\sqrt{-1}\,\theta}))$$

=>

$$h_f(e^{\sqrt{-1}\,\theta}) \leq \sup_{w\in\Gamma_\varepsilon} \text{Re}(we^{\sqrt{-1}\,\theta})$$

$$\leq H_f(e^{\sqrt{-1}\,\theta}) + \varepsilon$$

=>

$$h_f(e^{\sqrt{-1}\,\theta}) \leq H_f(e^{\sqrt{-1}\,\theta}).$$

As for the opposite direction, it suffices to work at $\theta = 0$, the claim being that

$$H_f(1) \le h_f(1).$$

But $\forall\ \varepsilon > 0$,

$$|f(t)| < \exp((h_f(1) + \varepsilon)t) \quad (t >> 0).$$

Therefore the integral

$$\int_0^\infty f(t)e^{-wt}dt$$

is a holomorphic function of w in the half-plane Re $w > h_f(1)$. Since $h_f(1) \le T(f)$, it follows from 18.13 that $\mathcal{B}_f$ has no singularities to the right of the line $x = h_f(1)$, so $H_f(1) \le h_f(1)$.

19.8 APPLICATION

- H_f convex => h_f convex
- h_f subharmonic => H_f subharmonic.

19.9 REMARK Any complex valued function with domain C which is subharmonic and homogeneous of degree 1 is necessarily convex.

19.10 LEMMA If $T(f) > 0$, then $T(f) = \tau(f)$ (cf. 17.3) and

$$\tau(f) = \sup_{0\le\theta\le 2\pi} h_f(e^{\sqrt{-1}\,\theta}).$$

19.11 LEMMA Assume that $f \not\equiv 0$ -- then $T(f) = 0$ iff $h_f = 0$.

PROOF If $T(f) = 0$, then $\mathcal{B}_f$ is holomorphic in the region $|w| > 0$, so $K_f = \{0\}$ (cf. 18.20), hence $H_f = 0$, hence $h_f = 0$. Conversely, if $h_f = 0$, then $T(f) = 0$

($T(f) > 0$ being ruled out by 19.10).

19.12 LEMMA If $f,g \in E_0$ and if g is an exponential polynomial, then

$$h_{fg} = h_f + h_g.$$

[Note: Recall that E_0 is an algebra (cf. 17.15), thus $fg \in E_0$.]

19.13 COROLLARY If $f,g \in E_0$, if g is an exponential polynomial, and if $\frac{f}{g}$ is entire, then $\frac{f}{g}$ is of exponential type (cf. 17.9) and

$$h_{\frac{f}{g}} = h_f - h_g.$$

19.14 THEOREM Suppose that $f \in E_0$ has the property that $h_f(\pm\sqrt{-1}) < \pi$. Assume further that $f(n) = 0$ for $n = 0, \pm 1, \pm 2, \ldots$ -- then $f \equiv 0$.

PROOF Let

$$\phi(z) = \frac{f(z)}{g(z)},$$

where $g(z) = \sin \pi z$ -- then $\phi \in E_0$. But g is an exponential polynomial, so

$$h_\phi = h_f - h_g$$

=>

$$\begin{aligned} h_\phi(\pm\sqrt{-1}) &= h_f(\pm\sqrt{-1}) - h_g(\pm\sqrt{-1}) \\ &= h_f(\pm\sqrt{-1}) - \pi \quad \text{(cf. 18.5)} \\ &< \pi - \pi = 0 \end{aligned}$$

=>

$$\phi \equiv 0 \quad \text{(cf. 18.21 } (h_\phi = H_\phi))$$

=>

$$f \equiv 0.$$

19.15 REMARK One cannot replace $h_f(\pm\sqrt{-1}) < \pi$ by $h_f(\pm\sqrt{-1}) = \pi$ (consider $\sin \pi z$).

19.16 LEMMA If $f \in E_0$, then $\forall$ complex constant c, $f_c \in E_0$ (cf. 17.16) and

$$K_f = K_{f_c}.$$

[Note: Here

$$f_c(z) = f(z + c).]$$

N.B. Therefore

$$H_f = H_{f_c}$$

or still,

$$h_f = h_{f_c}.$$

19.17 THEOREM Suppose that $f \in E_0$ has the property that $h_f(\pm\sqrt{-1}) < \pi$. Assume further that $f(n) = 0$ for $n = 0,1,2,\ldots$ -- then $f \equiv 0$.

PROOF

$$0 = f(n) = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \mathcal{B}_f(w)e^{nw}dw \quad \text{(cf. 18.19)}$$

=>

$$0 = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \mathcal{B}_f(w) \frac{1}{1 - ze^w} dw$$

=>

$$0 = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \mathcal{B}_f(w) \frac{z}{1 - ze^w} dw$$

=>

$$0 = -\frac{1}{2\pi\sqrt{-1}} \int_\Gamma \mathcal{B}_f(w)e^{-w}dw \quad (z \to \infty)$$

$\Rightarrow$

$$f(-1) = 0.$$

Now apply the same argument to f_{-1} to see that

$$f_{-1}(-1) = f(-2) = 0.$$

ETC. One may then quote 19.14.

[Note: In view of 19.16, $\forall$ n, $h_{f_n}(\pm \sqrt{-1}) < \pi$, and so $\forall\, w \in K_{f_n}$,

$$-\pi < -H_{f_n}(\sqrt{-1}) \leq \operatorname{Im} w \leq H_{f_n}(-\sqrt{-1}) < \pi,$$

as follows from 18.8.]

19.18 $\forall\, f \in E_0$,

$$h_{f'} \leq h_f.$$

[In fact,

$$K_{f'} \subset K_f \Rightarrow H_{f'} \leq H_f.]$$

§20. *DUALITY*

We shall provide here a description of the three standard realizations of the dual of the entire functions.

20.1 NOTATION E is the set of entire functions.

By definition, the C^0-topology on E is the topology of uniform convergence on compact subsets of C. Denote its dual by E^*. Since E is a closed subspace of $C^0(R^2)$, every continuous linear functional $\Lambda \in E^*$ extends to a continuous linear functional on $C^0(R^2)$, hence determines a compactly supported Radon measure.

20.2 DEFINITION The elements of E^* are called <u>analytic functionals</u>.

20.3 EXAMPLE The compactly supported Radon measures

$$F \to F(0)$$

and

$$F \to \frac{1}{2\pi\sqrt{-1}} \int_{|z|=1} \frac{F(z)}{z}\, dz$$

restrict to the same analytic functional.

20.4 REMARK The C^0-topology on E coincides with the C^∞-topology on E. Since E is a closed subspace of $C^\infty(R^2)$, every continuous linear functional $\Lambda \in E^*$ extends to a continuous linear functional on $C^\infty(R^2)$, hence determines a compactly supported distribution.

[Note: Recall that if $F_1, F_2, \ldots$ is a sequence in E and if $F_n \to F$ uniformly on compact subsets of C, then $F_n' \to F'$ uniformly on compact subsets of C.]

20.5 NOTATION M_0 is the set of compactly supported Radon measures on R^2.

20.6 DEFINITION Given $\mu \in M_0$, its FL-transform $\hat{\mu}$ is defined by

$$\hat{\mu}(z) = \int e^{zw} d\mu(w).$$

20.7 LEMMA $\hat{\mu}(z)$ is an entire function of exponential type.

PROOF To see that $\hat{\mu}$ is entire, simply observe that

$$\frac{d}{dz}\hat{\mu}(z) = \int (w) e^{zw} d\mu(w).$$

Next choose $R >> 0$: spt μ is contained in the circle of radius R centered at the origin -- then

$$|\hat{\mu}(z)| \leq \int |e^{zw}| |d\mu(w)|$$

$$\leq e^{R|z|} \int |d\mu(w)|.$$

20.8 NOTATION Given $\mu, \nu \in M_0$, write $\mu \sim \nu$ if $\hat{\mu} = \hat{\nu}$.

20.9 LEMMA $\mu \sim \nu$ iff $\forall F \in E$,

$$\langle F, \mu \rangle = \langle F, \nu \rangle.$$

Therefore $\sim$ is an equivalence relation on M_0.

20.10 EXAMPLE Take $d\mu = dz|\Gamma$, where Γ is a circle -- then

$$\hat{\mu}(z) = \int_\Gamma e^{zw} dw = 0.$$

So $\mu \sim 0$ but $\mu \neq 0$.

20.11 NOTATION Given $\mu \in M_0$, let $[\mu]$ be its associated equivalence class.

20.12 LEMMA The arrow

$$M_0/\sim \to E_0$$

that sends $[\mu]$ to $\hat{\mu}$ is a linear bijection.

PROOF Injectivity is manifest while surjectivity is an application of 18.19.

20.13 RAPPEL The arrow

$$\mathcal{B}:E_0 \to H_0(\infty)$$

that sends f to $\mathcal{B}_f$ is a linear bijection (cf. 18.23 and 18.24).

20.14 NOTATION Let $F \in E$.

- Given $f \in E_0$, put

$$\langle F,f\rangle = \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} F^{(n)}(0) \qquad (\gamma_n = f^{(n)}(0)).$$

- Given $\Phi \in H_0(\infty)$, put

$$\langle F,\Phi\rangle = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \Phi(w)F(w)dw.$$

- Given $[\mu] \in M_0/\sim$, put

$$\langle F,[\mu]\rangle = \int F(w)d\mu(w) \quad (= \langle F,\mu\rangle).$$

20.15 LEMMA Each of these prescriptions defines an analytic functional.

20.16 LEMMA Suppose given a triple $(f,\Phi,[\mu])$. Assume: $\Phi = \mathcal{B}_f$ and $\hat{\mu} = f$ -- then these three data points give rise to the same analytic functional.

PROOF By definition (cf. 20.6),

$$\hat{\mu}(z) = \int e^{zw}d\mu(w)$$

$$= \int \sum_{n=0}^{\infty} \frac{(zw)^n}{n!} \, d\mu(w)$$

$$= \sum_{n=0}^{\infty} \frac{\langle w^n, \mu\rangle}{n!} z^n$$

=>

$$\langle F, f\rangle = \langle F, \hat{\mu}\rangle = \sum_{n=0}^{\infty} \frac{\langle w^n, \mu\rangle}{n!} F^{(n)}(0)$$

$$= \langle \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} w^n, \mu\rangle$$

$$= \langle F, \mu\rangle = \langle F, [\mu]\rangle.$$

On the other hand,

$$\langle F, \mathcal{B}_{\hat{\mu}}\rangle = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \mathcal{B}_{\hat{\mu}}(w) F(w) \, dw$$

$$= \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \mathcal{B}_{\hat{\mu}}(w) \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} w^n dw$$

$$= \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \mathcal{B}_{\hat{\mu}}(w) w^n dw$$

$$= \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} (\hat{\mu})^{(n)}(0) \qquad \text{(cf. 18.19)}$$

$$= \sum_{n=0}^{\infty} \frac{(\hat{\mu})^{(n)}(0)}{n!} F^{(n)}(0)$$

$$= \langle F, \hat{\mu}\rangle = \langle F, f\rangle.$$

20.17 SCHOLIUM Each of the spaces E_0, $H_0(\infty)$, $M_0/\sim$ can be viewed as E^*.

[Note: If $\Lambda \in E^*$, then there is a $\mu \in M_0$: $\forall\, F \in E$,

$$<F,\Lambda> = <F,\mu>.$$

And if $\nu \in M_0$ has the same property, then $\mu \sim \nu$ (cf. 20.9).]

20.18 EXAMPLE Take $\mu = \delta_1$ -- then $\hat{\mu}(z) = e^z$ and $B_{\hat{\mu}}(w) = \frac{1}{w-1}$. Here

$$<F,\delta_1> = F(1)$$

while

$$<F,\hat{\mu}> = \sum_{n=0}^{\infty} \frac{(\hat{\mu})^{(n)}(0)}{n!} F^{(n)}(0)$$

$$= \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!}$$

$$= F(1)$$

and

$$\frac{1}{2\pi\sqrt{-1}} \int_\Gamma B_{\hat{\mu}}(w) F(w)\, dw$$

$$= \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \frac{F(w)}{w-1}\, dw$$

$$= F(1).$$

§21. *FOURIER TRANSFORMS*

Working on the real axis, the sign convention of the Fourier transform of an $f \in L^1(-\infty,\infty)$ is "plus":

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{\sqrt{-1}\,xt} dt.$$

[Note: From the point of view of harmonic analysis, the ambient Haar measure is $\frac{1}{\sqrt{2\pi}}$ times Lebesgue measure.]

21.1 LEMMA Let $f \in L^1(-\infty,\infty)$ -- then $\hat{f}(x)$ is a uniformly continuous function of x.

PROOF Write

$$|\hat{f}(x+y) - \hat{f}(x)|$$

$$= \frac{1}{\sqrt{2\pi}} \left|\int_{-\infty}^{\infty} f(t) e^{\sqrt{-1}\,xt} (e^{\sqrt{-1}\,yt} - 1) dt\right|$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(t)| |e^{\sqrt{-1}\,yt} - 1| dt$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(t)| (2(1 - \cos yt))^{1/2} dt$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(t)| 2 |\sin(\tfrac{yt}{2})| dt$$

$$= \frac{2}{\sqrt{2\pi}} \left[\int_{-\infty}^{-R} + \int_{R}^{\infty} + \int_{-R}^{R}\right] \quad \cdots$$

$$\leq \frac{2}{\sqrt{2\pi}} \left[\int_{-\infty}^{-R} + \int_{R}^{\infty} \right] |f(t)| dt$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} |f(t)| |yt| dt$$

$$\leq \frac{2}{\sqrt{2\pi}} \left[\int_{-\infty}^{-R} + \int_{R}^{\infty} \right] |f(t)| dt$$

$$+ \frac{|y|}{\sqrt{2\pi}} R \int_{-R}^{R} |f(t)| dt.$$

Given $\varepsilon > 0$, choose R large enough to render

$$\frac{2}{\sqrt{2\pi}} \left[\int_{-\infty}^{-R} + \int_{R}^{\infty} \right] |f(t)| dt < \frac{\varepsilon}{2} .$$

This done, choose y small enough to render

$$\frac{|y|}{\sqrt{2\pi}} R \int_{-R}^{R} |f(t)| dt < \frac{\varepsilon}{2} .$$

So, with these choices,

$$|\hat{f}(x+y) - \hat{f}(x)| < \varepsilon.$$

21.2 EXAMPLE Take $f(t) = e^{-|t|}$ -- then

$$\hat{f}(x) = \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{1+x^2} .$$

21.3 EXAMPLE Take $f(t) = e^{-\frac{1}{2} t^2}$ -- then

$$\hat{f}(x) = e^{-\frac{1}{2} x^2} .$$

21.4 EXAMPLE Take $f(t) = e^{-e^t} e^t$ -- then

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \Gamma(1 + \sqrt{-1}\, x).$$

21.5 NOTATION Let

$$C_0(-\infty,\infty)$$

stand for the set of continuous functions F on R such that

$$F(x) \to 0 \text{ as } |x| \to \infty.$$

[Note: When equipped with the supremum norm, $C_0(-\infty,\infty)$ is a Banach algebra and $C_c(-\infty,\infty)$ is a dense subalgebra.]

21.6 RIEMANN-LEBESGUE LEMMA Let $f \in L^1(-\infty,\infty)$ -- then $\hat{f} \in C_0(-\infty,\infty)$.

N.B. The arrow

$$L^1(-\infty,\infty) \to C_0(-\infty,\infty)$$

that sends f to $\hat{f}$ is a bounded linear transformation:

$$||\hat{f}||_\infty = \sup_{-\infty<x<\infty} |\hat{f}(x)| \le \frac{1}{\sqrt{2\pi}} ||f||_1.$$

21.7 REMARK Not every $F \in C_0(-\infty,\infty)$ is the Fourier transform of a function in $L^1(-\infty,\infty)$.

[Consider the function defined for $x \ge 0$ by the rule

$$F(x) = \begin{cases} x/e & (0 \le x \le e) \\ \dfrac{1}{\log x} & (x > e) \end{cases}$$

and put

$$F(x) = -F(-x) \quad (x \le 0).]$$

21.8 RAPPEL Let A be a subalgebra of $C_0(-\infty,\infty)$. Assume:

- A is selfadjoint: $F \in A \Rightarrow \bar{F} \in A$.
- A separates points: $\forall\, x,y \in R$ with $x \neq y$, $\exists\, F \in A$: $F(x) \neq F(y)$.
- A vanishes at no point: $\forall\, x \in R$, $\exists\, F \in A$: $F(x) \neq 0$.

Then A is dense in $C_0(-\infty,\infty)$.

21.9 NOTATION Let

$$A(-\infty,\infty)$$

stand for the set of all $\hat{f}$ $(f \in L^1(-\infty,\infty))$.

21.10 LEMMA $A(-\infty,\infty)$ is an algebra.

PROOF It is clear that $A(-\infty,\infty)$ is a vector space. If now $\hat{f},\hat{g} \in A(-\infty,\infty)$, then

$$\hat{f} \cdot \hat{g} = \frac{1}{\sqrt{2\pi}} (f * g)^{\wedge},$$

the $*$ being convolution.

21.11 THEOREM $A(-\infty,\infty)$ is dense in $C_0(-\infty,\infty)$.

PROOF

- $A(-\infty,\infty)$ is selfadjoint.

[Given $f \in L^1(-\infty,\infty)$,

$$\overline{(\hat{f})}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(t)}\, e^{-\sqrt{-1}\,xt} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-t)}\ e^{\sqrt{-1}\ xt} dt$$

$$= \hat{g}(x) \qquad (g(t) = \overline{f(-t)}).]$$

- $A(-\infty,\infty)$ separates points.

[In fact,

$$C_c^{\infty}(-\infty,\infty) \subset S(-\infty,\infty) \subset A(-\infty,\infty).]$$

- $A(-\infty,\infty)$ vanishes at no point (obvious).

21.12 THEOREM If $f_1, f_2 \in L^1(-\infty,\infty)$ and if $\hat{f}_1 = \hat{f}_2$ everywhere, then $f_1 = f_2$ almost everywhere.

In general, the Fourier transform $\hat{f}$ of f need not belong to $L^1(-\infty,\infty)$.

21.13 EXAMPLE Take

$$f(t) = \begin{cases} 1 & (|t| \leq 1) \\ 0 & (|t| > 1). \end{cases}$$

Then

$$\hat{f}(x) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\sin x}{x}$$

is not in $L^1(-\infty,\infty)$.

Accordingly, it cannot be expected that Fourier inversion will hold on the nose. Still, there are summability results.

21.14 THEOREM If $f \in L^1(-\infty,\infty)$, then for almost all t,

$$f(t) = \lim_{R \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \hat{f}(x)\left(1 - \frac{|x|}{R}\right) e^{-\sqrt{-1}\ tx} dx.$$

[Note: This relation is also valid at every continuity point of f.]

21.15 REMARK If $f \in L^1(-\infty,\infty)$, then as $R \to \infty$,

$$\frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \hat{f}(x)\left(1 - \frac{|x|}{R}\right) e^{-\sqrt{-1}\, tx} \to f(t)$$

in the L^1-norm.

21.16 THEOREM If $f \in L^1(-\infty,\infty)$ and if $\hat{f} \in L^1(-\infty,\infty)$, then

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(x) e^{-\sqrt{-1}\, tx} dx$$

almost everywhere.

21.17 THEOREM If $f \in L^1(-\infty,\infty)$ and if $\hat{f} \in L^1(-\infty,\infty)$, then

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(x) e^{-\sqrt{-1}\, tx} dx$$

everywhere provided f is continuous everywhere.

21.18 EXAMPLE Take

$$f(t) = \begin{cases} 1 - |t| & (|t| \le 1) \\ 0 & (|t| > 1). \end{cases}$$

Then

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \frac{\sin^2(x/2)}{(x/2)^2},$$

so here the assumptions of 21.17 are met, thus $\forall\, t$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{\sin^2(x/2)}{(x/2)^2} e^{-\sqrt{-1}\, tx} dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin^2(x/2)}{(x/2)^2} e^{\sqrt{-1}\, tx} dx$$

$$= \begin{cases} 1 - |t| & (|t| \le 1) \\ 0 & (|t| > 1). \end{cases}$$

In particular: At $t = 0$,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin^2(x/2)}{(x/2)^2} dx = 1$$

=>

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \pi.$$

21.19 EXAMPLE Take

$$f(t) = \begin{cases} te^{-t} & (t \ge 0) \\ 0 & (t < 0). \end{cases}$$

Then $f \in L^1(-\infty,\infty)$. Moreover,

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{(1 - \sqrt{-1}\, x)^2}$$

is also in $L^1(-\infty,\infty)$. Therefore at every t (cf. 21.17),

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(x) e^{-\sqrt{-1}\, tx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{(1 + \sqrt{-1}\, x)^2} e^{\sqrt{-1}\, tx} dx$$

$$= \hat{\phi}(t),$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{(1 + \sqrt{-1}\, x)^2}.$$

21.20 THEOREM If $f \in L^1(-\infty,\infty)$ is continuously differentiable and if $f' \in L^1(-\infty,\infty)$, then $\forall\, x$,

$$(f')^{\wedge}(x) = -\sqrt{-1}\, x\hat{f}(x).$$

PROOF Write

$$f(x) - f(0) = \int_0^x f'(t)dt.$$

Then

$$\begin{cases} \lim\limits_{x \to \infty} f(x) = f(0) + \int_0^{\infty} f'(t)dt = 0 \\ \\ \lim\limits_{x \to -\infty} f(x) = f(0) + \int_0^{-\infty} f'(t)dt = 0, \end{cases}$$

f being L^1. But for $x \neq 0$,

$$\int_{-R}^{R} f(t)e^{\sqrt{-1}\, xt}dt$$

$$= \frac{e^{\sqrt{-1}\, xt}}{\sqrt{-1}\, x} f(t) \Big|_{t = -R}^{t = R} - \int_{-R}^{R} \frac{e^{\sqrt{-1}\, xt}}{\sqrt{-1}\, x} f'(t)dt.$$

Therefore, upon letting $R \to \infty$, we have

$$\int_{-\infty}^{\infty} f(t)e^{\sqrt{-1}\, xt}dt = -\int_{-\infty}^{\infty} \frac{e^{\sqrt{-1}\, xt}}{\sqrt{-1}\, x} f'(t)dt$$

=>

$$-\sqrt{-1}\, x\hat{f}(x) = (f')^{\wedge}(x) \qquad (x \neq 0).$$

This relation is also valid at $x = 0$. In fact, both sides are continuous and the LHS is zero at $x = 0$ whereas the RHS at $x = 0$ equals

$$\int_{-\infty}^{\infty} f'(t)dt = f(\infty) - f(-\infty)$$

$$= 0 - 0 = 0.$$

[Note: By iteration, if f is continuously differentiable n times and if $f^{(k)} \in L^1(-\infty,\infty)$ $(0 \le k \le n)$, then $\forall$ x,

$$(f^{(n)})^{\wedge}(x) = (-\sqrt{-1}\, x)^n \hat{f}(x).]$$

21.21 RAPPEL If $0 < A < \infty$, then

$$L^2[-A,A] \subset L^1[-A,A]$$

but this is false if $A = \infty$: The function

$$f(x) = \frac{1}{1 + |x|}$$

is in $L^2(-\infty,\infty)$ but is not in $L^1(-\infty,\infty)$.

We shall now turn to the L^2-theory of the Fourier transform.

21.22 PLANCHEREL THEOREM If $f \in L^1(-\infty,\infty) \cap L^2(-\infty,\infty)$, then $\hat{f} \in L^2(-\infty,\infty)$ and $\wedge | L^1(-\infty,\infty) \cap L^2(-\infty,\infty)$ extends uniquely to an isometric isomorphism

$$\wedge : L^2(-\infty,\infty) \to L^2(-\infty,\infty).$$

It is of period 4 (i.e., $\wedge^4 = \text{id}$) and has pure point spectrum $1, \sqrt{-1}, -1, -\sqrt{-1}$.

[Note: For the record, given $f_1, f_2 \in L^2(-\infty,\infty)$,

$$\int_{-\infty}^{\infty} f_1(t)\overline{f_2(t)}dt = \int_{-\infty}^{\infty} \hat{f}_1(x)\overline{\hat{f}_2(x)}dx.$$

In particular: $\forall\ f \in L^2(-\infty,\infty)$,

$$||f||_2 = ||\hat{f}||_2.]$$

N.B. Computationally, if $f \in L^2(-\infty,\infty)$, then as $R \to \infty$,

$$\frac{1}{\sqrt{2\pi}} \int_{-R}^{R} f(t) e^{\sqrt{-1}\ xt} dt \to \hat{f}(x)$$

in the L^2-norm and

$$\frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \hat{f}(x) e^{-\sqrt{-1}\ tx} dx \to f(t)$$

in the L^2-norm.

21.23 REMARK Let

$$h_n(x) = (2^n n!)^{-1/2}\ \pi^{-1/4}\ e^{-x^2/2}\ H_n(x),$$

where

$$H_n(x) = (-1)^n\ e^{x^2} \frac{d^n}{dx^n} e^{-x^2/2}$$

is the n^{th} Hermite polynomial (cf. 8.17) ($n \geq 0$) -- then $\{h_n\}$ is an orthonormal basis for $L^2(-\infty,\infty)$ and

$$\wedge(h_n) = \hat{h}_n = (\sqrt{-1})^n h_n.]$$

21.24 RAPPEL If $f,g \in L^2(-\infty,\infty)$, then their convolution $f * g$ belongs to $C_0(-\infty,\infty)$ and

$$||f * g||_\infty \leq ||f||_2\ ||g||_2.$$

[Note: The same cannot be said if $f,g \in L^1(-\infty,\infty)$. For example, take

$$f(t) = \begin{cases} \frac{1}{\sqrt{t}} & (0 < t < 1) \\ 0 & (t \leq 0 \text{ or } t \geq 1) \end{cases}, \quad g(t) = \begin{cases} \frac{1}{\sqrt{1-t}} & (0 < t < 1) \\ 0 & (t \leq 0 \text{ or } t \geq 1). \end{cases}$$

Then

$$(f * g)(1) = \int_{-\infty}^{\infty} f(t)g(1-t)dt = \int_0^1 \frac{dt}{t}$$

is undefined.]

Let $f,g \in L^2(-\infty,\infty)$ -- then $f \cdot g \in L^1(-\infty,\infty)$ and

$$\int_{-\infty}^{\infty} f(t)g(t)dt = \int_{-\infty}^{\infty} \hat{f}(x)\hat{g}(-x)dx.$$

So, $\forall\ x_0$,

$$\int_{-\infty}^{\infty} f(t)g(t)e^{\sqrt{-1}\, x_0 t}dt$$

$$= \int_{-\infty}^{\infty} \hat{f}(x)\hat{g}(x_0 - x)dx = (\hat{f} * \hat{g})(x_0)$$

=>

$$(f \cdot g)\hat{} = \frac{1}{\sqrt{2\pi}} (\hat{f} * \hat{g}).$$

21.25 THEOREM $A(-\infty,\infty)$ consists precisely of the convolutions $F * G$, where $F,G \in L^2(-\infty,\infty)$.

PROOF Given $F,G \in L^2(-\infty,\infty)$, write

$$\begin{cases} F = \hat{f} \\ G = \hat{g} \end{cases} \qquad (f,g \in L^2(-\infty,\infty)).$$

Then

$$F * G = \hat{f} * \hat{g} = \sqrt{2\pi}\,(f \cdot g)^{\wedge} \in A(-\infty,\infty).$$

Conversely, every $\phi \in L^1(-\infty,\infty)$ is a product $f \cdot g$ with $f,g \in L^2(-\infty,\infty)$, thus matters can be turned around.

[Note: Let $f = \sqrt{|\phi|}$ and take $g = \phi/\sqrt{|\phi|}$ when f is not zero but take $g = 0$ when $f = 0$.]

21.26 THEOREM If $f \in L^2(-\infty,\infty)$, then for almost all t,

$$f(t) = \lim_{R \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \hat{f}(x)\left(1 - \frac{|x|}{R}\right) e^{-\sqrt{-1}\,tx}\,dx.$$

21.27 APPLICATION If $f_1 \in L^1(-\infty,\infty)$ and $f_2 \in L^2(-\infty,\infty)$ and if $\hat{f}_1 = \hat{f}_2$ almost everywhere, then $f_1 = f_2$ almost everywhere.

[Use the preceding result in conjunction with 21.14.]

21.28 LEMMA Let $f \in L^2(-\infty,\infty)$ -- then the restriction of f to [a,b] is L^2, hence is L^1, and

$$\int_a^b f(t)\,dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(x)\, \frac{e^{-\sqrt{-1}\,bx} - e^{-\sqrt{-1}\,ax}}{-\sqrt{-1}\,x}\,dx.$$

[If $\chi_{a,b}$ is the characteristic function of [a,b], then

$$\hat{\chi}_{a,b}(x) = \frac{1}{\sqrt{2\pi}} \frac{e^{\sqrt{-1}\,bx} - e^{\sqrt{-1}\,ax}}{\sqrt{-1}\,x}.]$$

21.29 THEOREM If $f \in L^2(-\infty,\infty)$ is continuously differentiable and if $f' \in L^2(-\infty,\infty)$, then

$$(f')^{\wedge}(x) = -\sqrt{-1}\,x\hat{f}(x)$$

almost everywhere (cf. 21.20).

PROOF Start by writing

$$f(t+h) - f(t) = \int_t^{t+h} f'(s)ds.$$

Next apply 21.28 to the integral on the right (replacing f by f'):

$$\int_t^{t+h} f'(s)ds = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f')^{\wedge}(x) \left(\frac{e^{-\sqrt{-1}\,hx}-1}{-\sqrt{-1}\,x}\right) e^{-\sqrt{-1}\,tx} dx.$$

On the other hand,

$$f(t+h) - f(t)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(x)(e^{-\sqrt{-1}\,hx}-1)e^{-\sqrt{-1}\,tx} dx$$

in the L^2-sense. But

$$(f')^{\wedge}(x) \in L^2(-\infty,\infty), \quad \frac{e^{-\sqrt{-1}\,hx}-1}{-\sqrt{-1}\,x} \in L^2(-\infty,\infty)$$

$$\Rightarrow$$

$$(f')^{\wedge}(x)\left(\frac{e^{-\sqrt{-1}\,hx}-1}{-\sqrt{-1}\,x}\right) \in L^1(-\infty,\infty).$$

Meanwhile

$$\hat{f}(x)\,(e^{-\sqrt{-1}\,hx}-1) \in L^2(-\infty,\infty).$$

Therefore (cf. 21.27)

$$(f')^{\wedge}(x)\left(\frac{e^{-\sqrt{-1}\,hx}-1}{-\sqrt{-1}\,x}\right) = \hat{f}(x)\,(e^{-\sqrt{-1}\,hx}-1)$$

almost everywhere. Take $h = 1$ and $x \neq 2\pi n$:

$$\Rightarrow$$

$$(f')^{\wedge}(x) = -\sqrt{-1}\,x\hat{f}(x)$$

almost everywhere.

[Note: It follows that $x\hat{f}(x)$ belongs to $L^2(-\infty,\infty)$.]

APPENDIX

Assuming that $\nu > -\frac{1}{2}$, take

$$f_\nu(t) = 0 \text{ if } |t| \geq 1$$

and take

$$f_\nu(t) = (1 - t^2)^{\nu - \frac{1}{2}} \text{ if } |t| < 1.$$

Then $f_\nu \in L^1(-\infty,\infty)$ and

$$\hat{f}_\nu(x) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^1 (1 - t^2)^{\nu - \frac{1}{2}} \cos xt\, dt$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \int_0^1 (1 - t^2)^{\nu - \frac{1}{2}}\, t^{2n}\, dt$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\, \frac{1}{2} \int_0^1 u^{n - \frac{1}{2}} (1 - u)^{\nu - \frac{1}{2}}\, du$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} B(n + \frac{1}{2}, \nu + \frac{1}{2})$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \frac{\Gamma(n + \frac{1}{2})\Gamma(\nu + \frac{1}{2})}{\Gamma(n + \nu + 1)}$$

$$= \frac{1}{\sqrt{2\pi}} \Gamma(\nu + \frac{1}{2}) \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \frac{\sqrt{\pi}\ (2n)!}{2^{2n}(n!)} \frac{1}{\Gamma(n + \nu + 1)}$$

$$= \frac{1}{\sqrt{2}} \Gamma(\nu + \frac{1}{2}) \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{x}{2})^{2n}}{n!\Gamma(n + \nu + 1)}$$

$$= \frac{1}{\sqrt{2}} \Gamma(\nu + \frac{1}{2}) (\frac{x}{2})^{-\nu} J_\nu(x) \qquad \text{(cf. 2.29)}.$$

EXAMPLE Take $\nu = \frac{1}{2}$ -- then

$$J_{1/2}(x) = (\frac{2}{\pi})^{1/2} \frac{\sin x}{\sqrt{x}},$$

so

$$\hat{f}_{1/2}(x) = \frac{1}{\sqrt{2}} \Gamma(1) (\frac{x}{2})^{-1/2} J_{1/2}(x)$$

$$= (\frac{2}{\pi})^{1/2} \frac{\sin x}{x},$$

in agreement with 21.13.

LEMMA If $\nu > 0$, then $f_\nu \in L^2(-\infty,\infty)$.

<u>N.B.</u>

$$f_0 \notin L^2(-\infty,\infty).$$

§22. *PALEY-WIENER*

Let

$$E_0(A) = \{f \in E_0 : T(f) \leq A\},$$

where $0 < A < \infty$.

22.1 NOTATION $PW(A)$ is the subset of $E_0(A)$ consisting of those f such that $f|R \in L^2(-\infty,\infty)$.

[Note: The elements of $PW(A)$ are called Paley-Wiener functions.]

N.B. The elements of $PW(A)$ are bounded on the real axis (cf. 17.29) and

$$f(x) \to 0 \text{ as } |x| \to \infty \quad \text{(cf. 17.34).}$$

22.2 LEMMA $PW(A)$ is a vector space.

22.3 LEMMA $PW(A)$ is an inner product space:

$$<f,g> = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx.$$

22.4 LEMMA $PW(A)$ is closed under differentiation (cf. 17.8 and 17.31).

[Note: If $f \in PW(A)$, then

$$||f'||_2 \leq ||f||_2\, T(f) \leq ||f||_2 A.$$

Therefore

$$\frac{d}{dz}: PW(A) \to PW(A)$$

is a bounded linear transformation (but it is not surjective).]

22.5 CONSTRUCTION Given $\phi \in L^2[-A,A]$ $(0 < A < \infty)$, put

put

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \phi(t) e^{\sqrt{-1}\, zt} dt.$$

Then $f \in E_0(A)$ (cf. 17.19). Taking z to be real and ϕ to be zero for $|t| > A$, it follows that $f|R = \hat{\phi}$, thus by Plancherel $||f|R||_2 = ||\phi||_2$, so $f \in PW(A)$. Therefore this procedure determines an isometric injection

$$L^2[-A,A] \to PW(A) \qquad \text{(cf. 21.11)}.$$

22.6 EXAMPLE Take

$$\phi(t) = \frac{1}{\sqrt{1 - t^2}} \quad (-1 < t < 1).$$

Then $\phi \in L^1[-1,1]$ but $\phi \notin L^2[-1,1]$. Moreover,

$$\int_{-1}^{1} \frac{e^{\sqrt{-1}\, xt}}{\sqrt{1 - t^2}} dt$$

is not square integrable on the real axis.

22.7 THEOREM The arrow

$$L^2[-A,A] \to PW(A)$$

that sends ϕ to

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \phi(t) e^{\sqrt{-1}\, zt} dt$$

is an isometric isomorphism.

PROOF On the basis of what has been said above, it remains to establish surjectivity. If $T(f) = 0$, then $f = 0$ (cf. 17.30), so in this case we can take

$\phi = 0$. Assume now that $T(f) > 0$ -- then

$$||f'||_2 \leq ||f||_2\, T(f) \quad \text{(cf. 17.31)},$$

thus by iteration

$$||f^{(n)}||_2 \leq ||f||_2\, T(f)^n$$

or still, passing to Fourier transforms (cf. 21.29),

$$\int_{-\infty}^{\infty} x^{2n} |\hat{f}(x)|^2 dx \leq ||\hat{f}||_2^2\, T(f)^{2n} \quad (n = 1,2,\ldots).$$

Fix $\varepsilon > 0$:

$$(T(f) + \varepsilon)^{2n} \int_{|x| \geq T(f)+\varepsilon} |\hat{f}(x)|^2 dx$$

$$\leq \int_{|x| \geq T(f)+\varepsilon} x^{2n} |\hat{f}(x)|^2 dx$$

$$\leq ||\hat{f}||_2^2\, T(f)^{2n}$$

=>

$$\left[\frac{T(f)+\varepsilon}{T(f)}\right]^{2n} \times \int_{|x| \geq T(f)+\varepsilon} |\hat{f}(x)|^2 dx \leq ||\hat{f}||_2^2$$

=>

$$\left[1 + \frac{\varepsilon}{T(f)}\right]^{2n} \times \int_{|x| \geq T(f)+\varepsilon} |\hat{f}(x)|^2 dx \leq ||\hat{f}||_2^2$$

=>

$$\int_{|x| \geq T(f)+\varepsilon} |\hat{f}(x)|^2 dx = 0 \quad (\text{send } n \text{ to } \infty).$$

Therefore $\hat{f}(x) = 0$ almost everywhere if $|x| \geq T(f) + \varepsilon$, hence $\hat{f}(x) = 0$ almost everywhere if $|x| \geq T(f)$. Consequently,

$$\hat{f} \in L^2[-T(f),T(f)] \subset L^2[-A,A].$$

And for almost all x (cf. 21.26),

$$f(x) = \lim_{R \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \hat{f}(t)\left(1 - \frac{|t|}{R}\right) e^{-\sqrt{-1}\, xt} dt$$

$$= \lim_{R \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \hat{f}(t)\left(1 - \frac{|t|}{R}\right) e^{-\sqrt{-1}\, xt} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \hat{f}(t) e^{-\sqrt{-1}\, xt} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \hat{f}(-t) e^{\sqrt{-1}\, xt} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \phi(t) e^{\sqrt{-1}\, xt} dt,$$

where $\phi(t) = \hat{f}(-t)$. But $f(z)$ is entire as is

$$\frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \phi(t) e^{\sqrt{-1}\, zt} dt.$$

Since they agree almost everywhere on the real line, they must agree everywhere in the complex plane.

22.8 EXAMPLE Let $f \in E_0(A)$. Assume: $\forall$ real x,

$$|f(x)| \leq M.$$

Then the function

$$\frac{f(z) - f(0)}{z} \ (z \neq 0), \quad f'(0)\ (z = 0),$$

belongs to $E_0(A)$ and its restriction to the real axis is square integrable. Therefore

$$f(z) = f(0) + \frac{z}{\sqrt{2\pi}} \int_{-A}^{A} \phi(t)e^{\sqrt{-1}\,zt}dt$$

for some $\phi \in L^2[-A,A]$.

22.9 ADDENDUM Assume that $\phi(t)$ does not vanish almost everywhere in any neighborhood of A (or $-A$) -- then $T(f) = A$ (hence f is of order 1 (cf. 17.3)).

[Suppose that $T(f) < A$, so $f \in E_0(B)$ with $B < A$ -- then

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-B}^{B} \psi(t)e^{\sqrt{-1}\,zt}dt,$$

where $\psi \in L^2[-B,B]$. Extend ψ to $[-A,A]$ by taking it to be zero in

$$\begin{cases} [-A,-B[& (-A \le t < -B) \\ ]B,A] & (B < t \le A). \end{cases}$$

Then still

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \psi(t)e^{\sqrt{-1}\,zt}dt.$$

Accordingly, by the uniqueness of Fourier transforms (cf. 21.12), $\phi(t) = \psi(t)$ almost everywhere in $[-A,A]$. In particular: $\phi(t) = 0$ almost everywhere in

$$\begin{cases} [-A,-B[& (-A \le t < -B) \\ ]B,A] & (B < t \le A), \end{cases}$$

a contradiction.]

22.10 THEOREM Let $f \in E_0$ ($f \not\equiv 0$). Assume: $f|R \in L^2(-\infty,\infty)$. Put

$$\left[\begin{array}{l} b = \overline{\lim\limits_{r \to \infty}} \dfrac{\log |f(-\sqrt{-1}\, r)|}{r} \equiv h_f(-\sqrt{-1}) \\ \\ -a = \overline{\lim\limits_{r \to \infty}} \dfrac{\log |f(\sqrt{-1}\, r)|}{r} \equiv h_f(\sqrt{-1}). \end{array}\right.$$

Then $b \geq a$ and

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_a^b \phi(t) e^{\sqrt{-1}\, zt} dt$$

for some $\phi \in L^2[a,b]$.

[Note: Since $f \not\equiv 0$, both a and b are finite (cf. 19.4).]

As will be seen below, this result is a consequence of 22.6 once the preliminaries are out of the way.

22.11 RAPPEL If A_1, A_2 are nonempty sets of real numbers which are bounded above and if

$$A_1 + A_2 = \{a_1 + a_2 : a_1 \in A_1,\ a_2 \in A_2\},$$

then

$$\sup(A_1 + A_2) = \sup A_1 + \sup A_2.$$

22.12 LEMMA Let $f \not\equiv 0$ be an entire function of exponential type -- then

$$h_f(\sqrt{-1}\, e^{\sqrt{-1}\,\theta}) + h_f(-\sqrt{-1}\, e^{\sqrt{-1}\,\theta}) \geq 0.$$

PROOF Work instead with H_f (cf. 19.7). Put

$$\left[\begin{array}{l} A_1 = \{\text{Re } (\sqrt{-1}\, e^{\sqrt{-1}\,\theta} w_1) : w_1 \in K_f\} \\ \\ A_2 = \{\text{Re } (-\sqrt{-1}\, e^{\sqrt{-1}\,\theta} w_2) : w_2 \in K_f\}, \end{array}\right.$$

so that by definition

$$\begin{cases} H_f(\sqrt{-1}\, e^{\sqrt{-1}\,\theta}) = \sup A_1 \\ H_f(-\sqrt{-1}\, e^{\sqrt{-1}\,\theta}) = \sup A_2. \end{cases}$$

Consider now $A_1 + A_2$, a generic element of which has the form

$$\text{Re}(\sqrt{-1}\, e^{\sqrt{-1}\,\theta} w_1) + \text{Re}(-\sqrt{-1}\, e^{\sqrt{-1}\,\theta} w_2).$$

In particular: $\forall\, w \in K_f$,

$$\text{Re}(\sqrt{-1}\, e^{\sqrt{-1}\,\theta} w) + \text{Re}(-\sqrt{-1}\, e^{\sqrt{-1}\,\theta} w)$$

$$= 0 \in A_1 + A_2.$$

Therefore

$$\sup(A_1 + A_2) \geq 0$$

=>

$$\sup A_1 + \sup A_2 = \sup(A_1 + A_2) \geq 0$$

=>

$$H_f(\sqrt{-1}\, e^{\sqrt{-1}\,\theta}) + H_f(-\sqrt{-1}\, e^{\sqrt{-1}\,\theta}) \geq 0.$$

22.13 APPLICATION Take $\theta = 0$ -- then

$$h_f(\sqrt{-1}) + h_f(-\sqrt{-1}) \geq 0,$$

i.e.,

$$h_f(-\sqrt{-1}) \geq -h_f(\sqrt{-1})$$

or still, $b \geq a$.

22.14 P-L-P Let F be holomorphic in Im z > 0 and continuous in Im z ≥ 0. Assume:

$$\log |F(z)| = O(|z|) \quad (|z| >> 0)$$

and

$$|F(x)| \le M \quad (-\infty < x < \infty)$$

and

$$\overline{\lim_{r \to \infty}} \frac{\log |F(\sqrt{-1}\, r)|}{r} = K.$$

Then for Im z ≥ 0,

$$|F(z)| \le Me^{K \operatorname{Im} z}.$$

Turning to the proof of 22.10, we have

$$\begin{cases} |f(z)| \le Me^{-a \operatorname{Im} z} & (\operatorname{Im} z \ge 0) \\ |f(z)| \le Me^{b|\operatorname{Im} z|} & (\operatorname{Im} z \le 0). \end{cases}$$

Put

$$g(z) = e^{-\sqrt{-1}\, cz} f(z) \quad (c = \frac{a+b}{2}).$$

Then

$$|g(z)| \le M \exp((1/2)(b-a)|\operatorname{Im} z|)$$

$$\Rightarrow$$

$$g \in E_0((1/2)(b-a))$$

if b > a (cf. infra). Setting

$$C = (1/2)(b-a),$$

it then follows from 22.7 that $\exists\ \psi \in L^2[-C,C]$:

$$g(z) = \frac{1}{\sqrt{2\pi}} \int_{-C}^{C} \psi(t) e^{\sqrt{-1}\, zt} dt$$

=>

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-C}^{C} \psi(t) e^{\sqrt{-1}\, z(t+c)} dt$$

=>

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{a}^{b} \phi(t) e^{\sqrt{-1}\, zt} dt,$$

where $\phi(t) = \psi(t-c)$.

[Note: If a = b, then g is bounded, hence is a constant, call it X:

$$X = e^{-\sqrt{-1}\, cz} f(z)$$

=>

$$f(x) = Xe^{\sqrt{-1}\, cx} \qquad (z = x + \sqrt{-1}\, 0)$$

=>

$$|f(x)| = X,$$

an impossibility ($f \not\equiv 0$ and $f|R \in L^2(-\infty,\infty)$).]

22.15 REMARK The indicator diagram K_f of f is a subset of $[\sqrt{-1}\, a, \sqrt{-1}\, b]$.

[Let $w \in K_f$ -- then

$$-H_f(-1) \le \text{Re } w \le H_f(1) \qquad (\text{cf. } 18.7)$$

or still,

$$-h_f(-1) \le \text{Re } w \le h_f(1) \qquad (\text{cf. } 19.7).$$

But

$$\left[\begin{aligned} h_f(1) &= \overline{\lim_{r \to \infty}} \frac{\log |f(re^{\sqrt{-1}\,0})|}{r} \\ h_f(-1) &= \overline{\lim_{r \to \infty}} \frac{\log |f(re^{\sqrt{-1}\,\pi})|}{r} \end{aligned}\right. .$$

And

$$\left[\begin{aligned} |f(re^{\sqrt{-1}\,0})| &= |f(r)| \leq M \\ |f(re^{\sqrt{-1}\,\pi})| &= |f(-r)| \leq M \end{aligned}\right.$$

=>

$$\left[\begin{aligned} h_f(1) &\leq 0 \\ h_f(-1) &\leq 0 \end{aligned}\right.$$

=>

$$0 \leq - h_f(-1) \leq \text{Re}\, w \leq h_f(1) \leq 0 \qquad (\text{cf. } 18.9).$$

Therefore w is necessarily pure imaginary. Finally

$$- H_f(\sqrt{-1}) \leq \text{Im}\, w \leq H_f(- \sqrt{-1}) \qquad (\text{cf. } 18.8)$$

or still,

$$- h_f(\sqrt{-1}) \leq \text{Im}\, w \leq h_f(- \sqrt{-1}) \qquad (\text{cf. } 19.7)$$

=>

$$a \leq \text{Im}\, w \leq b.]$$

[Note: If $\phi(t)$ does not vanish in any neighborhood of a and does not vanish in any neighborhood of b, then

$$K_f = [\sqrt{-1}\, a, \sqrt{-1}\, b].]$$

The functions

$$\frac{1}{\sqrt{2A}} \exp\left(- \frac{\sqrt{-1}\, tn\pi}{A}\right) \qquad (n = 0, \pm 1, \ldots)$$

constitute an orthonormal basis for $L^2[-A,A]$. Therefore the functions

$$\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2A}} \int_{-A}^{A} \exp\left(- \frac{\sqrt{-1}\, tn\pi}{A}\right) e^{\sqrt{-1}\, zt} dt$$

constitute an orthonormal basis for PW(A), i.e., the functions

$$\left(\frac{A}{\pi}\right)^{1/2} \frac{\sin(Az-n\pi)}{Az-n\pi}$$

constitute an orthonormal basis for PW(A).

[Note: Matters simplify when $A = \pi$: The functions

$$\frac{\sin \pi(z-n)}{\pi(z-n)}$$

constitute an orthonormal basis for $PW(\pi)$. In this connection, observe that if $f(z)$ belongs to PW(A), then $f(\frac{\pi z}{A})$ belongs to $PW(\pi)$.]

22.16 THEOREM Let $f \in PW(A)$ -- then there is an expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n \left(\frac{A}{\pi}\right)^{1/2} \frac{\sin(Az-n\pi)}{Az-n\pi}$$

in PW(A), where

$$c_n = \left(\frac{\pi}{A}\right)^{1/2} f\left(\frac{n\pi}{A}\right),$$

so

$$||f||^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{\pi}{A} \sum_{n=-\infty}^{\infty} |f(\frac{n\pi}{A})|^2.$$

N.B. Therefore

$$f(z) = \sum_{n=-\infty}^{\infty} f(\frac{n\pi}{A}) \frac{\sin(Az-n\pi)}{Az-n\pi} .$$

22.17 LEMMA The series

$$\sum_{n=-\infty}^{\infty} f(\frac{n\pi}{A}) \frac{\sin(Az-n\pi)}{Az-n\pi}$$

converges uniformly on every horizontal strip $|\text{Im } z| \leq h$.

22.18 EXAMPLE Take $A = \pi$ -- then

$$f(z) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(z-n)}{\pi(z-n)} .$$

Accordingly, if $f(n) = 0$ for $n = 0, \pm 1, \pm 2, \ldots$, then $f \equiv 0$ (cf. 19.14).

22.19 NOTATION ℓ^2 is the set of sequences $c_0, c_{\pm 1}, c_{\pm 2}, \ldots$ of complex numbers such that

$$\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty.$$

22.20 LEMMA The arrow

$$\ell^2 \to \text{PW}(\pi)$$

that sends $\{c_n\}$ to

$$f(z) = \sum_{n=-\infty}^{\infty} c_n \frac{\sin \pi(z-n)}{\pi(z-n)}$$

is an isometric isomorphism.

22.21 EXAMPLE Put

$$\begin{cases} c_n = 0 & (n \le 0) \\ c_n = \dfrac{(-1)^n}{n} & (n > 0) \end{cases}$$

and let

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{\sin \pi(z-n)}{\pi(z-n)} .$$

Then $f \in PW(\pi)$, yet the product $zf(z)$ does not belong to $PW(\pi)$ (but, of course, it does belong to $E_0(\pi)$ (cf. 17.15)).

[If $zf(z)$ was a Paley-Wiener function, then it would be bounded on the real axis (cf. 17.29), thus the same would be true of its derivative $zf'(z) + f(z)$ (cf. 17.24 (or quote 22.4)). But

$$f'(z) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{\pi^2(z-n)\cos \pi z - \pi\sin \pi z}{\pi^2(z-n)^2}$$

=>

$$kf'(k) = (-1)^k \sum_{\substack{n=1 \\ n \neq k}}^{\infty} (\frac{1}{n} - \frac{1}{n-k})$$

=>

$$|kf'(k)| = (1 + \frac{1}{2} + \cdots + \frac{1}{k}) - \frac{2}{k}$$

=>

$$|kf'(k)| \to \infty \text{ as } k \to \infty.$$

However

$$f(k) \to 0 \text{ as } k \to \infty.$$

Therefore

$$\{kf'(k) + f(k) : k = 1,2,\ldots\}$$

is not bounded.]

Moving on:

22.22 LEMMA $\forall$ real x,y:

$$\frac{\sin A(x-y)}{A(x-y)} = \sum_{n=-\infty}^{\infty} \frac{\sin(Ax-n\pi)}{Ax-n\pi} \cdot \frac{\sin(Ay-n\pi)}{Ay-n\pi}.$$

22.23 APPLICATION Let $f \in PW(A)$ -- then

$$f(x) = \frac{A}{\pi} \int_{-\infty}^{\infty} f(y) \frac{\sin A(x-y)}{A(x-y)} dy.$$

[Start with the RHS:

$$\frac{A}{\pi} \int_{-\infty}^{\infty} f(y) \frac{\sin A(x-y)}{A(x-y)} dy$$

$$= \frac{A}{\pi} \int_{-\infty}^{\infty} f(y) \sum_{n=-\infty}^{\infty} \frac{\sin(Ax-n\pi)}{Ax-n\pi} \cdot \frac{\sin(Ay-n\pi)}{Ay-n\pi} dy$$

$$= \sum_{n=-\infty}^{\infty} \frac{A}{\pi} \left(\int_{-\infty}^{\infty} f(y) \frac{\sin(Ay-n\pi)}{Ay-n\pi} dy\right) \frac{\sin(Ax-n\pi)}{Ax-n\pi}$$

$$= \sum_{n=-\infty}^{\infty} \frac{A}{\pi} \left(\left(\frac{\pi}{A}\right)^{1/2} \int_{-\infty}^{\infty} f(y) \left(\frac{A}{\pi}\right)^{1/2} \frac{\sin(Ay-n\pi)}{Ay-n\pi} dy\right) \frac{\sin(Ax-n\pi)}{Ax-n\pi}$$

$$= \sum_{n=-\infty}^{\infty} \frac{A}{\pi} \left(\left(\frac{\pi}{A}\right)^{1/2} c_n\right) \frac{\sin(Ax-n\pi)}{Ax-n\pi}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{A}{\pi}\right)^{1/2} c_n \frac{\sin(Ax-n\pi)}{Ax-n\pi}$$

$$= \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{A}\right) \frac{\sin(Ax-n\pi)}{Ax-n\pi}$$

$$= f(x).]$$

[Note: Consequently,

$$|f(x)| \leq \frac{A}{\pi} \int_{-\infty}^{\infty} |f(y)| \left|\frac{\sin A(x-y)}{A(x-y)}\right| dy$$

$$\leq \frac{A}{\pi} \left(\int_{-\infty}^{\infty} |f(y)|^2 dy\right)^{1/2} \left(\int_{-\infty}^{\infty} \left|\frac{\sin A(x-y)}{A(x-y)}\right|^2 dy\right)^{1/2}$$

$$= \frac{A}{\pi} ||f||_2 \frac{1}{\sqrt{A}} \left(\int_{-\infty}^{\infty} \frac{\sin^2 y}{y^2} dy\right)^{1/2}$$

$$= \frac{A}{\pi} ||f||_2 \frac{1}{\sqrt{A}} \sqrt{\pi} \quad \text{(cf. 21.18)}$$

$$= \left(\frac{A}{\pi}\right)^{1/2} ||f||_2.$$

Moreover, this estimate is sharp: Take $A = \pi$, $n = 0$, $f(z) = \frac{\sin \pi z}{\pi z}$ -- then for real x,

$$|f(x)| \leq 1 = ||f||_2,$$

and $f(0) = 1$.]

22.24 REMARK The following result is of importance in sampling theory:

$$\sum_{n=-\infty}^{\infty} \left|\frac{\sin \pi(x-n)}{\pi(x-n)}\right|^2 < 2.$$

[There is no loss of generality in imposing the restriction $-\frac{1}{2} < x \leq \frac{1}{2}$, hence

$$\sum_{n=-\infty}^{\infty} \left| \frac{\sin \pi(x-n)}{\pi(x-n)} \right|^2 \leq 1 + \sum_{n\neq 0} \frac{1}{\pi^2 |x-n|^2}$$

$$\leq 1 + \frac{1}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{1}{(n-x)^2} + \frac{1}{(n+x)^2} \right]$$

$$\leq 1 + \frac{1}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{1}{(n-\frac{1}{2})^2} + \frac{1}{(n+\frac{1}{2})^2} \right]$$

$$= 1 + \frac{1}{\pi^2} \left[\sum_{n=1}^{\infty} \frac{1}{(n+\frac{1}{2})^2} \right.$$

$$\left. + \frac{1}{(\frac{1}{2})^2} - \sum_{n=2}^{\infty} \frac{1}{(n-\frac{1}{2})^2} + 2 \sum_{n=2}^{\infty} \frac{1}{(n-\frac{1}{2})^2} \right]$$

$$= 1 + \frac{1}{\pi^2} \left[2^2 + 2 \sum_{n=2}^{\infty} \frac{1}{(n-\frac{1}{2})^2} \right]$$

$$< 1 + \frac{1}{\pi^2} \left[2^2 + 2 \int_1^{\infty} \frac{1}{(t-\frac{1}{2})^2} dt \right]$$

$$= 1 + \frac{1}{\pi^2} [2^2 + 2^2]$$

$$= 1 + 2\left(\frac{2}{\pi}\right)^2 < 1 + 1 = 2.]$$

22.25 THEOREM Let $f \in E_0(A)$. Assume: $\forall$ real x,

$$|f(x)| \leq M.$$

Then

$$f(z) = f'(0)\,\frac{\sin Az}{A} + f(0)\,\frac{\sin Az}{Az}$$

$$+ \sum_{n\neq 0} f\left(\frac{n\pi}{A}\right)\left(\frac{Az}{n\pi}\right)\frac{\sin(Az-n\pi)}{Az-n\pi}\,.$$

PROOF Apply 22.16 to the function figuring in 22.7, hence

$$\frac{f(z)-f(0)}{z} = f'(0)\,\frac{\sin Az}{Az}$$

$$+ \sum_{n\neq 0} \frac{f\left(\frac{n\pi}{A}\right)-f(0)}{\frac{n\pi}{A}}\,\frac{\sin(Az-n\pi)}{Az-n\pi}$$

=>

$$f(z) = f'(0)\,\frac{\sin Az}{A} + f(0)$$

$$+ \sum_{n\neq 0} f\left(\frac{n\pi}{A}\right)\left(\frac{Az}{n\pi}\right)\frac{\sin(Az-n\pi)}{Az-n\pi}$$

$$+ (-f(0))(\sin Az)\sum_{n\neq 0} (-1)^n\left(\frac{Az}{n\pi}\right)\frac{1}{Az-n\pi}\,.$$

But for w nonintegral,

$$\frac{\pi}{\sin \pi w} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n+w} = \frac{1}{w} + 2w\sum_{n=1}^{\infty}\frac{(-1)^n}{w^2-n^2}\,.$$

Therefore

$$\sum_{n\neq 0} (-1)^n\left(\frac{Az}{n\pi}\right)\frac{1}{Az-n\pi}$$

$$= 2Az\sum_{n=1}^{\infty}\frac{(-1)^n}{A^2z^2-n^2\pi^2}$$

$$= 2Az \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi^2((Az/\pi)^2-n^2)}$$

$$= \frac{2Az}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(Az/\pi)^2-n^2}$$

$$= \frac{1}{\pi} 2(\frac{Az}{\pi}) \sum_{n=1}^{\infty} \frac{(-1)^n}{(\frac{Az}{\pi})^2-n^2}$$

$$= \frac{1}{\pi} \left[\frac{\pi}{\sin \pi(\frac{Az}{\pi})} - \frac{1}{\frac{Az}{\pi}} \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi}{\sin Az} - \frac{\pi}{Az} \right]$$

$$= \frac{1}{\sin Az} - \frac{1}{Az} .$$

And so

$$f(0) + (- f(0))(\sin Az) \sum_{n\neq 0} (-1)^n (\frac{Az}{n\pi}) \frac{1}{Az-n\pi}$$

$$= f(0) + (- f(0))(\sin Az) \left[\frac{1}{\sin Az} - \frac{1}{Az} \right]$$

$$= f(0) - f(0) + f(0) \frac{\sin Az}{Az}$$

$$= f(0) \frac{\sin Az}{Az} .$$

Take $A = 1$ -- then the functions

$$\frac{1}{\sqrt{\pi}} \frac{\sin(z-n\pi)}{z-n\pi}$$

constitute an orthonormal basis for $PW(1)$ (the canonical choice...).

22.26 RAPPEL Let

$$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2-1)^n$$

be the n^{th} Legendre polynomial (cf. 8.17) -- then the functions

$$\sqrt{n + \frac{1}{2}}\, P_n(t) \quad (n = 0,1,\ldots)$$

constitute an orthonormal basis for $L^2[-1,1]$.

22.27 LEMMA We have

$$\frac{1}{\sqrt{2\pi}} \int_{-1}^{1} P_n(t) e^{\sqrt{-1}\, xt} dt = (\sqrt{-1})^n \frac{J_{n + \frac{1}{2}}(x)}{\sqrt{x}} .$$

22.28 EXAMPLE Take $n = 0$ -- then $P_0(t) = 1$ and

$$\frac{1}{\sqrt{2\pi}} \int_{-1}^{1} P_0(t) e^{\sqrt{-1}\, xt} = \left(\frac{2}{\pi}\right)^{1/2} \frac{\sin x}{x} = \frac{J_{\frac{1}{2}}(x)}{\sqrt{x}} .$$

22.29 SCHOLIUM The functions

$$\sqrt{n + \frac{1}{2}}\, (\sqrt{-1})^n \frac{J_{n + \frac{1}{2}}(z)}{\sqrt{z}}$$

constitute an orthonormal basis for $PW(1)$.

22.30 APPLICATION Let

$$\phi_n(t) = \sqrt{n + \frac{1}{2}}\, P_n(t).$$

Then in $L^2[-1,1]$,

$$\left[\begin{array}{l} \langle e^{\sqrt{-1}\,x\,-}, \phi_n\rangle = \int_{-1}^{1} e^{\sqrt{-1}\,xt}\phi_n(t)\,dt = \sqrt{2\pi}\,\hat{\phi}_n(x) \\ \langle e^{\sqrt{-1}\,y\,-}, \phi_n\rangle = \int_{-1}^{1} e^{\sqrt{-1}\,yt}\phi_n(t)\,dt = \sqrt{2\pi}\,\hat{\phi}_n(y). \end{array}\right.$$

Thus, by Parseval,

$$\langle e^{\sqrt{-1}\,x\,-}, e^{\sqrt{-1}\,y\,-}\rangle$$

$$= \sum_{n=0}^{\infty} \langle e^{\sqrt{-1}\,x\,-}, \phi_n\rangle \overline{\langle e^{\sqrt{-1}\,y\,-}, \phi_n\rangle}$$

$$= 2\pi \sum_{n=0}^{\infty} \hat{\phi}_n(x)\hat{\phi}_n(-y).$$

But

$$\langle e^{\sqrt{-1}\,x\,-}, e^{\sqrt{-1}\,y\,-}\rangle$$

$$= \int_{-1}^{1} e^{\sqrt{-1}\,(x-y)t}\,dt$$

$$= 2\,\frac{\sin(x-y)}{x-y}.$$

On the other hand,

$$2\pi \sum_{n=0}^{\infty} \hat{\phi}_n(x)\hat{\phi}_n(-y)$$

$$= 2\pi \sum_{n=0}^{\infty} \sqrt{n+\tfrac{1}{2}}\,(\sqrt{-1})^n \frac{J_{n+\frac{1}{2}}(x)}{\sqrt{x}} \sqrt{n+\tfrac{1}{2}}\,(\sqrt{-1})^n \frac{J_{n+\frac{1}{2}}(-y)}{\sqrt{-y}} \qquad \text{(cf. 22.27)}$$

$$= 2\pi \sum_{n=0}^{\infty} (n+\tfrac{1}{2})\,(\sqrt{-1})^{2n} \frac{J_{n+\frac{1}{2}}(x)}{\sqrt{x}}\,\frac{J_{n+\frac{1}{2}}(-y)}{\sqrt{-y}}$$

$$= 2\pi \sum_{n=0}^{\infty} (n + \tfrac{1}{2}) (\sqrt{-1})^{2n} (-1)^n \frac{J_{n+\frac{1}{2}}(x)}{\sqrt{x}} \frac{J_{n+\frac{1}{2}}(y)}{\sqrt{y}} .$$

And

$$(\sqrt{-1})^{2n}(-1)^n = ((\sqrt{-1})^2)^n(-1)^n$$

$$= (-1)^n(-1)^n$$

$$= (-1)^{2n} = 1.$$

Therefore

$$\frac{\sin(x-y)}{x-y} = \pi \sum_{n=0}^{\infty} (n + \tfrac{1}{2}) \frac{J_{n+\frac{1}{2}}(x)}{\sqrt{x}} \frac{J_{n+\frac{1}{2}}(y)}{\sqrt{y}} .$$

§23. *DISTRIBUTION FUNCTIONS*

Suppose given a function $F:R \to R$.

23.1 DEFINITION F is increasing if $F(x) \leq F(y)$ whenever $x \leq y$ and F is strictly increasing if $F(x) < F(y)$ whenever $x < y$.

Suppose given an increasing function $F:R \to R$.

23.2 NOTATION Write

$$\begin{cases} F(x^+) = \lim_{h \to 0} F(x + h) \\ F(x^-) = \lim_{h \to 0} F(x - h) \end{cases} \qquad (h > 0)$$

or still

$$\begin{cases} F(x^+) = \inf_{y > x} F(y) \\ F(x^-) = \sup_{y < x} F(y) \end{cases}$$

and put

$$\begin{cases} F(\infty) = \sup_{x \in R} F(x) \\ F(-\infty) = \inf_{x \in R} F(x). \end{cases}$$

23.3 DEFINITION F is continuous from the right if $\forall\, x$,

$$F(x^+) = F(x).$$

A <u>distribution function</u> is an increasing function $F:R \to R$ which is continuous from the right subject to

$$F(\infty) = 1, \ F(-\infty) = 0.$$

23.4 EXAMPLE The function

$$I(x) = \begin{cases} 0 & (x < 0) \\ 1 & (x \geq 1) \end{cases}$$

is a distribution function, the <u>unit step function</u>.

23.5 DEFINITION Suppose that F is a distribution function.

- A point x such that $F(x)\ (= F(x^+)) = F(x^-)$ is called a <u>continuity point</u> of F.

- A point x such that $F(x)\ (= F(x^+)) \neq F(x^-)$ is called a <u>discontinuity point</u> of F.

23.6 DEFINITION Suppose that F is a distribution function -- then the quantity

$$j_x = F(x^+) - F(x^-)$$

is called the <u>jump</u> of F at x.

[Note: j_x is positive at a discontinuity point and zero at a continuity point.]

23.7 LEMMA The set

$$\{x: j_x > 0\}$$

is at most countable.

Therefore the set of continuity points of a distribution function is dense in R.

23.8 REMARK There exist distribution functions whose set of discontinuity points is dense in R.

[Let $\{q_n : n = 1,2,\ldots\}$ be an enumeration of Q and consider

$$F(x) = \sum_{q_n \leq x} 2^{-n},$$

noting that $\sum_{n=1}^{\infty} 2^{-n} = 1$.]

23.9 NOTATION Bo(R) is the σ-algebra of Borel subsets of R.

23.10 LEMMA If f is a Lebesgue measurable function, then there exists a Borel measurable function g such that f = g almost everywhere.

22.11 CONSTRUCTION Let F be a distribution function -- then there exists a unique Borel measure μ_F on R characterized by the condition

$$\mu_F(]a,b]) = F(b) - F(a)$$

for all $a,b \in R$. Here

$$F(x) = \mu_F(]-\infty,x])$$

and

$$j_x = \mu_F(\{x\}).$$

Moreover,

$$1 = F(\infty) = \mu_F(R),$$

so μ_F is a probability measure on the line.

[Note: We have

$$\left[\begin{array}{l} \mu_F([a,b[) = F(b^-) - F(a^-) \\ \mu_F([a,b]) = F(b) - F(a^-) \\ \mu_F(]a,b[) = F(b^-) - F(a).] \end{array}\right.$$

23.12 EXAMPLE Take $F = I$ -- then $\mu_I = \delta_0$.

23.13 LEMMA Any bounded Borel measurable function on R is μ_F-integrable.

23.14 REMARK The considerations in 23.11 can be reversed. For suppose that μ is a probability measure on the line. Put

$$F_\mu(x) = \mu(]-\infty,x]).$$

Then F_μ is a distribution function and

$$\mu_{F_\mu} = \mu.$$

In fact,

$$]a,b] =]-\infty,b] -]-\infty,a],$$

thus

$$\begin{aligned} \mu_{F_\mu}(]a,b]) &= F_\mu(b) - F_\mu(a) \\ &= \mu(]-\infty,b]) - \mu(]-\infty,a]) \\ &= \mu(]-\infty,b] -]-\infty,a]) \\ &= \mu(]a,b]). \end{aligned}$$

[Note: In the other direction,

$$F_{\mu_F} = F.]$$

There are three kinds of "pure" distribution functions, viz.: discrete, absolutely continuous, and singular.

23.15 DEFINITION A distribution function F is said to be discrete if there is a sequence $\{x_n\} \subset R$ (possibly finite) and positive numbers j_n such that $\sum_n j_n = 1$ and

$$F(x) = \sum_n j_n I(x-x_n).$$

[Note: Accordingly,

$$\mu_F = \sum_n j_n \delta_{x_n}.]$$

23.16 LEMMA Suppose that F is a discrete distribution function -- then a Borel measurable function f is integrable with respect to μ_F iff

$$\sum_n j_n |f(x_n)| < \infty,$$

in which case

$$\int f d\mu_F = \sum_n j_n f(x_n).$$

23.17 RAPPEL An increasing function $\phi: R \to R$ is differentiable almost everywhere and its derivative ϕ' is Lebesgue measurable, nonnegative, and

$$\int_a^b \phi'(t)dt \le \phi(b) - \phi(a)$$

for all a and b.

23.18 APPLICATION Suppose that F is a distribution function -- then F is differentiable almost everywhere and its derivative F' is Lebesgue measurable, nonnegative, and integrable:

$$||F'||_1 = \int_{-\infty}^{\infty} F'(t)dt \le F(\infty) - F(-\infty) = 1.$$

23.19 DEFINITION A function $F:R \to R$ is <u>absolutely continuous</u> if $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ such that for any finite set of disjoint intervals $]a_1,b_1[,\ldots,]a_N,b_N[$,

$$\sum_{j=1}^{N} (b_j - a_j) < \delta \Rightarrow \sum_{j=1}^{n} |F(b_j) - F(a_j)| < \varepsilon.$$

[Note: An absolutely continuous function is necessarily uniformly continuous, the converse being false.]

23.20 EXAMPLE If F is everywhere differentiable and if F' is bounded, then F is absolutely continuous (use the mean value theorem).

23.21 RAPPEL If $f \in L^1(-\infty,\infty)$ and if $F(x) = \int_{-\infty}^{x} f(t)dt$, then F is absolutely continuous and $F' = f$ almost everywhere.

23.22 EXAMPLE The prescription

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$$

defines an absolutely continuous distribution function.

23.23 CRITERION Suppose that F is a distribution function -- then F is absolutely continuous iff μ_F is absolutely continuous with respect to the restriction of Lebesgue measure to $Bo(R)$.

So, under the assumption that F is absolutely continuous, the Radon-Nikodym theorem implies that μ_F admits a density $f \in L^1(-\infty,\infty)$:

$$\forall\ S \in Bo(R),\ \mu_F(S) = \int_S f.$$

Matters can then be made precise.

23.24 THEOREM If F is an absolutely continuous distribution function, then $\forall\, x,\ F(x) = \int_{-\infty}^{x} F'(t)dt$.

PROOF For $h > 0$,

$$\mu_F(]x,x+h]) = \begin{cases} F(x+h) - F(x) \\ \\ \int_x^{x+h} f \end{cases}$$

and

$$\mu_F(]x-h,x]) = \begin{cases} F(x) - F(x-h) \\ \\ \int_{x-h}^{x} f \end{cases}.$$

But on general grounds,

$$\begin{cases} \lim\limits_{h \to 0} \frac{1}{h} \int_x^{x+h} f = f(x) \\ \\ \lim\limits_{h \to 0} \frac{1}{h} \int_{x-h}^{x} f = f(x) \end{cases}$$

almost everywhere. Therefore

$$\begin{cases} \lim\limits_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x) \\ \\ \lim\limits_{h \to 0} \frac{F(x) - F(x-h)}{h} = f(x) \end{cases}$$

almost everywhere, hence $F'(x) = f(x)$ almost everywhere. Finally, $\forall\, x$,

$$F(x) = \mu_F(]-\infty,x]) = \int_{-\infty}^{x} f = \int_{-\infty}^{x} F'.$$

23.25 DEFINITION An increasing continuous function $F:R \to R$ is said to be singular if $F' = 0$ almost everywhere.

Trivially, a constant function is singular.

23.26 EXAMPLE There exist singular distribution functions.

[Let Θ denote the Cantor function on [0,1] and put $\Theta(x) = 0$ $(x < 0)$, $\Theta(x) = 1$ $(x > 1)$ -- then Θ is a singular distribution function. Therefore

$$\int_0^1 \Theta'(t)dt = 0 < 1 = \Theta(1) - \Theta(0) \quad \text{(cf. 23.17).]}$$

[Note: The Cantor function is increasing on [0,1] but there are refined versions of Θ that are strictly increasing on [0,1].]

23.27 LEMMA An absolutely continuous distribution function F cannot be singular.

PROOF For suppose F was singular -- then in view of 23.24, $\forall$ x,

$$F(x) = \int_{-\infty}^{x} F'(t)dt = 0,$$

an impossibility.

Given a distribution function F, let $\{x_n\}$ be its set of discontinuity points (which for this discussion we shall assume is not empty). Define $\Phi:R \to R$ by the prescription

$$\Phi(x) = \sum_n j_{x_n} I(x-x_n).$$

Then Φ is increasing, continuous from the right, and

$$\Phi(-\infty) = 0, \ \Phi(\infty) \equiv a \leq 1.$$

If $F \neq \Phi$, put

$$\Psi(x) = F(x) - \Phi(x).$$

Then Ψ is increasing, continuous, and

$$\Psi(-\infty) = 0, \ \Psi(\infty) \equiv b \leq 1.$$

23.28 NOTATION Let

$$\begin{cases} F_d(x) = \frac{1}{a}\Phi(x) \\ F_c(x) = \frac{1}{b}\Psi(x). \end{cases}$$

Therefore $\begin{cases} F_d \\ F_c \end{cases}$ are distribution functions and

$$F = aF_d + bF_c \quad (a + b = 1).$$

[Note: F_d is referred to as the discrete part of F while F_c is referred to as the continuous part of F. Here $0 \leq a \leq 1$, $0 \leq b \leq 1$, with the understanding that

$$\begin{cases} a = 1 \Longleftrightarrow F = F_d \\ b = 1 \Longleftrightarrow F = F_c. \end{cases}$$

N.B. More can be said about F_c (cf. infra).

Given a continuous distribution function F, there are two possibilities: Either $F' = 0$ almost everywhere (in which case F is singular) or else $F' \neq 0$ almost everywhere. Assuming that the second possibility is in force, define $\Phi:R \to R$ by the prescription

$$\Phi(x) = \int_{-\infty}^{x} F'(t)dt.$$

Then Φ is increasing, absolutely continuous, and

$$\Phi(-\infty) = 0, \ \Phi(\infty) \equiv u \leq 1.$$

If $F \neq \Phi$, put

$$\Psi(x) = F(x) - \Phi(x).$$

Then Ψ is increasing, continuous, and

$$\Psi(-\infty) = 0, \ \Psi(\infty) \equiv v \leq 1.$$

In addition, $\Phi' = F'$ almost everywhere, hence $\Psi' = 0$ almost everywhere, hence Ψ is singular.

23.29 NOTATION Let

$$\begin{cases} F_{ac}(x) = \frac{1}{u}\Phi(x) \\ F_s(x) = \frac{1}{v}\Psi(x). \end{cases}$$

Therefore $\begin{cases} F_{ac} \\ F_s \end{cases}$ are distribution functions and

$$F = uF_{ac} + vF_s \quad (u + v = 1).$$

[Note: F_{ac} is referred to as the absolutely continuous part of F while F_s is referred to as the singular part of F. Here $0 \leq u \leq 1$, $0 \leq v \leq 1$, with the understanding that

$$\begin{cases} u = 1 \iff F = F_{ac} \\ v = 1 \iff F = F_s. \end{cases}$$]

Now let F be an arbitrary distribution function, thus

$$F = aF_d + bF_c.$$

Since F_c is a continuous distribution function, the preceding discussion is applicable to it. Write

$$\begin{bmatrix} F_{ac} \text{ in place of } (F_c)_{ac} \\ F_s \text{ in place of } (F_c)_s. \end{bmatrix}$$

Then

$$F_c = uF_{ac} + vF_s$$

$$\Rightarrow$$

$$F = aF_d + b(uF_{ac} + vF_s).$$

And

$$a + bu + bv = a + b = 1.$$

23.30 SCHOLIUM Every distribution function F admits a (unique) decomposition

$$F = AF_d + BF_{ac} + CF_s,$$

where

$$A + B + C = 1 \quad (A \geq 0,\ B \geq 0,\ C \geq 0),$$

and F_d is a discrete distribution function, F_{ac} is an absolutely continuous distribution function, and F_s is a singular distribution function.

23.31 DEFINITION Let F_1, F_2 be distribution functions -- then their <u>convolution</u> is the function

$$F_1 * F_2(x) = \int_{-\infty}^{\infty} F_1(x-y)\, d\mu_{F_2}(y).$$

<u>N.B.</u> The integral defining $F_1 * F_2$ exists (cf. 23.13).

23.32 LEMMA The convolution $F_1 * F_2$ is a distribution function.

23.33 FORMALITIES We have

$$F_1 * F_2 = F_2 * F_1$$

and

$$F_1 * (F_2 * F_3) = (F_1 * F_2) * F_3.$$

Furthermore,

$$F = F * I = I * F.$$

23.34 THEOREM Suppose that $F = F_1 * F_2$.

- If F_1,F_2 are discrete, then F is discrete.
- If either F_1 or F_2 is continuous, then F is continuous.
- If either F_1 or F_2 is absolutely continuous, then F is absolutely continuous.
- If F_1 is discrete and F_2 is singular, then F is singular.
- If F_1,F_2 are singular, then F is continuous.

[Note: F might be singular, or F might be absolutely continuous, or F might be a mixture of both.]

APPENDIX

An <u>integrator</u> is an increasing function $F:R \to R$ which is continuous from the right. A distribution function is therefore an integrator but not conversely.

Every integrator F gives rise to a unique Borel measure μ_F characterized by the condition

$$\mu_F(]a,b]) = F(b) - F(a).$$

<u>N.B.</u> Given integrators F and G, $\mu_F = \mu_G$ iff F - G is a constant.

LEMMA If F is a continuously differentiable integrator, then $d\mu_F(x) = F'(x)dx$.

DEFINITION The completion $\bar{\mu}_F$ of μ_F is called the <u>Lebesgue-Stieltjes measure</u> associated with F.

EXAMPLE Take $F(x) = x$ -- then $\bar{\mu}_F$ is Lebesgue measure.

Denote by $\mathcal{A}_F \supset \text{Bo}(R)$ the domain of $\bar{\mu}_F$.

LEMMA If $X \in \mathcal{A}_F$, then there is a Borel set S and a $Z \in \mathcal{A}_F$ of Lebesgue-Stieltjes measure 0 such that $X = S \cup Z$.

Technically, one should distinguish between $\int f d\mu_F$ and $\int f d\bar{\mu}_F$ but this is unnecessary if f is Borel measurable.

NOTATION Write $\int_a^b$ in place of $\int_{[a,b]}$.

INTEGRATION BY PARTS If F,G are integrators, then

$$\int_a^b G(x^+)d\mu_F(x) + \int_a^b F(x^-)d\mu_G(x)$$

$$= F(b^+)G(b^+) - F(a^-)G(a^-).$$

[Note: G is continuous from the right so $G(x^+) = G(x)$ and $G(b^+) = G(b)$.]

§24. *CHARACTERISTIC FUNCTIONS*

Let $F:R \to R$ be a distribution function.

24.1 DEFINITION The <u>characteristic function</u> f of F is the Fourier transform of μ_F, i.e.,

$$f(x) = \int_{-\infty}^{\infty} e^{\sqrt{-1}\, xt} d\mu_F(t).$$

[Note: The integral defining f exists (cf. 23.13).]

Obviously,

$$f(0) = 1, \ |f(x)| \le 1, \ \overline{f(x)} = f(-x).$$

<u>N.B.</u> We have

$$\begin{cases} \text{Re } f(x) = \int_{-\infty}^{\infty} \cos(xt) d\mu_F(t) \\ \text{Im } f(x) = \int_{-\infty}^{\infty} \sin(xt) d\mu_F(t). \end{cases}$$

24.2 LEMMA f(x) is a uniformly continuous function of x (cf. 21.1).

24.3 DEFINITION A distribution function $F:R \to R$ is <u>symmetric</u> if $\forall$ x,

$$\mu_F(]-\infty,x]) = \mu_F([-x,\infty[).$$

Therefore

$$\mu_F(S) = \mu_F(-S)$$

for all $S \in \text{Bo}(R)$.

[Note: Write

$$]-\infty,-x[\ \cup \ [-x,\infty[\ = \ ]-\infty,\infty[$$

or still,

$$]-\infty,-x] - \{-x\}) \cup [-x,\infty[=]-\infty,\infty[.$$

Then

$$\mu_F(]-\infty,-x] - \{-x\}) + \mu_F([-x,\infty[) = \mu_F(]-\infty,\infty[)$$

$\Rightarrow$

$$\mu_F(]-\infty,-x]) - \mu_F(\{-x\}) + \mu_F([-x,\infty[) = 1$$

$\Rightarrow$

$$F(-x) - (F(-x) - F(-x^-)) + \mu_F([-x,\infty[) = 1$$

$\Rightarrow$

$$F(-x^-) + \mu_F([-x,\infty[) = 1$$

$\Rightarrow$

$$\mu_F([-x,\infty[) = 1 - F(-x^-).$$

Accordingly, F is symmetric iff $\forall$ x,

$$F(x) = 1 - F(-x^-).]$$

Given any distribution function F, the assignment $x \to 1 - F(-x^-)$ is a distribution function, call it $(-1)F$, thus

$$d\mu_{(-1)F}(t) = d\mu_F(-t)$$

and the characteristic function $(-1)f$ of $(-1)F$ is $f(-x)$ $(= \overline{f(x)})$.

[Note: F is symmetric iff $F = (-1)F$.]

24.4 REMARK Re f(x) is a characteristic function. Proof:

$$\text{Re } f(x) = \frac{1}{2}(f(x) + \overline{f(x)})$$

and

$$\frac{1}{2} F + \frac{1}{2} (-1)F$$

is a distribution function.

24.5 LEMMA F is symmetric iff f is real.

PROOF If F is symmetric, then $\mu_F = \mu_{(-1)F}$, so

$$f(x) = \int_{-\infty}^{\infty} e^{\sqrt{-1}\, xt} d\mu_F(t)$$

$$= \int_{-\infty}^{\infty} e^{-\sqrt{-1}\, xt} d\mu_F(-t)$$

$$= \int_{-\infty}^{\infty} e^{-\sqrt{-1}\, xt} d\mu_{(-1)F}(t)$$

$$= \int_{-\infty}^{\infty} e^{-\sqrt{-1}\, xt} d\mu_F(t)$$

$$= f(-x) = \overline{f(x)}.$$

I.e.: f is real. Conversely, if f is real, then F and (-1)F have the same characteristic function, hence F = (-1)F (cf. 24.16).

24.6 LEMMA We have

$$1 - \operatorname{Re} f(2x) \le 4(1 - \operatorname{Re} f(x))$$

and

$$|\operatorname{Im} f(x)| \le (\frac{1}{2} (1 - \operatorname{Re} f(2x)))^{1/2}.$$

PROOF Write

$$1 - \operatorname{Re} f(2x) = \int_{-\infty}^{\infty} (1 - \cos(2xt)) d\mu_F(t)$$

$$= \int_{-\infty}^{\infty} 2(1 - (\cos(xt))^2) d\mu_F(t)$$

$$\leq \int_{-\infty}^{\infty} 4(1 - \cos(xt))d\mu_F(t)$$

$$= 4(1 - \text{Re } f(x))$$

and

$$|\text{Im } f(x)| = \left|\int_{-\infty}^{\infty} \sin(xt)d\mu_F(t)\right|$$

$$\leq (\int_{-\infty}^{\infty} (\sin(xt))^2 d\mu_F(t))^{1/2}$$

$$= (\int_{-\infty}^{\infty} \frac{1}{2} (1 - \cos(2xt))d\mu_F(t))^{1/2}$$

$$= (\frac{1}{2} (1 - \text{Re } f(2x)))^{1/2}.$$

24.7 REMARK Elementary inequalities of this type (of which there are a number...) can be used to preclude a function from being a characteristic function. E.g.: The function

$$\exp(- |x|^{\alpha}) \quad (\alpha > 2)$$

is not a characteristic function since the first inequality above is violated for small x.

[Note: On the other hand, the function

$$\exp(- |x|^{\alpha}) \quad (0 < \alpha \leq 2)$$

is a characteristic function:

- $0 < \alpha \leq 1$ (apply 24.24)
- $\alpha = 2$ (immediate)
- $1 < \alpha < 2$ (trickier).]

24.8 ASYMTOTICS Let F be a distribution function, f its characteristic function.

• Suppose that F is discrete -- then

$$F(x) = \sum_n j_n I(x - x_n)$$

=>

$$\mu_F = \sum_n j_n \delta_{x_n}$$

=>

$$f(x) = \sum_n j_n e^{\sqrt{-1}\, x x_n}$$

=>

$$\overline{\lim_{|x| \to \infty}} |f(x)| = 1.$$

• Suppose that F is absolutely continuous -- then $F' \in L^1(-\infty,\infty)$ (cf. 23.18) and

$$F(x) = \int_{-\infty}^{x} F'(t)dt \quad \text{(cf. 23.24)}$$

=>

$$f(x) = \int_{-\infty}^{\infty} e^{\sqrt{-1}\, xt} F'(t)dt$$

$$\equiv \sqrt{2\pi}\, (F')^{\wedge}$$

=>

$$f \in C_0(-\infty,\infty) \quad \text{(cf. 21.6)}$$

=>

$$\lim_{|x| \to \infty} |f(x)| = 0.$$

- Suppose that F is singular -- then as can be seen by example,

$$\overline{\lim_{|x| \to \infty}} |f(x)|$$

might be 0 or it might be 1 or it might be between 0 and 1.

Put

$$S(A) = \int_0^A \frac{\sin t}{t} dt \quad (A \geq 0).$$

Then S(A) is bounded and

$$\int_0^A \frac{\sin t\theta}{t} dt = \operatorname{sgn} \theta \cdot S(A|\theta|).$$

[Note: Recall that

$$\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}.]$$

24.9 INVERSION FORMULA Let F be a distribution function, f its characteristic function -- then at any two continuity points a < b of F,

$$F(b) - F(a) = \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} \frac{e^{-\sqrt{-1}\, ax} - e^{-\sqrt{-1}\, bx}}{\sqrt{-1}\, x} f(x) dx.$$

PROOF Denoting by I_A the entity inside the limit, insert

$$f(x) = \int_{-\infty}^{\infty} e^{\sqrt{-1}\, xt} d\mu_F(t)$$

and write

$$I_A = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-A}^{A} \frac{e^{\sqrt{-1}\, x(t-a)} - e^{\sqrt{-1}\, x(t-b)}}{\sqrt{-1}\, x} dx\right) d\mu_F(t)$$

or still,

$$I_A = \int_{-\infty}^{\infty} \left[\frac{\operatorname{sgn}(t-a)}{\pi} S(A|t-a|) - \frac{\operatorname{sgn}(t-b)}{\pi} S(A|t-b|) \right] d\mu_F(t).$$

The integrand is bounded and converges as $A \to \infty$ to the function

$$\phi_{a,b}(t) = \begin{cases} 0 & (t < a) \\ 1/2 & (t = a) \\ 1 & (a < t < b) \\ 1/2 & (t = b) \\ 0 & (b < t). \end{cases}$$

Therefore

$$\lim_{A \to \infty} I_A = \int_{-\infty}^{\infty} \phi_{a,b}(t) d\mu_F(t)$$

$$= \frac{1}{2} \mu_F(\{a\}) + \mu_F(]a,b[) + \frac{1}{2} \mu_F(\{b\})$$

$$= \frac{1}{2}(F(a) - F(a^-)) + (F(b^-) - F(a)) + \frac{1}{2}(F(b) - F(b^-))$$

$$= F(b) - F(a).$$

24.10 REMARK Using similar methods, $\forall$ a,

$$j_a = \mu_F(\{a\}) = \lim_{A \to \infty} \frac{1}{2A} \int_{-A}^{A} e^{-\sqrt{-1}\, ax} f(x) dx.$$

24.11 THEOREM If $f \in L^1(-\infty,\infty)$, then F is continuous and its derivative F' exists. Moreover,

$$F'(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\sqrt{-1}\, tx} f(x) dx,$$

hence is continuous.

PROOF Since $f \in L^1(-\infty,\infty)$, the same is true of

$$\frac{e^{-\sqrt{-1}\,ax} - e^{-\sqrt{-1}\,bx}}{\sqrt{-1}\,x} f(x),$$

so per 24.9,

$$F(b) - F(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-\sqrt{-1}\,ax} - e^{-\sqrt{-1}\,bx}}{\sqrt{-1}\,x} f(x)dx.$$

To confirm that F is continuous, fix t and let δ be a positive parameter such that $a = t - \delta$, $b = t + \delta$ are continuity points of F -- then

$$F(t+\delta) - F(t-\delta)$$

$$= \frac{\delta}{\pi} \int_{-\infty}^{\infty} \frac{\sin \delta x}{\delta x} e^{-\sqrt{-1}\,tx} f(x)dx$$

=>

$$|F(t+\delta) - F(t-\delta)|$$

$$\leq \frac{\delta}{\pi} \int_{-\infty}^{\infty} \left|\frac{\sin \delta x}{\delta x}\right| |f(x)|dx$$

$$\leq \frac{\delta}{\pi} \int_{-\infty}^{\infty} |f(x)|dx.$$

Now let $\delta \to 0$, thus

$$F(t^+) - F(t^-) = 0,$$

so F is continuous at t. Next, for any h (positive or negative),

$$\frac{F(t+h) - F(t)}{h} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-\sqrt{-1}\,tx} - e^{-\sqrt{-1}\,(t+h)x}}{\sqrt{-1}\,hx} f(x)dx$$

=>

$$F'(t) = \lim_{h \to 0} \frac{F(t+h) - F(t)}{h}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{h \to 0} \frac{e^{-\sqrt{-1}\, tx} - e^{-\sqrt{-1}\,(t+h)x}}{\sqrt{-1}\, hx} f(x)dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\sqrt{-1}\, tx} f(x)dx.$$

[Note: $\forall\ t$,

$$|F'(t)| \leq \frac{1}{2\pi} ||f||_1 < \infty.$$

Therefore F is absolutely continuous (cf. 23.20).]

24.12 THEOREM Suppose that F_1,F_2 are distribution functions. Put $F = F_1 * F_2$ -- then

$$f = f_1 \cdot f_2.$$

[$\forall\ x$,

$$f(x) = \int_{-\infty}^{\infty} e^{\sqrt{-1}\, xt} d\mu_F(t)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\sqrt{-1}\, x(t_1+t_2)} d\mu_{F_1}(t_1) d\mu_{F_2}(t_2)$$

$$= \int_{-\infty}^{\infty} e^{\sqrt{-1}\, xt_1} d\mu_{F_1}(t_1) \cdot \int_{-\infty}^{\infty} e^{\sqrt{-1}\, xt_2} d\mu_{F_2}(t_2)$$

$$= f_1(x) \cdot f_2(x).]$$

24.13 EXAMPLE Given a distribution function F, consider the convolution

$$F * (-1)F.$$

Then its characteristic function is

$$f(x)f(-x) = f(x)\overline{f(x)} = |f(x)|^2.$$

24.14 RAPPEL $\forall\ t,\ \forall\ \sigma > 0,$

$$\int_{-\infty}^{\infty} \exp(-\sqrt{-1}\ xt - \frac{\sigma^2 x^2}{2})dx = \frac{\sqrt{2\pi}}{\sigma} \exp(-\frac{t^2}{2\sigma^2}).$$

N.B. Given real variables u, v, let

$$\phi(v) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{v^2}{2}).$$

Then

$$\Phi(u) = \int_{-\infty}^{u} \phi(v)dv$$

is an absolutely continuous distribution function with density $\phi(v)$ and characteristic function

$$\exp(-\frac{x^2}{2}).$$

So, $\forall\ \sigma > 0$, $\Phi_\sigma(u) \equiv \Phi(\frac{u}{\sigma})$ is an absolutely continuous distribution function with density $\phi_\sigma(v) \equiv \frac{1}{\sigma}\phi(\frac{v}{\sigma})$ and characteristic function

$$\exp(-\frac{1}{2}\sigma^2 x^2).$$

24.15 LEMMA Two distribution functions $\begin{bmatrix} F \\ G \end{bmatrix}$ that agree at all continuity points common to both agree everywhere.

PROOF Let $\begin{bmatrix} S \\ T \end{bmatrix}$ be the set of discontinuity points of $\begin{bmatrix} F \\ G \end{bmatrix}$ -- then $S \cup T$ is at most countable, hence its complement D is dense. And on D, F = G. If x_0

is arbitrary and if $x_n \in D$ approaches x_0 from the right, then

$$F(x_0) = \lim F(x_n) = \lim G(x_n) = G(x_0).$$

24.16 THEOREM Suppose that F_1, F_2 are distribution functions. Assume: $f_1 = f_2$ -- then $F_1 = F_2$.

PROOF Write

$$\left[\begin{array}{l} f_1(x) = \int_{-\infty}^{\infty} e^{\sqrt{-1}\, xs} d\mu_{F_1}(s) \\ \\ f_2(x) = \int_{-\infty}^{\infty} e^{\sqrt{-1}\, xs} d\mu_{F_2}(s). \end{array}\right.$$

Then $\forall\, t$, $\forall\, \sigma > 0$,

$$\int_{-\infty}^{\infty} f_1(x)\exp(-\sqrt{-1}\, xt - \frac{\sigma^2 x^2}{2})dx$$

$$= \int_{-\infty}^{\infty} f_2(x)\exp(-\sqrt{-1}\, xt - \frac{\sigma^2 x^2}{2})dx$$

or still,

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \exp(-\sqrt{-1}\, x(t-s) - \frac{\sigma^2 x^2}{2})dx\right] d\mu_{F_1}(s)$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \exp(-\sqrt{-1}\, x(t-s) - \frac{\sigma^2 x^2}{2})dx\right] d\mu_{F_2}(s)$$

or still,

$$\frac{\sqrt{2\pi}}{\sigma} \int_{-\infty}^{\infty} \exp(-\frac{(t-s)^2}{2\sigma^2}) d\mu_{F_1}(s)$$

$$= \frac{\sqrt{2\pi}}{\sigma} \int_{-\infty}^{\infty} \exp(-\frac{(t-s)^2}{2\sigma^2}) d\mu_{F_2}(s)$$

or still,

$$2\pi \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{(t-s)^2}{2\sigma^2}) d\mu_{F_1}(s)$$

$$= 2\pi \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{(t-s)^2}{2\sigma^2}) d\mu_{F_2}(s)$$

or still,

$$2\pi \int_{-\infty}^{\infty} \phi_\sigma(t-s) d\mu_{F_1}(s)$$

$$= 2\pi \int_{-\infty}^{\infty} \phi_\sigma(t-s) d\mu_{F_2}(s)$$

or still,

$$2\pi(\Phi_\sigma * F_1) = 2\pi(\Phi_\sigma * F_2)$$

=>

$$\Phi_\sigma * F_1 = \Phi_\sigma * F_2$$

=>

$$F_1 * \Phi_\sigma = F_2 * \Phi_\sigma$$

=>

$$\int_{-\infty}^{\infty} F_1(t-s) d\mu_{\Phi_\sigma}(s)$$

$$= \int_{-\infty}^{\infty} F_2(t-s) d\mu_{\Phi_\sigma}(s)$$

=>

$$\int_{-\infty}^{\infty} F_1(t-s)\exp(-\frac{s^2}{2\sigma^2})ds$$

$$= \int_{-\infty}^{\infty} F_2(t-s)\exp(-\frac{s^2}{2\sigma^2})ds$$

=>

$$\int_{-\infty}^{\infty} F_1(t-\sigma u)\exp(-\frac{u^2}{2})du$$

$$= \int_{-\infty}^{\infty} F_2(t-\sigma u)\exp(-\frac{u^2}{2})du.$$

Now let $\sigma \to 0$ and use dominated convergence to see that $F_1(t) = F_2(t)$ at all continuity points t common to both, so $F_1 = F_2$ period (cf. 24.15).

24.17 REMARK The demand is that $\hat{f}_1 = \hat{f}_2$ everywhere and this cannot be weakened to equality on some finite interval (cf. 24.26).

24.18 LEMMA If $f_1, f_2, \ldots$ is a sequence of characteristic functions that converges uniformly on compact subsets of R to a function f, then $\hat{f} \equiv f$ is a characteristic function.

24.19 EXAMPLE Let

$$F_n(t) = \begin{cases} 0 & (t < -n) \\ \frac{n+t}{2n} & (-n \le t < n) \\ 1 & (n \le t). \end{cases}$$

Then F_n is a distribution function whose characteristic function $\hat{f}_n$ is given by

$$f_n(x) = \frac{\sin xn}{xn} \quad (n = 1,2,\ldots).$$

Therefore

$$\lim_{n \to \infty} f_n(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ if } x \neq 0, \end{cases}$$

which shows that 24.18 can fail under the weaker assumption of mere pointwise convergence.

24.20 DEFINITION A continuous function $f:R \to C$ is said to be positive definite if for any finite sequence $x_1, x_2, \ldots, x_n$ of real numbers and for any finite sequence $\xi_1, \xi_2, \ldots, \xi_n$ of complex numbers,

$$\sum_{k=1}^{n} \sum_{\ell=1}^{n} f(x_k - x_\ell)\xi_k \bar{\xi}_\ell \geq 0.$$

E.g.: Every characteristic function f is positive definite. Proof:

$$\sum_{k=1}^{n} \sum_{\ell=1}^{n} f(x_k - x_\ell)\xi_k \bar{\xi}_\ell$$

$$= \sum_{k=1}^{n} \sum_{\ell=1}^{n} \left(\int_{-\infty}^{\infty} e^{\sqrt{-1}(x_k - x_\ell)t} d\mu_F(t)\right)\xi_k \bar{\xi}_\ell$$

$$= \int_{-\infty}^{\infty} \sum_{k=1}^{n} \sum_{\ell=1}^{n} e^{\sqrt{-1}\,(x_k - x_\ell)t} \xi_k \bar{\xi}_\ell d\mu_F(t)$$

$$= \int_{-\infty}^{\infty} \left(\sum_{k=1}^{n} e^{\sqrt{-1}\,x_k t} \xi_k\right)\left(\sum_{\ell=1}^{n} e^{-\sqrt{-1}\,x_\ell t} \bar{\xi}_\ell\right) d\mu_F(t)$$

$$= \int_{-\infty}^{\infty} \left|\sum_{k=1}^{n} e^{\sqrt{-1}\,x_k t} \xi_k\right|^2 d\mu_F(t)$$

$$\geq 0.$$

Conversely:

24.21 THEOREM A positive definite function $f:R \to C$ such that $f(0) = 1$ is a characteristic function.

We shall preface the proof with a lemma.

24.22 LEMMA Suppose that $\phi \in L^1[-A,A]$. Assume: ϕ is bounded, say $\sup|\phi| \leq M$, and

$$\Phi(x) = \int_{-A}^{A} e^{\sqrt{-1}\,xt}\phi(t)dt \geq 0.$$

Then $\Phi \in L^1[-\infty,\infty]$.

PROOF Put

$$G(X) = \int_{-X}^{X}\Phi.$$

Then G is increasing, thus it need only be shown that G is bounded. To this end, introduce

$$F(X) = \frac{1}{X}\int_{X}^{2X} G.$$

Then

$$F(X) \geq \frac{G(X)}{X}\int_{X}^{2X} 1 = G(X),$$

so it will be enough to prove that F is bounded.

- $G(X) = \int_{-X}^{X}\Phi$

$$= \int_{-X}^{X}\left(\int_{-A}^{A} e^{\sqrt{-1}\,xt}\phi(t)dt\right)dx$$

$$= \int_{-A}^{A}\left(\int_{-X}^{X} e^{\sqrt{-1}\,xt}dx\right)\phi(t)dt$$

$$= \int_{-A}^{A} \left(\frac{e^{\sqrt{-1}\, xt}}{\sqrt{-1}\, t} \Bigg|_{x=-X}^{x=X} \right) \phi(t)dt$$

$$= \int_{-A}^{A} \frac{e^{\sqrt{-1}\, Xt} - e^{-\sqrt{-1}\, Xt}}{\sqrt{-1}\, t} \phi(t)dt$$

$$= 2 \int_{-A}^{A} \frac{\sin Xt}{t} \phi(t)dt.$$

- $F(X) = \frac{1}{X} \int_{X}^{2X} G$

$$= \frac{2}{X} \int_{X}^{2X} \left(\int_{-A}^{A} \frac{\sin Yt}{t} \phi(t)dt \right) dY$$

$$= \frac{2}{X} \int_{-A}^{A} \left(\int_{X}^{2X} \frac{\sin Yt}{t} dY \right) \phi(t)dt$$

$$= \frac{2}{X} \int_{-A}^{A} \left(\frac{-\cos Yt}{t^2} \Bigg|_{Y=X}^{Y=2X} \right) \phi(t)dt$$

$$= \frac{2}{X} \int_{-A}^{A} \frac{\cos Xt - \cos 2Xt}{t^2} \phi(t)dt$$

$$= \frac{2}{X} \int_{-A}^{A} \frac{1 - 2\sin^2 \frac{Xt}{2} - (1 - 2\sin^2 Xt)}{t^2} \phi(t)dt$$

$$= \frac{4}{X} \int_{-A}^{A} \frac{\sin^2 Xt}{t^2} \phi(t)dt - \frac{4}{X} \int_{-A}^{A} \frac{\sin^2 \frac{Xt}{2}}{t^2} \phi(t)dt.$$

To bound the first term, write

$$\left| \frac{4}{X} \int_{-A}^{A} \frac{\sin^2 Xt}{t^2} \phi(t)dt \right|$$

$$\leq \frac{4M}{X} \int_{-A}^{A} \frac{\sin^2 Xt}{t^2} dt$$

$$\leq 4M \int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2} dt < \infty.$$

Ditto for the second term.

Passing to the proof of 24.21, let

$$f_A(x) = \frac{1}{\sqrt{2\pi}\, A} \int_0^A \int_0^A f(u-v) e^{\sqrt{-1}\, xu} e^{-\sqrt{-1}\, xv} du dv \quad (A > 0).$$

The fact that f is positive definite then implies by approximation that $f_A(x) \geq 0$. Now make the change of variable u = u, v = u-t to get

$$f_A(x) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} e^{\sqrt{-1}\, xt} (1 - \frac{|t|}{A}) f(t) dt.$$

This done, in 24.22 take

$$\phi(t) = (1 - \frac{|t|}{A}) f(t),$$

the conclusion being that $f_A \in L^1[-\infty,\infty]$. But then 21.17 is applicable, so

$$(1 - \frac{|t|}{A}) f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_A(x) e^{-\sqrt{-1}\, tx} dx,$$

i.e.,

$$(1 - \frac{|t|}{A}) f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_A(-x) e^{\sqrt{-1}\, tx} dx$$

if $|t| \leq A$. In particular:

$$1 = f(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_A(-x)\,dx.$$

Therefore

$$F_A(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} f_A(-y)\,dy$$

is a distribution function whose characteristic function is

$$\chi_{[-A,A]}(t)\left(1 - \frac{|t|}{A}\right)f(t).$$

Finally, put

$$\mathfrak{f}_n(t) = \chi_{[-n,n]}(t)\left(1 - \frac{|t|}{n}\right)f(t) \quad (n = 1,2,\ldots).$$

Then $\mathfrak{f}_n \rightarrow f$ uniformly on compact subsets of R, thus, as the $\mathfrak{f}_n$ are characteristic functions, the same is true of $\mathfrak{f} \equiv f$ (cf. 24.18).

24.23 EXAMPLE If $\mathfrak{f}$ is a characteristic function, then $e^{\mathfrak{f}-1}$ is a characteristic function.

24.24 POLYA CRITERION Suppose that $f:R \rightarrow R$ is continuous. Assume: $f(0) = 1$, $f(-x) = f(x)$,

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2} \quad (x_1, x_2 > 0),$$

and $\lim_{x \rightarrow \infty} f(x) = 0$ -- then f is the characteristic function of an absolutely continuous distribution function F.

PROOF Because f is a continuous, convex function, its derivative D_+f from the right exists for $x > 0$. As such, it is increasing and here

$$D_+f(x) \leq 0 \quad (x > 0), \quad \lim_{x \to \infty} D_+f(x) = 0.$$

In addition,

$$f(x) = f(0) + \int_0^x D_+f(y)dy$$

=>

$$0 = f(\infty) = f(0) + \lim_{x \to \infty} \int_0^x D_+f(y)dy$$

=>

$$1 = f(0) = -\lim_{x \to \infty} \int_0^x D_+f(y)dy.$$

Therefore D_+f is integrable on 0 to ∞. Put

$$\phi_X(t) = \frac{1}{2\pi} \int_{-X}^{X} f(x)e^{-\sqrt{-1}\,tx}dx.$$

Then

$$\phi_X(t) = \frac{1}{\pi} \int_0^X f(x)\cos tx\, dx$$

$$= \left(\frac{\sin Xt}{\pi t}\right)f(X) - \frac{1}{\pi t} \int_0^X D_+f(x)\sin tx\, dx.$$

So for $t \neq 0$,

$$\phi(t) \equiv \lim_{X \to \infty} \phi_X(t)$$

$$= -\frac{1}{\pi t} \int_0^\infty D_+f(x)\sin tx\, dx$$

$$= -\frac{1}{\pi t} \sum_{k=0}^{\infty} \int_{k\pi/t}^{(k+1)\pi/t} D_+f(x)\sin tx\, dx$$

$$= -\frac{1}{\pi t} \sum_{k=0}^{\infty} \int_0^{\pi/t} (-1)^k D_+f(x + (k\pi/t))\sin tx\, dx.$$

Since

$$\sum_{k=0}^{\infty} (-1)^k D_+f(x + (k\pi/t))$$

is an alternating series whose terms are decreasing in absolute value with

$$\lim_{k \to \infty} D_+f(x + (k\pi/t)) = 0,$$

it is boundedly convergent and since the first term is

$$D_+f(x) \le 0,$$

it follows that

$$\phi(t) = -\frac{1}{\pi t} \int_0^{\pi/t} \left(\sum_{k=0}^{\infty} (-1)^k D_+f(x + (k\pi/t))\sin tx\, dx \right.$$

$$\ge 0.$$

Now multiply $\phi(t)$ by cos xt and integrate with respect to t from 0 to T:

$$\int_0^T \phi(t)\cos xt\, dt$$

$$= -\frac{1}{\pi} \int_0^{\infty} D_+f(y)dy \int_0^T \frac{\cos xt \sin yt}{t}\, dt.$$

Next, let $T \to \infty$:

$$\lim_{T \to \infty} \int_0^T \frac{\cos xt \sin yt}{t}\, dt = \begin{cases} 0 & (|x| > y) \\ \frac{\pi}{4} & (|x| = y) \\ \frac{\pi}{2} & (|x| < y) \end{cases}$$

=>

$$\lim_{T \to \infty} \int_0^T \phi(t)\cos xt\, dt$$

$$= -\frac{1}{2}\int_x^\infty D_+f(y)dy$$

$$= -\frac{1}{2}\left(\int_0^\infty D_+f(y)dy - \int_0^x D_+f(y)dy\right)$$

$$= -\frac{1}{2}(1 - (f(x) - 1))$$

$$= \frac{1}{2}f(x).$$

In particular:

$$\lim_{T\to\infty}\int_0^T \phi(t)dt = \frac{1}{2}f(0) = \frac{1}{2},$$

so, being nonnegative, ϕ is integrable on 0 to ∞, or still, being even, ϕ is integrable on $-\infty$ to ∞. And

$$f(x) = \int_{-\infty}^\infty \phi(t)e^{\sqrt{-1}\,xt}dt,$$

thus to finish, let

$$F(x) = \int_{-\infty}^x \phi(t)dt.$$

24.25 EXAMPLE The function $e^{-|x|}$ satisfies the assumptions of 24.24 but the function $e^{-|x|^2}$ does not satisfy the assumptions of 24.24 (even though it is a characteristic function).

24.26 EXAMPLE The functions

$$\begin{cases} 1 - |x| & (0 \le x \le \frac{1}{2}) \\ \frac{1}{4|x|} & (|x| \ge \frac{1}{2}) \end{cases}, \quad \begin{cases} 1 - |x| & (|x| \le 1) \\ 0 & (|x| \ge 1) \end{cases}$$

satisfy the assumptions of 24.24.

[Note: This shows that distinct characteristic functions can coincide on a finite interval.]

§25. *HOLOMORPHIC CHARACTERISTIC FUNCTIONS*

Let $F:R \rightarrow R$ be a distribution function.

25.1 DEFINITION Let $k = 0,1,2,\ldots$.

- $\alpha_k = \int_{-\infty}^{\infty} t^k d\mu_F(t)$

is the moment of order k of F.

- $\beta_k = \int_{-\infty}^{\infty} |t|^k d\mu_F(t)$

is the absolute moment of order k of F.

[Note: α_k exists iff β_k exists.]

25.2 INEQUALITIES

$$\alpha_{2k} = \beta_{2k} \ (\alpha_0 = \beta_0 = 1), \ \alpha_{2k-1} \leq |\alpha_{2k-1}| \leq \beta_{2k-1},$$

$$\beta_{k-1}^2 \leq \beta_{k-2}\beta_k, \ \beta_1 \leq \beta_2^{1/2} \leq \cdots \leq \beta_k^{1/k}.$$

25.3 LEMMA If f has a derivative of order n at $x = 0$, then all the moments of F up to order n or up to order $n - 1$ exist according to whether n is even or odd.

25.4 EXAMPLE Take $n = 1$ (odd) -- then it can happen that $f'(x)$ exists and is continuous for all values of x, yet the first moment of F does not exist.

[Put

$$C = \sum_{j=2}^{\infty} \frac{1}{j^2 \log j}.$$

Then

$$F(t) = C^{-1} \sum_{j=2}^{\infty} \frac{1}{2j^2 \log j} [I(t-j) + I(t+j)]$$

is a distribution function whose characteristic function is

$$f(x) = C^{-1} \sum_{j=2}^{\infty} \frac{\cos jx}{j^2 \log j}.$$

To see the claim per $f'(x)$, note that

$$C^{-1} \sum_{j=2}^{\infty} \frac{\cos jx}{\log j}$$

is the Fourier series of an integrable function, hence on general grounds, the series

$$C^{-1} \sum_{j=2}^{\infty} \frac{-\sin jx}{j \log j}$$

is uniformly convergent (or proceed directly via the uniform Dirichlet test). On the other hand,

$$\int_{-\infty}^{\infty} |t| d\mu_F(t) = C^{-1} \sum_{j=2}^{\infty} \frac{1}{j \log j} = \infty.]$$

25.5 REMARK A characteristic function may be nowhere differentiable.

[The function

$$f(x) = \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} e^{\sqrt{-1}\, x 5^j}$$

is the characteristic function of

$$F(t) = \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} I(t-5^j).]$$

25.6 LEMMA If the moment α_k of order k of F exists, then f is k-times differentiable and

$$f^{(k)}(x) = (\sqrt{-1})^k \int_{-\infty}^{\infty} t^k e^{\sqrt{-1}\, xt} d\mu_F(t)$$

is a continuous function of x.

[Note: In particular,

$$f^{(k)}(0) = (\sqrt{-1})^k \alpha_k.]$$

25.7 SCHOLIUM The existence of the derivatives of all orders at the origin for f is equivalent to the existence of the moments of all orders for F.

25.8 DEFINITION A characteristic function f is said to be a <u>holomorphic characteristic function</u> if for some $\delta > 0$ it coincides with a function g which is holomorphic in the disk $|z| < \delta$.

25.9 THEOREM If f is a holomorphic characteristic function, then f is holomorphic in a strip containing the origin of the form $-\alpha < \text{Im } z < \beta$ ($\alpha > 0$, $\beta > 0$ (either α or β or both might be ∞)) and in that strip,

$$f(z) = \int_{-\infty}^{\infty} e^{\sqrt{-1}\, zt} d\mu_F(t).$$

PROOF It is clear that f has derivatives of all orders at the origin ($\forall$ n, $f^{(n)}(0) = g^{(n)}(0)$), hence F has moments of all orders (cf. 25.7). Moreover,

$$|f^{(2k)}(0)| = \alpha_{2k} = \beta_{2k}, \quad |f^{(2k-1)}(0)| = |\alpha_{2k-1}|.$$

Thus the series

$$\sum_{k=0}^{\infty} \frac{|\alpha_k|}{k!} r^k$$

is convergent if $0 \leq r < \delta$, thus the series

$$\sum_{k=0}^{\infty} \frac{\beta_{2k}}{(2k)!} r^{2k}$$

is convergent if $0 \le r < \delta$. It is also true that the series

$$\sum_{k=1}^{\infty} \frac{\beta_{2k-1}}{(2k-1)!} r^{2k-1}$$

is convergent if $0 \le r < \delta$. In fact, its radius of convergence R is

$$\varliminf_{k \to \infty} \left[\frac{\beta_{2k-1}}{(2k-1)!} \right]^{-1/(2k-1)}.$$

But

$$(\beta_{2k-1})^{1/(2k-1)} \le (\beta_{2k})^{1/2k} \quad \text{(cf. 25.2)}.$$

So

$$R \ge \varliminf_{k \to \infty} (\beta_{2k})^{-1/2k} [(2k-1)!]^{1/(2k-1)}$$

$$= \varliminf_{k \to \infty} (\beta_{2k})^{-1/2k} [(2k)!]^{1/(2k-1)} \left(\lim_{k \to \infty} (2k)^{1/(2k-1)} = 1 \right)$$

$$\ge \varliminf_{k \to \infty} \left[\frac{\beta_{2k}}{(2k)!} \right]^{-1/2k}.$$

Applying now the monotone convergence theorem, we have

$$\int_{-\infty}^{\infty} e^{r|t|} d\mu_F(t) = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{r^n |t|^n}{n!} d\mu_F(t)$$

$$= \sum_{n=0}^{\infty} \left(\int_{-\infty}^{\infty} |t|^n d\mu_F(t) \right) \frac{r^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\beta_n}{n!} r^n < \infty \quad (0 \le r < \delta).$$

And this implies that

$$\int_{-\infty}^{\infty} e^{rt} d\mu_F(t)$$

exists when $- \delta < r < \delta$. Put

$$\begin{cases} \alpha = \sup\{r \geq 0 : \int_{-\infty}^{\infty} e^{rt} d\mu_F(t) < \infty\} \\ \beta = \sup\{r \geq 0 : \int_{-\infty}^{\infty} e^{-rt} d\mu_F(t) < \infty\} \end{cases}$$

$$\Rightarrow \begin{cases} \alpha \geq \delta \\ \beta \geq \delta. \end{cases}$$

Then the integral

$$\int_{-\infty}^{\infty} e^{\sqrt{-1}\, zt} d\mu_F(t)$$

is defined if $- \alpha < \text{Im } z < \beta$, is a holomorphic function of z in this strip, and agrees with f on the real axis.

25.10 RAPPEL Suppose that the power series $f(z) \equiv \sum_{n=0}^{\infty} a_n z^n$ has a positive radius of convergence R. Assume: $\forall\, n \geq 0$, $a_n \geq 0$ -- then the point $z = R$ is a singularity for $f(z)$.

25.11 DEFINITION Let f be a holomorphic characteristic function and take α, β as in 25.9 -- then the strip $- \alpha < \text{Im } z < \beta$ is called the <u>strip of analyticity</u> of f.

25.12 ADDENDUM $- \sqrt{-1}\, \alpha$ (if α is finite) and $\sqrt{-1}\, \beta$ (if β is finite) are singularities for f, hence $-\alpha < \text{Im } z < \beta$ is the largest strip in which f is holomorphic.

[Put

$$\begin{cases} f_-(z) = \int_{-\infty}^{0} e^{zt} d\mu_F(t) \\ f_+(z) = \int_{0}^{\infty} e^{zt} d\mu_F(t). \end{cases}$$

Then

$$\int_{-\infty}^{\infty} e^{rt} d\mu_F(t) < \infty \quad (-\beta < r < \alpha)$$

=>

$$\begin{cases} \int_{0}^{\infty} e^{rt} d\mu_F(t) < \infty & (r < 0) \\ \int_{-\infty}^{0} e^{rt} d\mu_F(t) < \infty & (r > 0). \end{cases}$$

Therefore

$$\begin{cases} f_- \text{ is holomorphic in Re } z > -\beta \\ f_+ \text{ is holomorphic in Re } z < \alpha. \end{cases}$$

And

$$f(-\sqrt{-1}\, z) = f_+(z) + f_-(z) \quad (-\beta < \text{Re } z < \alpha).$$

Working now with f_+, we have

$$f_+^{(n)}(0) = \int_{0}^{\infty} t^n d\mu_F(t) \geq 0.$$

Consider the power series

$$f_+(z) = \sum_{n=0}^{\infty} \frac{f_+^{(n)}(0)}{n!} z^n.$$

Its radius of convergence is $\geq \alpha$ but it cannot be $> \alpha$ since otherwise $\exists\ \varepsilon > 0$:

$$\int_0^\infty e^{(\alpha+\varepsilon)t} d\mu_F(t) = \sum_{n=0}^\infty \frac{f_+^{(n)}(0)}{n!} (\alpha+\varepsilon)^n < \infty,$$

contradicting the definition of α. But its coefficients are ≥ 0, hence $z = \alpha$ is a singularity for $f_+(z)$ (cf. 25.10). Since

$$f(-\sqrt{-1}\, z) = f_+(z) + f_-(z) \quad (-\beta < \text{Re } z < \alpha)$$

and since f_- is holomorphic in $\text{Re } z > -\beta$, it follows that α is a singularity for $f(-\sqrt{-1}\, z)$ or still, $-\sqrt{-1}\, \alpha$ is a singularity for $f(z)$.]

[Note: To establish that $\sqrt{-1}\, \beta$ is a singularity for f, consider the characteristic function $(-1)f$ of $(-1)F$.]

25.13 REMARK There are characteristic functions which are not holomorphic characteristic functions, yet can be continued into regions other than strips.

[Consider $f(x) = e^{-|x|}$ -- then it can be continued into the half-planes $\text{Re } z \geq 0$ and $\text{Re } z \leq 0$, yet there is no continuation into a disk centered at the origin.]

Given a characteristic function f, put

$$I(r) = \int_{-\infty}^{\infty} e^{rt} d\mu_F(t) \quad (-\infty < r < \infty)$$

and let

$$\begin{cases} \underline{\alpha} = \underset{t \to \infty}{\underline{\lim}} - \dfrac{\log(1 - F(t))}{t} \\ \underline{\beta} = \underset{t \to \infty}{\underline{\lim}} - \dfrac{\log F(-t)}{t}. \end{cases}$$

N.B. Equivalently,

$$\begin{cases} \underline{\alpha} = - \overline{\lim}_{t \to \infty} \frac{\log(1 - F(t))}{t} \\ \underline{\beta} = - \overline{\lim}_{t \to \infty} \frac{\log F(-t)}{t} . \end{cases}$$

25.14 LEMMA $I(r)$ is defined for all points $r \in]-\underline{\beta},\underline{\alpha}[$, where it is understood that $\underline{\beta}$ (respectively $\underline{\alpha}$) is to be taken as infinite if $F(-t) = 0$ (respectively $1 - F(t) = 0$) for some $t > 0$.

PROOF Noting that $\underline{\alpha} \geq 0$, $\underline{\beta} \geq 0$, consider the interval $[0,\underline{\alpha}[$. Since $I(0) = 1$, take $\underline{\alpha} > 0$ and $0 < r < \underline{\alpha}$. Choose $r_0 : r < r_0 < \underline{\alpha}$ and then choose $T = T(r_0) > 0$:

$$t \geq T \Rightarrow - \frac{\log(1 - F(t))}{t} \geq r_0$$

or still,

$$t \geq T \Rightarrow 1 - F(t) \leq e^{-tr_0}.$$

There is no loss of generality in assuming that T is a continuity point of F ($\Rightarrow F(T^-) = F(T)$), so if $A > T$,

$$\int_T^A e^{rt} d\mu_{F-1}(t)$$

$$= e^{rA}(F(A^+)-1) - e^{rT}(F(T^-)-1)$$

$$- r\int_T^A (F(t^+)-1)e^{rt}dt$$

$$= e^{rA}(F(A)-1) - e^{rT}(F(T)-1)$$

$$- r\int_T^A (F(t)-1)e^{rt}dt$$

$$\leq e^{rT}(1-F(T)) + r\int_T^A e^{rt}(1-F(t))dt$$

$$\leq e^{rT}(1-F(T)) + r\int_T^A e^{rt}e^{-tr_0}dt,$$

hence sending A to ∞,

$$\int_T^\infty e^{rt}d\mu_F(t)$$

$$= \int_T^\infty e^{rt}d\mu_{F-1}(t)$$

$$\leq e^{rT}(1-F(T)) + r\int_T^\infty e^{(r-r_0)t}dt$$

$$< \infty.$$

Meanwhile

$$\int_{-\infty}^T e^{rt}d\mu_F(t) \leq e^{rT}F(T) < \infty.$$

Consequently, $I(r)$ is defined for all $r \in [0,\underline{\alpha}[$. And, analogously, $I(r)$ is defined for all $r \in]-\underline{\beta},0]$.

[Note: $I(r)$ is defined for all $r > 0$ if $1 - F(t) = 0$ for some $t > 0$ and for all $r < 0$ if $F(-t) = 0$ for some $t > 0$.]

25.15 REMARK $I(r)$ does not exist if $r > \underline{\alpha}$ ($\underline{\alpha}$ finite) or if $r < -\underline{\beta}$ ($\underline{\beta}$ finite).

E.g.: Suppose that for some $r > 0$, $\int_{-\infty}^\infty e^{rs}d\mu_F(s) = C < \infty$ -- then $\forall\ t > 0$,

$$e^{rt}(1 - F(t)) \leq \int_t^\infty e^{rs}d\mu_F(s) \leq C$$

=>

$$\underline{\lim_{t \to \infty}} - \frac{\log(1 - F(t))}{t} \geq r,$$

i.e., $r \leq \underline{\alpha}$.

[Note: In general, nothing can be said about the existence of $I(r)$ when $r = \underline{\alpha}$ or when $r = -\underline{\beta}$.]

25.16 THEOREM If $\underline{\alpha} > 0$, $\underline{\beta} > 0$, then f is a holomorphic characteristic function.

PROOF On the basis of 25.14, the integral

$$\int_{-\infty}^{\infty} e^{\sqrt{-1}\, zt} d\mu_F(t)$$

is defined and holomorphic in the region $-\underline{\alpha} < \operatorname{Im} z < \underline{\beta}$ and coincides with $f(z)$ on the real axis.

25.17 REMARK If f is a holomorphic characteristic function, then

$$\left[\begin{array}{l} \alpha = \underline{\alpha} \\ \beta = \underline{\beta}, \end{array}\right.$$

where, by definition (cf. 25.9),

$$\left[\begin{array}{l} \underline{\alpha} = \sup\{r \geq 0 : \int_{-\infty}^{\infty} e^{rt} d\mu_F(t) < \infty\} \\ \underline{\beta} = \sup\{r \geq 0 : \int_{-\infty}^{\infty} e^{-rt} d\mu_F(t) < \infty\}. \end{array}\right.$$

25.18 RAIKOV CRITERION Suppose there exists a positive constant R such that $\forall\ 0 < r < R$:

$$\left[\begin{array}{l} 1 - F(t) = O(e^{-rt}) \\ F(-t) = O(e^{-rt}). \end{array}\right. \qquad (t \to \infty)$$

Then f is a holomorphic characteristic function and its strip of analyticity (cf. 25.11) contains the strip $|\text{Im } z| < R$.

[In view of the foregoing, this is immediate.]

25.19 LEMMA Let f be a holomorphic characteristic function -- then

$$|f(z)| \leq f(\sqrt{-1}\ \text{Im } z) \quad (-\alpha < \text{Im } z < \beta).$$

[In the strip $-\alpha < \text{Im } z < \beta$,

$$f(z) = \int_{-\infty}^{\infty} e^{\sqrt{-1}\ zt} d\mu_F(t).]$$

25.20 APPLICATION A holomorphic characteristic function f has no zeros on the segment of the imaginary axis inside its strip of analyticity.

[For such a zero would force f to vanish on a horizontal line within its strip of analyticity which in turn would imply that $f \equiv 0$.]

25.21 LEMMA Let f be a holomorphic characteristic function -- then $\log f(\sqrt{-1}\ r)$ is convex as a function of the real variable $-\alpha < r < \beta$.

PROOF Bearing in mind that $f(\sqrt{-1}\ r) > 0$, consider the second derivative of $\log f(\sqrt{-1}\ r)$:

$$\frac{f(\sqrt{-1}\ r) \cdot f''(\sqrt{-1}\ r) - (f'(\sqrt{-1}\ r))^2}{f(\sqrt{-1}\ r)^2}.$$

Then

$$f(\sqrt{-1}\ r) \cdot f''(\sqrt{-1}\ r) - (f'(\sqrt{-1}\ r))^2$$

$$= \int_{-\infty}^{\infty} e^{-rt} d\mu_F(t) \cdot \int_{-\infty}^{\infty} t^2 e^{-rt} d\mu_F(t)$$

$$- \left(\int_{-\infty}^{\infty} te^{-rt} d\mu_F(t)\right)^2,$$

which is nonnegative (Schwarz inequality applied to the measure $e^{-rt} d\mu_F(t)$).

25.22 APPLICATION For any holomorphic characteristic function f, the function

$$\frac{\log f(\sqrt{-1}\, r)}{r}$$

is an increasing function of the real variable $0 < r < \beta$.

[In fact, $\log f(\sqrt{-1}\, r)$ is convex in $[0,\beta[$ and $\log f(\sqrt{-1}\, 0) = \log f(0) = \log 1 = 0$.]

§26. *ENTIRE CHARACTERISTIC FUNCTIONS*

A holomorphic characteristic function $\hat{f}$ is said to be entire if its strip of analyticity is the complex plane, i.e., if $\alpha = \infty$, $\beta = \infty$.

26.1 RAPPEL

$$\begin{cases} \underline{\alpha} = \underset{t \to \infty}{\underline{\lim}} - \dfrac{\log(1 - F(t))}{t} \\ \\ \underline{\beta} = \underset{t \to \infty}{\underline{\lim}} - \dfrac{\log F(-t)}{t}. \end{cases}$$

26.2 SCHOLIUM A characteristic function $\hat{f}$ is entire iff $\underline{\alpha} = \infty$, $\underline{\beta} = \infty$ (cf. 25.17).

26.3 SUBLEMMA Suppose that $\hat{f}$ is an entire characteristic function -- then

$$M(r;\hat{f}) = \max(\hat{f}(\sqrt{-1}\, r), \hat{f}(-\sqrt{-1}\, r)).$$

PROOF For all real x and y,

$$|\hat{f}(x + \sqrt{-1}\, y)| \le \hat{f}(\sqrt{-1}\, y) \quad \text{(cf. 25.19)}.$$

26.4 LEMMA Suppose that $\hat{f}$ is an entire characteristic function -- then $\forall\, t > 0$,

$$M(r;\hat{f}) \ge \frac{1}{2} e^{rt}(1 - F(t) + F(-t)).$$

PROOF

$$\begin{aligned} M(r;\hat{f}) &= \max(\hat{f}(\sqrt{-1}\, r), \hat{f}(-\sqrt{-1}\, r)) \\ &\ge (\hat{f}(\sqrt{-1}\, r) + \hat{f}(-\sqrt{-1}\, r))/2 \\ &= \frac{1}{2}\left(\int_{-\infty}^{\infty} e^{-rs} d\mu_F(s) + \int_{-\infty}^{\infty} e^{rs} d\mu_F(s)\right) \end{aligned}$$

$$= \int_{-\infty}^{\infty} \cosh(rs)\,d\mu_F(s)$$

$$\geq \int_{|s|\geq t} \cosh(rs)\,d\mu_F(s)$$

$$\geq (\cosh rt) \int_{|s|\geq t} d\mu_F(s)$$

$$\geq \frac{1}{2} e^{rt} \int_{|s|\geq t} d\mu_F(s)$$

But

$$\int_{|s|\geq t} d\mu_F(s) = \mu_F([t,\infty[) + \mu_F(]-\infty,-t])$$

$$= \mu_F([t,\infty[) + F(-t).$$

And

$$[t,\infty[= R -]-\infty,t[$$

$$\Rightarrow$$

$$\mu_F([t,\infty[) = 1 - \mu_F(]-\infty,t[)$$

$$\geq 1 - \mu_F([-\infty,t])$$

$$= 1 - F(t).$$

26.5 THEOREM The order of an entire characteristic function $\hat{f}$ cannot be less than one except for the case when $\hat{f} \equiv 1$ (i.e., when $F = I$ (cf. 23.4)).

PROOF If $F \neq I$, then

$$1 - F(a) + F(-a) > 0$$

for some $a > 0$. Now take $t = a$ in 26.4.

[Note: It can be shown that there exist entire characteristic functions of any order ≥ 1 (including ∞).]

26.6 TERMINOLOGY Let F be a distribution function.

- F is bounded to the left if $F(a) = 0$ for some real a. When this is so, one puts

$$\text{lext}[F] = \sup\{a: F(a) = 0\}$$

and calls lext[F] the left extremity of F.

- F is bounded to the right if $F(b) = 1$ for some real b. When this is so, one puts

$$\text{rext}[F] = \inf\{b: F(b) = 1\}$$

and calls rext[F] the right extremity of F.

26.7 DEFINITION A distribution function F such that $F(a) = 0$ and $F(b) = 1$ for some real a and b is said to be finite.

26.8 THEOREM Let f be an entire characteristic function. Assume: f is of exponential type -- then its distribution function F is finite. Moreover,

$$\begin{cases} \text{rext}[F] = \overline{\lim_{r \to \infty}} \dfrac{\log |f(-\sqrt{-1}\, r)|}{r} \\ \\ \text{lext}[F] = -\overline{\lim_{r \to \infty}} \dfrac{\log |f(\sqrt{-1}\, r)|}{r} . \end{cases}$$

PROOF It will be enough to deal with lext[F]. So choose $M > 0$, $K > 0$:

$$|f(z)| \leq Me^{K|z|}.$$

Then

$$\log |f(\sqrt{-1}\, r)| \leq \log M + Kr$$

=>

$$\overline{\lim_{r \to \infty}} \frac{\log |f(\sqrt{-1}\, r)|}{r} \leq K$$

or still,

$$\overline{\lim_{r \to \infty}} \frac{\log f(\sqrt{-1}\, r)}{r} \leq K \quad \text{(cf. 25.19)}$$

or still,

$$\overline{\lim_{r \to \infty}} \frac{\log f(\sqrt{-1}\, r)}{r} \leq K \quad \text{(cf. 25.22)}.$$

Denote this limit by $-a$, hence

$$\frac{\log f(\sqrt{-1}\, r)}{r} \leq -a$$

for all $r > 0$. Given an arbitrary $\varepsilon > 0$, let $t_1 < t_2 = a - \varepsilon$, thus

$$e^{-rt_2}(F(t_2) - F(t_1))$$

$$= e^{-rt_2} \mu_F(]t_1, t_2])$$

$$\leq e^{-rt_2} \mu_F([t_1, t_2])$$

$$= e^{-rt_2} \int_{t_1}^{t_2} d\mu_F(t)$$

$$= \int_{t_1}^{t_2} e^{-rt_2} d\mu_F(t)$$

$$\leq \int_{t_1}^{t_2} e^{-rt} d\mu_F(t)$$

$$\leq f(\sqrt{-1}\, r) \leq e^{-\, ar}$$

=>

$$F(t_2) - F(t_1) \leq e^{-\, \varepsilon r}$$

=>

$$F(t_2) - F(t_1) = 0 \quad (\text{let } r \to \infty)$$

=>

$$F(t_2) = 0 \quad (\text{let } t_1 \to -\infty)$$

=>

$$F(a - \varepsilon) = 0$$

=>

$$\text{lext}[F] \geq a.$$

To reverse this, put

$$\lambda_F = \text{lext}[F].$$

Then

$$f(\sqrt{-1}\, r) = \int_{\lambda_F}^{\infty} e^{-\, rt} d\mu_F(t)$$

$$\leq e^{-\, \lambda_F r}$$

=>

$$a = -\lim_{r \to \infty} \frac{\log f(\sqrt{-1}\, r)}{r} \geq \lambda_F.$$

Therefore

$$a = \lambda_F = \text{lext}[F],$$

the contention.

N.B. It is a corollary that the distribution function of an entire characteristic function of order 1 and of maximal type is not finite.

26.9 REMARK Compare the above result with that of 22.10.

A degenerate distribution function is, by definition, of the form

$$F(t) = I(t - C),$$

C a real constant.

N.B. The associated characteristic function is

$$f(x) = e^{\sqrt{-1}\, Cx},$$

hence is entire of exponential type, hence further is of order 1 and type $|C|$ provided $C \neq 0$.

26.10 LEMMA If F is degenerate, then F is finite and

$$\text{rext}[F] = \text{lext}[F].$$

PROOF

$$\begin{cases} \text{rext}[F] = \lim\limits_{r \to \infty} \dfrac{\log e^{Cr}}{r} = C \\[2ex] \text{lext}[F] = - \lim\limits_{r \to \infty} \dfrac{\log e^{-Cr}}{r} = - (- C) = C. \end{cases}$$

26.11 CONSTRUCTION Suppose that $F \neq I$ is a finite distribution function. Let

$$\begin{cases} a = \text{lext}[F] \\ b = \text{rext}[F]. \end{cases}$$

Then

$$f(x) = \int_{-\infty}^{\infty} e^{\sqrt{-1}\, xt} d\mu_F(t)$$

$$= \int_a^b e^{\sqrt{-1}\, xt} d\mu_F(t).$$

But the integral

$$\int_a^b e^{\sqrt{-1}\, zt} d\mu_F(t)$$

represents an entire function, thus f is an entire function of exponential type (cf. 17.19), thus is of order 1 (cf. 26.5).

<u>N.B.</u>

$$T(f) = \max(-a,b).$$

For, by definition,

$$T(f) = \overline{\lim_{r \to \infty}} \frac{\log M(r;f)}{r}.$$

On the other hand,

$$a = -\lim_{r \to \infty} \frac{\log f(\sqrt{-1}\, r)}{r}$$

and

$$b = \lim_{r \to \infty} \frac{\log f(-\sqrt{-1}\, r)}{r}.$$

And

$$M(r;f) = \max(f(\sqrt{-1}\, r),\ f(-\sqrt{-1}\, r)) \qquad \text{(cf. 26.3)}$$

=>

$$T(f) \geq \max(-a,b).$$

In the other direction,

$$f(\sqrt{-1}\, r) \le e^{-ar} \text{ and } f(-\sqrt{-1}\, r) \le e^{br}$$

=>

$$M(r;f) \le \max(e^{-ar}, e^{br})$$

=>

$$T(f) \le \max(-a,b).]$$

26.12 EXAMPLE If

$$F(t) = I(t - C) \quad (C \ne 0),$$

then

$$a = b = C.$$

- $a > 0 \Rightarrow \max(-a,a) = a = C$
- $a < 0 \Rightarrow \max(-a,a) = -a = -C = |C|$.

I.e.: $T(f) = |C|$ in agreement with what has been said earlier.

26.13 REMARK There is no entire characteristic function of order 1 and of minimal type (apply 17.18).

26.14 LEMMA If F is a finite distribution function and if F is nondegenerate, then its characteristic function f has an infinity of zeros (they need not be real).

PROOF Since f is bounded on the real axis, the conclusion that f has finitely many zeros is untenable (cf. §7).

26.15 REMARK An infinitely divisible entire characteristic function has no zeros.†

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† E. Lukacs, *Characteristic Functions*, Griffin, 1970, pp. 258-259.

26.16 NOTATION Given a distribution function F, let

$$T(t) = 1 - F(t) + F(-t) \quad (t > 0).$$

Let K and α be positive constants.

26.17 SUBLEMMA The integral

$$I(z) = \int_0^\infty \exp(\sqrt{-1}\, zt - Kt^{1+\alpha})dt$$

defines an entire function of order $1 + \frac{1}{\alpha}$.

[Consider the expansion

$$I(z) = \sum_{n=0}^{\infty} c_n z^n,$$

where

$$c_n = \frac{(\sqrt{-1})^n}{n!} \Gamma(\frac{n+1}{1+\alpha}) \frac{1}{(1+\alpha)K^{(n+1)/(1+\alpha)}} .]$$

[Note: To within a constant factor, I(z) is an entire characteristic function. Accordingly,

$$M(r;I) = \max(I(\sqrt{-1}\, r), I(-\sqrt{-1}\, r)) \quad \text{(cf. 26.3)}$$

$$= \int_0^\infty \exp(rt - Kt^{1+\alpha})dt.]$$

26.18 LEMMA Let F be a distribution function. Assume: $\exists$ A > 0 such that

$$t \geq A \Rightarrow T(t) \leq \exp(-Kt^{1+\alpha}).$$

Then the associated characteristic function f is entire (cf. 25.18) and its order is $\leq 1 + \frac{1}{\alpha}$.

PROOF Take $A > 0$ to be a continuity point of F and let $R > A$ -- then for $r > 0$:

$$\int_A^R e^{rt} d\mu_F(t) = \int_A^R e^{rt} d\mu_{F-1}(t)$$

$$= e^{rR}(F(R^+)-1) - e^{rA}(F(A^-)-1)$$

$$- r \int_A^R (F(t^+)-1)e^{rt} dt$$

$$= e^{rR}(F(R)-1) - e^{rA}(F(A)-1)$$

$$- r \int_A^R (F(t)-1)e^{rt} dt$$

$$\leq e^{rA}(1 - F(A)) + r \int_A^R e^{rt}(1 - F(t))dt$$

=>

$$\int_A^\infty e^{rt} d\mu_F(t) \leq e^{rA}(1 - F(A)) + r \int_A^\infty e^{rt}(1 - F(t))dt$$

$$\leq e^{rA}(1 - F(A)) + r \int_A^\infty \exp(rt - Kt^{1+\alpha})dt$$

$$\leq e^{rA}(1 - F(A)) + r \int_0^\infty \exp(rt - Kt^{1+\alpha})dt.$$

But

$$\int_{-\infty}^A e^{rt} d\mu_F(t) \leq e^{rA} F(A).$$

Therefore

$$\int_{-\infty}^\infty e^{rt} d\mu_F(t) \leq e^{rA} + r \int_0^\infty \exp(rt - Kt^{1+\alpha})dt.$$

And analogously,

$$\int_{-\infty}^{\infty} e^{-rt} d\mu_F(t) \le e^{rA} + r \int_0^{\infty} \exp(rt - Kt^{1+\alpha}) dt.$$

These estimates then enable one to estimate $M(r;f)$:

$$M(r;f) = \max(f(\sqrt{-1}\, r), f(-\sqrt{-1}\, r)) \quad \text{(cf. 26.3)}$$

$$\le e^{rA} + r \int_0^{\infty} \exp(rt - Kt^{1+\alpha}) dt$$

$$= M(r;e^{zA}) + M(r;zI(z)).$$

The order of e^{zA} is 1 whereas the order of $I(z)$ is $1 + \frac{1}{\alpha}$ (cf. 26.17), hence the order of $zI(z)$ is also $1 + \frac{1}{\alpha}$ (cf. 2.36), thus for any $\varepsilon > 0$,

$$M(r;e^{zA}) + M(r;zI(z)) < \exp(r^{1 + \frac{1}{\alpha} + \varepsilon}) \quad (r >> 0),$$

which implies that the order of f is $\le 1 + \frac{1}{\alpha}$.

26.19 THEOREM The characteristic function f of a distribution function F is entire of order 1 and of maximal type iff

$$t > 0 \Rightarrow T(t) > 0$$

and

$$\lim_{t \to \infty} \frac{\log \log \frac{1}{T(t)}}{\log t} = \infty.$$

PROOF

- Necessity It is clear that the first condition

$$t > 0 \Rightarrow T(t) > 0$$

holds (simply note that F is not finite). To see that the second condition holds, let $\varepsilon > 0$ be given and choose R:

$$r \geq R \Rightarrow \exp(r^{1+\varepsilon}) \geq M(r;f).$$

But $\forall\, t > 0$,

$$M(r;f) \geq \frac{1}{2} e^{rt} T(t) \qquad \text{(cf. 26.4)}.$$

Therefore

$$T(t) \leq 2\exp(-\, rt + r^{1+\varepsilon}).$$

Choosing $t \geq 2R^{\varepsilon}$ and taking $r = (\frac{t}{2})^{1/\varepsilon}$, we have

$$T(t) \leq 2\exp(-\, (\tfrac{t}{2})^{1 + (1/\varepsilon)})$$

=>

$$\underline{\lim}_{t \to \infty} \frac{\log \log \frac{1}{T(t)}}{\log t} \geq 1 + (1/\varepsilon)$$

=>

$$\lim_{t \to \infty} \frac{\log \log \frac{1}{T(t)}}{\log t} = \infty,$$

ε being arbitrary.

- <u>Sufficiency</u> Given $\varepsilon > 0$,

$$\frac{\log \log \frac{1}{T(t)}}{\log t} \quad 1 + \frac{1}{\varepsilon} \quad (t >> 0)$$

=>

$$T(t) \leq \exp(-\, t^{1 + \frac{1}{\varepsilon}}) \quad (t >> 0).$$

Therefore f is entire of order

$$\leq 1 + \frac{1}{\frac{1}{\varepsilon}} = 1 + \varepsilon \quad \text{(cf. 26.18)}.$$

But $F \neq I$, hence $\rho(f) = 1$ (cf. 26.5). Now f cannot be of minimal type (cf. 26.13) nor can f be of intermediate type (cf. 26.8 (F is not finite due to the assumption on T)), thus f must be of maximal type.

While a discussion of entire characteristic functions of order > 1 will be omitted, there is an important result of a negative nature.

26.20 THEOREM If p is a polynomial of degree > 2, then e^p is not a characteristic function.

APPENDIX

Let $F:R \to R$ -- then F is an <u>NBV function</u> if F is of bounded variation, if F is continuous from the right, and if $F(-\infty) = 0$.

NOTATION T_F is the total variation function associated with an NBV function F. So:

- T_F is increasing.
- T_F is continuous from the right.
- $T_F(-\infty) = 0$, $T_F(\infty) < \infty$.

RAPPEL The distribution functions F are in a one-to-one correspondence with the probability measures on the line: $F \to \mu_F$.

This can be generalized: The NBV functions F are in a one-to-one correspondence with the finite signed measures on the line: $F \to \mu_F$.

NOTATION $|\mu_F|$ is the total variation measure associated with an NBV function F. So

- $|\mu_F|\ (R) < \infty$.
- $|\mu_F| = \mu_{T_F}$.

N.B. For the record,

$$F(t) = \mu_F(]-\infty,t])$$

and

$$T_F(t) = \mu_{T_F}(]-\infty,t]) = |\mu_F|\ (]-\infty,t]).$$

EXAMPLE

$$\mu_{T_F}/\mu_{T_F}(R)$$

is a probability measure on the line.

LEMMA Any bounded Borel measurable function on R is μ_F-integrable (cf. 23.13).

DEFINITION Given an NBV function F, put

$$f(x) = \int_{-\infty}^{\infty} e^{\sqrt{-1}\, xt} d\mu_F(t),$$

the Fourier transform of μ_F.

Obviously,

$$|f(x)| \leq |\mu_F|(R) < \infty.$$

DEFINITION An NBV function F is constant outside a finite interval [T',T''] if

$$\begin{cases} F(t) = 0 & (t < T') \\ F(t) = C & (t > T'') \end{cases}$$

for some real number C.

<u>N.B.</u> Under these circumstances,

$$\int_{-\infty}^{\infty} e^{\sqrt{-1}\, zt} d\mu_F(t) = \int_{T'}^{T''} e^{\sqrt{-1}\, zt} d\mu_F(t)$$

and the integral on the right is defined for all complex z, thus f admits a continuation as an entire function and, as such, is of exponential type.

[Put

$$\mathfrak{T}_f(z) = \int_{-\infty}^{\infty} e^{\sqrt{-1}\, zt} d\mu_{T_F}(t),$$

the "characteristic function" of T_F -- then

$$M(r;\mathfrak{T}_f) = \max(\mathfrak{T}_f(\sqrt{-1}\, r), \mathfrak{T}_f(-\sqrt{-1}\, r)) \quad \text{(cf. 26.3)}.$$

On the other hand,

$$\begin{aligned} |f(x + \sqrt{-1}\, y)| &= \left| \int_{-\infty}^{\infty} e^{\sqrt{-1}\, zt} d\mu_F(t) \right| \\ &\leq \int_{-\infty}^{\infty} e^{-yt} d\mu_{T_F}(t) \\ &= \mathfrak{T}_f(\sqrt{-1}\, y) \end{aligned}$$

=>

$$M(r;f) \leq M(r;\mathfrak{T}_f).$$

But

$$\mathfrak{T}_f(\sqrt{-1}\, r) \leq e^{-T'r} \mu_{T_F}(R)$$

and

$$\mathcal{C}_f(-\sqrt{-1}\,r) \le e^{T''r}\mu_{T_F}(R).$$

Therefore

$$M(r;f) \le \exp(\max(|T'|,|T''|)r),$$

so f is of exponential type.]

THEOREM Suppose that F is an NBV function. Assume: $\hat{f}$ can be extended into the complex plane as an entire function of exponential type. Let

$$\begin{cases} a = -\overline{\lim_{r\to\infty}}\,\dfrac{\log|\hat{f}(\sqrt{-1}\,r)|}{r} \\[2ex] b = \overline{\lim_{r\to\infty}}\,\dfrac{\log|\hat{f}(-\sqrt{-1}\,r)|}{r}. \end{cases}$$

Then a and b are finite (sic). Moreover, F is constant outside a finite interval and in fact [a,b] is the smallest finite interval outside of which F is constant.

PROOF We shall work initially with b and show that F is constant to the right of b. To this end, note that for any pair $t_1 < t_2$ of continuity points of F:

$$F(t_2) - F(t_1) = \lim_{r\to\infty}\int_{-r}^{r}\frac{e^{-\sqrt{-1}\,t_1x} - e^{-\sqrt{-1}\,t_2x}}{2\pi\sqrt{-1}\,x}\hat{f}(x)dx \quad \text{(cf. 24.9)}.$$

Now specialize and take $b < t_1 < t_2$ (t_2 arbitrary) and let $2\varepsilon = t_1 - b > 0$ ($\Rightarrow b < b + \varepsilon = t_1 - \varepsilon < t_1$). Put

$$f(z) = (1 - e^{-\sqrt{-1}\,(t_2 - t_1)z})\hat{f}(z)e^{-\sqrt{-1}\,(b+\varepsilon)z}.$$

Then

- f is entire of exponential type.
- f is bounded on the real axis.
- $f(-\sqrt{-1}\, r)$ $(0 \le r < \infty)$ is bounded.

Therefore (...) f is bounded in the lower half-plane: $|f| \le M$. And

$$2\pi\sqrt{-1}\,(F(t_2) - F(t_1)) = \lim_{r \to \infty} \int_{-r}^{r} \frac{f(x)}{x} \cdot e^{-\sqrt{-1}\,\varepsilon x} dx.$$

Since the integrand is entire $(f(0) = 0)$, the integration interval can be replaced by a semi-circular arc of radius r centered at the origin and situated in the lower half-plane, hence

$$\left| \int_{-r}^{r} \frac{f(x)}{x} \cdot e^{-\sqrt{-1}\,\varepsilon x} dx \right|$$

$$\le \int_{\pi}^{2\pi} |f(re^{\sqrt{-1}\,\theta})| e^{\varepsilon r \sin \theta} d\theta$$

$$\le M \int_{0}^{\pi} e^{-\varepsilon r \sin \theta} d\theta$$

$$\le 2M \int_{0}^{\pi/2} e^{-\varepsilon r \sin \theta} d\theta$$

$$\le 2M \int_{0}^{\pi/2} e^{-(2\varepsilon r \theta)/\pi} d\theta$$

$$\to 0 \quad (r \to \infty)$$

=>

$$\lim_{r \to \infty} \int_{-r}^{r} \frac{f(x)}{x} \cdot e^{-\sqrt{-1}\,\varepsilon x} dx = 0$$

=>

$$F(t_2) - F(t_1) = 0$$

=>

$$F(t_2) = F(t_1) = F(b + 2\varepsilon),$$

proving that F is constant to the right of b. By a similar argument, one finds that F is constant to the left of a, thus equals $F(-\infty) = 0$ there. Finally, if $[T',T'']$ is a finite interval outside of which F is constant, then $T' \le a$, $b \le T''$. E.g.:

$$|f(\sqrt{-1}\, r)| \le \mathfrak{C}_f(\sqrt{-1}\, r)$$

$$\le e^{-T'r} \mu_{T_F}(R)$$

=>

$$a = -\overline{\lim_{r \to \infty}} \frac{\log |f(\sqrt{-1}\, r)|}{r} \ge T'.$$

§27. *ZERO THEORY: BERNSTEIN FUNCTIONS*

Let $B_0(A)$ be the subset of $E_0(A)$ consisting of those f which are bounded on the real axis.

[Note: The elements of $B_0(A)$ are called <u>Bernstein functions</u>.]

<u>N.B.</u> If $f \in B_0(A)$ and if $T(f) = 0$, then f is a constant (cf. 17.18).

[Note: Accordingly, if $f \in B_0(A)$ is not a constant, then $T(f) > 0$ and $\rho(f) = 1$ (with $T(f) = \tau(f)$) (cf. 17.3).]

27.1 EXAMPLE Take $A = 1$ -- then $e^{\sqrt{-1}\, z} \in B_0(1)$.

27.2 EXAMPLE Suppose that $F \neq I$ is a finite distribution function -- then its characteristic function $f \in B_0(A)$, where $A = \max(-a,b)$ (cf. 26.11).

[Note: Take

$$F(t) = I(t-1).$$

Then $f(z) = e^{\sqrt{-1}\, z}$.]

27.3 LEMMA $PW(A)$ is a subset of $B_0(A)$ (cf. 17.29).

27.4 LEMMA $B_0(A)$ is a vector space (under pointwise addition and scalar multiplication) and, when equipped with the supremum norm, is a Banach space (cf. 17.17).

27.5 LEMMA $B_0(A)$ is closed under differentiation (cf. 17.24).

27.6 LEMMA If $f \in B_0(A)$ is not a constant, then $n(r) = O(r)$, i.e., $\frac{n(r)}{r}$

remains bounded as $r \to \infty$ (cf. 4.31).

27.7 NOTATION Given $f \in B_0(A)$, let $z_n = r_n e^{\sqrt{-1}\,\theta_n}$ $(n = 1,2,\ldots)$ be the nonzero zeros of f repeated according to multiplicity with

$$0 < |z_1| \leq |z_2| \leq \ldots .$$

[Note:

$$\frac{1}{z_n} = \frac{e^{-\sqrt{-1}\,\theta_n}}{r_n} = \frac{\cos\theta_n}{r_n} - \sqrt{-1}\,\frac{\sin\theta_n}{r_n} .]$$

27.8 LEMMA If $f \in B_0(A)$ is not a constant, then

$$S(r) = \sum_{|z_n| \leq r} \frac{1}{z_n}$$

remains bounded as $r \to \infty$.

[One can extract a proof from the material in §6. To proceed directly, assume for convenience that $|f(0)| = 1$ and choose $K > 0 : n(r) \leq Kr$ (cf. 27.6) -- then

$$|S(r) - S(R)| \leq 2K \quad (R \leq r \leq 2R)$$

=>

$$\int_R^{2R} S(r) r\,dr = \frac{3}{2} R^2 S(R) + O(R^2).$$

Under the supposition that $f(z)$ is zero free on $|z| = r$, write

$$S(r) = \frac{1}{2\pi\sqrt{-1}} \int_C \frac{f'(z)}{f(z)} \cdot \frac{1}{z}\,dz - \frac{f'(0)}{f(0)}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\partial}{\partial x} - \sqrt{-1}\,\frac{\partial}{\partial y}\right) \log\,|f(re^{\sqrt{-1}\,\theta})|\,d\theta - \frac{f'(0)}{f(0)}$$

=>

$$\frac{3}{2} R^2 S(R) = \int_R^{2R} S(r)rdr + O(R^2)$$

$$= \frac{1}{2\pi} \iint_{R \le |z| \le 2R} (\frac{\partial}{\partial x} - \sqrt{-1} \frac{\partial}{\partial y}) \log |f(z)| dxdy + O(R^2)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (2R \log |f(2Re^{\sqrt{-1}\,\theta})| - R \log |f(Re^{\sqrt{-1}\,\theta})|) e^{-\sqrt{-1}\,\theta} d\theta + O(R^2)$$

=>

$$\frac{3}{2} R^2 |S(R)|$$

$$\le \frac{R}{2\pi} \int_0^{2\pi} (2|\log |f(2Re^{\sqrt{-1}\,\theta})|| + |\log |f(Re^{\sqrt{-1}\,\theta})||) d\theta + O(R^2).$$

Estimating the integral in the usual way gives rise to another $O(R^2)$, so in the end

$$\frac{3}{2} R^2 |S(R)| \le O(R^2)$$

=>

$$|S(R)| \le O(1) \quad (R \to \infty).]$$

27.9 CARLEMAN FORMULA Suppose that $f(z)$ is holomorphic for $\text{Im } z \ge 0$ and let $z_k = r_k e^{\sqrt{-1}\,\theta_k}$ $(k = 1,\ldots,n)$ be its zeros in the region

$$\{z : \text{Im } z \ge 0,\ 1 \le |z| \le R\}.$$

Then

$$\sum_{k=1}^{n} (\frac{1}{r_k} - \frac{r_k}{R^2}) \sin \theta_k$$

$$= \frac{1}{\pi R} \int_0^{\pi} \log |f(Re^{\sqrt{-1}\,\theta})| \sin \theta \, d\theta$$

$$+ \frac{1}{2\pi} \int_1^R \left(\frac{1}{x^2} - \frac{1}{R^2}\right) \log |f(x)f(-x)| dx + A(R),$$

where $A(R)$ is a bounded function of R.

[Note: Replace 1 by $\rho > 0$ -- then $A(R)$ depends on ρ and

$$A(\rho,R) = - \text{Im} \frac{1}{2\pi} \int_0^\pi \log f(\rho e^{\sqrt{-1}\,\theta}) \left(\frac{\rho e^{\sqrt{-1}\,\theta}}{R^2} - \frac{e^{-\sqrt{-1}\,\theta}}{\rho}\right) d\theta,$$

thus if $f(0) = 1$,

$$\lim_{\rho \to 0} A(\rho,R) = \frac{1}{2} \text{Im}\, f'(0),$$

so

$$\sum_{r_k \le R} \left(\frac{1}{r_k} - \frac{r_k}{R^2}\right) \sin \theta_k$$

$$= \frac{1}{\pi R} \int_0^\pi \log |f(Re^{\sqrt{-1}\,\theta})| \sin \theta \, d\theta$$

$$+ \frac{1}{2\pi} \int_0^R \left(\frac{1}{x^2} - \frac{1}{R^2}\right) \log |f(x)f(-x)| dx + \frac{1}{2} \text{Im}\, f'(0).]$$

27.10 THEOREM If $f \in B_0(A)$ is not a constant, then the series

$$\sum_{n=1}^\infty \frac{\sin \theta_n}{r_n}$$

is absolutely convergent.

PROOF Apply 27.9 to $f(z)$, $f(-z)$ and add the results. In this way we are led to

$$\sum_{k=1}^n \left(\frac{1}{r_k} - \frac{r_k}{R^2}\right) \sin \theta_k \quad (0 \le \theta_k \le \pi)$$

$$+ \sum_{\ell=1}^m \left(\frac{1}{r_\ell} - \frac{r_\ell}{R^2}\right) \sin(\theta_\ell + \pi) \; (- \pi \le \theta_\ell \le 0).$$

But $\sin \theta_k = |\sin \theta_k|$, $\sin(\theta_\ell + \pi) = -\sin \theta_\ell = |\sin \theta_\ell|$, hence

$$\sum_{r_n \leq R} \left(1 - \frac{r_n^2}{R^2}\right) \frac{|\sin \theta_n|}{r_n} < C \quad (R >> 0)$$

for some constant $C > 0$. And this implies that

$$\sum_{r_n \leq R/2} \left(1 - \frac{1}{4}\right) \frac{|\sin \theta_n|}{r_n} < C.$$

Now send R to ∞.

[Note: The zeros on the real axis do not figure in the calculation.]

N.B. Restated, 27.10 says that

$$\sum_{n=1}^{\infty} \left|\text{Im} \frac{1}{z_n}\right| < \infty.$$

[Note: In traditional terminology, an entire function f of exponential type is said to be class A if

$$\sum_{n=1}^{\infty} \left|\text{Im} \frac{1}{z_n}\right| < \infty.$$

Characterization: f is class A iff

$$\sup_{R>1} \int_1^R \frac{\log |f(x)f(-x)|}{x^2} \, dx < \infty.]$$

27.11 APPLICATION Given $\varepsilon > 0$, let $\Omega(\varepsilon)$ be the sector

$$|\arg z| < \varepsilon \cup |\arg z - \pi| < \varepsilon.$$

Then

$$\sum_{k=1}^{\infty} \frac{1}{|z_{n_k}|} < \infty,$$

where z_{n_k} runs through the zeros of f which are not in $\Omega(\varepsilon)$.

27.12 THEOREM If $f \in B_0(A)$ is not a constant, then

$$\lim_{r \to \infty} \frac{n(r)}{r} = \frac{h_f(\sqrt{-1}) + h_f(-\sqrt{-1})}{\pi}.$$

[This is a substantial reinforcement of 27.6. For a proof, consult B. Levin[†] (see also P. Koosis[††]).]

27.13 REMARK One can say more. Thus let $n_+(r)$ be the number of zeros of f with real part ≥ 0 and modulus $\leq r$ and let $n_-(r)$ be the number of zeros of f with real part < 0 and modulus $\leq r$ -- then

$$n(r) = n_+(r) + n_-(r).$$

Moreover, it can be shown that

$$\lim_{r \to \infty} \frac{n_+(r)}{r} = \frac{h_f(\sqrt{-1}) + h_f(-\sqrt{-1})}{2\pi}$$

and

$$\lim_{r \to \infty} \frac{n_-(r)}{r} = \frac{h_f(\sqrt{-1}) + h_f(-\sqrt{-1})}{2\pi}.$$

27.14 EXAMPLE Take $f(z) = e^{\sqrt{-1}\, z}$ -- then $n(r) \equiv 0$. On the other hand,

$$h_f(\sqrt{-1}) = \overline{\lim_{r \to \infty}} \frac{\log |e^{\sqrt{-1}(\sqrt{-1}\, r)}|}{r} = \overline{\lim_{r \to \infty}} \frac{\log e^{-r}}{r} = -1$$

[†] *Lectures on Entire Functions*, A.M.S., 1996, pp. 127-130.

[††] *The Logarithmic Integral* I, Cambridge University Press, 1988, pp. 69-76.

and

$$h_f(-\sqrt{-1}) = \overline{\lim_{r \to \infty}} \frac{\log |e^{\sqrt{-1}(-\sqrt{-1}\,r)}|}{r} = \overline{\lim_{r \to \infty}} \frac{\log e^r}{r} = 1.$$

Therefore

$$h_f(\sqrt{-1}) + h_f(-\sqrt{-1}) = -1 + 1 = 0.$$

27.15 LEMMA[†] If $f \in B_0(A)$ is not a constant, then

$$H_f(1) = 0 \text{ and } H_f(-1) = 0$$

or still,

$$h_f(1) = \overline{\lim_{r \to \infty}} \frac{\log |f(r)|}{r} = 0$$

and

$$h_f(-1) = \overline{\lim_{r \to \infty}} \frac{\log |f(-r)|}{r} = 0.$$

[Note: This result is a consequence of "Ahlfors-Heins theory" and is valid for any entire function f of exponential type in the Cartwright class, i.e., such that

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(x)|}{1 + x^2} dx < \infty.]$$

27.16 COROLLARY The indicator diagram K_f of f is a segment of the imaginary axis (or a point) (cf. 18.9).

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† R. Boas, *Entire Functions*, Academic Press, 1954, p. 116.

27.17 LEMMA Let $K = [\sqrt{-1}\,A, \sqrt{-1}\,B]$ $(A \le B)$ -- then

$$H_K(e^{\sqrt{-1}\,\theta}) = a|\sin\theta| + b\sin\theta,$$

where

$$a = \frac{B-A}{2},\ b = \frac{-B-A}{2}\ .$$

27.18 EXAMPLE Take A = B, call it C -- then

$$a = \frac{C-C}{2} = 0,\ b = \frac{-C-C}{2} = -C$$

and

$$H_K(e^{\sqrt{-1}\,\theta}) = -C\sin\theta \quad \text{(cf. 18.2)}.$$

27.19 EXAMPLE Take $A = -c$, $B = c$ with $c > 0$ -- then

$$a = \frac{c-(-c)}{2} = c,\ b = \frac{-c+c}{2} = 0$$

and

$$H_K(e^{\sqrt{-1}\,\theta}) = a|\sin\theta| \quad \text{(cf. 18.5)}.$$

27.20 RAPPEL If $f \in B_0(A)$ is not a constant, then

$$T(f) = \tau(f) = \sup_{0\le\theta\le 2\pi} h_f(e^{\sqrt{-1}\,\theta}) \quad \text{(cf. 19.10)}.$$

Recalling that $H_f (= H_{K_f}$ (cf. 18.17)) $= h_f$ (cf. 19.7), we have

$$\sup_{0\le\theta\le 2\pi} h_f(e^{\sqrt{-1}\,\theta})$$

$$= \sup_{0\le\theta\le 2\pi} (a|\sin\theta| + b\sin\theta)$$

$$= \max(a+b, a-b) = a + |b|.$$

But

$$\begin{cases} a + b = h_f(\sqrt{-1}) \\ a - b = h_f(-\sqrt{-1}). \end{cases}$$

Therefore

$$T(f) = \max(h_f(\sqrt{-1}), h_f(-\sqrt{-1})).$$

27.21 SCHOLIUM If $h_f(\sqrt{-1}) = h_f(-\sqrt{-1})$, then

$$\lim_{r \to \infty} \frac{n(r)}{r} = \frac{h_f(\sqrt{-1}) + h_f(-\sqrt{-1})}{\pi} \qquad \text{(cf. 27.12)}$$

$$= 2\,\frac{T(f)}{\pi}.$$

27.22 LEMMA

$$K_f = [\sqrt{-1}\ (-h_f(\sqrt{-1}),\ \sqrt{-1}\ h_f(-\sqrt{-1}).]$$

PROOF Writing $K_f = [\sqrt{-1}\ A, \sqrt{-1}\ B]$, it is a question of explicating A and B.

But

$$\begin{cases} a + b = h_f(\sqrt{-1}) \\ a - b = h_f(-\sqrt{-1}). \end{cases}$$

And

$$a = \frac{B-A}{2},\ b = \frac{-B-A}{2}$$

=>

$$\begin{cases} \dfrac{B-A}{2} + \dfrac{-B-A}{2} = -A \\ \dfrac{B-A}{2} - \dfrac{-B-A}{2} = B \end{cases}$$

=>

$$\begin{cases} -A = h_f(\sqrt{-1}) \\ B = h_f(-\sqrt{-1}) \end{cases}$$

=>

$$K_f = [\sqrt{-1}(-h_f(\sqrt{-1}), \sqrt{-1}\, h_f(-\sqrt{-1}).]$$

27.23 APPLICATION K_f reduces to a point iff

$$h_f(\sqrt{-1}) + h_f(-\sqrt{-1}) = 0,$$

hence K_f reduces to a point iff

$$\lim_{r \to \infty} \frac{n(r)}{r} = 0.$$

27.24 EXAMPLE Suppose that $c \neq 0$ is real and let $f(z) = e^{\sqrt{-1}\, cz}$ -- then

$$h_f(e^{\sqrt{-1}\,\theta}) = -c \sin \theta \quad \text{(cf. 19.2)}$$

=>

$$\begin{cases} h_f(\sqrt{-1}) = -c \\ h_f(-\sqrt{-1}) = c \end{cases} \Rightarrow K_f = \{\sqrt{-1}\, c\}.$$

And $T(f) = |c|$.

27.25 EXAMPLE Suppose that $F \neq I$ is a finite distribution function, f its characteristic function (cf. 27.2) -- then

$$\left[\begin{array}{l} \text{rext}[F] = h_f(-\sqrt{-1}) \\ \\ \text{lext}[F] = -h_f(\sqrt{-1}) \end{array}\right. \qquad \text{(cf. 26.8)}$$

and

$$-h_f(\sqrt{-1}) \leq h_f(-\sqrt{-1})$$

in agreement with 27.22 (cf. 22.13).

[Note: Recall too that

$$T(f) = \max(-\text{lext}[F], \text{rext}[F]) \qquad \text{(cf. 26.11).]}$$

27.26 EXAMPLE Given $\phi \in L^1[-A,A]$ $(0 < A < \infty)$, put

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \phi(t) e^{\sqrt{-1}\, zt} dt.$$

Then $f \in B_0(A)$ (cf. 17.19). Assume further that $\phi(t)$ does not vanish almost everywhere in any neighborhood of A (or -A) -- then

$$\left[\begin{array}{l} A = h_f(-\sqrt{-1}) \\ \\ -A = -h_f(\sqrt{-1}) \end{array}\right. \qquad \Rightarrow T(f) = A$$

=>

$$\lim_{r \to \infty} \frac{n(r)}{r} = 2\,\frac{T(f)}{\pi} \qquad \text{(cf. 27.21)}$$

$$= 2\,\frac{A}{\pi}.$$

27.27 NOTATION Put

$$D = \frac{h_f(\sqrt{-1}) + h_f(-\sqrt{-1})}{\pi}.$$

27.28 DEFINITION The zeros of f have a density if $D > 0$.

27.29 RAPPEL Take $\alpha > 0$ -- then the series

$$\sum_{n=1}^{\infty} \frac{1}{r_n^{\alpha}}$$

converges iff the integral

$$\int_0^{\infty} \frac{n(t)}{t^{\alpha+1}}\, dt$$

converges.

27.30 LEMMA If the zeros of f have a density, then the series

$$\sum_{n=1}^{\infty} \frac{1}{r_n}$$

is divergent.

[In 27.29, take $\alpha = 1$:

$$\int_0^{\infty} \frac{n(t)}{t^2}\, dt = \int_0^{\infty} \frac{n(t)}{t} \cdot \frac{dt}{t}$$

$$= \int_0^{\infty} \frac{(n(t)/t)}{D} \cdot D \frac{dt}{t}$$

is divergent (cf. 27.12).]

[Note: The convergence exponent is equal to 1 (cf. 4.10). Therefore f is of divergence class (cf. 4.24).]

27.31 THEOREM If $f \in B_0(A)$ is not a constant and if the zeros of f have a density, then the series

$$\sum_{n=1}^{\infty} \frac{\cos \theta_n}{r^n}$$

is convergent.

27.32 REMARK According to 27.10, the series

$$\sum_{n=1}^{\infty} \frac{\sin \theta_n}{r^n}$$

is absolutely convergent. On the other hand, in view of 27.30, the series

$$\sum_{n=1}^{\infty} \frac{\cos \theta_n}{r_n}$$

is not absolutely convergent.

Before tackling the proof, we shall first set up the relevant generalities.

27.33 RAPPEL Given a sequence $a_1, a_2, \ldots,$ put

$$\sigma_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Assume: $\lim\limits_{n \to \infty} a_n = 0$ -- then $\lim\limits_{n \to \infty} \sigma_n = 0$.

27.34 APPLICATION If $a_n \to L$, then $\sigma_n \to L$.

[In fact, $a_n - L \to 0$, so

$$\frac{(a_1 - L) + (a_2 - L) + \cdots + (a_n - L)}{n} \to 0$$

or still, $\sigma_n - L \to 0$.]

27.35 RAPPEL Given an infinite series $\sum\limits_1^{\infty} a_n$, let s_n denote its n^{th} partial sum and put

$$\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}.$$

Assume: $\{\sigma_n\}$ converges to S and $a_n = O(\frac{1}{n})$ -- then $\{s_n\}$ converges to S.

[Note: In other words, if $\sum_1^\infty a_n$ is (C,1) summable to S and if $a_n = O(\frac{1}{n})$, then $\sum_1^\infty a_n$ is convergent to S.]

N.B.

$$\frac{\cos \theta_n}{r_n} = O(\frac{1}{n}).$$

[For

$$\frac{n(r_n)}{r_n} = \frac{n}{r_n} \to D.]$$

27.36 JENSEN FORMULA Suppose that f(z) is holomorphic in $|z| < R$ with $f(0) = 1$ -- then

$$\int_0^r \frac{n(t)}{t}\, dt = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{\sqrt{-1}\,\theta})|d\theta \quad (0 < r < R).$$

27.37 CARLEMAN FORMULA (bis) Suppose that f(z) is holomorphic for Re $z \ge 0$ and let $z_k = r_k e^{\sqrt{-1}\,\theta_k}$ $(k = 1,\ldots,n)$ be its zeros in the region

$$\{z:\text{Re } z \ge 0,\ 1 \le |z| \le R\}.$$

Then

$$\sum_{k=1}^{n} (\frac{1}{r_k} - \frac{r_k}{R^2}) \cos \theta_k$$

$$= \frac{1}{\pi R} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(Re^{\sqrt{-1}\,\theta})|\cos \theta\, d\theta$$

$$+ \frac{1}{2\pi} \int_1^R (\frac{1}{x^2} - \frac{1}{R^2}) \log |f(\sqrt{-1}\,x)f(-\sqrt{-1}\,x)|dx + A(R),$$

where A(R) is a bounded function of R.

[Note: If f(0) = 1, then

$$\sum_{r_k \leq R} \left(\frac{1}{r_k} - \frac{r_k}{R^2}\right) \cos \theta_k$$

$$= \frac{1}{\pi R} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(Re^{\sqrt{-1}\,\theta})| \cos \theta \, d\theta$$

$$+ \frac{1}{2\pi} \int_0^R \left(\frac{1}{x^2} - \frac{1}{R^2}\right) \log |f(\sqrt{-1}\,x) f(-\sqrt{-1}\,x)| dx - \frac{1}{2} \operatorname{Re} f'(0).]$$

Proceeding to the proof of 27.31, it will be assumed that f(0) = 1.

[Note: Zeros of f(z) on the imaginary axis do not participate $(\cos(\pm \frac{\pi}{2}) = 0)$.]

Step 1: In the formula

$$\sum_{r_k \leq R} \left(\frac{1}{r_k} - \frac{r_k}{R^2}\right) \cos \theta_k + \frac{1}{2} \operatorname{Re} f'(0) = \cdots,$$

replace f(z) by f(-z) to get

$$\sum_{r_\ell \leq R} \left(\frac{1}{r_\ell} - \frac{r_\ell}{R^2}\right) \cos(\theta_\ell + \pi) - \frac{1}{2} \operatorname{Re} f'(0)$$

$$= \frac{1}{\pi R} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(-\,Re^{\sqrt{-1}\,\theta})| \cos \theta \, d\theta$$

$$+ \frac{1}{2\pi} \int_0^R \left(\frac{1}{x^2} - \frac{1}{R^2}\right) \log |f(-\sqrt{-1}\,x) f(\sqrt{-1}\,x)| dx$$

or still,

$$- \sum_{r_\ell \leq R} \left(\frac{1}{r_\ell} - \frac{r_\ell}{R^2}\right) \cos \theta_\ell - \frac{1}{2} \operatorname{Re} f'(0) = \cdots.$$

Therefore

$$\sum_{r_k \leq R} \left(\frac{1}{r_k} - \frac{r_k}{R^2}\right) \cos \theta_k + \frac{1}{2} \operatorname{Re} f'(0)$$

$$+ \sum_{r_\ell \leq R} \left(\frac{1}{r_\ell} - \frac{r_\ell}{R^2}\right) \cos \theta_\ell + \frac{1}{2} \operatorname{Re} f'(0)$$

$$= \frac{1}{\pi R} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(Re^{\sqrt{-1}\,\theta})| \cos \theta \, d\theta$$

$$- \frac{1}{\pi R} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(-Re^{\sqrt{-1}\,\theta})| \cos \theta \, d\theta.$$

Step 2:

$$- \frac{1}{\pi R} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(-Re^{\sqrt{-1}\,\theta})| \cos \theta \, d\theta$$

$$= - \frac{1}{\pi R} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(Re^{\sqrt{-1}(\theta+\pi)})| \cos \theta \, d\theta$$

$$= - \frac{1}{\pi R} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \log |f(Re^{\sqrt{-1}\,\theta})| \cos (\theta-\pi) d\theta$$

$$= \frac{1}{\pi R} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \log |f(Re^{\sqrt{-1}\,\theta})| \cos \theta \, d\theta.$$

Step 3: Therefore

$$\sum_{r_k \leq R} \left(\frac{1}{r_k} - \frac{r_k}{R^2}\right) \cos \theta_k + \frac{1}{2} \operatorname{Re} f'(0)$$

$$+ \sum_{r_\ell \le R} \left(\frac{1}{r_\ell} - \frac{r_\ell}{R^2}\right) \cos \theta_\ell + \frac{1}{2} \operatorname{Re} f'(0)$$

$$= \frac{1}{\pi R} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(Re^{\sqrt{-1}\,\theta})| \cos \theta \, d\theta$$

$$+ \frac{1}{\pi R} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \log |f(Re^{\sqrt{-1}\,\theta})| \cos \theta \, d\theta$$

$$= \frac{1}{\pi R} \int_{-\frac{\pi}{2}}^{0} \log |f(Re^{\sqrt{-1}\,\theta})| \cos \theta \, d\theta$$

$$+ \frac{1}{\pi R} \int_{0}^{\frac{\pi}{2}} \log |f(Re^{\sqrt{-1}\,\theta})| \cos \theta \, d\theta$$

$$+ \frac{1}{\pi R} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \log |f(Re^{\sqrt{-1}\,\theta})| \cos \theta \, d\theta$$

$$= \frac{1}{\pi R} \int_{\frac{3\pi}{2}}^{2\pi} \log |f(Re^{\sqrt{-1}(\theta-2\pi)})| \cos(\theta-2\pi) d\theta$$

$$+ \frac{1}{\pi R} \int_{0}^{\frac{3\pi}{2}} \log |f(Re^{\sqrt{-1}\,\theta})| \cos \theta \, d\theta$$

$$= \frac{1}{\pi R} \int_{0}^{2\pi} \log |f(Re^{\sqrt{-1}\,\theta})| \cos \theta \, d\theta.$$

Summary:

$$\sum_{r_n \le r} \left(\frac{1}{r_n} - \frac{r_n}{r^2}\right) \cos \theta_n + \operatorname{Re} f'(0)$$

$$= \frac{1}{\pi r} \int_{0}^{2\pi} \log |f(re^{\sqrt{-1}\,\theta})| \cos \theta \, d\theta.$$

Step 4:

$$\int_0^r \frac{n(t)}{t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} \log\,|f(re^{\sqrt{-1}\,\theta})|\,d\theta$$

=>

$$\frac{1}{r}\int_0^r \frac{n(t)}{t}\,dt = \frac{1}{2\pi r}\int_0^{2\pi} \log\,|f(re^{\sqrt{-1}\,\theta})|\,d\theta$$

=>

$$\lim_{r\to\infty} \frac{1}{r}\int_0^r \frac{n(t)}{t}\,dt = D = \lim_{r\to\infty} \frac{1}{2\pi r}\int_0^{2\pi} \log\,|f(re^{\sqrt{-1}\,\theta})|\,d\theta.$$

[Given $\varepsilon > 0$, choose t_0:

$$t > t_0 \Rightarrow D - \varepsilon < \frac{n(t)}{t} < D + \varepsilon.$$

Write

$$\frac{1}{r}\int_0^r \frac{n(t)}{t}\,dt = \frac{1}{r}\int_0^{t_0} \frac{n(t)}{t}\,dt + \frac{1}{r}\int_{t_0}^r \frac{n(t)}{t}\,dt \quad (r > t_0).$$

Then

$$\frac{(r-t_0)(D-\varepsilon)}{r} < \frac{1}{r}\int_{t_0}^r \frac{n(t)}{t}\,dt < \frac{(r-t_0)(D+\varepsilon)}{r}$$

=> $(r \to \infty)$

$$D - \varepsilon \le \lim_{r\to\infty} \frac{1}{r}\int_{t_0}^r \frac{n(t)}{t}\,dt \le D + \varepsilon.]$$

Step 5: We have

$$h_f(e^{\sqrt{-1}\,\theta}) = a|\sin\theta| + b\sin\theta$$

$$= \frac{h_f(\sqrt{-1}) + h_f(-\sqrt{-1})}{2}\,|\sin\theta| + \frac{h_f(\sqrt{-1}) - h_f(-\sqrt{-1})}{2}\,\sin\theta$$

$$= \frac{\pi D}{2} |\sin \theta| + b \sin \theta$$

=>

$$\frac{1}{2\pi} \int_0^{2\pi} h_f(e^{\sqrt{-1}\,\theta})d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{\pi D}{2} |\sin \theta| d\theta$$

$$= \frac{D}{4} \int_0^{2\pi} |\sin \theta| d\theta$$

$$= D.$$

<u>Step 6</u>: Given $\varepsilon > 0$, choose r_0:

$$r > r_0 \Rightarrow$$

$$-2\varepsilon < \int_0^{2\pi} (h_f(e^{\sqrt{-1}\,\theta}) + \varepsilon - \frac{1}{r} \log |f(re^{\sqrt{-1}\,\theta})|)d\theta < 2\varepsilon.$$

But for $r_0 >> 0$,

$$\frac{1}{r} \log |f(re^{\sqrt{-1}\,\theta})| < h_f(e^{\sqrt{-1}\,\theta}) + \varepsilon$$

uniformly in θ (inspect the first part of the proof of 19.7), thus

$$-2\varepsilon < \int_0^{2\pi} (h_f(e^{\sqrt{-1}\,\theta}) + \varepsilon - \frac{1}{r} \log |f(re^{\sqrt{-1}\,\theta})|)\cos \theta \, d\theta < 2\varepsilon$$

and so

$$\lim_{r \to \infty} \frac{1}{r} \int_0^{2\pi} \log |f(re^{\sqrt{-1}\,\theta})| \cos \theta \, d\theta$$

$$= \int_0^{2\pi} h_f(e^{\sqrt{-1}\,\theta}) \cos \theta \, d\theta.$$

Step 7:

- $\int_0^{\pi} |\sin\theta|\cos\theta\, d\theta = \int_0^{\pi} \sin\theta\cos\theta\, d\theta$

$$= \frac{1}{2}\int_0^{\pi} \sin 2\theta\, d\theta$$

$$= \frac{1}{2} - \frac{\cos 2\theta}{2}\Big|_0^{\pi} = \frac{1}{4}(-\cos 2\pi + \cos 0)$$

$$= 0.$$

- $\int_{\pi}^{2\pi} \sin\theta\cos\theta\, d\theta = \frac{1}{2}\int_{\pi}^{2\pi} \sin 2\theta\, d\theta$

$$= \frac{1}{2} - \frac{\cos 2\theta}{2}\Big|_{\pi}^{2\pi} = \frac{1}{4}(-\cos 4\pi + \cos 2\pi)$$

$$= 0.$$

Consequently,

$$\frac{1}{\pi}\int_0^{2\pi} h_f(e^{\sqrt{-1}\,\theta})\cos\theta\, d\theta = 0,$$

which implies that

$$\lim_{r\to\infty} \frac{1}{\pi r}\int_0^{2\pi} \log|f(re^{\sqrt{-1}\,\theta})|\cos\theta\, d\theta = 0.$$

Summary:

$$\lim_{r\to\infty} \sum_{r_n \le r} \left(\frac{1}{r_n} - \frac{r_n}{r^2}\right)\cos\theta_n = -\operatorname{Re} f'(0).$$

Step 8: Let r take the values m/D, where m is an integer -- then

$$|m - n(\tfrac{m}{D})| = o(m) \quad (m \to \infty)$$

=>

$$\lim_{m\to\infty} \sum_{n=1}^{m} \frac{\cos\theta_n}{r_n}\left(1 - \frac{r_n^2 D^2}{m^2}\right) = -\operatorname{Re} f'(0).$$

Step 9: Let

$$\gamma_m = \sum_{n=1}^{m} \frac{\cos\theta_n}{r_n}\left(1 - \frac{r_n^2 D^2}{m^2}\right).$$

Then

$$(m+1)^2\gamma_{m+1} - m^2\gamma_m$$

$$= (2m+1)\sum_{n=1}^{m}\frac{\cos\theta_n}{r_n}$$

$$+ \frac{\cos\theta_{m+1}}{r_{m+1}}\left((m+1)^2 - D^2 r_{m+1}^2\right).$$

[Starting from the LHS,

$$(m+1)^2\gamma_{m+1} - m^2\gamma_m$$

$$= \sum_{n=1}^{m+1}\frac{\cos\theta_n}{r_n}(m^2 + 2m+1 - D^2 r_n^2)$$

$$- \sum_{n=1}^{m}\frac{\cos\theta_n}{r_n}(m^2 - D^2 r_n^2)$$

$$= \sum_{n=1}^{m}\frac{\cos\theta_n}{r_n}m^2 - \sum_{n=1}^{m}\frac{\cos\theta_n}{r_n}m^2 + \frac{\cos\theta_{m+1}}{r_{m+1}}m^2$$

$$+ \sum_{n=1}^{m+1}\frac{\cos\theta_n}{r_n}(2m+1 - D^2 r_n^2)$$

$$+ \sum_{n=1}^{m}\frac{\cos\theta_n}{r_n}D^2 r_n^2$$

$$= (2m+1)\sum_{n=1}^{m}\frac{\cos\theta_n}{r_n} + \frac{\cos\theta_{m+1}}{r_{m+1}}(2m+1) + \frac{\cos\theta_{m+1}}{r_{m+1}}m^2$$

$$- \sum_{n=1}^{m} \frac{\cos\theta_n}{r_n} D^2 r_n^2 + \sum_{n=1}^{m} \frac{\cos\theta_n}{r_n} D^2 r_n^2 - \frac{\cos\theta_{m+1}}{r_{m+1}} D^2 r_n^2$$

$$= (2m+1) \sum_{n=1}^{m} \frac{\cos\theta_n}{r_n}$$

$$+ \frac{\cos\theta_{m+1}}{r_{m+1}} (m^2 + 2m+1 - D^2 r_n^2) .]$$

<u>Step 10</u>: Write

$$\sum_{n=1}^{m} \frac{\cos\theta_n}{r_n} = \frac{(m+1)^2 \gamma_{m+1} - m^2 \gamma_m}{2m+1} + A_m,$$

where

$$A_m = - \frac{\frac{\cos\theta_{m+1}}{r_{m+1}} ((m+1)^2 - D^2 r_{m+1}^2)}{2m+1} .$$

Claim:

$$\lim_{m \to \infty} A_m = 0.$$

[Take absolute values:

$$|A_m| = \left|\frac{\cos\theta_{m+1}}{r_{m+1}} \cdot \frac{1}{2m+1} \cdot ((m+1)^2 - D^2 r_{m+1}^2)\right|$$

$$\leq \frac{1}{r_{m+1}} \left| \frac{1}{2m+1} (m^2 + 2m+1 - D^2 r_{m+1}^2) \right|$$

$$= \left|\frac{m^2}{2m+1} \frac{1}{r_{m+1}} + \frac{1}{r_{m+1}} - \frac{D^2 r_{m+1}}{2m+1}\right| .$$

- $$\frac{m^2}{2m+1} \frac{1}{r_{m+1}} = \frac{m^2}{2m+1} \frac{1}{m+1} \frac{m+1}{r_{m+1}} \to \frac{D}{2} \ (m \to \infty) .$$

•

$$\frac{1}{r_{m+1}} = \frac{1}{m+1}\,\frac{m+1}{r_{m+1}}$$

$$\to OD = 0 \quad (m \to \infty).$$

•

$$-\frac{D^2 r_{m+1}}{2m+1} = -D^2\,\frac{r_{m+1}}{m+1}\,\frac{m+1}{2m+1}$$

$$\to -D^2\,\frac{1}{D}\,\frac{1}{2} = -\frac{D}{2} \quad (m \to \infty).$$

Step 11: Form

$$\frac{1}{p}\sum_{m=1}^{p}\left(\sum_{n=1}^{m}\frac{\cos\theta_n}{r_n}\right)$$

$$= \frac{1}{p}\sum_{m=1}^{p}\left(\frac{(m+1)^2\gamma_{m+1} - m^2\gamma_m}{2m+1} + A_m\right)$$

$$= \frac{1}{p}\left(-\frac{\gamma_1}{3} + \sum_{m=2}^{p}\frac{2m^2}{4m^2-1}\gamma_m + \frac{(p+1)^2}{2p+1}\gamma_{p+1} + \sum_{m=1}^{p} A_m\right)$$

$$= \frac{1}{p}\left(-\gamma_1 + \sum_{m=1}^{p}\frac{2m^2}{4m^2-1}\gamma_m + \frac{(p+1)^2}{2p+1}\gamma_{p+1} + \sum_{m=1}^{p} A_m\right).$$

Step 12: The series

$$\sum_{n=1}^{\infty}\frac{\cos\theta_n}{r_n}$$

is (C,1) summable to - Re f'(0), hence the series

$$\sum_{n=1}^{\infty}\frac{\cos\theta_n}{r_n}$$

is convergent to - Re f'(0) (cf. 27.35).

[Let $p \to \infty$ in the expression above and see what happens. First, $-\frac{\gamma_1}{p} \to 0$ $(p \to \infty)$. Second,

$$\left[\begin{array}{l} \gamma_m \to -\operatorname{Re} f'(0) \quad (m \to \infty) \\ \\ \dfrac{2m^2}{4m^2-1} \to \dfrac{1}{2} \quad (m \to \infty) \end{array}\right.$$

=>

$$\frac{1}{p} \sum_{m=1}^{p} \frac{2m^2}{4m^2-1} \gamma_m \to -\frac{1}{2} \operatorname{Re} f'(0) \quad (p \to \infty) \qquad (\text{cf. } 27.34).$$

Third,

$$\frac{1}{p} \frac{(p+1)^2}{2p+1} \gamma_{p+1} \to -\frac{1}{2} \operatorname{Re} f'(0) \quad (p \to \infty).$$

Fourth,

$$\frac{1}{p} \sum_{m=1}^{p} A_m \to 0 \quad (p \to \infty) \qquad (\text{cf. } 27.33).]$$

This completes the proof of 27.31 which, as a bonus, serves to establish that

$$\sum_{n=1}^{\infty} \frac{\cos \theta_n}{r_n} = -\operatorname{Re} f'(0) \quad (f(0) = 1).$$

On the other hand, the series

$$\sum_{n=1}^{\infty} \frac{\sin \theta_n}{r_n}$$

is absolutely convergent (cf. 27.10), thus is convergent, the only new wrinkle being that

$$\frac{1}{\pi} \int_0^{2\pi} h_f(e^{\sqrt{-1}\,\theta}) \sin \theta \, d\theta$$

$$= \frac{1}{\pi} \int_0^{2\pi} (a|\sin\theta| + b\sin\theta)\sin\theta \, d\theta$$

is equal to

$$b = \frac{h_f(\sqrt{-1}) - h_f(-\sqrt{-1})}{2} \equiv b_f$$

and this might not vanish (cf. 27.25). The upshot, therefore, is that

$$\sum_{n=1}^{\infty} \frac{\sin\theta_n}{r_n} = \text{Im } f'(0) + b_f \quad (f(0) = 1).$$

27.38 SCHOLIUM If $f(0) = 1$ and $b_f = 0$, then

$$\sum_{n=1}^{\infty} \frac{1}{z_n} = \sum_{n=1}^{\infty} \frac{\cos\theta_n}{r_n} - \sqrt{-1} \sum_{n=1}^{\infty} \frac{\sin\theta_n}{r_n}$$

$$= - \text{Re } f'(0) - \sqrt{-1}\, f'(0)$$

$$= - f'(0).$$

[Note: When $f(0) \neq 1$ (but $f(0) \neq 0$), the formula becomes

$$\sum_{n=1}^{\infty} \frac{1}{z_n} = - \frac{f'(0)}{f(0)} \text{ .]}$$

27.39 REMARK Write

$$f(z) = f(0)e^{cz} \prod_{n=1}^{\infty} (1 - \frac{z}{z_n})e^{z/z_n}.$$

Then

$$c = - \frac{f'(0)}{r(0)}$$

and

$$f(z) = f(0) \lim_{R \to \infty} \prod_{|z_n|<R} (1 - \frac{z}{z_n}),$$

the convergence of the product being conditional.

27.40 EXAMPLE Take

$$f(z) = \frac{(e^{\sqrt{-1}\, z} - 1)(e^{-\sqrt{-1}\, z} + \sqrt{-1})}{\sqrt{-1}\, z}$$

Then

$$f(0) = \sqrt{-1} + 1, \; f'(0) = \frac{(\sqrt{-1} - 1)}{2} \sqrt{-1}$$

$$\Rightarrow \frac{f'(0)}{f(0)} = -\frac{1}{2}$$

and the theory predicts that

$$\sum_{n=1}^{\infty} \frac{\cos \theta_n}{r_n} = \frac{1}{2} .$$

To establish this, note that the zeros of $f(z)$ are at

$$\pm 2\pi, \; \pm 4\pi, \ldots$$

and at

$$\frac{\pi}{2}, -\frac{3\pi}{2}, \frac{5\pi}{2}, -\frac{7\pi}{2}, \ldots .$$

Those of the first kind make no contribution (since the corresponding terms of the series cancel in pairs) but there is a contribution from those of the second kind, viz.

$$\frac{2}{\pi} (1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots) = \frac{1}{2} .$$

[Note: As regards

$$\sum_{n=1}^{\infty} \frac{\sin \theta_n}{r_n},$$

it is clear that $\sin \theta_n = 0 \ \forall \ n$. To see that here $b_f = 0$, work on [- 1,1] and let

$$\phi(t) = \begin{cases} 1 & (-1 \leq t \leq 0) \\ \sqrt{-1} & (0 < t \leq 1). \end{cases}$$

Then

$$f(z) = \int_{-1}^{1} \phi(t) e^{\sqrt{-1}\, zt} dt,$$

hence

$$\begin{cases} 1 = h_f(-\sqrt{-1}) \\ -1 = -h_f(\sqrt{-1}) \end{cases} \qquad \text{(cf. 27.26)}$$

=>

$$b_f = \frac{1-1}{2} = 0.]$$

§28. ZERO THEORY: PALEY-WIENER FUNCTIONS

Recall that $PW(A)$ is the subset of $E_0(A)$ consisting of those f such that $f|R \in L^2(-\infty,\infty)$ (cf. 22.1).

28.1 EXAMPLE Take $A = \pi$ -- then

$$(1 - \frac{\sin \pi z}{\pi z})/(\pi z)^2 \in PW(\pi)$$

has no real zeros.

28.2 EXAMPLE Take $A = \pi$ -- then

$$(1 - \frac{\sin \pi z}{\pi z})/\pi z \in PW(\pi)$$

has exactly one real zero.

28.3 EXAMPLE Take $A = 1$ -- then

$$\frac{e^{\sqrt{-1}\, z} - 1}{z} \in PW(1)$$

and has infinitely many real zeros.

28.4 RAPPEL The elements $f \in PW(A)$ have the form

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \phi(t) e^{\sqrt{-1}\, zt} dt \quad (0 < A < \infty)$$

for some $\phi \in L^2[-A,A]$ (cf. 22.7).

[Note: The prescription

$$\phi(t) = \lim_{R \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} f(x) e^{-\sqrt{-1}\, tx} dx \quad (L^2)$$

computes ϕ in terms of f.]

28.5 DEFINITION Suppose that $f \in PW(A)$ -- then f is called a <u>band-pass</u> function if there exists an interval $[-B,B]$ $(0 < B < A)$ in which $\phi = 0$ almost everywhere.

28.6 LEMMA If $f \not\equiv 0$ is a real integrable band-pass function, then f has at least one real zero.

PROOF Take $\phi \equiv 0$ in $[-B,B]$, hence $\int_{-\infty}^{\infty} f(x)dx = 0$, so f must change sign somewhere in R.

More is true.

28.7 THEOREM If $f \not\equiv 0$ is a real band-pass function, then f has infinitely many real zeros.

[The point of departure is the following observation: $\forall\, g \in PW(B)$ $(\subset PW(A))$,

$$\langle g,f\rangle = \langle \psi,\phi\rangle,$$

where

$$g(z) = \frac{1}{\sqrt{2\pi}} \int_{-B}^{B} \psi(t)e^{\sqrt{-1}\, zt}dt.$$

With this in mind, assume that f has but finitely many real zeros. One then arrives at a contradiction by exhibiting a real $g \in PW(B)$ such that $\langle g,f\rangle \neq 0$.

- $f(x)$ is of constant sign: Take

$$g(z) = \left(\frac{1}{z} \sin\left(\frac{B}{2} z\right)\right)^2.$$

- $f(x)$ is not of constant sign, thus has zeros of odd order, say $x_1,\ldots,x_n$ (these are the zeros at which f changes sign). Now construct a real $g \in PW(B)$ whose real zeros are precisely the x_k $(k = 1,\ldots,n)$, each x_k being of order 1 (per g). Therefore $g(x)f(x) \geq 0 \ \forall \ x$ or $g(x)f(x) \leq 0 \ \forall \ x$, so $\langle g,f\rangle \neq 0$.]

28.8 RAPPEL Let f be a continuously differentiable complex valued function on [a,b]. Assume: $f(a) = f(b) = 0$ -- then

$$\int_a^b |f(x)|^2 dx \leq \left(\frac{b-a}{\pi}\right)^2 \int_a^b |f'(x)|^2 dx$$

with equality iff

$$f(x) = C \sin\left(\pi \frac{x-a}{b-a}\right).$$

[This is known as Wirtinger's inequality[†].]

28.9 THEOREM Let $f \in PW(A)$ be nonzero -- then $|f| > 0$ on at least one open interval of the real axis of length $> \frac{\pi}{A}$.

PROOF One need only consider the situation when f has infinitely many real zeros. So suppose that $a < b$ are two consecutive zeros of f and that, moreover, $b - a \leq \frac{\pi}{A}$. Since f is not a sine function on any interval,

$$\int_a^b |f(x)|^2 dx < \left(\frac{b-a}{\pi}\right)^2 \int_a^b |f'(x)|^2 dx$$

$$\leq \left(\frac{1}{A}\right)^2 \int_a^b |f'(x)|^2 dx,$$

which implies by addition that

$$||f||_2 < \frac{1}{A} ||f'||_2.$$

But

$$||f'||_2 \leq ||f||_2 \, T(f) \qquad (\text{cf. } 17.31).$$

[†] G. Folland, *Real Analysis*, Wiley-Interscience, 1984, p. 247.

Therefore

$$||f||_2 < \frac{T(f)}{A} ||f||_2$$

$$\Rightarrow$$

$$A < T(f),$$

a contradiction.

28.10 EXAMPLE The Paley-Wiener function

$$\frac{\sin Ax}{Ax}$$

has just one zero free open interval of length > $\frac{\pi}{A}$, namely $]-\frac{\pi}{A}, \frac{\pi}{A}[$.

§29. *INTERMEZZO*

Given $\phi \in L^1[a,b]$, let

$$f(z) = \int_a^b \phi(t)e^{\sqrt{-1}\, zt}dt.$$

Then f(z) is a Bernoulli function and subject to suitable restrictions on ϕ, the overall program is to study the position of the zeros of f(z).

N.B. It is sometimes convenient to "normalize" the interval and take [a,b] = [0,1] or [a,b] = [- 1,1].

- Thus

$$\int_a^b \phi(t)e^{\sqrt{-1}\, zt}dt$$

$$= (b-a)e^{\sqrt{-1}\, az} \int_0^1 \phi(a + (b-a)t)e^{\sqrt{-1}(b-a)zt}dt.$$

- Thus

$$\int_a^b \phi(t)e^{\sqrt{-1}\, zt}dt$$

$$= \frac{1}{2}(b-a)e^{\frac{1}{2}(a+b)\sqrt{-1}\, z} \int_{-1}^1 \phi(\frac{1}{2}(b+a) + \frac{1}{2}(b-a)t)e^{\frac{1}{2}(b-a)\sqrt{-1}\, zt}dt.$$

The theory developed in §27 is applicable under the following conditions.

- Assume: $f(0) \neq 0$.

[Note: Nothing of substance is lost in so doing. For if f(0) = 0, then

$$\frac{f(z)}{z} = -\sqrt{-1}\int_a^b \psi(t)e^{\sqrt{-1}\, zt}dt,$$

where

$$\psi(t) = \int_a^t f(s)ds.]$$

- Assume: There is no $\alpha > a$ such that

$$\int_a^\alpha |\phi(t)|dt = 0$$

and there is no $\beta < b$ such that

$$\int_\beta^b |\phi(t)|dt = 0.$$

[Note: Accordingly,

$$a = -h_f(\sqrt{-1}),\ b = h_f(-\sqrt{-1}),$$

and

$$T(f) = \max(h_f(\sqrt{-1}),\ h_f(-\sqrt{-1})).]$$

Therefore in review:

1. $\lim\limits_{r \to \infty} \frac{n(r)}{r} = \frac{b-a}{\pi} \equiv D > 0.$

2. $\sum\limits_{n=1}^{\infty} \frac{\sin \theta_n}{r_n}$ is absolutely convergent and has sum

$$\text{Im}\,\frac{f'(0)}{f(0)} - \frac{(a+b)}{2}\ .$$

3. $\sum\limits_{n=1}^{\infty} \frac{\cos \theta_n}{r_n}$ is conditionally convergent and has sum

$$-\ \text{Re}\,\frac{f'(0)}{f(0)}\ .$$

N.B. Matters simplify if $a = -A$, $b = A$.

29.1 EXAMPLE The zeros of $f(z)$ which lie on the imaginary axis constitute a "thin" set (if there are any at all) (cf. 27.11). Still, their number may be infinite.

[Working on [0,1], choose constants $0 < \mu < \frac{1}{2}$, $\nu > 2$, and put $\alpha = \nu/\mu$.

Define $\phi \in L^1[0,1]$ by letting

$$\phi(t) = (-\alpha)^k e^{-\nu^k} (\mu^k - \alpha^{-k} < t \le \mu^k) \quad (k = 1,2,\ldots)$$

and taking $\phi(t) = 0$ elsewhere on $[0,1]$. Given any positive integer n, we have

$$\left|\int_0^{\mu^{n+1}} \phi(t)e^{-\alpha^n t}dt\right|$$

$$\le \int_0^{\mu^{n+1}} |\phi(t)|dt$$

$$= \sum_{k=n+1}^{\infty} e^{-\nu^k}$$

$$< e^{-\nu^{n+1}} \sum_{j=0}^{\infty} e^{-\nu^j}$$

$$= e^{-\nu^{n+1}} \int_0^1 |\phi(t)|dt$$

and

$$\left|\int_{\mu^{n-1}-\alpha^{-n+1}}^{1} \phi(t)e^{-\alpha^n t}dt\right|$$

$$\le e^{-\alpha^n(\mu^{n-1}-\alpha^{-n+1})} \int_0^1 |\phi(t)|dt$$

$$= e^{-\nu^n/\mu+\alpha} \int_0^1 |\phi(t)|dt$$

and

$$\int_{\mu^n-\alpha^{-n}}^{\mu^n} \phi(t)e^{-\alpha^n t}dt$$

$$= (-1)^n(e-1)e^{-2\nu^n}.$$

Therefore

$$\left| e^{2\nu^n} \int_0^1 \phi(t) e^{-\alpha^n t} dt - (e-1)(-1)^n \right|$$

$$< (e^{\nu^n(2-\nu)} + e^{\nu^n(2-1/\mu)+\alpha}) \int_0^1 |\phi(t)| dt.$$

So for $n >> 0$,

$$\operatorname{sgn} \int_0^1 \phi(t) e^{-\alpha^n t} dt = \operatorname{sgn} (-1)^n,$$

thus at some $x_0 : - \alpha^{n+1} \leq x_0 \leq - \alpha^n$,

$$\int_0^1 \phi(t) e^{x_0 t} dt = 0$$

or still,

$$f\left(\frac{x_0}{\sqrt{-1}}\right) = 0.]$$

29.2 NOTATION Let

$$F(z) = \int_a^b \phi(t) e^{zt} dt.$$

Then

$$f(z) = F(\sqrt{-1}\, z).$$

29.3 LEMMA Take $[a,b] = [-1,1]$ -- then

$$F(re^{\sqrt{-1}\,\theta}) = o(e^{r|\cos\theta|}) \quad (r \to \infty)$$

uniformly with respect to θ.

PROOF Assume first that $\theta = 0$ and write

$$|F(r)| = \left|\int_{-1}^{1} \phi(t)e^{rt}dt\right|$$

$$= \left|\int_{-1}^{1-\delta} \phi(t)e^{rt}dt + \int_{1-\delta}^{1} \phi(t)e^{rt}dt\right|$$

$$\le e^{(1-\delta)r} \int_{-1}^{1-\delta} |\phi(t)|dt + e^{r} \int_{1-\delta}^{1} |\phi(t)|dt.$$

Given $\varepsilon > 0$, choose $\delta > 0$:

$$\int_{1-\delta}^{1} |\phi(t)|dt < \frac{\varepsilon}{2}$$

and then choose $r_0 >> 0$:

$$e^{-\delta r} \int_{-1}^{1-\delta} |\phi(t)|dt < \frac{\varepsilon}{2} \quad (r > r_0).$$

Therefore

$$|F(r)| < \varepsilon e^{r} \quad (r > r_0).$$

I.e.: $F(r) = o(e^{r})$ $(\cos 0 = 1)$. Next

$$F(\sqrt{-1}\,x) = \int_{-1}^{1} \phi(t)\cos xt\, dt$$

$$+ \sqrt{-1} \int_{-1}^{1} \phi(t)\sin xt\, dt$$

and the two integrals on the right approach 0 as $x \to \infty$ (Riemann-Lebesgue lemma). These facts, in conjunction with Phragmén-Lindelöf, then imply that the function $e^{-z}F(z)$ tends uniformly to zero in the sector $0 \le \theta \le \frac{\pi}{2}$ which gives the result in this range. And so on... .

29.4 RAPPEL If ϕ is absolutely continuous on [a,b], then its derivative ϕ' exists almost everywhere. Moreover, $\phi' \in L^1[a,b]$ and

$$\phi(t) = \phi(a) + \int_a^t \phi'(s)ds \quad (a \leq t \leq b).$$

29.5 THEOREM Take [a,b] = [- 1,1] and assume that ϕ is absolutely continuous with $\phi(1) = \phi(-1) = 1$ -- then the zeros of f(z) are determined asymptotically by the formula

$$z = \pm m\pi + \varepsilon_m,$$

where m is a positive integer and $\varepsilon_m \to 0$ $(m \to \infty)$.

PROOF We shall work instead with F(z), thereby shifting the claim to $\pm m\pi\sqrt{-1} + \varepsilon_m$. So $\forall\ z \neq 0$, integrate by parts and write

$$F(z) = \frac{e^z - e^{-z}}{z} - \frac{1}{z}\int_{-1}^1 \phi'(t)e^{zt}dt$$

or still,

$$zF(z) = e^z - e^{-z} - \int_{-1}^1 \phi'(t)e^{zt}dt,$$

a relation that is valid $\forall$ z. Since ϕ' is integrable, 29.3 is applicable (replace the ϕ there by ϕ'), hence

$$\int_{-1}^1 \phi'(t)e^{zt}dt = o(e^{r|\cos\theta|}) \quad (r \to \infty)$$

uniformly with respect to θ. If generically, ε_r is a function of r and θ which tends to 0 uniformly in θ as $r \to \infty$, then at a zero of F(z),

$$e^z(1 + \varepsilon_r) = e^{-z}(1 + \varepsilon_r)$$

=>

$$e^{2z} = 1 + \varepsilon_r$$

=>

$$2z = \pm\ 2m\pi \sqrt{-1} + \varepsilon_m$$

=>

$$z = \pm\ m\pi \sqrt{-1} + \varepsilon_m.$$

To reverse this, note that sinh z has exactly one zero at each point $\pm\ m\pi \sqrt{-1}$. Choosing $\delta > 0$ small, surround each of these points by a circle of radius δ, thus on the circle

$$|\sinh z| > K(\delta) > 0$$

and

$$zF(z) = \sinh z\ (1 + \varepsilon_m),$$

where $\varepsilon_m > 0$ $(m > \infty)$. So for large m, zF(z) has the same number of zeros inside the circle as sinh z, i.e., one.

29.6 REMARK The supposition that $\phi(1) = \phi(-1) = 1$ is not unduly restrictive at least if $\phi(1)$, $\phi(-1)$ are real and positive: Consider

$$\psi(t) = \left[\frac{\phi(-1)}{\phi(1)} \right]^{t/2} \frac{\phi(t)}{\sqrt{\phi(1)\phi(-1)}}$$

and define w by the relation

$$z = w + \frac{1}{2} \log \frac{\phi(-1)}{\phi(1)}\ .$$

Then

$$f(z) = \sqrt{\phi(1)\phi(-1)} \int_{-1}^{1} \psi(t) e^{wt} dt$$

$$\equiv \sqrt{\phi(1)\phi(-1)}\ g(w)$$

and ψ is absolutely continuous with $\psi(1) = \psi(-1) = 1$.

29.7 EXAMPLE The situation can be different if $\phi(-1) = 0$ and $\phi(1) = 0$. To see this, let

$$\phi(t) = \begin{cases} 1 - t & (0 < t \le 1) \\ 1 + t & (-1 \le t \le 0). \end{cases}$$

Then

$$\phi(t) = \int_{-1}^{t} \phi'(s)ds$$

is absolutely continuous and

$$F(z) = \frac{4 \sinh^2(\frac{z}{2})}{z^2} .$$

However, the zeros are at the points $\pm\ 2m\pi \sqrt{-1}$, hence the pattern has changed.

29.8 THEOREM Take $[a,b] = [-1,1]$ and assume that ϕ is of bounded variation and continuous at 1 and -1 with $\phi(1) = \phi(-1) = 1$ -- then the zeros of $f(z)$ lie within a horizontal strip $|\text{Im } z| \le C$.

PROOF An equivalent assertion is that the zeros of $F(z)$ lie within a vertical strip $|\text{Re } z| \le C$. Thus let $\text{Re } z = x > 0$, and for $\delta > 0$ small, write

$$zF(z) = e^z - e^{-z} - \int_{-1}^{1-\delta} e^{zt}d\phi - \int_{1-\delta}^{1} e^{zt}d\phi.$$

Then

$$\left|\int_{-1}^{1-\delta} e^{zt}d\phi\right|$$

$$\leq e^{x(1-\delta)} \int_{-1}^{1-\delta} |d\phi|$$

$$< Ke^{x(1-\delta)}$$

and

$$\left|\int_{1-\delta}^{1} e^{zt} d\phi\right|$$

$$\leq e^{x} \max_{1-\delta<t_1<t_2\leq 1} |\phi(t_2) - \phi(t_1)|$$

$$= e^{x} M(\delta).$$

Therefore

$$|zF(z)| \geq e^{x}(1 - e^{-2x} - Ke^{-\delta x} - M(\delta)).$$

Bearing in mind that $\phi(t)$ is continuous at $t = 1$, choose δ so small that $M(\delta) < \frac{1}{4}$. This done, choose x so large that

$$e^{-2x} + Ke^{-\delta x} < \frac{1}{4}.$$

Then

$$e^{x}(1 - e^{-2x} - Ke^{-\delta x} - M(\delta)) > e^{x}(1 - \frac{1}{2})$$

$$= \frac{e^{x}}{2} > 0.$$

Consequently, for $x > > 0$, $F(z)$ has no zeros. And, analogously, for $x < < 0$, $F(z)$ has no zeros.

29.9 REMARK The result goes through if the assumption on ϕ at the endpoints is weakened to $\phi(1^{-}) \neq 0$, $\phi(-1^{+}) \neq 0$.

29.10 EXAMPLE Let ϕ be defined on $]0,1[$. Suppose that ϕ is positive and increasing and

$$\begin{cases} \phi(1^-) < \infty \\ \phi(0^+) > 0. \end{cases}$$

Then ϕ can be extended to a function of bounded variation on [0,1]. Taking [a,b] = [0,1], write

$$\int_0^1 \phi(t)e^{\sqrt{-1}\, zt}dt$$

$$= \frac{1}{2}\, e^{\frac{1}{2}\sqrt{-1}\, z} \cdot \int_{-1}^{1} \phi\left(\frac{1+t}{2}\right) e^{\frac{1}{2}\sqrt{-1}\, zt} dt$$

to conclude that the zeros of f(z) lie within a horizontal strip $|\text{Im } z| \leq C$.

29.11 RAPPEL Suppose that $\phi \in C[a,b]$. Given $\delta > 0$, let $\omega(\delta)$ be the supremum of $|\phi(t_2) - \phi(t_1)|$ computed over all points t_1, t_2 in [a,b] such that $|t_2 - t_1| < \delta$ -- then $\omega(\delta)$ is called the <u>modulus of continuity</u> of ϕ. As a function of δ, ω is continuous and increasing and $\lim_{\delta \to 0} \omega(\delta) = 0$. In addition, $\omega(\delta) \geq A\delta$ for some $A > 0$ provided ϕ is not a constant.

29.12 THEOREM Take [a,b] = [- 1,1] and let $\phi \in C[-1,1]$, where $\phi(\pm 1) = 1$ -- then all the zeros of

$$F(z) = \int_{-1}^{1} \phi(t)e^{zt}dt$$

which are sufficiently large in modulus lie in the set

$$|x| \leq Kr\omega\left(\frac{1}{r}\right) \quad (x = \text{Re } z,\ r = |z|).$$

PROOF It can be assumed that ϕ is not a constant (since otherwise F(z) is proportional to $\frac{\sinh z}{z}$ and there is nothing to prove). Proceeding, subdivide [- 1,1] into 2m equal parts and write

$$\phi(t) = \phi(\tfrac{j}{m}) - \psi_j(t) \quad (\tfrac{j-1}{m} \le t \le \tfrac{j}{m}).$$

Then

$$|\psi_j(t)| \le \omega(\tfrac{1}{m}).$$

There are now two cases: $x > 0$ or $x < 0$, and it will be enough to consider the first of these. To begin with,

$$F(z) = \sum_{j=-m+1}^{m} \int_{(j-1)/m}^{j/m} (\phi(\tfrac{j}{m}) - \psi_j(t))e^{zt}dt$$

$$= \sum_{j=-m+1}^{m} \phi(\tfrac{j}{m}) \int_{(j-1)/m}^{j/m} e^{zt}dt - \sum_{j=-m+1}^{m} \int_{(j-1)/m}^{j/m} \psi_j(t)e^{zt}dt$$

$$= I_1 + I_2.$$

- $$|I_2| \le \sum_{j=-m+1}^{m} \int_{(j-1)/m}^{j/m} e^{xt}\omega(\tfrac{1}{m})dt$$

$$= \omega(\tfrac{1}{m}) \int_{-1}^{1} e^{xt}dt$$

$$= \omega(\tfrac{1}{m}) \frac{e^x - e^{-x}}{x}.$$

- $$I_1 = \sum_{j=0}^{2m-1} \phi(1 - \tfrac{j}{m}) \frac{e^{z(1-j/m)} - e^{z(1-(j+1)/m)}}{z}$$

$$= \frac{e^z}{z} + \frac{e^z}{z}\sum_{j=1}^{2m-1} \phi(1 - \frac{j}{m})(e^{-zj/m} - e^{-z(j+1)/m}) - \frac{e^z}{z} e^{-z/m}$$

$$= \frac{e^z}{z} + \frac{e^z}{z}\sum_{j=1}^{2m-1} (\phi(1 - \frac{j}{m}) - \phi(1 - \frac{j-1}{m}))e^{-zj/m} - \phi(-1 + \frac{1}{m}) \frac{e^{-z}}{z}$$

$$= \frac{e^z}{z} + \frac{e^z}{z} I_3 - \phi(-1 + \frac{1}{m}) \frac{e^{-z}}{z} .$$

•

$$|I_3| \le \sum_{j=1}^{\infty} \omega(\frac{1}{m}) e^{-jx/m}$$

$$= \omega(\frac{1}{m}) \frac{e^{-x/m}}{1 - e^{-x/m}}$$

$$\le \omega(\frac{1}{m}) \frac{m}{x} .$$

[Note: For $\alpha > 0$,

$$1 + \alpha \le e^{\alpha} \Rightarrow \alpha \le e^{\alpha} - 1$$

$$\Rightarrow \alpha \le \frac{1 - e^{-\alpha}}{e^{-\alpha}}$$

$$\Rightarrow \alpha e^{-\alpha} \le 1 - e^{-\alpha}$$

$$\Rightarrow \frac{e^{-\alpha}}{1-e^{-\alpha}} \le \frac{1}{\alpha} .]$$

Setting $m = [r]$, we have

$$\omega(\frac{1}{[r]}) \le 2\omega(\frac{1}{r}) \quad (r >> 0).$$

Therefore

$$zF(z) = zI_1 + zI_2$$

$$= z\left(\frac{e^z}{z} + \frac{e^z}{z} I_3 - \phi\left(-1 + \frac{1}{[r]}\right) \frac{e^{-z}}{z}\right) + zI_2$$

$$= e^z\left(1 + I_3 - \phi\left(-1 + \frac{1}{[r]}\right) e^{-2z}\right) + zI_2$$

$$= e^z\left(1 + O\left(\frac{r\omega(1/r)}{x}\right) - (1 + o(1))e^{-2z}\right) + zI_2,$$

where $o(1) \to 0$ $(r \to \infty)$. Next

$$zI_2 = e^z e^{-z} zI_2.$$

And

$$|e^{-z}zI_2| \le e^{-x}r|I_2|$$

$$\le e^{-x}r\omega\left(\frac{1}{[r]}\right) \frac{e^x - e^{-x}}{x}$$

$$\le 2r\omega\left(\frac{1}{r}\right) \frac{1 - e^{-2x}}{x}$$

$$= O\left(\frac{r\omega(1/r)}{x}\right).$$

So in summary: $\forall\ r >> 0$,

$$zF(z) = e^z\left(1 + O\left(\frac{r\omega(1/r)}{x}\right) - (1 + o(1))e^{-2z}\right).$$

If $K > 0$ and if $x > Kr\omega(\frac{1}{r})$, then $x > AK$ (cf. 29.11), thus if K is sufficiently large

$$\left|O\left(\frac{r\omega(1/r)}{x}\right) - (1 + o(1))e^{-2z}\right| \le \frac{1}{2} \quad (r >> 0).$$

But this implies that

$$1 + O\left(\frac{r\omega(1/r)}{x}\right) - (1 + o(1))e^{-2z}$$

is bounded away from 0, hence $F(z)$ does not vanish in the region $x > Kr\omega(\frac{1}{r})$.

29.13 REMARK The condition $\phi(\pm 1) = 1$ can be replaced by the condition $\phi(\pm 1) \neq 0$.

29.14 DEFINITION A step function ϕ on $[0,1]$ of the form

$$\phi(t) = c_j \quad (t_j < t < t_{j+1}),$$

where

$$0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$$

and

$$0 < c_0 < c_1 < \cdots < c_n,$$

is said to be exceptional if the t_j are rational numbers.

29.15 NOTATION Write $E(1,0)$ for the set of exceptional step functions on $[0,1]$.

29.16 THEOREM If $\phi \in L^1[0,1]$ is positive and increasing on $]0,1[$ and if $\phi \notin E(1,0)$, then the zeros of $f(z)$ lie in the open upper half-plane.

[We shall postpone the proof until later (cf. 34.2).]

[Note: In terms of $F(z)$, the conclusion is that its zeros lie in the open left half-plane.]

29.17 EXAMPLE The zeros of the real entire function

$$z \rightarrow \int_0^z e^{-t^2} dt$$

with the exception of $z = 0$ lie inside the region $\text{Re } z^2 < 0$ (a spiral in the complex plane).

[Write

$$\int_0^z e^{-t^2} dt = \frac{z}{2} \int_0^1 \frac{1}{\sqrt{t}} e^{-z^2 t} dt$$

$$= \frac{z}{2} \int_0^1 \frac{1}{\sqrt{1-t}} e^{-z^2(1-t)} dt$$

$$= \frac{z}{2} e^{-z^2} \int_0^1 \frac{1}{\sqrt{1-t}} e^{z^2 t} dt.]$$

[Note: The <u>error function</u> is defined by

$$\text{erf } z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

and the <u>complementary error function</u> is defined by

$$\text{erf}_c\, z = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt.$$

Therefore

$$\text{erf } z + \text{erf}_c\, z = 1.$$

The <u>Fresnel integrals</u> are defined by

$$\begin{cases} C(z) = \int_0^z \cos(\frac{\pi}{2} t^2) dt \\ S(z) = \int_0^z \sin(\frac{\pi}{2} t^2) dt. \end{cases}$$

Accordingly, in terms of the error function,

$$C(z) + \sqrt{-1}\, S(z) = \frac{1 + \sqrt{-1}}{2} \text{erf}\left(\frac{\sqrt{\pi}}{2} (1 - \sqrt{-1}) z\right).]$$

Consider a step function ϕ per 29.14 -- then

$$f(z) = \sum_{j=0}^{n} c_j \int_{t_j}^{t_j+1} e^{\sqrt{-1}\, zt} dt \quad (\Rightarrow f(0) > 0)$$

$\Rightarrow$

$$\sqrt{-1}\, zf(z) = c_0(e^{\sqrt{-1}\, zt_1} - e^{\sqrt{-1}\, zt_0}) + c_1(e^{\sqrt{-1}\, zt_2} - e^{\sqrt{-1}\, zt_1})$$

$$+ \cdots + c_n(e^{\sqrt{-1}\, zt_{n+1}} - e^{\sqrt{-1}\, zt_n})$$

$$= c_n e^{\sqrt{-1}\, z} - c_0 - e^{\sqrt{-1}\, zt_1}(c_1 - c_0) - \cdots - e^{\sqrt{-1}\, zt_n}(c_n - c_{n-1})$$

$\Rightarrow$

$$|\sqrt{-1}\, xf(x)| \geq c_n - c_0 - (c_1 - c_0) - \cdots - (c_n - c_{n-1}) = 0.$$

29.18 LEMMA If for some $x \neq 0$,

$$|\sqrt{-1}\, xf(x)| = 0,$$

then $\phi \in E(1,0)$.

PROOF The assumption implies that

$$e^{\sqrt{-1}\, x} = 1,\ e^{\sqrt{-1}\, xt_1} = 1, \ldots,\ e^{\sqrt{-1}\, xt_n} = 1,$$

from which the existence of integers q, $p_1, \ldots, p_n$ such that

$$x = 2\pi q,\ xt_1 = 2\pi p_1, \ldots, xt_n = 2\pi p_n,$$

so

$$t_j = \frac{p_j}{q}.$$

And this shows that $\phi \in E(1,0)$.

[Note: If x is positive, then q and the p_j are positive but if x is negative, then q and the p_j are negative and we write

$$t_j = \frac{-p_j}{-q}\ .]$$

If ϕ is a step function and if $\phi \notin E(1,0)$, then

$$x \neq 0 \Rightarrow |\sqrt{-1}\, xf(x)| > 0,$$

thus $f(z)$ has no real zeros. Now fix $y < 0$ and consider

$$f(z) = f(x + \sqrt{-1}\, y) = \int_0^1 \phi(t)e^{\sqrt{-1}\,(x + \sqrt{-1}\, y)}dt$$

$$= \int_0^1 (\phi(t)e^{-yt})e^{\sqrt{-1}\, x}dt.$$

Since y is negative, the function $\phi(t)e^{-yt}$ is positive and increasing on $]0,1[$ and it is obviously not in $E(1,0)$. Therefore, on the basis of 29.16,

$$\int_0^1 (\phi(t)e^{-yt})e^{\sqrt{-1}\, x}dt$$

does not vanish on the real axis, so $f(z)$ does not vanish on the line $\text{Im } z = y$.

29.19 SCHOLIUM If ϕ is a step function and if $\phi \notin E(1,0)$, then the zeros of $f(z)$ lie in the open upper half-plane.

[Note: This is an important point of principle: If ϕ is a step function, then it either is in $E(1,0)$ or it isn't and if it isn't, then the truth of 29.16 for those ϕ which are not step functions implies the truth of 29.16 for those step functions $\phi \notin E(1,0)$.]

29.20 LEMMA If $\phi \in E(1,0)$, then $f(z)$ has a real zero.

PROOF Let

$$t_1 = \frac{p_1}{q_1}\ (q_1 > 0),\ t_2 = \frac{p_2}{q_2}\ (q_2 > 0),\ldots,\ t_n = \frac{p_n}{q_n}\ (q_n > 0).$$

Put

$$q = q_1\cdots q_n,\ a_j = \frac{p_j q}{q_j}\ (\Rightarrow t_j = \frac{a_j}{q}\ (j = 1,\ldots,n))$$

and set $x = 2\pi q$ -- then

$$e^{\sqrt{-1}\,x} = e^{\sqrt{-1}\,2\pi q} = 1$$

and

$$e^{\sqrt{-1}\,xt_j} = e^{\sqrt{-1}\,2\pi q t_j} = e^{\sqrt{-1}\,2\pi a_j} = 1\ (j = 1,\ldots,n).$$

Therefore

$$\begin{aligned}
&\sqrt{-1}\,(2\pi q) f(2\pi q) \\
&= c_n e^{\sqrt{-1}\,2\pi q} - c_0 - e^{\sqrt{-1}\,2\pi q t_1}(c_1 - c_0) - \cdots - e^{\sqrt{-1}\,2\pi q t_n}(c_n - c_{n-1}) \\
&= c_n - c_0 - (c_1 - c_0) - \cdots - (c_n - c_{n-1}) \\
&= 0
\end{aligned}$$

$$\Rightarrow f(x) = f(2\pi q) = 0.$$

29.21 THEOREM If $\phi \in E(1,0)$, then $f(z)$ has an infinity of real zeros.

PROOF Write

$$\sqrt{-1}\,zf(z) = P(e^{\sqrt{-1}\,z/q}),$$

where P is a polynomial of degree q -- then $P(1) = 0$ (set $z = 0$), hence

$$\sqrt{-1}\,zf(z) = (e^{\sqrt{-1}\,z/q} - 1)P_1(e^{\sqrt{-1}\,z/q}).$$

Therefore

$$\pm 2\pi q,\ \pm 4\pi q, \ldots$$

are zeros of f(z).

Let $u = e^{\sqrt{-1}\, z/q}$ -- then

$$\sqrt{-1}\, zf(z) = c_0(u^{a_1} - 1) + c_1(u^{a_2} - u^{a_1}) + \cdots + c_n(u^q - u^{a_n})$$

$$= (u-1)(c_0 + c_0 u + \cdots + c_0 u^{a_1 - 1} + c_1 u^{a_1} + \cdots + c_n u^{q-1})$$

$$= (u-1)P_1(u).$$

Thanks to wellknown generalities (explicated in §30 (cf. 30.13)), the structure of the coefficients of P_1 confines the zeros of P_1 to the closed unit disk $|u| \leq 1$, thus, in terms of z:

$$|e^{\sqrt{-1}\, z/q}| \leq 1 \Rightarrow |e^{\sqrt{-1}(x + \sqrt{-1}\, y)/q}| \leq 1$$

$$\Rightarrow |e^{(\sqrt{-1}\, x - y)/q}| \leq 1 \Rightarrow e^{-y/q} \leq 1$$

$$\Rightarrow - y/q \leq 0 \Rightarrow y \leq 0.$$

[Note: Any zero of P_1 on the unit circle $|u| = 1$ is necessarily simple, so the real zeros of f(z) are simple.]

29.22 LEMMA If $\phi \in E(1,0)$, then the zeros of f(z) lie on a finite set of horizontal straight lines $\text{Im } z = b_k$ ($b_k \geq 0$, $1 \leq k \leq s$, $s \leq q$).

[In terms of the distinct roots $w_1 = 1, w_2, \ldots, w_s$ of P,

$$b_k = - q \log |w_k|.]$$

[Note: These lines are not necessarily distinct. E.g., if $w_k = \sqrt{-1}$, the associated horizontal straight line is the real axis and the zeros are situated at

$$q\,\frac{\pi}{2},\ q(\frac{\pi}{2} \pm 2\pi),\ q(\frac{\pi}{2} \pm 4\pi), \ldots\ .]$$

Here is an application of 29.16.

29.23 THEOREM If $\phi \in L^1[0,1]$ is positive and differentiable on $]0,1[$ with

$$\alpha \leq -\frac{\phi'(t)}{\phi(t)} \leq \beta \quad (0 < t < 1)$$

and if

$$\phi(t) \neq Ce^{-\alpha t}, Ce^{-\beta t},$$

then the zeros of

$$F(z) = \int_0^1 \phi(t)e^{zt}dt$$

are confined to the open strip $\alpha < \text{Re } z < \beta$.

PROOF Write

$$F(z) = \int_0^1 e^{\beta t}\phi(t)e^{(z-\beta)t}dt.$$

Then

$$\frac{d}{dt}(e^{\beta t}\phi(t)) = e^{\beta t}\phi(t)\,(\frac{\phi'(t)}{\phi(t)} + \beta) \geq 0.$$

Therefore the zeros of $F(z)$ are restricted by the relation

$$\text{Re}(z-\beta) < 0 \quad (\text{cf. } 29.16).$$

Write

$$F(z) = e^z \int_0^1 e^{-\alpha t}\phi(1-t)e^{(\alpha-z)t}dt.$$

Then

$$\frac{d}{dt}(e^{-\alpha t}\phi(1-t)) = e^{-\alpha t}\phi(1-t)\left(- \frac{\phi'(1-t)}{\phi(1-t)} - \alpha\right) \geq 0.$$

Therefore the zeros of $F(z)$ are resticted by the relation

$$\mathrm{Re}(\alpha-z) < 0 \quad (\text{cf. } 29.16).$$

But

$$\left[\begin{array}{l} \mathrm{Re}(z-\beta) < 0 \\ \\ \mathrm{Re}(\alpha-z) < 0 \end{array}\right. \Rightarrow \alpha < \mathrm{Re}\, z < \beta.$$

29.24 EXAMPLE Take $\phi(t) = \exp(-e^t)$ -- then

$$-\frac{\phi'(t)}{\phi(t)} = e^t$$

and

$$1 \leq e^t \leq e \quad (0 < t < 1).$$

Consequently, $\forall\, \varepsilon > 0$, the zeros of

$$F(z) = \int_0^1 \exp(-e^t)e^{zt}dt$$

are confined to the open strip

$$1 - \varepsilon < \mathrm{Re}\, z < e + \varepsilon$$

or still, to the closed strip

$$1 \leq \mathrm{Re}\, z \leq e.$$

29.25 EXAMPLE Given a complex parameter μ, let

$$E(z;\mu) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu+n)},$$

an entire function of z. In particular:

$$e^z = E(z;1),\ ze^z = E(z;0)$$

and

$$z^{1-\mu}e^{z} = E(z;\mu) \quad (\mu = -1, -2, \ldots).$$

<u>Differential Equations</u>:

- $$(\mu-1)E(z;\mu) + zE'(z;\mu) = E(z;\mu-1)$$
- $$E(z;\mu) - E'(z;\mu) = (\mu-1)E(z;\mu+1)$$

Suppose now that $\mu > 1$ -- then

$$E(z;\mu) = \int_0^1 \phi(t)e^{zt}dt,$$

where

$$\phi(t) = \frac{(1-t)^{\mu-2}}{\Gamma(\mu-1)},$$

thus

$$-\frac{\phi'(t)}{\phi(t)} = \frac{\mu-2}{1-t} \quad (0 < t < 1)$$

=>

$$\begin{cases} -\dfrac{\phi'(t)}{\phi(t)} \leq \mu-2 & (1 < \mu < 2) \\[2ex] -\dfrac{\phi'(t)}{\phi(t)} \geq \mu-2 & (\mu > 2). \end{cases}$$

So, the zeros of $E(z;\mu)$ lie in the region $\operatorname{Re} z < \mu-2$ if $1 < \mu < 2$ and in the region $\operatorname{Re} z > \mu-2$ if $\mu > 2$.

<u>$1 < \mu < 2$</u>: The zeros of $E(z;\mu)$ are simple. In fact, if $E(z;\mu)$ had a multiple zero z_0, then

$$E(z_0;\mu+1) = 0.$$

But

$$\mu + 1 > 2 \Rightarrow \operatorname{Re} z_0 > (\mu+1) - 2 = \mu - 1 > 0$$

in contradiction to

$$\text{Re } z_0 < \mu - 2 < 0.$$

<u>$2 \leq \mu \leq 3$:</u> First

$$E(z;2) = \frac{e^z - 1}{z}$$

and its zeros are simple and lie on the imaginary axis. Assume, therefore, that $2 < \mu \leq 3$ -- then the zeros of $E(z;\mu)$ are also simple. For at a multiple zero z_0, we would have

$$E(z_0;\mu-1) = 0$$

from which

$$\text{Re } z_0 \leq \mu - 1 - 2 \leq 3 - 3 = 0,$$

contradicting

$$\text{Re } z_0 > \mu - 2 > 0.$$

29.26 EXAMPLE The <u>incomplete gamma function</u> is defined by the rule

$$\gamma(\alpha,z) = \int_0^z e^{-t} t^{\alpha-1} dt \qquad (\text{Re } \alpha > 0).$$

As a function of z, $\gamma(\alpha,z)$ is holomorphic with the potential exception of a branch point at the origin, the principal branch being determined by introducing a cut along the negative real t axis and requiring $t^{\alpha-1}$ to have its principal value. Expanding e^{-t} and integrating gives

$$\gamma(\alpha,z) = z^\alpha \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!\,(n+\alpha)},$$

the right hand side providing an extension of the left hand side to all $\alpha \neq 0$,

-1, -2,... . Put

$$\gamma^*(\alpha,z) = \frac{\gamma(\alpha,z)}{z^\alpha \Gamma(\alpha)} .$$

Then $\gamma^*(\alpha,z)$ is entire and

$$\gamma^*(\alpha,z) = e^{-z} \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha+n+1)}$$

or still,

$$\gamma^*(\alpha,z) = e^{-z} E(z;1+\alpha).$$

Specializing what has been said in 29.25, we can thus say the following.

- For $0 < \alpha < 1$, all the zeros of $\gamma^*(\alpha,z)$ lie in the region $\text{Re } z < \alpha - 1$.
- For $\alpha > 1$, all the zeros of $\gamma^*(\alpha,z)$ lie in the region $\text{Re } z > \alpha - 1$.
- For $0 < \alpha \leq 2$, all the zeros of $\gamma^*(\alpha,z)$ are simple.

[Note:

$$\gamma^*(0,z) \equiv 1 \text{ and } \gamma^*(-n,z) = z^n (n = 1,2,\ldots).]$$

29.27 EXAMPLE Consider the error function

$$\text{erf } z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \quad (\text{cf. } 29.17).$$

Then erf z has a simple zero at $z = 0$ and no other real zeros. Since

$$\text{erf } z = \frac{1}{\sqrt{\pi}} \gamma(\tfrac{1}{2}, t^2),$$

the nonreal zeros of erf z coincide with the zeros of $\gamma^*(\frac{1}{2}, z^2)$, these lying in the region $\text{Re } z^2 < -\frac{1}{2}$ (which, when explicated, is seen to consist of two curvilinear sectors placed symmetrically with respect to the real axis and bounded by the components of the hyperbola $y^2 - x^2 = \frac{1}{2}$ $(z = x + \sqrt{-1}\, y)$.

[Note: It can be shown that the zeros of erf z are simple. In addition, the nonreal zeros of erf z are comprised of two sequences z_n^+, z_n^- $(n = \pm 1, \pm 2, \ldots)$ which are symmetric with respect to the real axis and contained in the region $y^2 - x^2 > \frac{1}{2}$. And asymptotically,

$$(z_n^{\pm})^2 = 2\pi n\sqrt{-1} - \frac{1}{2}\log|n| - \sqrt{-1}\,\frac{\pi}{4}\,\operatorname{sgn} n - \log(\pi\sqrt{2}) + O\left(\frac{\log|n|}{|n|}\right) \quad (n \to \infty).]$$

§30. *TRANSFORM THEORY: JUNIOR GRADE*

If $\phi \in L^1[0,1]$, then by definition

$$f(z) = \int_0^1 \phi(t)e^{\sqrt{-1}\, zt}dt$$

or still,

$$f(z) = C(z) + \sqrt{-1}\, S(z),$$

where

$$C(z) = \int_0^1 \phi(t)\cos zt\, dt, \quad S(z) = \int_0^1 \phi(t)\sin zt\, dt.$$

30.1 EXAMPLE Take $\phi(t) = \frac{1}{\sqrt{1-t^2}}$ $(0 \leq t < 1)$ -- then

$$\frac{2}{\pi} \int_0^1 \frac{\cos zt}{\sqrt{1-t^2}}\, dt = J_0(z).$$

Extend ϕ to an even function $\tilde{\phi}$ on $[-1,1]$ and let

$$\tilde{C}(z) = \int_{-1}^1 \tilde{\phi}(t)\cos zt\, dt,$$

thus

$$\tilde{C}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{2n!} \int_{-1}^1 \tilde{\phi}(t)t^{2n}dt.$$

30.2 RAPPEL The n^{th} Appell polynomial J_n^* associated with a real entire function f is defined by

$$J_n^*(f;z) = \sum_{k=0}^{n} \binom{n}{k}\gamma_k z^{n-k} \quad \text{(cf. 12.4).}$$

30.3 LEMMA We have

$$J_n^*(\tilde{C};z) = \int_{-1}^{1} \tilde{\phi}(t)(z + \sqrt{-1}\, t)^n dt.$$

PROOF Expand the RHS:

$$\int_{-1}^{1} \tilde{\phi}(t)(z + \sqrt{-1}\, t)^n dt = \int_{-1}^{1} \tilde{\phi}(t)(\sqrt{-1}\, t + z)^n dt$$

$$= \sum_{k=0}^{n} \binom{n}{k} (\sqrt{-1})^k \left(\int_{-1}^{1} \tilde{\phi}(t) t^k dt\right) z^{n-k}$$

$$= \sum_{k=0}^{[n/2]} \binom{n}{2k} (-1)^k \left(\int_{-1}^{1} \tilde{\phi}(t) t^{2k} dt\right) z^{n-2k}.$$

On the other hand, from the definitions,

$$\gamma_0 = \int_{-1}^{1} \tilde{\phi}(t)dt, \ \gamma_1 = 0,$$

$$\gamma_2 = - \int_{-1}^{1} \tilde{\phi}(t)t^2 dt, \ \gamma_3 = 0,$$

$$\gamma_4 = \int_{-1}^{1} \tilde{\phi}(t)t^4 dt, \ \gamma_5 = 0,$$

$$\vdots$$

30.4 RAPPEL The n^{th} Jensen polynomial J_n associated with a real entire function f is defined by

$$J_n(f;z) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k z^k \quad \text{(cf. 12.1).}$$

30.5 LEMMA We have

$$J_n(\tilde{C};z) = \int_{-1}^{1} \tilde{\phi}(t)(1 + \sqrt{-1}\, zt)^n dt.$$

PROOF In fact,

$$J_n(\tilde{C};z) = z^n J_n^*(\tilde{C};\frac{1}{z})$$

$$= z^n \int_{-1}^{1} \tilde{\phi}(t) (\frac{1}{z} + \sqrt{-1}\ t)^n dt$$

$$= z^n \int_{-1}^{1} \tilde{\phi}(t) (\frac{1 + \sqrt{-1}\ zt}{z})^n dt$$

$$= \int_{-1}^{1} \tilde{\phi}(t)(1 + \sqrt{-1}\ zt)^n dt.$$

30.6 EXAMPLE Take $\phi(t) = (1 - t^{2p})^\lambda$, where $p = 1,2,\ldots$, and $\lambda > -1$ -- then the real polynomial

$$\int_{-1}^{1} (1 - t^{2p})^\lambda (1 + \sqrt{-1}\ zt)^n dt \quad (n > 1)$$

has real zeros only, hence the real entire function

$$\int_0^1 (1 - t^{2p})^\lambda \cos zt\, dt$$

has real zeros only (being in $L - P$ (cf. 12.14)).

[Note: It is known that for $\nu > -\frac{1}{2}$,

$$J_\nu(z) = \frac{2}{\sqrt{\pi}\ \Gamma(\nu + \frac{1}{2})} (\frac{z}{2})^\nu \int_0^1 (1 - t^2)^{\nu - \frac{1}{2}} \cos zt\, dt.$$

But then $\nu - \frac{1}{2} > -1$, so the zeros of $J_\nu(z)$ are real (cf. 12.33) (matters there require only that $\nu > -1$).]

30.7 REMARK Let $\lambda = k = 1,2,\ldots$, and replace z by $zk^{1/2p}$:

$$\int_0^1 (1 - t^{2p})^k \cos zk^{1/2p}t\, dt.$$

Then make the change of variable $t = xk^{-1/2p}$:

$$k^{-1/2p} \int_0^{k^{1/2p}} (1 - \frac{x^{2p}}{k})^k \cos zx \, dx.$$

Now replace x by t and form

$$\lim_{k \to \infty} \int_0^{k^{1/2p}} (1 - \frac{t^{2p}}{k})^k \cos zt \, dt$$

to see that the real entire function

$$\Phi_{2p}(z) = \int_0^{\infty} \exp(- t^{2p}) \cos zt \, dt$$

has real zeros only (cf. 12.34).

30.8 THEOREM Suppose that $\phi(t)$ is positive, strictly increasing, and continuous on $[0,1[$ and

$$\int_0^1 \phi(t)dt = \lim_{\varepsilon \to 0} \int_0^{1-\varepsilon} \phi(t)dt$$

exists -- then the real entire function

$$C(z) = \int_0^1 \phi(t) \cos zt \, dt$$

has real zeros only.

<u>N.B.</u> Accordingly,

$$\lim_{n \to \infty} \frac{\phi(\frac{1}{n}) + \phi(\frac{2}{n}) + \cdots + \phi(\frac{n-1}{n})}{n} = \int_0^1 \phi(t)dt.$$

[The expression on the left (sans the limit) is bounded from below by

$$\int_0^{1 - \frac{1}{n}} \phi(t)dt$$

and from above by

$$\int_{\frac{1}{n}}^{1} \phi(t)dt.]$$

30.9 REMARK The assumptions on ϕ can be weakened (cf. 31.1) but the methods utilized in arriving at 30.8 are instructive and can be employed in other situations as well.

30.10 LEMMA Suppose given polynomials

$$\begin{cases} P(z) = a_n(z - z_1)(z - z_2)\cdots(z - z_n) \\ Q(z) = \bar{a}_n(1 - \bar{z}_1 z)(1 - \bar{z}_2 z)\cdots(1 - \bar{z}_n z). \end{cases}$$

Assume: The zeros of $P(z)$ lie in the region $|z| \geq 1$ -- then the zeros of

$$P(z) + \gamma z^k Q(z) \quad (|\gamma| = 1,\ k = 1,2,\ldots)$$

lie on the unit circle $|z| = 1$.

PROOF There are two points.

- If $|w| > 1$, then

$$\left|\frac{z - w}{1 - \bar{w}z}\right| \begin{matrix} > \\ = \\ < \end{matrix} 1 \text{ for } |z| \begin{matrix} < \\ = \\ > \end{matrix} 1.$$

- If $|\omega| = 1$, then

$$\left|\frac{z - \omega}{1 - \bar{\omega}z}\right| = \left|\frac{z - \omega}{\omega - z}\right| \text{ for } |z| \begin{matrix} < \\ = \\ > \end{matrix} 1.$$

Therefore the equality is possible only when $|z| = 1$.

30.11 REMARK If $|z_i| > 1$ $(i = 1,\ldots,n)$, then the zeros of

$$P(z) + \gamma z^k Q(z)$$

are simple.

[Let $p(z) = P(z)$, $q(z) = -\gamma z^k Q(z)$ and suppose that z_0 is a multiple zero of $p(z) - q(z)$ -- then

$$\begin{cases} p(z) = q(z_0) \\ p'(z_0) = q'(z_0). \end{cases}$$

Since $p(z)$ and $q(z)$ do not vanish on $|z| = 1$, it follows that

$$\frac{p'}{p}(z_0) = \frac{q'}{q}(t_0)$$

or still,

$$\sum_{i=1}^{n} \frac{1}{z_0 - z_i} = \sum_{i=1}^{n} \frac{1}{z_0 - 1/\bar{z}_i} + \frac{k}{z_0}$$

or still,

$$\sum_{i=1}^{n} \frac{1}{1 - z_i/z_0} = \sum_{i=1}^{n} \frac{1}{1 - 1/\bar{z}_i z_0} + k.$$

But

$$\begin{cases} |w| < 1 \Rightarrow \mathrm{Re}\,\frac{1}{1-w} > \frac{1}{2} \\ |w| > 1 \Rightarrow \mathrm{Re}\,\frac{1}{1-w} < \frac{1}{2}. \end{cases}$$

Therefore

$$\mathrm{Re}\left(\sum_{i=1}^{n} \frac{1}{1 - z_i/z_0}\right) < \frac{n}{2}$$

while

$$\mathrm{Re}\left(\sum_{i=1}^{n} \frac{1}{1 - 1/\bar{z}_i z_0}\right) > \frac{n}{2},$$

from which the evident contradiction.]

Let

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n$$

be a real polynomial whose zeros lie in the region $|z| \geq 1$. Put $\zeta = e^{\sqrt{-1}\,z}$ -- then

$$\begin{cases} P(\zeta) = a_0 + a_1\zeta + \cdots + a_n\zeta^n \\ Q(\zeta) = a_0\zeta^n + a_1\zeta^{n-1} + \cdots + a_n \end{cases}$$

and

$$P(\zeta) + \zeta^n Q(\zeta) = 0$$

$$\Rightarrow |\zeta| = 1 \text{ (cf. 30.10)} \Rightarrow z \in \mathbf{R}.$$

30.12 LEMMA The trigonometric polynomial

$$\sum_{k=0}^{n} a_{n-k} \cos kz$$

has real zeros only.

PROOF Write

$$\zeta^{-n}(P(\zeta) + \zeta^n Q(\zeta))$$

$$= 2a_n + a_{n-1}(\zeta + \zeta^{-1}) + \cdots + a_0(\zeta^n + \zeta^{-n})$$

$$= 2(a_n + a_{n-1} \cos z + \cdots + a_0 \cos nz)$$

$$= 2 \sum_{k=0}^{n} a_{n-k} \cos kz.$$

30.13 ENESTRÖM-KAKEYA CRITERION Let

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n,$$

where

$$a_0 > a_1 > \cdots > a_n > 0.$$

Then the zeros of p lie in the region $|z| > 1$.

PROOF Assuming that $|z| \le 1$ $(z \ne 1)$, we have

$$|(1 - z)(a_0 + a_1 z + \cdots + a_n z^n)|$$

$$= |a_0 - (a_0 - a_1)z - \cdots - (a_{n-1} - a_n)z^n - a_n z^{n+1}|$$

$$\ge a_0 - |(a_0 - a_1)z + \cdots + (a_{n-1} - a_n)z^n + a_n z^{n+1}|$$

$$> a_0 - ((a_0 - a_1) + \cdots + (a_{n-1} - a_n) + a_n) = 0.$$

[Note: If instead

$$a_0 \ge a_1 \ge \cdots \ge a_n > 0,$$

then the zeros of p lie in the region $|z| \ge 1$.]

30.14 APPLICATION If

$$0 < a_0 < a_1 < \cdots < a_n$$

and if

$$P(z) = \sum_{k=0}^{n} a_{n-k} z^k,$$

then the zeros of P lie in the region $|z| > 1$, thus the zeros of the trigonometric polynomial

$$\sum_{k=0}^{n} a_k \cos kz$$

are real (and simple (cf. 30.11)).

30.15 FACT For any continuous function f(t) on [0,1],

$$\lim_{n \to \infty} \frac{\phi(\frac{1}{n})f(\frac{1}{n}) + \phi(\frac{2}{n})f(\frac{2}{n}) + \cdots + \phi(\frac{n-1}{n})f(\frac{n-1}{n})}{n} = \int_0^1 \phi(t)f(t)dt.$$

PROOF Given $\varepsilon > 0$, choose $\delta > 0$:

$$\int_{1-\delta}^{1} \phi(t)dt < \varepsilon.$$

Then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{[(1-\delta)n]} \phi(\frac{k}{n})f(\frac{k}{n}) = \int_0^{1-\delta} \phi(t)f(t)dt.$$

On the other hand, with $M = \sup_{[0,1]} |f|$, we have

$$\left| \frac{1}{n} \sum_{k=[(1-\delta)n]+1}^{n-1} \phi(\frac{k}{n})f(\frac{k}{n}) \right|$$

$$\leq \frac{M}{n} \sum_{k=[(1-\delta)n]+1}^{n-1} \phi(\frac{k}{n})$$

$$\leq M \int_{1-\delta}^{1} \phi(t)dt \leq M\varepsilon.$$

With these preliminaries established, the proof of 30.8 is straightforward.

Indeed, for n = 1,2,...,

$$0 < \phi(0) < \phi(\frac{1}{n}) < \cdots < \phi(\frac{n-1}{n}),$$

so a specialization of the preceding generalities implies that the zeros of the trigonometric polynomial

$$\phi(0) + \phi(\frac{1}{n})\cos z + \cdots + \phi(\frac{n-1}{n})\cos(n-1)z$$

are real, as are the zeros of the trigonometric polynomial

$$\phi(0) + \phi(\frac{1}{n})\cos\frac{z}{n} + \cdots + \phi(\frac{n-1}{n})\cos\frac{(n-1)}{n}z.$$

But (cf. 30.15)

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(\frac{k}{n})\cos\frac{k}{n}z = \int_0^1 \phi(t)\cos zt\, dt,$$

the convergence being uniform on compact subsets of $\mathbf{C}$, thereby terminating the proof of 30.8.

[Note: The zeros of

$$\sum_{k=0}^{n-1} \phi(\frac{k}{n})\cos(\frac{k}{n}z)$$

are not only real but they are also simple (cf. 30.14). Still, additional argument is needed in order to conclude that the zeros of

$$C(z) = \int_0^1 \phi(t)\cos zt\, dt$$

are simple (cf. 31.1).]

30.16 REMARK Work instead with

$$\zeta^{-n}(P(\zeta) - \zeta^n Q(\zeta))$$

to see that the trigonometric polynomial

$$2\sqrt{-1} \sum_{k=0}^{n} a_{n-k} \sin kz$$

has real zeros only. Pass now to

$$\phi(\tfrac{1}{n})\sin z + \cdots + \phi(\tfrac{n-1}{n})\sin(n-1)z$$

and proceed as above, the bottom line being that the zeros of the real entire function

$$S(z) = \int_0^1 \phi(t)\sin zt\, dt$$

are real.

30.17 EXAMPLE The zeros of

$$\frac{\cos z}{z^2}(\tan z - z) = \int_0^1 t \sin zt\, dt$$

are real.

[Note: Consequently, tan z - z has real zeros only.]

16.18 EXAMPLE The zeros of

$$J_1(z) = - J_0'(z) = \frac{2}{\pi}\int_0^1 \frac{t}{\sqrt{1-t^2}} \sin zt\, dt$$

are real (cf. 12.33).

16.19 EXAMPLE Consider

$$\int_0^1 (1 - t^2)\cos zt\, dt.$$

Then its zeros are real (cf. 30.6).

[Since $1 - t^2$ is decreasing, this is not a special case of 30.8. But

$$\int_0^1 (1 - t^2)\cos zt\, dt = \frac{2}{z} \int_0^1 t \sin zt\, dt,$$

so it is a special case of 30.16.]

[Note: In detail,

$$\int_0^1 t \sin zt\, dt = -\frac{1}{2} \int_0^1 \sin zt\, d(1-t^2)$$

$$= -\frac{1}{2} (\sin zt)(1-t^2)\Big|_0^1 + \frac{z}{2} \int_0^1 \cos zt(1-t^2)dt$$

$$= \frac{z}{2} \int_0^1 \cos zt(1-t^2)dt.]$$

30.20 REMARK If in 30.8, the assumption that $\phi(t)$ is positive, strictly increasing, and continuous on $[0,1[$ is replaced by the assumption that $\phi(t)$ is positive, strictly decreasing, and continuous on $[0,1]$, then $C(z)$ may have nonreal zeros.

[Consider

$$\int_0^1 e^{-t} \cos zt\, dt = \frac{(z \sin z - \cos z) + 1}{e(z^2+1)} .]$$

§31. *TRANSFORM THEORY: SENIOR GRADE*

The following result supercedes 30.8.

31.1 THEOREM If $\phi \in L^1[0,1]$ is positive and increasing on $]0,1[$, then the zeros of

$$C(z) = \int_0^1 \phi(t)\cos zt\,dt$$

are real and simple. Furthermore, the positive zeros of $C(z)$ lie in the intervals

$$]\frac{\pi}{2}, \frac{3\pi}{2}[\ ,\ ]\frac{3\pi}{2}, \frac{5\pi}{2}[\ ,\ ]\frac{5\pi}{2}, \frac{7\pi}{2}[\ ,\ \ldots$$

and only in these intervals. Finally, each of these intervals contains exactly one zero of $C(z)$.

[Note: $C(z)$ is even, hence $C(z_0) = 0$ iff $C(-z_0) = 0$.]

The proof is spelled out in the lines below.

Step 1:

$$C(\frac{\pi}{2}) = \int_0^1 \phi(t)\cos\frac{\pi}{2}t\,dt > 0.$$

Step 2:

- $C(\frac{\pi}{2} + 2\pi n) > 0 \quad (n = 1,2,\ldots)$.

[We have

$$\int_0^1 \phi(t)\cos(2\pi n + \frac{\pi}{2})t\,dt$$

$$= \int_0^{1/(4n+1)} \phi(t)\cos(4n+1)\frac{\pi}{2}t\,dt + \sum_{k=0}^{n} \int_{\frac{4k+1}{4n+1}}^{\frac{4k+5}{4n+1}} \phi(t)\cos(4n+1)\frac{\pi}{2}t\,dt$$

$$\geq \int_0^{1/(4n+1)} \phi(t)\cos(4n+1)\frac{\pi}{2}t\,dt > 0.]$$

- $C(\frac{3\pi}{2} + 2\pi n) < 0 \quad (n = 0,1,2,\ldots).$

[We have

$$\int_0^1 \phi(t)\cos(4n+3)\frac{\pi}{2}\, t\, dt$$

$$= \int_0^{2/(4n+3)} \phi(t)\cos(4n+3)\frac{\pi}{2}\, t\, dt + \int_{2/(4n+3)}^{3/(4n+3)} \phi(t)\cos(4n+3)\frac{\pi}{2}\, t\, dt$$

$$+ \sum_{k=0}^{n} \int_{\frac{4k+3}{4n+3}}^{\frac{4k+7}{4n+3}} \phi(t)\cos(4n+3)\frac{\pi}{2}\, t\, dt$$

$$\leq \int_{2/(4n+3)}^{3/(4n+3)} \phi(t)\cos(4n+3)\frac{\pi}{2}\, t\, dt < 0.]$$

So far then

$$C(\tfrac{\pi}{2}) > 0,\ C(\tfrac{3\pi}{2}) < 0,\ C(\tfrac{5\pi}{2}) > 0,\ C(\tfrac{7\pi}{2}) < 0\ \ldots,$$

which implies that each of the intervals

$$]\tfrac{\pi}{2}, \tfrac{3\pi}{2}[\ ,\ ]\tfrac{3\pi}{2}, \tfrac{5\pi}{2}[\ ,\ ]\tfrac{5\pi}{2}, \tfrac{7\pi}{2}[\ ,\ \ldots$$

contains at least one zero of $C(z)$, as do the intervals symmetric to them. The objective now is to show that any such interval contains but one zero of $C(z)$, that said zero is simple, and that there are no other zeros.

To move forward, assume without loss of generality that $C(0) = 1$.

31.2 RAPPEL

$$\int_0^r \frac{n(t)}{t}\, dt = \frac{1}{2\pi}\int_0^{2\pi} \log|C(re^{\sqrt{-1}\,\theta})|\, d\theta \qquad \text{(cf. 27.36)}.$$

Let $n^*(t)$ denote the number of points $\pm(\frac{\pi}{2} + \pi n)$ $(n = 1,2,\ldots)$ in the interval

$]-t,t[$ $(t > 0)$, thus $n^*(t) = 0$ for $|t| < \frac{3\pi}{2}$ and

$$n^*(t) = 2k \text{ if } \frac{\pi}{2} + \pi k < t < \frac{\pi}{2} + \pi(k+1) \quad (k = 1,2,\ldots).$$

To derive a contradiction, suppose that $C(z_0) = 0$ ($\Rightarrow C(-z_0) = 0$), where z_0 is either not in one of the intervals above or is a multiple zero of one thereof. Choose $K > 0$:

$$n(t) \geq n^*(t) \quad (0 < t < K), \quad n(t) \geq n^*(t) + 2 \quad (t > K).$$

Step 3: Take $r = \pi n + \frac{3\pi}{2}$ -- then

$$\int_0^r \frac{n(t)}{t}\,dt \geq \sum_{k=1}^{n} (2k+2) \int_{\frac{\pi}{2}+\pi k}^{\frac{\pi}{2}+\pi(k+1)} \frac{dt}{t} + O(1)$$

$$= 2 \sum_{k=1}^{n} (k+1)\log\left(1 + \frac{1}{k+\frac{1}{2}}\right) + O(1)$$

$$= 2 \sum_{k=1}^{n} (k+1)\left(1 + \frac{1}{k+\frac{1}{2}} - \frac{1}{2(k+\frac{1}{2})^2}\right) + O(1)$$

$$= 2 \sum_{k=1}^{n} 1 + \sum_{k=1}^{n} \frac{1}{k+\frac{1}{2}} - \sum_{k=1}^{n} \frac{k+1}{(k+\frac{1}{2})^2} + O(1)$$

$$= 2n + O(1) = 2\frac{r}{\pi} + O(1).$$

Step 4: Since

$$C(x) \to 0 \text{ as } x \to \pm\infty$$

and since the exponential type of $C(z)$ is ≤ 1,

$$\frac{|C(re^{\sqrt{-1}\,\theta})|}{e^{|r \sin \theta|}} \to 0 \qquad (r \to \infty)$$

uniformly in θ. Therefore

$$\frac{1}{2\pi} \int_0^{2\pi} \log|C(re^{\sqrt{-1}\,\theta})|d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \log\left|\frac{C(re^{\sqrt{-1}\,\theta})}{e^{|r \sin \theta|}} \cdot e^{|r \sin \theta|}\right|d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \log\left|\frac{C(re^{\sqrt{-1}\,\theta})}{e^{|r \sin \theta|}}\right|d\theta + \frac{1}{2\pi} \int_0^{2\pi} |r \sin \theta|d\theta$$

$$\leq \log o(1) + 2\,\frac{r}{\pi}\,.$$

<u>Step 5</u>: Combine the data:

$$\log o(1) + 2\,\frac{r}{\pi} \geq \frac{1}{2\pi} \int_0^{2\pi} \log|C(re^{\sqrt{-1}\,\theta})|d\theta$$

$$= \int_0^r \frac{n(t)}{t}\,dt \geq 2\,\frac{r}{\pi} + O(1)$$

=>

$$\log o(1) \geq O(1),$$

an impossibility.

31.3 THEOREM If $\phi \in L^1[0,1]$ is positive and increasing on $]0,1[$ and is not exceptional (cf. 29.14), then the zeros of

$$S(z) = \int_0^1 \phi(t)\sin zt\,dt$$

are real and simple. Furthermore, the positive zeros of $S(z)$ lie in the intervals

$$]\pi, 2\pi[\ ,\ ]2\pi, 3\pi[\ ,\ ]3\pi, 4\pi[\ , \ldots$$

and only in these intervals. Finally, each of these intervals contains exactly one zero of $S(z)$.

[Note: $S(z)$ is odd, hence $S(z_0) = 0$ iff $S(-z_0) = 0$.]

The proof is spelled out in the lines below.

Step 1:

$$S(0) = \int_0^1 \phi(t) \sin 0t \, dt = 0.$$

And

$$S'(z) = \int_0^1 \phi(t) t \cos zt \, dt$$

=>

$$S'(0) = \int_0^1 \phi(t) t \cos 0t \, dt$$

$$= \int_0^1 \phi(t) t \, dt > 0.$$

Therefore 0 is a simple zero of $S(z)$.

Step 2:

$$S(\pi) = \int_0^1 \phi(t) \sin \pi t \, dt > 0.$$

Step 3:

- $S(\pi + 2\pi n) > 0 \quad (n = 1,2,\ldots).$

[We have

$$\int_0^1 \phi(t) \sin(2n+1)\pi t \, dt$$

$$= \int_0^{1/(2n+1)} \phi(t)\sin(2n+1)\pi t\ dt + \sum_{k=0}^{n-1} \int_{\frac{2k+1}{2n+1}}^{\frac{2k+3}{2n+1}} \phi(t)\sin(2n+1)\pi t\ dt$$

$$\geq \int_0^{1/(2n+1)} \phi(t)\sin(2n+1)\pi t\ dt > 0.]$$

- $S(2\pi n) < 0 \quad (n = 1,2,\ldots).$

[We have

$$\int_0^1 \phi(t)\sin 2\pi nt\ dt$$

$$= \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \phi(t)\sin 2\pi nt\ dt$$

$$= \sum_{k=0}^{n-1} \int_0^{1/n} \phi(t + \tfrac{k}{n})\sin 2\pi nt\ dt$$

$$= \sum_{k=0}^{n-1} \int_0^{1/2n} (\phi(t + \tfrac{k}{n}) - \phi(\tfrac{k+1}{n} - t))\sin 2\pi nt\ dt$$

$$< 0.]$$

[Note: The function $\sin 2\pi nt$ is positive on $]0, \frac{1}{2n}[$ and

$$\phi(t + \tfrac{k}{n}) - \phi(\tfrac{k+1}{n} - t) \quad (0 < t < \tfrac{1}{2n})$$

is nonpositive and increasing, thus a priori

$$\sum_{k=0}^{n-1} \int_0^{1/2n} (\phi(t + \tfrac{k}{n}) - \phi(\tfrac{k+1}{n} - t))\sin 2\pi nt\ dt$$

$$\leq 0,$$

with equality only if $\forall$ k

$$\phi(t + \frac{k}{n}) - \phi(\frac{k+1}{n} - t) = 0$$

almost everywhere and this means zero on $]0, \frac{1}{2n}[$ (if negative anywhere on $]0, \frac{1}{2n}[$, then it is negative from there to the left giving a negative integral), hence $\phi(t)$ would be a constant in each of the intervals $\frac{k}{n} < t < \frac{k+1}{n}$ $(k = 0,\ldots,n-1)$, a scenario excluded by the assumption $\phi \notin E(1,0)$.]

So far then

$$S(\pi) > 0, \ S(2\pi) < 0, \ S(3\pi) > 0, \ S(4\pi) < 0, \ldots$$

which implies that each of the intervals

$$]\pi, 2\pi[\ , \ ]2\pi, 3\pi[\ , \ ]3\pi, 4\pi[\ , \ \ldots$$

contains at least one zero of $S(z)$, as do the intervals symmetric to them (recall too that 0 is a simple zero of $S(z)$). The remaining details are similar to those figuring in 31.1 and will be omitted.

31.4 LEMMA If $\phi \in L^1[0,1]$ is positive and increasing on $]0,1[$ and if $\phi \notin E(1,0)$, then $C(z)$ and $S(z)$ have no common zeros.

PROOF The zeros of

$$f(z) = \int_0^1 \phi(t) e^{\sqrt{-1}\, zt} \, dt$$

$$= C(z) + \sqrt{-1}\, S(z)$$

lie in the open upper half-plane (cf. 29.16). On the other hand, as has been seen above, the zeros of $C(z)$ and $S(z)$ are real, so

$$\left[\begin{array}{l} C(x_0) = 0 \\ \\ S(x_0) = 0 \end{array}\right. \quad \Rightarrow f(x_0) = 0,$$

which cannot be.]

§32. *APPLICATION OF INTERPOLATION*

Let $f \in B_0(A)$ and assume that f is not a constant, hence $T(f) > 0$.

32.1 RAPPEL (cf. 17.22) $\forall$ real x,

$$f'(x) = \frac{4T(f)}{\pi^2} \sum_{k=-\infty}^{\infty} (-1)^k \frac{1}{(2k+1)^2} f(x + \frac{2k+1}{2T(f)} \pi),$$

the convergence being uniform on compact subsets of R.

32.2 THEOREM $\forall$ $x,\alpha \in R$, there is an expansion

$$\sin \alpha \cdot f'(x) - A \cos \alpha \cdot f(x)$$

$$= A \sin^2\alpha \sum_{k=-\infty}^{\infty} \frac{(-1)^{k-1}}{(\alpha - k\pi)^2} f(x + \frac{k\pi - \alpha}{A}),$$

the convergence being uniform on compact subsets of R.

[Note: Replace k by k + 1 and take $\alpha = \frac{\pi}{2}$, $A = T(f)$ to recover 31.1.]

PROOF Write

$$f(z) = f(0) + \frac{z}{\sqrt{2\pi}} \int_{-A}^{A} \phi(t) e^{\sqrt{-1}\, zt} dt$$

for some $\phi \in L^2[-A,A]$ (cf. 22.8), so

$$\sin \alpha \cdot f'(x) - A \cos \alpha \cdot f(x)$$

$$= - A \cos \alpha \cdot f'(0)$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \phi(t) \frac{\partial}{\partial t}(e^{\sqrt{-1}\, xt}(t \sin \alpha + \sqrt{-1}\, A \cos \alpha))dt.$$

Now develop

$$- \sqrt{-1}\, e^{\sqrt{-1}\,\frac{\alpha}{A}\,t} (t \sin \alpha + \sqrt{-1}\, A \cos \alpha)$$

into a Fourier series:

$$A \sin^2\alpha \sum_{k=-\infty}^{\infty} (-1)^k \frac{1}{(\alpha-k\pi)^2} e^{\frac{\sqrt{-1}\, k\pi}{A} t}$$

=>

$$\sin \alpha \cdot f'(x) - A \cos \alpha \cdot f(x)$$

$$= - A \cos \alpha \cdot f(0)$$

$$+ \frac{\sqrt{-1}\, A \sin^2\alpha}{\sqrt{2\pi}} \int_{-A}^{A} \phi(t) \frac{\partial}{\partial t}\Big(\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(\alpha-k\pi)^2} \exp(\sqrt{-1}\, t(x + \frac{k\pi-\alpha}{A}))\Big)dt$$

$$= - A \cos \alpha \cdot f(0)$$

$$- A \sin^2\alpha \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(\alpha-k\pi)^2} (f(x + \frac{k\pi-\alpha}{A}) - f(0))$$

$$= A \sin^2\alpha \sum_{k=-\infty}^{\infty} \frac{(-1)^{k-1}}{(\alpha-k\pi)^2} f(x + \frac{k\pi-\alpha}{A}),$$

since

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(\alpha-k\pi)^2} = - \frac{d}{d\alpha} \frac{1}{\sin \alpha} = \frac{\cos \alpha}{\sin^2\alpha} .$$

32.3 APPLICATION $\forall\, B \in R$,

$$\sin A(x-B) \cdot f'(x) - A \cos A(x-B) \cdot f(x)$$

$$= A \sin^2 A(x-B) \sum_{k=-\infty}^{\infty} \frac{(-1)^{k-1}}{(A(x-B) - k\pi)^2} f(\frac{k\pi}{A} + B).$$

[Replace α by $A(x-B)$ in 32.2.]

N.B. If $f(\frac{k\pi}{A} + B) = 0 \ \forall\ k$, then

$$f(x) = C \sin A(x-B) \quad (C \neq 0)$$

and its zeros are at the points $\frac{k\pi}{A} + B$.

32.4 NOTATION $RB_0(A)$ is the subset of $B_0(A)$ consisting of those nonconstant f which are real on the real axis.

32.5 DEFINITION Let $f \in RB_0(A)$ -- then f is standard of level B if $\exists\ n = 0$ or 1 and $B \in R$ such that $\forall\ k \in Z$,

$$(-1)^{n+k} f(\frac{k\pi}{A} + B) \geq 0.$$

[Note: If f is standard of level B, then $-f$ is standard of level B.]

32.6 EXAMPLE Take $A = 1$, $B = 0$ -- then if $n = 0$,

$$\ldots f(-2\pi) \geq 0,\ f(-\pi) \leq 0,\ f(0) \geq 0,\ f(\pi) \leq 0,\ f(2\pi) \geq 0 \ldots,$$

with a reversal of signs if $n = 1$.

32.7 EXAMPLE Take $A = 1$, $B = \frac{\pi}{2}$ -- then if $n = 0$,

$$\ldots f(-\frac{5\pi}{2}) \leq 0,\ f(-\frac{3\pi}{2}) \geq 0,\ f(-\frac{\pi}{2}) \leq 0,\ f(\frac{\pi}{2}) \geq 0,\ f(\frac{3\pi}{2}) \leq 0,\ f(\frac{5\pi}{2}) \geq 0 \ldots,$$

with a reversal of signs if $n = 1$.

32.8 LEMMA If $f \in RB_0(A)$ is standard of level B, then $\forall\ x \in R$,

$$\sin A(x-B) \cdot f'(x) - A \cos A(x-B) \cdot f(x)$$

$$= (-1)^{n-1} A \sin^2 A(x-B) \sum_{k=-\infty}^{\infty} \frac{1}{(A(x-B) - k\pi)^2} |f(\frac{k\pi}{A} + B)|.$$

32.9 THEOREM If $f \in RB_0(A)$ is standard of level B, then $\forall\, p \in Z$, the <u>ambient interval</u>

$$I_p =]\frac{(p-1)\pi}{A} + B, \frac{p\pi}{A} + B[$$

contains at most one zero of f and if there is one, then it must be simple.

PROOF Suppose that for some $p \in Z$, $f(x_0) = 0$ ($x_0 \in I_p$) -- then $\exists\, k \in Z$ such that $f(\frac{k\pi}{A} + B) \neq 0$, hence

$$\sin A(x_0-B) \cdot f'(x_0)$$

$$= (-1)^{n-1} A \sin^2 A(x_0-B) M(x_0) \quad (M(x_0) > 0)$$

=>

$$f'(x_0) = (-1)^{n-1} A \sin A(x_0-B) M(x_0)$$

$$= (-1)^{n-1} (-1)^{p-1} A |\sin A(x_0-B)| M(x_0)$$

=>

$$(-1)^{n+p} f'(x_0) > 0,$$

which implies that x_0 is simple. If now $f(x_1) = 0$, $f(x_2) = 0$ with $x_1 < x_2$ and $f(x) \neq 0$ ($x_1 < x < x_2$), then we shall arrive at a contradiction by showing that there would be another zero of f between x_1 and x_2. To see this, choose a small $h > 0$ with the property that $f(x)$ and $f'(x)$ have the same sign in $]x_1, x_1+h[$ and opposite signs in $]x_2-h, x_2[$ (=> $x_1+h < x_2-h$).

• <u>n + p even</u>: Therefore $f'(x_1) > 0$, $f'(x_2) > 0$ and it can be assumed that $f'(x)$ is positive in $]x_1,x_1+h[$ and $]x_2-h,x_2[$. But then

$$\begin{cases} x_1 < x < x_1 + h \Rightarrow f(x) > 0 \\ x_2 - h < x < x_2 \Rightarrow f(x) < 0. \end{cases}$$

• <u>n + p odd</u>: Therefore $f'(x_1) < 0$, $f'(x_2) < 0$ and it can be assumed that $f'(x)$ is negative in $]x_1,x_1+h[$ and $]x_2-h,x_2[$. But then

$$\begin{cases} x_1 < x < x_1 + h \Rightarrow f(x) < 0 \\ x_2 - h < x < x_2 \Rightarrow f(x) > 0. \end{cases}$$

32.10 LEMMA If $f \in RB_0(A)$ is standard of level B, then

$$\sup_{x \in R} x^2|f(x)| = \infty.$$

PROOF Assuming this is false, let

$$g(z) = f(z)(z-x_0)^2 \quad (x_0 \in I_1 =]B, \frac{\pi}{A} + B[).$$

Then $g \in RB_0(A)$ is standard of level B. But x_0 is a zero of g of multiplicity ≥ 2, an impossibility (cf. 32.9).

32.11 THEOREM If $f \in RB_0(A)$ is standard of level B, then all the zeros of f are real.

PROOF Suppose that $f(z_0) = 0$ for some $z_0 \in C - R$. Since f is real, $f(\bar{z}_0) = 0$ and the function

$$g(z) = \frac{f(z)}{(z-z_0)(z-\bar{z}_0)}$$

belongs to $RB_0(A)$. As such, it is standard of level B and

$$\sup_{x\in R} x^2|g(x)| < \infty,$$

which contradicts 32.10.

32.12 EXAMPLE Given $\phi \in L^1[0,1]$ real $\not\equiv 0$, let

$$C(z) = \int_0^1 \phi(t)\cos zt\, dt.$$

Then $C \in RB_0(1)$. Assume: $\forall\, k \in Z$,

$$(-1)^k C(k\pi) > 0.$$

Then all the zeros of C are real and each ambient interval I_p contains a single zero and it is simple.

We have yet to examine what happens at the endpoints of an I_p.

32.13 THEOREM If $f \in RB_0(A)$ is standard of level B and if for some $p \in Z$,

$$f(\frac{p\pi}{A} + B) = 0,$$

then

$$x_p \equiv \frac{p\pi}{A} + B$$

is a zero of multiplicity ≤ 2 and f cannot have zeros in both ambient intervals I_p and I_{p+1}. Moreover, if x_p is a zero of multiplicity 2, then

$$(-1)^{n+p} f''(x_p) < 0$$

and

$$(-1)^{n+p} f(x) < 0 \quad (x \in I_p \cup I_{p+1}),$$

while if x_{p-1} (or x_{p+1}) is a zero, then x_{p-1} (or x_{p+1}) must be simple.

PROOF This is elementary, albeit detailed.

- If $f(x_p) = 0$, $f'(x_p) = 0$,

then

$$(-1)^{n+p} f''(x_p) < 0,$$

hence in particular, x_p is a zero of multiplicity ≤ 2. Thus let

$$g(z) = \frac{f(z)}{(z-x_p)^2}.$$

Then $g \in RB_0(A)$ and we claim that g is standard of level B if

$$(-1)^{n+p} f''(x_p) \geq 0.$$

For it is clear that

$$(-1)^{n+k} g(\frac{k\pi}{A} + B) \geq 0$$

$\forall\, k \neq p$, so take $k = p$ and consider

$$(-1)^{n+p} g(\frac{p\pi}{A} + B)$$

or still,

$$(-1)^{n+p} g(x_p)$$

or still,

$$\lim_{h \to 0} (-1)^{n+p} g(x_p + h)$$

or still,

$$\lim_{h \to 0} (-1)^{n+p} \frac{f(x_p+h)}{(x_p+h - x_p)^2}$$

or still,

$$\lim_{h \to 0} (-1)^{n+p} \frac{f(x_p+h)}{h^2}$$

or still,

$$\lim_{h \to 0} (-1)^{n+p} \frac{f'(x_p+h)}{2h}$$

or still,

$$\lim_{h \to 0} (-1)^{n+p} \frac{f''(x_p+h)}{2}$$

or still,

$$\frac{1}{2} (-1)^{n+p} f''(x_p) \geq 0.$$

Therefore g is standard of level B. But

$$\sup_{x \in R} x^2 |g(x)| < \infty,$$

contradicting 32.10. Accordingly, the supposition

$$(-1)^{n+p} f''(x_p) \geq 0$$

is untenable, leaving

$$(-1)^{n+p} f''(x_p) < 0.$$

• To see that f cannot have zeros in both intervals I_p and I_{p+1}, assume the opposite:

$$\begin{cases} f(x_1) = 0 & (x_1 \in I_p) \\ f(x_2) = 0 & (x_2 \in I_{p+1}). \end{cases}$$

Then x_1 is the only zero of f in I_p and it is simple, whereas x_2 is the only zero of f in I_{p+1} and it is simple (cf. 32.9). Now form

$$g(z) = \frac{f(z)(z-x_p)^2}{(z-x_1)(z-x_2)}.$$

Then $g \in RB_0(A)$ and g is standard of level B: $\forall\, k \in Z$,

$$(-1)^{n+k} g(\frac{k\pi}{A} + B).$$

Here the point is slightly subtle and explains the presence of two factors in the denominator rather than just one factor. For

$$\frac{(p-1)\pi}{A} + B < x_1 < x_2,$$

so

$$\frac{k\pi}{A} + B \le \frac{(p-1)\pi}{A} + B$$

=>

$$\frac{k\pi}{A} + B - x_1 < 0, \frac{k\pi}{A} + B - x_2 < 0$$

=>

$$(\frac{k\pi}{A} + B - x_1)(\frac{k\pi}{A} + B - x_2) > 0.$$

What remains is obvious and one then comes to a contradiction, x_p being a zero of g of multiplicity > 2.

- Suppose that x_p is a zero of multiplicity 2 -- then f has no zeros in $I_p \cup I_{p+1}$. E.g.: Let $x_1 \in I_p$ be a zero of f and put

$$g(z) = \frac{f(z)}{(z-x_1)(z-x_p)}$$

Then $g \in RB_0(A)$ is standard of level B. On the other hand,

$$\sup_{x \in R} x^2 |g(x)| < \infty,$$

which is incompatible with 32.10. Bearing in mind that

$$(-1)^{n+p} f''(x_p) < 0,$$

it then follows that

$$(-1)^{n+p} f(x) < 0 \quad (x \in I_p \cup I_{p+1}).$$

Thus choose a small $h > 0$ with the property that

$$(-1)^{n+p} \begin{bmatrix} f(x) \\ \\ f'(x) \end{bmatrix} \quad \text{and} \quad (-1)^{n+p} \begin{bmatrix} f'(x) \\ \\ f''(x) \end{bmatrix}$$

have the same sign in $]x_p, x_p+h[$ and opposite signs in $]x_p-h, x_p[$. Working first with $]x_p, x_p+h[$ and assuming, as we may, that

$$x \in]x_p, x_p+h[\Rightarrow (-1)^{n+p} f''(x) < 0,$$

thence

$$x \in]x_p, x_p+h[\Rightarrow (-1)^{n+p} f'(x) < 0$$

$$\Rightarrow (-1)^{n+p} f(x) < 0.$$

But f has no zeros in I_{p+1}, so

$$(-1)^{n+p} f(x) < 0 \quad (x \in I_{p+1}).$$

As for $]x_p-h, x_p[$, it can be assumed that

$$x \in]x_p-h, x_p[\Rightarrow (-1)^{n+p} f''(x) < 0,$$

thence

$$x \in]x_p - h, x_p[\Rightarrow (-1)^{n+p} f'(x) > 0$$

$$\Rightarrow (-1)^{n+p} f(x) < 0.$$

But f has no zeros in I_p, so

$$(-1)^{n+p} f(x) < 0 \quad (x \in I_p).$$

- That x_p and x_{p-1} cannot both be zeros of multiplicity 2 is ruled out by consideration of

$$g(z) = \frac{f(z)}{(z-x_{p-1})(z-x_p)}.$$

The zero theory for f' can be reduced to that for f. To begin with, matters are trivial if

$$f(x) = C \sin A(x-B) \quad (C \neq 0),$$

so this case can be ignored. Suppose, therefore, that $f(\frac{k\pi}{A} + B) \neq 0$ for some k and in 32.8 take

$$x = \frac{p\pi}{A} + \frac{\pi}{2A} + B \quad (p \in Z).$$

Then

$$\cos A(\frac{p\pi}{A} + \frac{\pi}{2A} + B - B)$$

$$= \cos(p\pi + \frac{\pi}{2}) = \cos p\pi \cos \frac{\pi}{2} - \sin p\pi \sin \frac{\pi}{2}$$

$$= 0$$

and

$$\sin A(\frac{p\pi}{A} + \frac{\pi}{2A} + B - B)$$

$$= \sin(p\pi + \frac{\pi}{2}) = \sin p\pi \cos \frac{\pi}{2} + \sin \frac{\pi}{2} \cos p\pi$$

$$= (-1)^p$$

=>

$$(-1)^p f'(\frac{p\pi}{A} + \frac{\pi}{2A} + B)$$

$$= (-1)^{n-1} M(p) \quad (M(p) > 0)$$

=>

$$(-1)^{n-1}(-1)^p f'(\frac{p\pi}{A} + \frac{\pi}{2A} + B) > 0$$

=>

$$(-1)^{n'}(-1)^p f'(\frac{p\pi}{A} + \frac{\pi}{2A} + B) > 0,$$

where

$$\begin{cases} n' = 0 \text{ if } n = 1 \\ n' = 1 \text{ if } n = 0. \end{cases}$$

I.e.: f' is standard of level $\frac{\pi}{2A} + B$.

N.B. The ambient interval per f' is

$$I'_p = \left]\frac{(p-1)\pi}{A} + \frac{\pi}{2A} + B, \frac{p\pi}{A} + \frac{\pi}{2A} + B\right[.$$

32.14 LEMMA The zeros of f' are real (cf. 32.11).

32.15 LEMMA The zeros of f' are simple.

PROOF The only possibility for a nonsimple zero is at an endpoint of an ambient interval (cf. 32.9) and at such an endpoint, f' does not vanish.

32.16 LEMMA $\forall\, p \in Z$, f' has a zero in the ambient interval I'_p (it being necessarily unique).

PROOF We have

$$(-1)^{n'}(-1)^{p-1}f'\left(\frac{(p-1)\pi}{A}+\frac{\pi}{2A}+B\right) > 0$$

and

$$(-1)^{n'}(-1)^{p}f'\left(\frac{p\pi}{A}+\frac{\pi}{2A}+B\right) > 0.$$

- <u>p even</u>: Then

$$(-1)^{n'}f'\left(\frac{(p-1)\pi}{A}+\frac{\pi}{2A}+B\right) < 0$$

while

$$(-1)^{n'}f'\left(\frac{p\pi}{A}+\frac{\pi}{2A}+B\right) > 0.$$

- <u>p odd</u>: Then

$$(-1)^{n'}f\left(\frac{(p-1)\pi}{A}+\frac{\pi}{2A}+B\right) > 0$$

while

$$(-1)^{n'}f\left(\frac{p\pi}{A}+\frac{\pi}{2A}+B\right) < 0.$$

But this means that f' has a zero in I_p'.

32.17 EXAMPLE Take C per 32.12 (=> A = 1, B = 0) -- then C' is standard of level $\frac{\pi}{2}$ and n = 0 => n' = 1

=>

$$(-1)^{1}(-1)^{k}C'\left(k\pi+\frac{\pi}{2}\right) > 0.$$

And all the zeros of C' are real, each ambient interval I_p' contains a single zero and this zero is simple.

There is another situation which arises in the applications.

32.18 DEFINITION Let $f \in RB_0(A)$ -- then f is <u>semi-standard</u> of level B if $\exists\ n = 0$ or 1 and $B \in R$ such that $\forall\ k \in Z$,

$$\begin{cases} (-1)^{n+k} f(\frac{k\pi}{A} + B) \le 0 & (k \ge 1) \\ (-1)^{n+k} f(\frac{k\pi}{A} + B) \ge 0 & (k \le 0). \end{cases}$$

[Note: A fundamental class of examples is dealt with in the next §.]

Suppose that f is semi-standard of level B. Fix $x_0 \in I_1 =]B, \frac{\pi}{A} + B[$ and let

$$g(z) = (x_0 - z) f(z).$$

Impose the condition

$$\sup_{x \in R} |xf(x)| < \infty.$$

Then g is standard of level B. But $g(x_0) = 0$, thus g has a unique zero in I_1, viz. x_0. Therefore

$$x \in I_1 \Rightarrow f(x) \neq 0.$$

In addition, however,

$$(-1)^{n+1} g'(x_0) > 0 \quad \text{(cf. 32.9)}.$$

So

$$f(x_0) = g'(x_0)$$

=>

$$\begin{aligned} (-1)^n f(x_0) &= (-1)^n (-1)^1 g'(x_0) \\ &= (-1)^{n+1} g'(x_0) \\ &> 0. \end{aligned}$$

Therefore

$$x \in I_1 \Rightarrow (-1)^n f(x) > 0.$$

32.19 THEOREM Suppose that f is semi-standard of level B and

$$\sup_{x \in R} |xf(x)| < \infty.$$

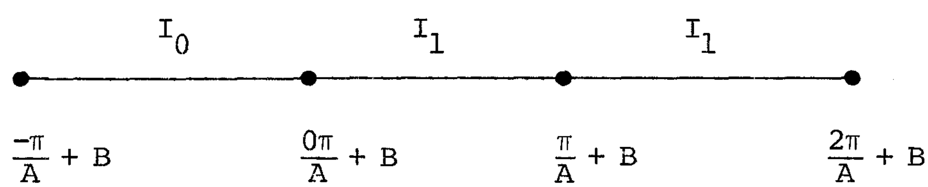
Then all the zeros of f are real (cf. 32.11). Furthermore, the ambient interval

$$I_p =]\frac{(p-1)\pi}{A} + B, \frac{p\pi}{A} + B[\quad (p \in Z, p \neq 1)$$

contains at most one zero of f and if there is one, then it must be simple. Finally,

$$x \in I_1 \Rightarrow (-1)^n f(x) > 0.$$

Picture:



32.30 THEOREM Suppose that f is semi-standard of level B and

$$\sup_{x \in R} |xf(x)| < \infty.$$

- If $f(B) = 0$, then its multiplicity is equal to 1 and there are no zeros of f in $I_0 \cup I_1$.

[Apply 32.13 to

$$g(z) = (B-z)f(z).$$

Then per g, B is a zero of multiplicity 2, hence (p = 0)

$$(-1)^n g(x) < 0 \quad (x \in I_0 \cup I_1)$$

=>

$$(-1)^n(B-x)f(x) < 0 \quad (x \in I_0)$$

=>

$$(-1)^n f(x) < 0 \quad (x \in I_0).$$

On the other hand, a priori,

$$(-1)^n f(x) > 0 \quad (x \in I_1).$$

- If $f(\frac{\pi}{A} + B) = 0$, then its multiplicity is equal to 1 and there are no zeros of f in $I_1 \cup I_2$.

[Apply 32.13 to

$$g(z) = (\frac{\pi}{A} + B - z)f(z).$$

Then per g, $\frac{\pi}{A} + B$ is a zero of multiplicity 2, hence (p = 1)

$$(-1)^{n+1}g(x) < 0 \quad (x \in I_1 \cup I_2)$$

=>

$$(-1)^{n+1}(\frac{\pi}{A} + B - x)f(x) < 0 \quad (x \in I_2)$$

=>

$$(-1)^n(x - \frac{\pi}{A} - B)f(x) < 0 \quad (x \in I_2)$$

=>

$$(-1)^n f(x) < 0 \quad (x \in I_2).$$

On the other hand, a priori,

$$(-1)^n f(x) > 0 \quad (x \in I_1).]$$

32.21 REMARK The condition

$$\sup_{x \in R} |xf(x)| < \infty$$

is not automatic (consider sin A(x-B)).

§33. *ZEROS OF* $W_{A,\alpha}$

Working on $]0,A[$ $(A > 0)$, suppose that ϕ is defined on $]0,A[$ and is integrable on $[0,A]$. Assume further that ϕ is positive and increasing on $]0,A[$.

33.1 NOTATION Given $\alpha \in [0,\pi[$, let

$$W_{A,\alpha}(z) = \int_0^A \phi(t)\sin(zt + \alpha)dt,$$

thus

$$W_{A,\alpha}(z) = (\sin \alpha)C_A(z) + (\cos \alpha)S_A(z),$$

where

$$C_A(z) = \int_0^A \phi(t)\cos zt\ dt,\ S_A(z) = \int_0^A \phi(z)\sin zt\ dt.$$

It is clear that $W_{A,\alpha} \in RB_0(A)$.

33.2 LEMMA $W_{A,\alpha}$ is semi-standard of level $-\frac{\alpha}{A}$.

PROOF In 32.18, take $n = 0$, the issue being $\forall\ k \in Z$ the inequalities

$$\begin{cases} (-1)^k W_{A,\alpha}(\frac{k\pi-\alpha}{A}) \leq 0 & (k \geq 1) \\ (-1)^k W_{A,\alpha}(\frac{k\pi-\alpha}{A}) \geq 0 & (k \leq 0). \end{cases}$$

- <u>$k = 0$</u>: Here

$$W_{A,\alpha}(-\frac{\alpha}{A}) = \int_0^A \phi(t)\sin(\frac{\alpha(A-t)}{A})dt \geq 0$$

and

$$W_{A,\alpha}(-\frac{\alpha}{A}) = 0$$

iff $\alpha = 0$.

• <u>k = 1,2,...:</u> Here

$$W_{A,\alpha}(\frac{k\pi-\alpha}{A}) = \frac{A}{k\pi-\alpha}\int_{\alpha}^{k\pi} \phi(\frac{A(s-\alpha)}{k\pi-\alpha})\sin s\, ds$$

and

$$\frac{A}{k\pi-\alpha} > 0.$$

• <u>—>:k odd</u> Split the interval of integration $[\alpha,k\pi]$ into the closed subintervals $[\alpha,\pi]$, $[\pi,3\pi]$,..., $[k\pi-2\pi, k\pi]$ -- then the integral over each of these subintervals is nonnegative, hence

$$(-1)^k W_{A,\alpha}(\frac{k\pi-\alpha}{A}) \leq 0.$$

• <u>—>:k even</u> Split the interval of integration $[\alpha,k\pi]$ into the closed subintervals $[\alpha,2\pi]$, $[2\pi,4\pi]$,..., $[k\pi-2\pi, k\pi]$ -- then the integral over each of these subintervals is nonpositive, hence

$$(-1)^k W_{A,\alpha}(\frac{k\pi-\alpha}{A}) \leq 0.$$

• <u>k = -1, -2,...:</u> Here

$$W_{A,\alpha}(\frac{k\pi-\alpha}{A}) = \frac{A}{k\pi-\alpha}\int_{-\alpha}^{-k\pi} \phi(\frac{A(s+\alpha)}{\alpha-k\pi})\sin s\, ds$$

and

$$\frac{A}{k\pi-\alpha} < 0.$$

• <u>—>:k odd</u> Split the interval of integration $[-\alpha, -k\pi]$ into the closed subintervals $[-\alpha,\pi]$, $[\pi,3\pi]$,..., $[-k\pi-2\pi, -k\pi]$ -- then the integral over each of these subintervals is nonpositive, hence

$$(-1)^k W_{A,\alpha}(\frac{k\pi-\alpha}{A}) \geq 0.$$

• <u>$\Longrightarrow$:k even</u> Split the interval of integration $[-\alpha, -k\pi]$ into the closed subintervals $[-\alpha,0]$, $[0, 2\pi], \ldots, [-k\pi - 2\pi, -k\pi]$ -- then the integral over each of these subintervals is nonpositive, hence

$$(-1)^k W_{A,\alpha}\left(\frac{k\pi-\alpha}{A}\right) \geq 0.$$

33.3 APPLICATION If ϕ is bounded on $]0,A[$, then all the zeros of $W_{A,\alpha}$ are real. Furthermore, the ambient interval

$$I_p = \left]\frac{(p-1)\pi-\alpha}{A}, \frac{p\pi-\alpha}{A}\right[\qquad (p \in Z,\ p \neq 1)$$

contains at most one zero of $W_{A,\alpha}$ and if there is one, then it must be simple. Finally,

$$x \in I_1 \Rightarrow (-1)^n W_{A,\alpha}(x) > 0$$

$$\Rightarrow W_{A,\alpha}(x) > 0 \qquad (n = 0).$$

[In fact,

$$\sup_{x\in R} |xW_{A,\alpha}(x)| \leq 2 \lim_{t\uparrow A} \phi(t) < \infty,$$

so one can quote 32.19.]

A finer analysis will lead to more precise results.

• <u>$k \geq 1$ (k odd):</u> Suppose that

$$W_{A,\alpha}\left(\frac{k\pi-\alpha}{A}\right) = 0.$$

Then there exist constants

$$0 < c_0 \leq c_1 \leq \cdots \leq c_{(k-1)/2}$$

and points

$$t_{-1} = 0,\ t_j = A\frac{(2j+1)\pi-\alpha}{k\pi-\alpha}$$

such that

$$\phi(t) = c_j \ (t_{j-1} < t < t_j)\,(0 \le j \le \frac{k-1}{2}).$$

Therefore

$$W_{A,\alpha}(x) = \frac{2}{x}\sin(\frac{A\pi x}{k\pi-\alpha}) \sum_{j=0}^{(k-1)/2} c_j \sin(\frac{2j\pi-\alpha}{k\pi-\alpha}Ax + \alpha).$$

- <u>k ≥ 1 (k even)</u>: Suppose that

$$W_{A,\alpha}(\frac{k\pi-\alpha}{A}) = 0.$$

Then there exist constants

$$0 < c_0 \le c_1 \le \dots \le c_{(k-2)/2}$$

and points

$$t_{-1} = 0,\ t_j = A\,\frac{(2j+2)\pi-\alpha}{k\pi-\alpha}$$

such that

$$\phi(t) = c_j \ (t_{j-1} < t < t_j) \quad (0 \le j \le \frac{k-2}{2}).$$

Therefore

$$W_{A,\alpha}(x) = \frac{2}{x}\sin(\frac{A\pi x}{k\pi-\alpha}) \sum_{j=0}^{(k-2)/2} c_j \sin(\frac{(2j+1)\pi-\alpha}{k\pi-\alpha}Ax + \alpha).$$

- <u>k ≤ -1 (k odd)</u>: Suppose that

$$W_{A,\alpha}(\frac{k\pi-\alpha}{A}) = 0.$$

Then there exist constants

$$0 < c_0 \le c_1 \le \dots \le c_{(-k-1)/2}$$

and points

$$t_{-1} = 0,\ t_j = A\,\frac{(2j+1)\pi+\alpha}{\alpha-k\pi}$$

such that

$$\phi(t) = c_j \quad (t_{j-1} < t < t_j) \qquad (0 \le j \le \frac{-k-1}{2}).$$

Therefore

$$W_{A,\alpha}(x) = \frac{2}{x}\sin\left(\frac{A\pi x}{\alpha-k\pi}\right) \sum_{j=0}^{(-k-1)/2} c_j \sin\left(\frac{2j\pi+\alpha}{\alpha-k\pi}Ax + \alpha\right).$$

- <u>$k \le -1$ (k even)</u>: Suppose that

$$W_{A,\alpha}\left(\frac{k\pi-\alpha}{A}\right) = 0.$$

Then there exist constants

$$0 < c_0 \le c_1 \le \dots \le c_{-k/2}$$

and points

$$t_{-1} = 0,\ t_j = A\,\frac{2j\pi+\alpha}{\alpha-k\pi}$$

such that

$$\phi(t) = c_j \quad (t_{j-1} < t < t_j) \qquad (0 \le j \le -\frac{k}{2}).$$

Therefore

$$W_{A,\alpha}(x) = \frac{2}{x}\sin\left(\frac{A\pi x}{\alpha-k\pi}\right) \sum_{j=0}^{-k/2} c_j \sin\left(\frac{(2j-1)\pi+\alpha}{\alpha-k\pi}Ax + \alpha\right).$$

33.4 NOTATION Write

$$E(A,\alpha,k)$$

for the set of those ϕ such that

$$W_{A,\alpha}(\frac{k\pi-\alpha}{A}) = 0$$

for some $k \in Z - \{0\}$ and put

$$E(A,\alpha) = \underset{k}{\cup} E(A,\alpha,k).$$

[Note: In general,

$$E(A,\alpha,k_1) \cap E(A,\alpha,k_2) \neq \emptyset.]$$

33.5 RECONCILIATION Take $A = 1$, $\alpha = 0$, hence

$$W_{1,0}(z) = \int_0^1 \phi(t)\sin zt\, dt.$$

Recall now the definition of "exceptional" from 29.14 and the notation $E(1,0)$ from 29.15 -- then the claim is that the two possible meanings of $E(1,0)$ are one and the same. To see this, consider

$$W_{1,0}(\frac{k\pi-\alpha}{A}) \equiv W_{1,0}(k\pi) \quad (k = \pm 1, \pm 2,\ldots),$$

there being no loss of generality in assuming that $k = 1,2,\ldots$.

- <u>k odd</u>: Here

$$W_{1,0}(k\pi) > 0 \quad (k = 1,3,\ldots) \quad (\text{cf. } 31.3).$$

Therefore

$$E(1,0,\ k \text{ odd}) = \emptyset.$$

- <u>k even</u>: Suppose that

$$W_{1,0}(2n\pi) = 0 \text{ for some } n = 1,2,\ldots .$$

I.e.:

$$\int_0^1 \phi(t)\sin 2n\pi t\, dt = 0.$$

But this implies that ϕ is exceptional (look at the proof of 31.3). Therefore

$$E(1,0,\ k\ \text{even})$$

is comprised of exceptional ϕ, so

$$\bigcup_{n=1}^{\infty} E(1,0,2n)$$

is contained in the $E(1,0)$ per 29.15. To turn matters around, take an exceptional ϕ and write

$$f(z) = \int_0^1 \phi(t)e^{\sqrt{-1}\, zt}dt$$

$$= C(z) + \sqrt{-1}\, S(z)$$

where, of course,

$$S(z) \equiv W_{1,0}(z).$$

Then in the notation of 29.20,

$$f(2\pi q) = 0$$

$$\Rightarrow$$

$$C(2\pi q) + \sqrt{-1}\, S(2\pi q) = 0$$

$$\Rightarrow$$

$$S(2\pi q) = 0 \Rightarrow W_{1,0}(2\pi q) = 0$$

$$\Rightarrow$$

$$\phi \in E(1,0,2q).$$

Conclusion:

$$E(1,0) \subset \bigcup_{n=1}^{\infty} E(1,0,2n) \subset E(1,0).$$

33.6 REMARK If $\phi \in E(A,\alpha)$, then

$$\sup_{x\in R} |xW_{A,\alpha}(x)| < \infty.$$

[Note: Accordingly, all the particulars of the semi-standard theory developed at the end of §32 are in force but the detailed explication thereof will be left to the reader.]

33.7 LEMMA If $\phi \notin E(A,\alpha)$, then

$$\begin{cases} (-1)^k W_{A,\alpha}(\frac{k\pi-\alpha}{A}) < 0 & (k \geq 1) \\ (-1)^k W_{A,\alpha}(\frac{k\pi-\alpha}{A}) > 0 & (k \leq -1) \end{cases}$$

and at $k = 0$,

$$W_{A,\alpha}(-\frac{\alpha}{A}) > 0 \quad (0 < \alpha < \pi).$$

33.8 LEMMA If $\phi \notin E(A,\alpha)$ and if

$$\sup_{x\in R} |xW_{A,\alpha}(x)| < \infty,$$

then all the zeros of $W_{A,\alpha}$ are real (cf. 32.11) and simple (cf. infra).

PROOF The ambient interval

$$I_p =]\frac{(p-1)\pi}{A} - \frac{\alpha}{A}, \frac{p\pi}{A} - \frac{\alpha}{A}[\quad (p \in Z, p \neq 0,1)$$

contains exactly one zero of $W_{A,\alpha}$ and it is simple (cf. 32.19).

- <u>p = 0</u>: $I_0 =]-\frac{\pi}{A} - \frac{\alpha}{A}, -\frac{\alpha}{A}[$. If $0 < \alpha < \pi$, then

$$(-1)^1 W_{A,\alpha}(-\frac{\pi}{A} - \frac{\alpha}{A}) > 0$$

=>

$$W_{A,\alpha}(-\frac{\pi}{A} - \frac{\alpha}{A}) < 0.$$

Meanwhile,

$$W_{A,\alpha}(-\frac{\alpha}{A}) > 0.$$

So $W_{A,\alpha}$ has a (unique) zero in I_0 and it is simple (cf. 32.19). If $\alpha = 0$, then $W_{A,0}(-\frac{0}{A}) = 0$ and its multiplicity is equal to 1 and there are no zeros of $W_{A,0}$ in $I_0 \cup I_1$ (cf. 32.20).

- $\underline{p = 1}$: $I_1 =]-\frac{\alpha}{A}, \frac{\pi}{A} - \frac{\alpha}{A}[$. In this situation,

$$x \in I_1 \Rightarrow W_{A,\alpha}(x) > 0 \quad (n = 0),$$

thus in I_1, $W_{A,\alpha}$ is zero free.

[Note: $\frac{k\pi-\alpha}{A}$ is a zero of $W_{A,\alpha}$ only when $k = 0$, $\alpha = 0$.]

33.9 THEOREM If $\phi \notin E(A,\alpha)$, then all the zeros of $W_{A,\alpha}$ are real and simple.

PROOF The idea is to reduce things to the bounded case, i.e., to 33.8. To this end, for $n > 1$, let

$$\phi_n(t) = \phi(t) \quad (0 < t \leq A - \frac{1}{n})$$

and

$$\phi_n(t) = \phi(A - \frac{1}{n}) + t - A + \frac{1}{n} \quad (A - \frac{1}{n} \leq t < A).$$

Then $\phi_n \notin E(A,\alpha)$ and

$$\int_0^A |\phi(t) - \phi_n(t)|dt$$

$$= \int_{A - \frac{1}{n}}^{A} |\phi(t) - \phi_n(t)|dt$$

$$\leq \int_{A - \frac{1}{n}}^{A} |\phi(t)|dt + \frac{1}{2n^2}$$

$$\to 0 \ (n \to \infty).$$

Put

$$W_{A,\alpha,n}(z) = \int_0^A \phi_n(t)\sin(zt + \alpha)dt.$$

Then $W_{A,\alpha,n} \to W_{A,\alpha}$ uniformly on compact subsets of $\mathbb{C}$. On the other hand, ϕ_n is bounded on $]0,A[$, hence

$$\sup_{x\in R} |xW_{A,\alpha,n}(x)| < \infty \quad \text{(cf. 33.3)}.$$

Therefore all the zeros of $W_{A,\alpha,n}$ are real and simple (cf. 33.8), so all the zeros of $W_{A,\alpha}$ are real and it remains to establish their simplicity.

- <u>$0 < \alpha < \pi$</u>: Given $p \in Z$, let D_p be the rectangle

$$\{z: |\text{Im } z| \leq 1, \frac{(p-1)\pi}{A} - \frac{\alpha}{A} \leq \text{Re } z \leq \frac{p\pi}{A} - \frac{\alpha}{A}\}.$$

Then for $z \in \partial D_p$ and $n >> 0$,

$$|W_{A,\alpha,n}(z) - W_{A,\alpha}(z)|$$

$$< \min_{\partial Dp} |W_{A,\alpha}| \leq |W_{A,\alpha}(z)|.$$

But this implies by Rouché that $W_{A,\alpha}$ and $W_{A,\alpha,n}$ have the same number of zeros inside D_p.

• $\underline{0 = \alpha}$: At level 0,1, work with $D_0 \cup D_1$ rather than D_0 and D_1 separately.

Implicit in the foregoing is a description of the position of the zeros of $W_{A,\alpha}$ (what was said in the proof of 33.8 is valid in general).

33.10 EXAMPLE By definition,

$$W_{1,\frac{\pi}{2}}(z) = \int_0^1 \phi(t)\cos zt\, dt.$$

Assuming that $\phi \notin E(1,0)$ (a restriction that is actually unnecessary...), the theory predicts that all the zeros of $W_{1,\frac{\pi}{2}}$ are real. As for their position, $W_{1,\frac{\pi}{2}}$ has a zero in each of the ambient intervals

$$I_2 =]\frac{\pi}{2}, \frac{3\pi}{2}[\ , \ I_3 =]\frac{3\pi}{2}, \frac{5\pi}{2}[\ , \ I_4 =]\frac{5\pi}{2}, \frac{7\pi}{2}[, \ \ldots$$

and this zero is unique and simple. Moreover,

$$C(\frac{\pi}{2}) > 0,\ C(\frac{3\pi}{2}) < 0,\ C(\frac{5\pi}{2}) > 0,\ C(\frac{7\pi}{2}) < 0 \ \ldots$$

and $I_1 =]-\frac{\pi}{2}, \frac{\pi}{2}[$ is zero free. All the positive zeros of $W_{1,\frac{\pi}{2}}$ are thereby accounted for so 31.1 has been recovered.

33.11 LEMMA We have

$$\int_0^A \phi(t)\cos(zt + \alpha)dt = \begin{cases} W_{A,\alpha + \frac{\pi}{2}} & (0 \le \alpha < \frac{\pi}{2}) \\ \\ -W_{A,\alpha - \frac{\pi}{2}} & (\frac{\pi}{2} \le \alpha < \pi). \end{cases}$$

§34. *ZEROS OF* f_A

34.1 NOTATION Given $\phi \in L^1[0,A]$, put

$$f_A(z) = \int_0^A \phi(t)e^{\sqrt{-1}\, zt}dt,$$

thus

$$f_A(z) = C_A(z) + \sqrt{-1}\, S_A(z),$$

where

$$C_A(z) = \int_0^A \phi(t)\cos zt\, dt,\ S_A(z) = \int_0^A \phi(t)\sin zt\, dt.$$

[Note: To be in agreement with §30, drop the "A" if $A = 1$.]

34.2 THEOREM If $\phi \in L^1[0,A]$ is positive and increasing on $]0,A[$ and if ϕ is not a step function, then the zeros of $f_A(z)$ lie in the open upper half-plane.

N.B. Since ϕ is not a step function, it follows that $\forall\ \alpha$,

$$\phi \notin E(A,\alpha).$$

Therefore all the zeros of $W_{A,\alpha}$ are real and simple (cf. 33.9) and this persists to all $\alpha \in R$ (elementary verification).

34.3 REMARK Take $A = 1$ -- then this result implies 29.16 (granted 29.19).

Let P and Q be nonconstant real entire functions.

34.4 CHEBOTAREV CRITERION Assume:

- P and Q have no common zeros.
- $\forall\ \mu,\nu \in R$, $\mu^2 + \nu^2 \neq 0$, the combination $\mu P + \nu Q$ has no zeros in $C - R$.

- $\exists\, x_0 \in R$ such that

$$P(x_0)Q'(x_0) - Q(x_0)P'(x_0) > 0.$$

Then

$$F(z) = P(z) + \sqrt{-1}\, Q(z)$$

has all its zeros in the open upper half-plane.

[Note: It is an a posteriori conclusion that $\forall\, x \in R$,

$$P(x)Q'(x) - Q(x)P'(x) > 0.]$$

34.5 REMARK Compare the above with what has been said in §16: There it was a question of nonconstant real polynomials and zeros in the open lower half-plane, hence the sign switch to

$$Q(x_0)P'(x_0) - P(x_0)Q'(x_0) > 0.$$

N.B. It is clear that $F(z)$ has no zeros on the real axis:

$$F(x_0) = P(x_0) + \sqrt{-1}\, Q(x_0) = 0$$

$$\Rightarrow P(x_0) = 0,\ Q(x_0) = 0.$$

Proceeding to the proof, begin by noting that the mereomorphic function

$$\theta(z) = \frac{Q(z)}{P(z)}$$

does not take on real values for $\operatorname{Im} z \neq 0$, thus it maps the open upper half-plane either onto itself or onto the open lower half-plane. But

$$P(x_0)Q'(x_0) - Q(x_0)P'(x_0) > 0$$

$$\Rightarrow \theta'(x_0) > 0,$$

so $\theta(z)$ maps the open upper half-plane onto itself. Since

$$\frac{P + \sqrt{-1}\,Q}{P - \sqrt{-1}\,Q} = \frac{1 + \sqrt{-1}\,\theta}{1 - \sqrt{-1}\,\theta},$$

it then follows that

$$\operatorname{Im} z > 0 \Rightarrow \left|\frac{P(z) + \sqrt{-1}\,Q(z)}{P(z) - \sqrt{-1}\,Q(z)}\right| < 1.$$

Next

$$\begin{cases} P(\bar{z}) = \overline{P(z)} \\ Q(\bar{z}) = \overline{Q(z)}, \end{cases}$$

hence

$$P(z_0) + \sqrt{-1}\,Q(z_0) = 0$$

$$\Rightarrow$$

$$P(\bar{z}_0) - \sqrt{-1}\,Q(\bar{z}_0) = 0.$$

Accordingly, it need only be shown that $P - \sqrt{-1}\,Q$ has no zeros in the open upper half-plane. However

$$\frac{P + \sqrt{-1}\,Q}{P - \sqrt{-1}\,Q}$$

is unbounded near any zero of $P - \sqrt{-1}\,Q$ which is not a zero of $P + \sqrt{-1}\,Q$. And this means that any zero of $P - \sqrt{-1}\,Q$ in the open upper half-plane must be a zero of $P + \sqrt{-1}\,Q$. But

$$\begin{cases} P(z_0) - \sqrt{-1}\,Q(z_0) = 0 \\ P(z_0) + \sqrt{-1}\,Q(z_0) = 0 \end{cases} \qquad (\operatorname{Im} z_0 > 0)$$

$$\Rightarrow \begin{cases} 2P(z_0) = 0 & \Rightarrow P(z_0) = 0 \\ -2\sqrt{-1}\,Q(z_0) = 0 & \Rightarrow Q(z_0) = 0, \end{cases}$$

contradicting the assumption that P and Q have no common zeros.

Having dispensed with the preparation, we are now in a position to give the proof of 34.2. Bearing in mind that

$$f_A(z) = C_A(z) + \sqrt{-1}\,S_A(z),$$

start by writing

$$W_{A,\alpha}(z) = (\sin\alpha)C_A(z) + (\cos\alpha)S_A(z).$$

Then there are three items to be checked.

1. C_A and S_A have no common zeros. To see this, observe that

$$W_{A,\frac{\pi}{2}}(z) = C_A(z),\ W_{A,0}(z) = S_A(z),$$

so the zeros of $C_A(z)$ and $S_A(z)$ are real and simple. If $C_A(x_0) = 0$, $S_A(x_0) = 0$ for some $x_0 \in R$, then $C_A'(x_0) \neq 0$, $S_A'(x_0) \neq 0$ and taking

$$\alpha = \arc\tan\left(-\frac{S_A'(x_0)}{C_A'(x_0)}\right),$$

we have

$$\begin{aligned} W_{A,\alpha}'(x_0) &= (\sin\alpha)C_A'(x_0) + (\cos\alpha)S_A'(x_0) \\ &= 0 \end{aligned}$$

for a suitable choice of arc tan. But this implies that x_0 is a zero of $W_{A,\alpha}$ of multiplicity ≥ 2 which cannot be.

2. $\forall\ \mu,\nu \in R$, $\mu^2 + \nu^2 \neq 0$, the combination $\mu C_A + \nu S_A$ has no zeros in $C - R$.

The cases $\mu \neq 0$, $\nu = 0$ and $\mu = 0$, $\nu \neq 0$ being obvious, consider the remaining four possibilities.

- <u>$\mu > 0$, $\nu > 0$:</u> Write

$$\mu C_A + \nu S_A = \sqrt{\mu^2 + \nu^2}\ \left(\frac{\mu}{\sqrt{\mu^2 + \nu^2}} C_A + \frac{\nu}{\sqrt{\mu^2 + \nu^2}} S_A\right)$$

and determine α by

$$\sin\alpha = \frac{\mu}{\sqrt{\mu^2 + \nu^2}}, \quad \cos\alpha = \frac{\nu}{\sqrt{\mu^2 + \nu^2}}.$$

- <u>$\mu < 0$, $\nu < 0$:</u> Write

$$\mu C_A + \nu S_A = -\sqrt{\mu^2 + \nu^2}\ \left(\frac{-\mu}{\sqrt{\mu^2 + \nu^2}} C_A + \frac{-\nu}{\sqrt{\mu^2 + \nu^2}} S_A\right)$$

and determine α by

$$\sin\alpha = \frac{-\mu}{\sqrt{\mu^2 + \nu^2}}, \quad \cos\alpha = \frac{-\nu}{\sqrt{\mu^2 + \nu^2}}.$$

- <u>$\mu < 0$, $\nu > 0$:</u> Write

$$\mu C_A + \nu S_A = \sqrt{\mu^2 + \nu^2}\ \left(-\frac{-\mu}{\sqrt{\mu^2 + \nu^2}} C_A + \frac{\nu}{\sqrt{\mu^2 + \nu^2}} S_A\right)$$

$$= \sqrt{\mu^2 + \nu^2}\ ((-\sin\alpha) C_A + (\cos\alpha) S_A)$$

$$= \sqrt{\mu^2 + \nu^2}\ ((\sin -\alpha) C_A + (\cos -\alpha) S_A).$$

- <u>$\mu > 0$, $\nu < 0$:</u> Write

$$\mu C_A + \nu S_A = \sqrt{\mu^2 + \nu^2}\ \left(\frac{\mu}{\sqrt{\mu^2 + \nu^2}} C_A - \frac{-\nu}{\sqrt{\mu^2 + \nu^2}} S_A\right)$$

$$= \sqrt{\mu^2 + \nu^2}\ ((\sin \alpha) C_A - (\cos \alpha) S_A)$$

$$= \sqrt{\mu^2 + \nu^2}\ (- (\sin -\alpha) C_A - (\cos -\alpha) S_A)$$

$$= - \sqrt{\mu^2 + \nu^2}\ ((\sin -\alpha) C_A + (\cos -\alpha) S_A).$$

3. $\exists\ x_0 \in R$ such that

$$C_A(x_0) S_A'(x_0) - S_A(x_0) C_A'(x_0) \neq 0.$$

In fact,

$$C_A(0) S_A'(0) - S_A(0) C_A'(0)$$

$$= C_A(0) S_A'(0)$$

$$= (\int_0^1 \phi(t)dt)(\int_0^1 \phi(t) t\ dt)$$

$$> 0.$$

34.6 REMARK If ϕ is a step function and if $\phi \in E(A,\alpha)$, then $f_A(z)$ has an infinity of real zeros (cf. 29.21) (all of which are simple) and there is an analog of 29.22.

34.7 NOTATION Given $\phi \in L^1[0,A]$, let

$$\begin{cases} \mathfrak{C}_A(z) = \int_0^A \phi(A-t)\cos zt\ dt \\ \mathfrak{S}_A(z) = \int_0^A \phi(A-t)\sin zt\ dt. \end{cases}$$

34.8 IDENTITIES

$$f_A(z)e^{-\sqrt{-1}\,Az} = \mathfrak{C}_A(z) - \sqrt{-1}\,\mathfrak{S}_A(z)$$

and

$$\begin{cases} C_A(z) = \mathfrak{C}_A(z)\cos Az + \mathfrak{S}_A(z)\sin Az \\ S_A(z) = \mathfrak{C}_A(z)\sin Az - \mathfrak{S}_A(z)\cos Az. \end{cases}$$

34.9 RAPPEL If 0 and A are the effective limits of integration (thus excluding the possibility that $\phi = 0$ almost everywhere), then $f_A(z)$ has an infinity of zeros (see the initial comments in §29).

34.10 LEMMA Put

$$H(s) = -\frac{y}{\pi(y^2 + s^2)} \quad (y \in R).$$

Then

$$\int_{-\infty}^{\infty} e^{\sqrt{-1}\,st} H(s)\,ds = e^{y|t|}.$$

34.11 THEOREM If $\phi \in L^1[0,A]$ is real and if

$$\mathfrak{C}_A(x) \geq 0 \quad (x \in R),$$

then $f_A(z)$ has no zeros in the open lower half-plane.

PROOF Let $z = x + \sqrt{-1}\,y$ $(y < 0)$ and write

$$f_A(z)e^{-\sqrt{-1}\,Az}$$

$$= \int_0^A \phi(t)e^{\sqrt{-1}\,zt}e^{-\sqrt{-1}\,Az}dt$$

$$= \int_0^A \phi(t) e^{\sqrt{-1}\, z(t-A)} dt$$

$$= \int_0^A \phi(t) e^{-\sqrt{-1}\, z(A-t)} dt$$

$$= \int_0^A \phi(A-t) e^{-\sqrt{-1}\, xt} e^{yt} dt$$

$$= \int_0^A \phi(A-t) e^{-\sqrt{-1}\, xt} \left(\int_{-\infty}^{\infty} e^{\sqrt{-1}\, st} H(s)\, ds\right)$$

$$= \int_{-\infty}^{\infty} H(s) \left(\int_0^A e^{\sqrt{-1}(s-x)t} \phi(A-t)\, dt\right) ds$$

$$= \int_{-\infty}^{\infty} H(s+x) (\mathfrak{C}_A(s) + \sqrt{-1}\, \mathfrak{S}_A(s))\, ds.$$

But $\mathfrak{C}_A \not\equiv 0$ (consult the Appendix below), hence

$$\operatorname{Re}(f_A(z) e^{-\sqrt{-1}\, Az})$$

$$= - \int_{-\infty}^{\infty} \frac{1}{\pi(y^2 + (s+x)^2)} \mathfrak{C}_A(s)\, ds$$

$$> 0.$$

34.12 REMARK Any real zero of $f_A(z)$ (if there is one) is necessarily simple.

34.13 EXAMPLE If $\phi \in C[0,A]$ is real, $\phi(0) = 0$, $\phi(A) > 0$, and the function

$$t \rightarrow \phi((A - |t|)_+)$$

is positive definite on R, then

$$\mathfrak{C}_A(x) \geq 0 \quad (x \in R),$$

so 34.11 is applicable.

APPENDIX

MÜNTZ CRITERION If $\lambda_1, \lambda_2, \ldots$ is a strictly increasing sequence of real numbers such that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty,$$

then the set

$$\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}$$

is total in $C[0,1]$.

EXAMPLE The set

$$\{t^0, t^2, t^4, \ldots\}$$

is total in $C[0,1]$.

APPLICATION If $\psi \in L^1[0,1]$ and if

$$\int_0^1 \psi(t)\,dt = 0, \int_0^1 t^{2k}\psi(t)\,dt = 0 \quad (k = 1,2,\ldots),$$

then $\psi = 0$ almost everywhere.

[Let

$$\Psi(t) = \int_0^t \psi(s)\,ds.$$

Then Ψ is absolutely continuous and $\Psi(0) = 0$, $\Psi(1) = 0$. Now integrate by parts to get

$$0 = \int_0^1 t^{2k}\psi(t)\,dt$$

$$= -2k \int_0^1 t^{2k-1}\Psi(t)\,dt \quad (k = 1,2,\ldots).$$

Therefore

$$\int_0^1 t^0 (t\Psi(t))dt = 0 \quad (k = 1)$$

$$\int_0^1 t^2 (t\Psi(t))dt = 0 \quad (k = 2)$$

$$\int_0^1 t^4 (t\Psi(t))dt = 0 \quad (k = 3)$$

$$\vdots$$

Define a bounded linear functional μ on $C[0,1]$ by the rule

$$\mu(g) = \int_0^1 g(t)(t\Psi(t))dt.$$

Then

$$\mu(t^{2k}) = 0 \quad (k = 0,1,2,\ldots)$$

$$\Rightarrow$$

$$\mu \equiv 0$$

$$\Rightarrow t\Psi(t) = 0 \ (0 \le t \le 1) \Rightarrow \Psi(t) = 0 \ (0 \le t \le 1).$$

But this implies that $\psi = 0$ almost everywhere.]

THEOREM If $C_A(z) \equiv 0$, then $\phi = 0$ almost everywhere ($\Rightarrow f_A(z) \equiv 0$).

PROOF Consider the expansion

$$\int_0^A \phi(t)\cos zt \, dt$$

$$= \int_0^A \phi(t) \sum_{k=0}^{\infty} \frac{(-1)^k (zt)^{2k}}{(2k)!} dt$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\int_0^A t^{2k} \phi(t) dt\right) z^{2k},$$

hence

$$\int_0^A t^{2k}\phi(t)dt = 0 \quad (k = 0,1,2,\ldots)$$

or still (letting $t = sA$),

$$A^{2k+1} \int_0^1 s^{2k}\phi(sA)ds = 0 \quad (k = 0,1,2,\ldots).$$

Consequently, $\phi(sA)$ vanishes almost everywhere $(0 \le s \le 1)$, so $\phi(t)$ vanishes almost everywhere $(0 \le t \le A)$.

N.B. If $\mathfrak{C}_A(z) \equiv 0$, then $\phi = 0$ almost everywhere ($\Rightarrow f_A(z) \equiv 0$) (argue analogously).

REMARK If $f_A(z) \equiv 0$, then $\phi = 0$ almost everywhere.

[In fact,

$$\begin{aligned} C_A(z) &= \int_0^A \phi(t)\cos zt\,dt \\ &= \int_0^A \phi(t)\,\frac{e^{\sqrt{-1}\,zt} + e^{-\sqrt{-1}\,zt}}{2}\,dt \\ &= \frac{f_A(z) + f_A(-z)}{2} \equiv 0.] \end{aligned}$$

§35. *MISCELLANEA*

Here there will be found a number of complements, some theoretical, others disguised as "examples".

35.1 LEMMA If $\phi \in L^1[0,A]$ is real valued and continuously differentiable and if $\phi(A) \neq 0$, then

$$C_A(z) = \int_0^A \phi(t)\cos zt\, dt$$

has an infinite number of real zeros.

PROOF In fact,

$$xC_A(x) = \phi(A)\sin(xA) - \int_0^A \phi'(t)\sin(xt)\,dt$$

$$= \phi(A)\sin(xA) + o(1) \quad (|x| \to \infty).$$

35.2 CHAKALOV CRITERION† Suppose given a sequence

$$\cdots < a_{-2} < a_{-1} < a_0 < a_1 < a_2 < \cdots$$

and real numbers

$$\ldots, A_{-2}, A_{-1}, A_0, A_1, A_2, \ldots,$$

where

$$A_k \neq 0,\ k = 0, \pm 1, \pm 2, \ldots .$$

Assume: $\exists$ integers p and q with $p < q$ such that A_k and A_{k+1} have the same sign for $k < p$ and for $k \geq q$. Put

$$R_n(z) = \sum_{k=-n+1}^{n} \frac{A_K}{z-a_k}$$

† Списание ЬАН 36 (1927), pp. 51–92.

and impose the condition that

$$R(z) = \lim_{n \to \infty} R_n(z)$$

uniformly on compact subsets of $C - \{a_k\}_{-\infty}^{\infty}$ -- then $R(z)$ has no more than $q - p$ nonreal zeros.

Maintaining the setup of 35.1, introduce the meromorphic function

$$R(z) = \frac{C_A(z)}{\cos(zA)}$$

and put

$$R_n(z) = \sum_{k=-n+1}^{n} (-1)^k \frac{C_A\left(\frac{(k-\frac{1}{2})\pi}{A}\right)}{z - \frac{(k-\frac{1}{2})\pi}{A}}.$$

Abbreviate

$$\frac{(k-\frac{1}{2})\pi}{A} \text{ to } a_k.$$

35.3 LEMMA We have

$$R(z) = \lim_{n \to \infty} R_n(z)$$

uniformly on compact subsets of $C - \{a_k\}_{-\infty}^{\infty}$.

Next

$$\lim_{k \to \pm\infty} (-1)^k a_k C_A(a_k)$$

$$= \phi(A) \lim_{k \to \pm\infty} (-1)^k \sin(a_k A)$$

$$= \phi(A) \lim_{k \to \pm\infty} (-1)^k \sin\left(\frac{(k - \frac{1}{2})\pi}{A} A\right)$$

$$= \phi(A) \lim_{k \to \pm\infty} (-1)^k (-1)(-1)^k$$

$$= -\phi(A) \neq 0.$$

If now

$$A_k \equiv (-1)^k C_A(a_k),$$

then the sequence

$$\ldots, A_{-2}, A_{-1}, A_0, A_1, A_2, \ldots$$

has but a finite number of sign changes.

[E.g.: Suppose that $L \equiv -\phi(A)$ is positive and send k to $+\infty$ -- then from some point on, A_k is also positive:

$$k >> 0 \Rightarrow |a_k A_k - L| < \frac{L}{2}$$

$$\Rightarrow \frac{L}{2} < a_k A_k < \frac{3L}{2}$$

$$\Rightarrow 0 < \frac{L}{2a_k} < A_k.]$$

[Note: These considerations also serve to show that the number of k for which $A_k = 0$ is finite.]

35.4 LEMMA If $\phi \in L^1[0,A]$ is real valued and continuously differentiable and if $\phi(A) \neq 0$, then

$$C_A(z) = \int_0^A \phi(t) \cos zt \, dt$$

has at most a finite number of nonreal zeros.

[Thanks to what has been said above, one has only to invoke 35.2.]

N.B. Therefore

$$C_A \in * - L - P \quad \text{(cf. 10.36)}.$$

35.5 EXAMPLE Take $\phi(t) = e^{-t}$ — then the zeros of

$$C_A(z) = \int_0^A e^{-t}\cos zt\,dt$$

$$= \frac{e^{-A}(z \sin Az - \cos Az) + 1}{z^2 + 1}$$

$$= \frac{\sqrt{-1}}{2}\left[\frac{e^{A(-1 - \sqrt{-1}\,z)} - 1}{z - \sqrt{-1}} - \frac{e^{A(-1 + \sqrt{-1}\,z)} - 1}{z + \sqrt{-1}}\right]$$

lie in the horizontal strip

$$-1 < y < 1 \quad \left(\text{cf. } 29.23 \left(\left|\frac{\phi'(t)}{\phi(t)}\right| = 1\right)\right).$$

The number of real zeros is infinite (cf. 35.1) while the number of nonreal zeros is finite (cf. 35.4). And the estimate $-1 < y < 1$ cannot be improved provided A is allowed to vary, i.e., given $\varepsilon > 0$, in

$$-1 < y < -1 + \varepsilon \cup 1 - \varepsilon < y < 1$$

there is a zero if $A >> 0$. Finally, any compact subset S of $-1 < y < 1$ is zero free for $A >> 0$. Proof: In S,

$$\lim_{A \to \infty} \int_0^A e^{-t}\cos zt\,dt = \frac{1}{z^2+1}$$

and the function on the right has no zeros there.

[Note: As a function of A, the number of nonreal zeros is unbounded.]

35.6 NOTATION (cf. 34.1) Given $\phi \in L^1(-\infty,\infty)$, put

$$f_\infty(z) = \int_{-\infty}^{\infty} \phi(t) e^{\sqrt{-1}\, zt} dt,$$

thus

$$f_\infty(z) = C_\infty(z) + \sqrt{-1}\, S_\infty(z),$$

where

$$C_\infty(z) = \int_{-\infty}^{\infty} \phi(t) \cos zt\, dt, \quad S_\infty(z) = \int_{-\infty}^{\infty} \phi(t) \sin zt\, dt.$$

N.B. If ϕ is real and even (odd), then one can work instead with

$$C_\infty(z) \equiv \int_0^{\infty} \phi(t) \cos zt\, dt \quad (S_\infty(z) \equiv \int_0^{\infty} \phi(t) \sin zt\, dt).$$

35.7 EXAMPLE Suppose that 2n is an even positive integer and take

$$\phi(t) = \exp(-t^{2n}) \quad (n = 1,2,\ldots).$$

Then

$$\int_{-\infty}^{\infty} \exp(-t^2) e^{\sqrt{-1}\, zt} dt = \sqrt{\pi} \exp(-\frac{z^2}{4})$$

has no zeros but

$$\int_{-\infty}^{\infty} \exp(-t^{4,6,\ldots}) e^{\sqrt{-1}\, zt} dt$$

has an infinity of real zeros though it has no complex zeros (cf. 12.34).

[Note: Put

$$f_n(z) = \int_{-\infty}^{\infty} \exp(-t^{2n}) e^{\sqrt{-1}\, zt} dt \quad (n = 1,2,\ldots).$$

Then $f_n \in \mathcal{L} - \mathcal{P}$ is transcendental and satisfies the differential equation

$$f_n^{(2n-1)}(z) = \frac{(-1)^n}{2n} z f_n(z).$$

Therefore all the zeros of f_n are simple (see the Appendix to §13).]

35.8 REMARK Consider

$$\int_0^A \exp(-t^2)\cos zt\, dt.$$

Then 35.1 and 35.4 are applicable and there is an A with the property that

$$\int_0^A \exp(-t^2)\cos zt\, dt$$

has a nonreal zero (but no characterization is known of those A for which this happens) (the situation in 35.5 is simpler although a complete explication is lacking there too).

35.9 EXAMPLE The zeros of

$$\int_{-\infty}^{\infty} \exp(-t^{4,6,\ldots}) e^{t} e^{\sqrt{-1}\, zt} dt$$

lie on the line Im $z = 1$.

[If $z = a + \sqrt{-1}\, b$ is a zero, write

$$e^{t} e^{\sqrt{-1}\, zt} = e^{\sqrt{-1}(-\sqrt{-1} + z)t},$$

hence $-\sqrt{-1} + z$ is real, so $b = 1$.]

35.10 EXAMPLE Fix $\alpha > 1$, $\alpha \neq 2n$ $(n = 1,2,\ldots)$, take $\phi(t) = \exp(-t^{\alpha})$, and put

$$\Phi_\alpha(z) = \int_0^{\infty} \exp(-t^{\alpha})\cos zt\, dt.$$

Then Φ_α has an infinite number of nonreal zeros and a finite number of real zeros, there being at least $2\left[\frac{\alpha}{2}\right]$ of the latter if $\alpha > 2$.

35.11 LEMMA We have

$$\lim_{x \to \infty} x^{\alpha+1}\Phi_\alpha(x) = \Gamma(\alpha+1)\sin(\tfrac{\pi\alpha}{2}).$$

PROOF There are seven steps.

Step 1: Integrate by parts to get

$$x^{\alpha+1}\Phi_\alpha(x) = x^\alpha \int_0^\infty \sin xt \cdot \alpha t^{\alpha-1}e^{-t^\alpha}dt.$$

Step 2: Make the change of variable $u = x^\alpha t^\alpha$, hence

$$x^{\alpha+1}\Phi_\alpha(x) = \int_0^\infty \sin u^{1/\alpha} \cdot e^{-x^{-\alpha}u}du,$$

a.k.a. the Laplace transform of $\sin u^{1/\alpha}$ at $x^{-\alpha}$.

Step 3: Rewrite the right hand side in terms of a complex exponential, so

$$x^{\alpha+1}\Phi_\alpha(x) = \operatorname{Im} \int_0^\infty \exp(\sqrt{-1}\, u^{1/\alpha} - x^{-\alpha}u)du.$$

Step 4: Move the contour of integration up to a straight line going from 0 to ∞ placed at a "small" angle θ to the positive real axis, call it ℓ_θ.

Step 5: By Jordan's lemma, the integral around the curved part is small when $s = x^{-\alpha} > 0$ is small and on ℓ_θ the integrand is bounded by an absolutely integrable function, thus the result is continuous as a function of s all the way to 0 (dominated convergence). Therefore

$$\lim_{x \to \infty} x^{\alpha+1}\Phi_\alpha(x) = \operatorname{Im} \int_0^{\infty,\theta} \exp(\sqrt{-1}\, u^{1/\alpha})du,$$

the symbol $\int_0^{\infty,\theta} \cdots$ being an abbreviation for the integral along ℓ_θ.

Step 6: Now change the variable and let $u = v \exp(\frac{\sqrt{-1}\pi\alpha}{2})$:

$$\text{Im} \int_0^\infty \exp(\sqrt{-1}\ v^{1/a} \exp(\frac{\sqrt{-1}\ \pi}{2})) \cdot \exp(\frac{\sqrt{-1}\ \pi a}{2})dv$$

$$= \text{Im}(\exp(\frac{\sqrt{-1}\ \pi a}{2}) \int_0^\infty \exp(-\ v^{1/a})dv)$$

$$= \sin(\frac{\pi a}{2}) \int_0^\infty \exp(-\ v^{1/a})dv.$$

[Note: Strictly speaking, this is a rotation of contours, not a change of variable.]

Step 7: In

$$\int_0^\infty \exp(-\ v^{1/a})dv,$$

let

$$w = v^{1/a},\ \text{so}\ dw = \frac{1}{a} v^{\frac{1}{a}} v^{-1}dv$$

$$= \frac{1}{a} w \cdot w^{-a}dv$$

$$= \frac{1}{a} w^{1-a}dv$$

=>

$$\int_0^\infty \exp(-\ v^{1/a})dv$$

$$= a\int_0^\infty \exp(-\ w)w^{a-1}dw$$

$$= a\Gamma(a) = \Gamma(a+1).$$

Returning to 35.10, the assumption on α implies that $\sin(\frac{\pi\alpha}{2}) \neq 0$.

Consequently, Φ_α cannot have an infinite number of real zeros. But Φ_α does have an infinite number of zeros (cf. §7), from which it follows that Φ_α has an infinite number of nonreal zeros.

There remains the claim that the number (finite) of real zeros of Φ_α is $\geq 2\left[\frac{\alpha}{2}\right]$ if $\alpha > 2$. To this end, choose $m \geq 1$:

$$2m < \alpha < 2m + 2.$$

Write

$$\frac{2}{\pi} \int_0^\infty \Phi_\alpha(x) \cos xt = e^{-t^\alpha},$$

differentiate 2m times with respct to t, and then put t = 0:

=>

$$\int_0^\infty \Phi_\alpha(x) x^2 dx = 0$$
$$\vdots$$
$$\int_0^\infty \Phi_\alpha(x) x^{2m} dx = 0.$$

Accordingly,

$$\int_0^\infty \Phi_\alpha(x) x^2 P(x^2) dx = 0,$$

where P is any polynomial of degree $\leq m - 1$.

For sake of argument, suppose now that $\Phi_\alpha(x)$ changes sign at most $k \leq m - 1$ times $(x > 0)$, e.g., at

$$0 < x_1 < x_2 < \cdots < x_k.$$

Introduce

$$P(x^2) = (x_1^2 - x^2)(x_2^2 - x^2) \cdots (x_k^2 - x^2).$$

Then

$$\Phi_\alpha(x)x^2P(x)$$

is never negative ($\Phi_\alpha(0)$ is positive) while

$$\int_0^\infty \Phi_\alpha(x)x^2P(x^2)dx = 0,$$

a contradiction.

So in conclusion, $\Phi_\alpha(x)$ changes sign at least $m = \left[\frac{\alpha}{2}\right]$ times $(x > 0)$, thus being even, the number of real zeros of Φ_α is $\geq 2\left[\frac{\alpha}{2}\right]$ if $\alpha > 2$.

N.B. This analysis breaks down if $1 < \alpha < 2$. However, in this case it can be shown that Φ_α has no real zeros.[†]

[Note: A crucial preliminary to the proof is the fact that

$$e^{-|t|^\alpha}$$

is the characteristic function of an absolutely continuous distribution function (which is definitely not an "elementary" function).]

35.12 REMARK Take $\phi \in L^1(0,\infty)$ real valued and twice continuously differentiable -- then under appropriate decay conditions on ϕ, ϕ', ϕ'', the assumption that $\phi'(0) \neq 0$ implies that

$$C_\infty(z) = \int_0^\infty \phi(t)\cos zt\, dt$$

has an infinite number of nonreal zeros and a finite number of real zeros (if any at all).

[†] A. Wintner, *American J. Math.* 58 (1936), pp. 64-66.

[Supposing that $C_\infty(z)$ is of order < 2, consider the formula

$$x^2 C_\infty(x) = -\phi'(0) + \int_0^\infty \phi''(t)\cos xt\,dt$$

that arises upon a double integration by parts.]

[Note: Since

$$\frac{d}{dt}\exp(-t^\alpha) = \exp(-t^\alpha)(-\alpha t^{\alpha-1})$$

vanishes at t = 0, this fact cannot be used to circumvent the analysis in 35.10.]

35.13 EXAMPLE The zeros of the function

$$\int_{-\infty}^\infty \exp(-t^{4n} + t^{2n} + t^2)e^{\sqrt{-1}\,zt}dt \quad (n = 1,2,\ldots)$$

are real.

35.14 DEFINITION Let $\phi \in L^1(-\infty,\infty)$ subject to

$$\phi(-t) = \overline{\phi(t)}.$$

Then ϕ is said to be of regular growth if

$$\phi(t) = O(e^{-|t|^b}) \quad (|t| \to \infty)$$

for some constant b > 2.

35.15 LEMMA Suppose that ϕ is of regular growth -- then f_∞ is a real entire function of order

$$\leq \frac{b}{b-1} < 2.$$

PROOF The computation

$$\overline{f_\infty(x)} = \int_{-\infty}^\infty \overline{\phi(t)}e^{-\sqrt{-1}\,xt}dt$$

$$= \int_{-\infty}^{\infty} \phi(-t) e^{-\sqrt{-1}\, xt} dt$$

$$= \int_{-\infty}^{\infty} \phi(t) e^{\sqrt{-1}\, xt} dt = f_{\infty}(x)$$

shows that f_∞ is real. Define now $\beta > 0$ by writing $b = 2 + \beta$, hence

$$|\phi(t)| \le Me^{-|t|^{2+\beta}} \qquad (M > 0)$$

=>

$$|f_\infty(z)| \le 2M \int_0^\infty e^{-|t|^{2+\beta}} e^{|z|t} dt$$

$$= 2M \int_0^\infty \exp(|z|t - |t|^{2+\beta}) dt.$$

But

$$|z|t - |t|^{2+\beta} < |z|t$$

if

$$0 < t < 2 |z|^{\frac{1}{1+\beta}}$$

and

$$|z|t - |t|^{2+\beta} < \left(\frac{t}{2}\right)^{1+\beta} t - t^{2+\beta}$$

$$< -\frac{1}{2} t^{2+\beta}$$

if

$$|t| > 2|z|^{\frac{1}{1+\beta}}.$$

Therefore

$$|f_\infty(z)| \le 2M \left[\int_0^{2|z|^{\frac{1}{1+\beta}}} + \int_{2|z|^{\frac{1}{1+\beta}}}^{\infty} \right] \exp(|z|t - |t|^{2+\beta}) dt$$

$$\leq 2M\left[\,|z|^{-1}\exp(2|z|^{\frac{2+\beta}{1+\beta}})\,\right] + \int_0^\infty \exp(-\tfrac{1}{2}t^{2+\beta})dt.$$

And so the integral defining $f_\infty(z)$ is an entire function of order

$$\leq \frac{2+\beta}{1+\beta} = \frac{b}{b-1} < 2.$$

N.B.

$$\underline{\text{gen}}\ f_\infty \leq \rho(f_\infty) < 2 \quad (\text{cf. } 6.2)$$

=>

$$\underline{\text{gen}}\ f_\infty = 0 \text{ or } \underline{\text{gen}}\ f_\infty = 1.$$

35.16 RAPPEL Suppose that the real polynomial

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n$$

has real zeros only -- then $\forall\ f \in L - P$, the function

$$P(\tfrac{d}{dz})f(z) \equiv a_0 f(z) + a_1 f'(z) + \cdots + a_n f^{(n)}(z)$$

is in $L - P$ (easy extension of 12.10).

35.17 PROPAGATION PRINCIPLE If ϕ is of regular growth and if

$$f_\infty(z) = \int_{-\infty}^{\infty} \phi(t)e^{\sqrt{-1}\,zt}dt$$

has real zeros only, then $\forall\ f \in L - P$, the function

$$\int_{-\infty}^{\infty} \phi(t)f(\sqrt{-1}\,t)e^{\sqrt{-1}\,zt}dt$$

has real zeros only.

PROOF Per §12, write

$$f(z) = \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} z^n.$$

Then on compact subsets of $\mathbf{C}$,

$$P_n(z) \equiv J_n(f;\frac{z}{n}) \to f(z)$$

uniformly (cf. 12.9). Moreover, $\exists\ K > 0$: $\forall\ n$,

$$|J_n(f;\frac{z}{n})| < \exp(K(|z|^2 + 1)).$$

The preliminaries in place, by hypothesis $f_\infty \in L - P$, thus

$$P_n(\frac{d}{dz})f_\infty \in L - P \quad (\text{cf. } 35.16).$$

But

$$(P_n(\frac{d}{dz})f_\infty)(z) = \int_{-\infty}^{\infty} \phi(t)P_n(\sqrt{-1}\ t)e^{\sqrt{-1}\ zt}dt$$

$$\to \int_{-\infty}^{\infty} \phi(t)f(\sqrt{-1}\ t)e^{\sqrt{-1}\ zt}dt \qquad (n \to \infty).$$

35.18 EXAMPLE Take $f(z) = (z + \alpha)^n$ $(n = 1,2,\ldots)$ (α real) -- then

$$f(\sqrt{-1}\ t) = (\sqrt{-1}\ t + \alpha)^n.$$

Therefore the zeros of the function

$$\int_{-\infty}^{\infty} \phi(t)(\sqrt{-1}\ t + \alpha)^n e^{\sqrt{-1}\ zt}dt$$

are real if $f_\infty \in L - P$.

35.19 EXAMPLE Take $f(z) = e^{bz}$ (b real) -- then

$$f(\sqrt{-1}\ t) = e^{b\sqrt{-1}\ t} = \cos bt + \sqrt{-1} \sin bt.$$

Therefore the zeros of the function

$$\int_{-\infty}^{\infty} \phi(t)(\cos bt + \sqrt{-1} \sin bt)e^{\sqrt{-1}\ zt}dt$$

are real if $f \in L - P$.

35.20 EXAMPLE Take $f(z) = e^{az^2}$ (a real and < 0) -- then

$$f(\sqrt{-1}\,t) = e^{a(\sqrt{-1}\,t)^2} = e^{-at^2} = e^{\lambda t^2} \quad (\lambda = -a).$$

Therefore the zeros of the function

$$\int_{-\infty}^{\infty} \phi(t) e^{\lambda t^2} e^{\sqrt{-1}\,zt} dt \quad (\lambda > 0)$$

are real if $f_\infty \in L - P$.

35.21 RAPPEL Suppose that f is a real entire function of genus 0 or 1 and write

$$|f(x + \sqrt{-1}\,y)|^2 = \sum_{n=0}^{\infty} \Lambda_n(f)(x) y^{2n} \quad (\text{cf. } 13.8)$$

or still,

$$|f(x + \sqrt{-1}\,y)|^2 = \sum_{n=0}^{\infty} L_n(f)(x) y^{2n} \quad (\text{cf. } 13.9).$$

Then $f \in L - P$ iff $\forall\, n \geq 0$ and $\forall\, x \in R$,

$$L_n(f)(x) \geq 0 \quad (\text{cf. } 13.7).$$

35.22 APPLICATION $f_\infty \in L - P$ iff $\forall\, n \geq 0$ and $\forall\, x \in R$,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(s)\phi(t) e^{\sqrt{-1}(s+t)x} (s - t)^{2n} \, ds\, dt \geq 0.$$

[In fact,

$$|f_\infty(x + \sqrt{-1}\,y)|^2 = f_\infty(x + \sqrt{-1}\,y) f_\infty(x - \sqrt{-1}\,y)$$

$$= \sum_{n=0}^{\infty} \frac{y^{2n}}{n!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(s)\phi(t) e^{\sqrt{-1}(s+t)x} (s - t)^{2n} \, ds\, dt.]$$

35.23 EXAMPLE Take

$$\phi(t) = \exp(-t^{2k})\ (k \geq 2) \qquad (\text{cf. } 35.7).$$

Then is it obvious that $\forall\, n \geq 0$ and $\forall\, x \in R$, the expression

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(s)\phi(t)e^{\sqrt{-1}(s+t)x}\ (s - t)^{2n}\ ds\ dt$$

is nonnegative?

35.24 RAPPEL Suppose that f is a real entire function of genus 0 or 1 -- then $f \in L - P$ iff

$$\frac{\partial^2}{\partial y^2}\ |f(x + \sqrt{-1}\ y)|^2 \geq 0.$$

[Examine the proof of 13.12.]

35.25 APPLICATION $f_\infty \in L - P$ iff $\forall\, x,y \in R$,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(s)\phi(t)e^{\sqrt{-1}(s+t)x}\ e^{(s-t)y}\ (s - t)^2\ ds\ dt \geq 0.$$

[Differentiate

$$|f_\infty(x + \sqrt{-1}\ y)|^2 = f_\infty(x + \sqrt{-1}\ y)f_\infty(x - \sqrt{-1}\ y)$$

twice with respect to y.]

One can employ 35.24 to ascertain that the zeros of certain real entire functions are real.

35.26 EXAMPLE We have

$$\begin{cases} |\sin z|^2 = \sin^2 x + \sinh^2 y \\ |\cos z|^2 = \cos^2 x + \sinh^2 y. \end{cases}$$

And

$$\begin{cases} \frac{\partial^2}{\partial y^2} |\sin(x + \sqrt{-1}\, y)|^2 = 2(\cosh^2 y + \sinh^2 y) \geq 2 > 0 \\ \\ \frac{\partial^2}{\partial y^2} |\cos(x + \sqrt{-1}\, y)|^2 = 2(\cosh^2 y + \sinh^2 y) \geq 2 > 0. \end{cases}$$

Therefore the zeros of sin z and cos z are real (...).

[Note: It is a corollary that the zeros of

$$\begin{cases} J_{\frac{1}{2}}(z) = (\frac{2}{\pi z})^{\frac{1}{2}} \sin z \\ \\ J_{-\frac{1}{2}}(z) = (\frac{2}{\pi z})^{\frac{1}{2}} \cos z \end{cases}$$

are real.

35.27 EXAMPLE Recall from 12.33 that the zeros of the Bessel function $J_\nu(z)$ $(\nu > -1)$ are real. This important point can also be established via 35.24. Thus put

$$\mathcal{J}_\nu(z) = z^{-\nu} J_\nu(z).$$

Then it can be shown that

$$\frac{\partial^2}{\partial y^2} |\mathcal{J}_\nu(x + \sqrt{-1}\, y)|^2 \geq 4(\nu+1) |\mathcal{J}_{\nu+1}(x)|^2,$$

from which the contention.

In terms of the modified Bessel functions, let

$$K_z(\alpha) = \frac{\pi}{2} \frac{I_{-z}(\alpha) - I_z(\alpha)}{\sin \pi z} \quad (\alpha > 0).$$

Then

$$K_z(\alpha) = \int_0^\infty e^{-\alpha \cosh t} \cosh zt \, dt$$

or still,

$$K_{\sqrt{-1}\, z}(\alpha) = \int_0^\infty e^{-\alpha \cosh t} \cosh \sqrt{-1}\, zt \, dt$$

$$= \int_0^\infty e^{-\alpha \cosh t} \cos zt \, dt.$$

35.28 EXAMPLE Take $\phi(t) = e^{-\alpha \cosh t}$ — then ϕ is of regular growth and the claim is that all the zeros of

$$C_\infty(z) = \int_0^\infty e^{-\alpha \cosh t} \cos zt \, dt$$

are real.

[A "special function" manipulation leads to the relation

$$|K_{\sqrt{-1}\, z}(\alpha)|^2 = |K_{\sqrt{-1}\, x}(\alpha)|^2$$

$$+ y^2 \int_0^1 t^{y-1} \, {}_2F_1 \left[\begin{matrix} y+1, \ y+1 \\ 2 \end{matrix} ; 1-t \right] (K_{\sqrt{-1}\, x}(\frac{\alpha}{\sqrt{t}}))^2 dt.$$

Therefore

$$\frac{\partial^2}{\partial y^2} |K_{\sqrt{-1}\, z}(\alpha)|^2$$

$$= \int_0^1 \frac{\partial^2}{\partial y^2} f_t(y) (K_{\sqrt{-1}\, x}(\frac{\alpha}{\sqrt{t}}))^2 \frac{dt}{t},$$

where

$$f_t(y) = y^2 t^y \, {}_2F_1 \left[\begin{matrix} y+1, \; y+1 \\ 2 \end{matrix} ; 1-t \right].$$

But $f_t(y)$ is an (even) absolutely monotonic function of y when $0 < t < 1$, hence

$$\frac{\partial^2}{\partial y^2} f_t(y) \geq 0 \qquad (0 < t < 1).]$$

35.29 RAPPEL If $f \in L - P$, then $\forall \lambda \in R$, either $f_\lambda \in L - P$ or $f_\lambda \equiv 0$ (cf. 14.9).

35.30 EXAMPLE Take

$$f(z) = K_{\sqrt{-1}\, z}(\alpha) \qquad (\alpha > 0).$$

Then $\forall \lambda \in R$, the real entire function

$$K_{\sqrt{-1}(z + \sqrt{-1}\,\lambda)}(\alpha) + K_{\sqrt{-1}(z - \sqrt{-1}\,\lambda)}(\alpha)$$

$$= 2 \int_0^\infty e^{-\alpha \cosh t} \cosh(\lambda t) \cos zt \, dt$$

has real zeros only.

[Note: Since

$$\cosh(\lambda t) = \cos(\sqrt{-1}\,\lambda t),$$

one could also quote 35.17.]

§36. *LOCATION, LOCATION, LOCATION*

Let $f \not\equiv 0$ be a real entire function -- then for any real number λ,

$$f_\lambda(z) = f(z + \sqrt{-1}\,\lambda) + f(z - \sqrt{-1}\,\lambda) \qquad \text{(cf. 14.1)}.$$

36.1 NOTATION Given $A \geq 0$ $(A < \infty)$, put

$$A_\lambda = (\max(A^2 - \lambda^2, 0))^{1/2}.$$

36.2 RAPPEL Let $f \in A - L - P$ and take $\lambda > 0$ -- then

$$f_\lambda \in A \ - L - P \qquad \text{(cf. 15.8)}.$$

36.3 THEOREM Suppose that ϕ is of regular growth and

$$f_\infty(z) = \int_{-\infty}^{\infty} \phi(t) e^{\sqrt{-1}\,zt} dt$$

is in $A - L - P$ -- then for $\lambda > 0$,

$$(f_\infty)_\lambda(z) = \int_{-\infty}^{\infty} \phi(t)\,(e^{\lambda t} + e^{-\lambda t}) e^{\sqrt{-1}\,zt} dt$$

is in $A_\lambda - L - P$.

[Note: Specialize to $A = 0$ and in 35.17, take

$$f(z) = \cos \lambda z.$$

Then

$$f(\sqrt{-1}\,t) = \cos \sqrt{-1}\,\lambda t = \cosh \lambda t = \frac{e^{\lambda t} + e^{-\lambda t}}{2},$$

so a priori,

$$(f_\infty)_\lambda \in L - P.]$$

36.4 LEMMA Suppose that ϕ is of regular growth and

$$f_\infty(z) = \int_{-\infty}^{\infty} \phi(t)e^{\sqrt{-1}\,zt}dt$$

is in $A - L - P$ -- then for $\lambda_1 > 0, \lambda_2 > 0, \ldots, \lambda_N > 0$, the zeros of

$$(\ldots((f_\infty)_{\lambda_1})_{\lambda_2}\ldots)_{\lambda_N}$$

$$= \int_{-\infty}^{\infty} \phi(t) \prod_{k=1}^{N} (e^{\lambda_k t} + e^{-\lambda_k t})e^{\sqrt{-1}\,zt}dt$$

are in the strip

$$|\text{Im } z| \le (\max(A^2 - \sum_{k=1}^{N} \lambda_k^2, 0))^{1/2}.$$

36.5 THEOREM Suppose that ϕ is of regular growth and

$$f_\infty(z) = \int_{-\infty}^{\infty} \phi(t)e^{\sqrt{-1}\,zt}dt$$

is in $A - L - P$ -- then the function

$$\int_{-\infty}^{\infty} \phi(t)e^{\frac{1}{2}\lambda^2t^2} e^{\sqrt{-1}\,zt}dt \qquad (\lambda > 0)$$

is in $A_\lambda - L - P$.

PROOF Given a positive integer N, the zeros of the function

$$\int_{-\infty}^{\infty} \phi(t)(\cosh \frac{\lambda t}{N})^{N^2} e^{\sqrt{-1}\,zt}dt$$

lie in the strip

$$|\text{Im } z| \le (\max(A^2 - (\frac{\lambda}{N})^2N^2, 0))^{1/2}$$

$$= (\max(A^2 - \lambda^2, 0))^{1/2} \quad (\text{cf. } 36.4).$$

But

$$\int_{-\infty}^{\infty} \phi(t) (\cosh \frac{\lambda t}{N})^{N^2} e^{\sqrt{-1}\, zt} dt$$

$$\to \int_{-\infty}^{\infty} \phi(t) e^{\frac{1}{2}\lambda^2 t^2} e^{\sqrt{-1}\, zt} dt \quad (N \to \infty)$$

uniformly on compact subsets of $\mathbf{C}$.

[Note: To supply the details for this contention, use the inequality

$$\cosh r \le \exp(\frac{r^2}{2}) \quad (-\infty < r < \infty)$$

to get

$$C(N,t) \equiv (\cosh \frac{\lambda t}{N})^{N^2}$$

$$\le \exp(\frac{1}{2}\lambda^2 t^2).$$

We then claim that

$$\lim_{N \to \infty} C(N,t) = \exp(\frac{1}{2}\lambda^2 t^2)$$

or still,

$$N^2 \log \cosh \frac{\lambda t}{N} \to \frac{\lambda^2 t^2}{2} \quad (N \to \infty)$$

or still,

$$(\frac{N}{\lambda t})^2 \log \cosh \frac{\lambda t}{N} \to \frac{1}{2} \quad (N \to \infty).$$

But letting $s = \frac{\lambda t}{N}$,

$$\lim_{s \to 0} \frac{\log \cosh s}{s^2} = \frac{1}{2}$$

by L'Hospital. Now fix a compact subset S of C and let K > 0 be a bound for the $|\text{Im } z|$ $(z \in S)$ -- then

$$|\phi(t)(C(N,t) - \exp(\tfrac{1}{2}\lambda^2 t^2))e^{\sqrt{-1}\, zt}|$$

$$\leq |\phi(t)||C(N,t) - \exp(\tfrac{1}{2}\lambda^2 t^2)|e^{K|t|}$$

$$\leq Me^{-|t|^b}(\exp(\tfrac{1}{2}\lambda^2 t^2) - C(N,t))e^{K|t|}$$

$$\leq Me^{-|t|^b}\exp(\tfrac{1}{2}\lambda^2 t^2)e^{K|t|}$$

$$\in L^1(-\infty,\infty) \quad (b > 2),$$

so dominated convergence is applicable.]

<u>N.B.</u> For use below, subject the data to a relabeling: $f_\infty \in A - L - P$ implies that the function

$$\int_{-\infty}^{\infty} \phi(t)e^{\lambda t^2} e^{\sqrt{-1}\, zt} \quad (\lambda > 0)$$

is in

$$A_{\sqrt{2\lambda}} - L - P,$$

where

$$A_{\sqrt{2\lambda}} = (\max(A^2 - 2\lambda, 0))^{1/2} \quad \text{(cf. 35.20)}.$$

36.6 NOTATION Put

$$f_\infty(z;\lambda) = \int_{-\infty}^{\infty} \phi(t)e^{\lambda t^2} e^{\sqrt{-1}\, zt}dt \qquad (\lambda \in R),$$

thus in particular,

$$f_\infty(z;0) = f_\infty(z).$$

36.7 LEMMA For every real number λ,

$$\phi(t;\lambda) \equiv \phi(t)e^{\lambda t^2}$$

is of regular growth.

PROOF By definition, for some $\beta > 0$,

$$e^{|t|^{2+\beta}} \phi(t)$$

stays bounded as $|t| \to \infty$. Let $\beta' = \frac{\beta}{2}$ and consider

$$e^{|t|^{2+\beta'}} e^{\lambda t^2} |\phi(t)|$$

$$= e^{t^2(\lambda + |t|^{\beta'})} |\phi(t)|$$

which is eventually

$$\leq e^{|t|^{2+\beta}} |\phi(t)|$$

once

$$\lambda + |t|^{\beta'} < |t|^{\beta}.$$

36.8 APPLICATION If $\lambda_1 < \lambda_2$ and if the zeros of $f_\infty(z;\lambda_1)$ lie in the strip $\{z:|\text{Im } z| \leq A\}$, then the zeros of $f_\infty(z;\lambda_2)$ lie in the strip

$$\{z: |\mathrm{Im}\ z| \le \frac{A}{\sqrt{2(\lambda_2 - \lambda_1)}}\}.$$

[Simply write

$$f_\infty(z;\lambda_2) = \int_{-\infty}^{\infty} \phi(t)e^{\lambda_2 t^2} e^{\sqrt{-1}\ zt}dt$$

$$= \int_{-\infty}^{\infty} \phi(t)e^{\lambda_1 t^2} e^{(\lambda_2 - \lambda_1)t^2} e^{\sqrt{-1}\ zt}dt$$

$$= \int_{-\infty}^{\infty} \phi(t;\lambda_1)e^{(\lambda_2 - \lambda_1)t^2} e^{\sqrt{-1}\ zt}dt$$

and use the assumption that the zeros of

$$f_\infty(z;\lambda_1) = \int_{-\infty}^{\infty} \phi(t;\lambda_1)e^{\sqrt{-1}\ zt}dt$$

lie in the strip $\{z: |\mathrm{Im}\ z| \le A\}$.]

36.9 SCHOLIUM If the zeros of $f_\infty(z)$ lie in the strip $\{z: |\mathrm{Im}\ z| \le A\}$, then the zeros of $f_\infty(z;\lambda)$ $(\lambda > 0)$ are real when $A^2 - 2\lambda \le 0$, i.e., provided $\frac{A^2}{2} \le \lambda$.

36.10 SCHOLIUM If the zeros of $f_\infty(z;\lambda_1)$ are real and if $\lambda_1 < \lambda_2$, then the zeros of $f_\infty(z;\lambda_2)$ are real.

There is more to be said but before so doing we shall install some machinery.

36.11 NOTATION Given a complex constant γ and an entire function f of order < 2, let

$$e^{\gamma D^2} f(z) = \sum_{n=0}^{\infty} \frac{\gamma^n}{n!} f^{(2n)}(z)$$

or, equivalently,

$$e^{\gamma D^2} f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} e^{\gamma D^2} z^n.$$

36.12 EXAMPLE Suppose that ϕ is of regular growth — then f_∞ is a real entire function of order < 2 (cf. 35.15) and

$$f_\infty(z;\lambda) = e^{-\lambda D^2} f_\infty(z).$$

36.13 LEMMA Either series defining $e^{\gamma D^2} f(z)$ converges absolutely and uniformly on compact subsets of $\mathbf{C}$, hence represents an entire function.

36.14 LEMMA $\forall$ complex constant c,

$$e^{c^2 D^2/2} f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} f(z + ct)dt.$$

PROOF Bearing in mind that

$$\int_{-\infty}^{\infty} e^{-t^2/2} t^{2n}dt = \sqrt{2\pi} \frac{(2n)!}{2^n n!}$$

and

$$\int_{-\infty}^{\infty} e^{-t^2/2} t^{2n+1}dt = 0$$

for n = 0,1,2,..., we have

$$e^{c^2 D^2/2} f(z) = \sum_{n=0}^{\infty} \frac{c^{2n}}{2^n n!} f^{(2n)}(z)$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} e^{-t^2/2} \frac{f^{(k)}(z)}{k!} (ct)^k dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} \left(\sum_{k=0}^{\infty} \frac{f^{(k)}(z)}{k!} (ct)^k\right) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} f(z + ct)dt.$$

[Note: The interchange of summation and integration is legal.]

36.15 LEMMA The order of

$$e^{\gamma D^2} f(z)$$

is < 2.

PROOF For $\varepsilon > 0$ and sufficiently small,

$$f(z) = O(e^{|z|^{\rho+\varepsilon}}) \qquad (\rho = \rho(f)),$$

where $\rho + \varepsilon < 2$, so $\exists$ a constant $C > 0$:

$$|f(z)| \leq C \exp(|z|^{\rho+\varepsilon}).$$

Choose c such that $\gamma = \frac{c^2}{2}$ --- then

$$e^{\gamma D^2} f(z) = e^{c^2 D^2/2} f(z)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} f(z + ct)dt \qquad \text{(cf. 36.14)}.$$

Therefore

$$|e^{\gamma D^2} f(z)|$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} |f(z + ct)| dt$$

$$\leq \frac{C}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} \exp(|z + ct|^{\rho+\varepsilon}) dt$$

$$\leq \frac{C}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} \exp((|z| + |ct|)^{\rho+\varepsilon}) dt$$

$$\leq \frac{C}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} \exp(2^{\rho+\varepsilon}(|z|^{\rho+\varepsilon} + |ct|^{\rho+\varepsilon})) dt$$

$$\leq \frac{C}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} e^{-t^2/2} \exp(2^{\rho+\varepsilon}|ct|^{\rho+\varepsilon}) dt\right) \exp(2^{\rho+\varepsilon}|z|^{\rho+\varepsilon})$$

$$\leq \frac{C}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} \ldots\right) \exp(4|z|^{\rho+\varepsilon}),$$

from which the assertion.

36.16 LEMMA Given complex constants μ and ν,

$$e^{\mu D^2} e^{\nu D^2} f(z) = e^{(\mu+\nu) D^2} f(z) = e^{\nu D^2} e^{\mu D^2} f(z).$$

[Note: Thanks to 36.15, it makes sense to apply $e^{\mu D^2}$ to $e^{\nu D^2} f(z)$ and $e^{\nu D^2}$ to $e^{\mu D^2} f(z)$.]

36.17 RAPPEL Define polynomials $\tilde{H}_n(z)$ by the rule

$$\tilde{H}_n(z) = (-1)^n e^{z^2/2} \frac{d^n}{dz^n} e^{-z^2/2} \qquad (n = 0,1,2,\ldots).$$

Then the zeros of the $\tilde{H}_n(z)$ are real and simple.

[Note: This is but one of several variations on the definition of "Hermite polynomial" (cf. 8.17).]

36.18 SUBLEMMA Given a nonzero complex constant c,

$$e^{-c^2D^2/2} z^n = c^n \tilde{H}_n(\frac{z}{c}) \qquad (n = 0,1,2,\ldots).$$

36.19 LEMMA Suppose that f(z) has a multiple zero at the origin — then there is a positive constant λ_1 such that for all $\lambda \in]0,\lambda_1[$, $e^{\lambda D^2} f(z)$ has a nonreal zero.

PROOF Write

$$f(z) = \sum_{n=k}^{\infty} c_n z^n,$$

where $k \geq 2$ and $c_k \neq 0$. Take c positive and consider

$$e^{c^2D^2/2} f(z) = \sum_{n=k}^{\infty} c_n e^{c^2D^2/2} z^n$$

$$= \sum_{n=k}^{\infty} c_n (\sqrt{-1}\, c)^n \tilde{H}_n(\frac{-\sqrt{-1}\, z}{c}).$$

Now replace z by cw and instead consider

$$F_c(w) = (\sqrt{-1}\, c)^{-k} e^{c^2D^2/2} f(cw)$$

$$= \sum_{n=k}^{\infty} c_n (\sqrt{-1}\, c)^{n-k} \tilde{H}_n(-\sqrt{-1}\, w).$$

The point then is that $\tilde{H}_k(-\sqrt{-1}\,w)$ has a nonreal zero, thus if $c > 0$ is sufficiently small, the same holds for $F_c(w)$ (quote Rouché). And this suffices... .

36.20 THEOREM If the zeros of $f_\infty(z)$ lie in the strip $\{z:|\text{Im } z| \leq A\}$, then the zeros of $f_\infty(z;\lambda)$ $(\lambda > 0)$ are real when $A^2 - 2\lambda \leq 0$, i.e., provided $\frac{A^2}{2} \leq \lambda$ (cf. 36.9), and are simple when $A^2 - 2\lambda < 0$, i.e., provided $\frac{A^2}{2} < \lambda$.

PROOF The issue is simplicity. So suppose that

$$f_\infty(z;\lambda) = e^{-\lambda D^2} f_\infty(z) \qquad \text{(cf. 36.12)}$$

has a multiple zero at $z = a$. Without essential loss of generality, take $a = 0$ and apply 36.19 to $f_\infty(z;\lambda)$ and secure $\varepsilon > 0$:

$$e^{\varepsilon D^2} e^{-\lambda D^2} f(z)$$

has a nonreal zero, imposing simultaneously the restriction

$$A^2 < 2(\lambda-\varepsilon).$$

But

$$\begin{aligned} e^{\varepsilon D^2} e^{-\lambda D^2} f_\infty(z) &= e^{-(\lambda-\varepsilon)D^2} f_\infty(z) \qquad \text{(cf. 36.16)} \\ &= f_\infty(z;\lambda-\varepsilon), \end{aligned}$$

a function with real zeros only. Contradiction.

36.21 REMARK Take $A = 0$, thus $f_\infty(z)$ is in $L - P$, as is $f_\infty(z;\lambda)$ $(\lambda > 0)$ and its zeros are simple.

36.22 LEMMA Let f be a real entire function of order < 2. Assume:

$f \in A - L - P$ -- then

$$e^{-\lambda D^2} f(z) \qquad (\lambda > 0)$$

is in $A_{\sqrt{2\lambda}} - L - P$ (cf. 36.5).

PROOF Let T^γ be the translation operator:

$$T^\gamma f(z) = f(z+\gamma).$$

Then

$$e^{-\lambda D^2} f(z) = e^{(\sqrt{-1}\,\sqrt{2\lambda})^2 D^2/2} f(z)$$

$$= \lim_{N \to \infty} 2^{-N}(T^{\sqrt{-1}\,\sqrt{2\lambda}/\sqrt{N}} + T^{-\sqrt{-1}\,\sqrt{2\lambda}/\sqrt{N}})^N f(z),$$

the convergence being uniform on compact subsets of $\mathbf{C}$. But $\forall$ N, the function

$$(T^{\sqrt{-1}\,\sqrt{2\lambda}/\sqrt{N}} + T^{-\sqrt{-1}\,\sqrt{2\lambda}/\sqrt{N}})^N f(z)$$

is in

$$A_{\sqrt{2\lambda}} = (\max(A^2 - 2\lambda, 0))^{1/2} \qquad \text{(cf. 36.2)}.$$

<u>N.B.</u> In general, this estimate cannot be improved as can be seen by taking $f(z) = z^2 + A^2$:

$$e^{-\lambda D^2} f(z) = z^2 + A^2 - 2\lambda.$$

36.23 LEMMA Let f be a real entire function of order < 2. Assume: $f \in A - L - P$ and $A^2 < 2\lambda$ -- then all the zeros of

$$e^{-\lambda D^2} f(z)$$

are real and simple.

[From the above, reality is clear and the simplicity can be established as in 36.20.]

36.24 NOTATION

- S - L - P denotes the subclass of L - P whose zeros are simple.
- * - S - L - P denotes the subclass of * - L - P consisting of all real entire functions which are the product of a real polynomial and a function in S - L - P.

36.25 LEMMA S - L - P and * - S - L - P are closed under differentiation.

36.26 NOTATION Given complex constants γ,c and an entire function F of order < 2, define $\Gamma_{\gamma,c}F(z)$ by the prescription

$$\Gamma_{\gamma,c}F(z) = (z-c)F(z) - 2\gamma F'(z).$$

<u>N.B.</u> The order of $\Gamma_{\gamma,c}F(z)$ is < 2 (cf. 2.25 and 2.31).

36.27 LEMMA $\forall\ \gamma,\ \forall\ c$,

$$e^{-\gamma D^2}\ ((z-c)F(z)) = \Gamma_{\gamma,c}\ e^{-\gamma D^2}\ F(z).$$

[Note: The order of

$$e^{-\gamma D^2}\ F(z)$$

is < 2 (cf. 36.15).]

LEMMA $\forall\ \gamma \neq 0,\ \forall\ c$,

$$\Gamma_{\gamma,c}F(z) = -2\gamma \exp(\frac{(z-c)^2}{4\gamma}) \frac{d}{dz} (\exp(-\frac{(z-c)^2}{4\gamma})F(z).$$

36.29 APPLICATION Given $\lambda > 0$ and a real, the class $* - S - L - P$ is closed under the operator $\Gamma_{\lambda,a}$.

[If $f(z)$ is in $* - S - L - P$, then

$$\exp(-\frac{(z-a)^2}{4\lambda})f(z)$$

is in $* - S - L - P$ (a being real), as is its derivative (cf. 36.25), so all but a finite number of zeros of the latter are real and simple. The same then holds for $\Gamma_{\lambda,a}f(z)$, itself a real entire function of order < 2.]

36.30 LEMMA Suppose that λ is positive and c is nonreal. Let f be a real entire function of order < 2 and assume that

$$e^{-\lambda D^2} f(z) \in * - S - L - P.$$

Then

$$e^{-\lambda D^2}((z-c)(z-\bar{c})f(z)) \in * - S - L - P.$$

PROOF Write

$$(z-c)(z-\bar{c}) = z^2 - (c+\bar{c})z + c\bar{c}$$

$$= z^2 - 2az + a^2 + b^2,$$

where $c = a + \sqrt{-1}\, b$. With

$$P(z) = z^2 + b^2 \quad (b \neq 0),$$

we thus have

$$\begin{aligned}(T^{-a}P)(z) &= P(z-a)\\ &= (z-a)^2 + b^2\\ &= z^2 - 2az + a^2 + b^2\\ &= (z-c)(z-\bar{c}).\end{aligned}$$

But on the basis of the definitions, $e^{-\lambda D^2}$ commutes with the translation operators T^{γ}, hence

$$\begin{aligned}&e^{-\lambda D^2}((z-c)(z-\bar{c}))f(z))\\ &= e^{-\lambda D^2}((T^{-a}P)(z)f(z))\\ &= e^{-\lambda D^2}(T^{-a}P \cdot T^{-a+a}f)\\ &= e^{-\lambda D^2}(T^{-a}(P \cdot T^{a}f))\\ &= T^{-a}(e^{-\lambda D^2}(P \cdot T^{a}f)).\end{aligned}$$

Since $* - S - L - P$ is closed under translation by a real constant, matters therefore reduce to showing that

$$e^{-\lambda D^2}(P \cdot T^{a}f) \in * - S - L - P$$

or still, to showing that

$$e^{-\lambda D^2}((z - \sqrt{-1}\ |b|)(z + \sqrt{-1}\ |b|)T^{a}f(z) \in * - S - L - P$$

or still, to showing that

$$\Gamma_{\lambda,\sqrt{-1}\,|b|} \circ \Gamma_{\lambda,-\sqrt{-1}\,|b|}(e^{-\lambda D^2} T^a f(z)) \in * - S - L - P \quad \text{(cf. 36.27).}$$

And for this, cf. 36.31 and 36.32 infra.

36.31 SUBLEMMA Fix positive constants λ and β -- then

$$\Gamma_{\lambda,\sqrt{-1}\,\sqrt{\beta}} \circ \Gamma_{\lambda,-\sqrt{-1}\,\sqrt{\beta}} = \Gamma^2_{\lambda,0} + \beta.$$

PROOF

$$\Gamma_{\lambda,-\sqrt{-1}\,\sqrt{\beta}} F(z) = (z + \sqrt{-1}\sqrt{\beta})F(z) - 2\lambda F'(z)$$

$$\Rightarrow$$

$$\Gamma_{\lambda,\sqrt{-1}\,\sqrt{\beta}} \circ \Gamma_{\lambda,-\sqrt{-1}\,\sqrt{\beta}} F(z)$$

$$= (z - \sqrt{-1}\sqrt{\beta})((z + \sqrt{-1}\,\sqrt{\beta}\,F(z) - 2\lambda F'(z))$$

$$- 2\lambda(F(z) + (z + \sqrt{-1}\sqrt{\beta})F'(z) - 2\lambda F''(z))$$

$$= (z^2 + \beta)F(z) - 2\lambda(z - \sqrt{-1}\,\sqrt{\beta} + z + \sqrt{-1}\,\sqrt{\beta}\,)F'(z)$$

$$- 2\lambda F(z) + 4\lambda^2 F''(z)$$

$$= z^2 F(z) - 2\lambda(2zF'(z) + F(z)) + 4\lambda^2 F''(z) + \beta F(z).$$

Meanwhile

$$\Gamma^2_{\lambda,0} F(z) = \Gamma_{\lambda,0} \circ \Gamma_{\lambda,0} F(z)$$

$$= \Gamma_{\lambda,0}(zF(z) - 2\lambda F'(z))$$

$$= z(zF(z) - 2\lambda F'(z))$$

$$- 2\lambda(zF'(z) + F(z) - 2\lambda F''(z))$$

$$= z^2F(z) - 2\lambda(2zF'(z) + F(z)) + 4\lambda^2F''(z).$$

36.32 LEMMA Fix positive constants λ and β -- then $* - S - L - P$ is closed under the operator

$$\Gamma^2_{\lambda,0} + \beta \quad (\lambda > 0, \beta > 0).$$

[We shall relegate the proof of this to the Appendix of this §.]

36.33 THEOREM Suppose that $\forall\ \varepsilon > 0$, all but a finite number of zeros of $f_\infty(z)$ lie in the strip $|\text{Im } z| \leq \varepsilon$ -- then $\forall\ \lambda > 0$, the function

$$f_\infty(z;\lambda) = \int_{-\infty}^{\infty} \phi(t)e^{\lambda t^2} e^{\sqrt{-1}\, zt}dt$$

belongs to $* - S - L - P$.

PROOF Fix $\lambda > 0$ and choose $\varepsilon > 0:\varepsilon^2 < 2\lambda$. By assumption, there are only a finite number of zeros of $f_\infty(z)$ outside the strip $|\text{Im } z| \leq \varepsilon$, hence

$$f_\infty(z) = (z-c_1)(z-\bar{c}_1)\ldots(z-c_n)(z-\bar{c}_n)f(z),$$

where

$$|\text{Im } c_k| > \varepsilon \quad (k = 1,\ldots,n)$$

and $f(z)$ is a real entire function of order < 2 whose zeros lie in the strip $|\text{Im } z| \leq \varepsilon$, thus the zeros of $e^{-\lambda D^2}f(z)$ lie in the strip

$$(\max(\varepsilon^2 - 2\lambda,0))^{1/2} \quad (\text{cf. } 36.22).$$

But ε^2 is less than 2λ, so all the zeros of $e^{-\lambda D^2}f(z)$ are real and simple (cf. 36.23) or still,

$$e^{-\lambda D^2}f(z) \in S - L - P.$$

Therefore

$$f_\infty(z;\lambda) = e^{-\lambda D^2}f_\infty(z) \qquad \text{(cf. 36.12)}$$

$$= e^{-\lambda D^2}((z-c_1)(z-\bar{c}_1)\ldots(z-c_n)(z-\bar{c}_n)f(z))$$

$$\in * - S - L - P$$

via iteration of 36.30.

N.B. In consequence, all but a finite number of the zeros of $f_\infty(z;\lambda)$ are real and simple and in particular $f_\infty(z;\lambda)$ has at most a finite number of nonreal zeros.

36.34 REMARK The result remains valid if f_∞ is replaced by an arbitrary real entire function f of order < 2, the role of $f_\infty(z;\lambda)$ being played by $e^{-\lambda D^2}f(z)$.

36.35 THEOREM Let f be a real entire function of order < 2. Assume: Given any $\lambda_0 > 0$, $\forall\ \varepsilon > 0$, all but a finite number of zeros of $e^{-\lambda_0 D^2}f(z)$ lie in the strip $|\text{Im } z| \leq \varepsilon$ -- then $\forall\ \lambda > 0$, all but a finite number of zeros of $e^{-\lambda D^2}f(z)$ are real and simple.

PROOF Take $\lambda_0 = \frac{\lambda}{2}$ and put

$$f_0(z) = e^{-\lambda_0 D^2} f(z),$$

a real entire function of order < 2 (cf. 36.15). Now write

$$e^{-\lambda D^2} f(z) = e^{-(\lambda_0 + \lambda_0)D^2} f(z)$$

$$= e^{-\lambda_0 D^2} e^{-\lambda_0 D^2} f(z) \qquad \text{(cf. 36.16)}$$

$$= e^{-\lambda_0 D^2} f_0(z)$$

and apply 36.34.

36.36 LEMMA Let f be a real entire function of order < 2. Assume: f has 2K nonreal zeros — then $\forall\ \lambda > 0$, $e^{-\lambda D^2} f$ has at most 2K nonreal zeros.

[Work first with f_λ (use 16.5).]

36.37 THEOREM Let f be a real entire function of order < 2. Assume: f has 2K nonreal zeros and K is ≤ the number of real zeros of f. Fix $A > 0 : f \in A - \mathcal{L} - \mathcal{P}$ — then

$$e^{-\lambda D^2} f(z) \qquad (0 < 2\lambda < A^2)$$

is in $\underline{A} - \mathcal{L} - \mathcal{P}$ for some $\underline{A} < (A^2 - 2\lambda)^{1/2}$.

PROOF $e^{-\lambda D^2} f$ has at most 2K nonreal zeros and they lie in the strip

$$\{z : |\text{Im } z| \leq (A^2 - 2\lambda)^{1/2}\} \quad \text{(cf. 36.22)},$$

thus it will be enough to show that $e^{-\lambda D^2}f$ does not vanish on the line

$$\{z:\text{Im } z = (A^2 - 2\lambda)^{1/2}\}$$

if $0 < 2\lambda < A^2$. Write

$$f(z) = (z-a_1)\ldots(z-a_K)g(z),$$

where $a_1,\ldots,a_K$ are real zeros of f and g (like f) is a real entire function of order < 2 -- then f and g have the same nonreal zeros, hence $e^{-\lambda D^2}g$ has at most K nonreal zeros in the open upper half-plane, these being subject to the restriction that their imaginary parts are positive and $\leq (A^2 - 2\lambda)^{1/2}$. Set $h_0 = e^{-\lambda D^2}g$ and define $h_1,\ldots,h_K$ by

$$h_k = \Gamma_{\lambda,a_k}h_{k-1} \quad (k = 1,\ldots,K).$$

Then $h_0,h_1,\ldots,h_K$ are real entire functions of order < 2. And (cf. 36.27)

$$h_1 = \Gamma_{\lambda,a_1}h_0$$

$$= \Gamma_{\lambda,a_1} e^{-\lambda D^2}g$$

$$= e^{-\lambda D^2}((z-a_1)g),$$

so in the end

$$h_K = e^{-\lambda D^2}f.$$

If now h_K has a zero z_K on the line

$$\{z:\text{Im } z = (A^2 - 2\lambda)^{1/2}\},$$

then there are complex numbers $z_0,\ldots,z_{K-1}$ in the open upper half-plane such that

$h_k(z_k) = 0$ and

$$|z_{k+1} - \mathrm{Re}\, z_k| \le \mathrm{Im}\, z_k \quad (k = 0,1,\ldots,K-1) \text{ (Jensen...)}.$$

Therefore $\mathrm{Im}\, z_{k+1} \le \mathrm{Im}\, z_k$ and $\mathrm{Im}\, z_{k+1} = \mathrm{Im}\, z_k$ iff $z_{k+1} = z_k$. Since $h_0(z_0) = 0$, it follows that $\mathrm{Im}\, z_0 \le (A^2 - 2\lambda)^{1/2}$ from which

$$\mathrm{Im}\, z_K = (A^2 - 2\lambda)^{1/2}$$

$$\le \mathrm{Im}\, z_{K-1} \le \ldots \le \mathrm{Im}\, z_0 \le (A^2 - 2\lambda)^{1/2}$$

$$\Rightarrow$$

$$z_0 = z_1 = \ldots = z_K$$

and we claim that z_0 is a zero of h_0 of multiplicity $> K$. First

$$0 = h_1(z_1) = h_1(z_0)$$

$$= (z_0 - a_1)h_0(z_0) - 2\lambda h_0'(z_0)$$

$$= -2\lambda h_0'(z_0)$$

$$\Rightarrow$$

$$h_0'(z_0) = 0.$$

Next

$$0 = h_2(z_2) = h_2(z_1)$$

$$= (z_0 - a_2)h_1(z_1) - 2\lambda h_1'(z_1)$$

$$= -2\lambda h_1'(z_1)$$

$$= -2\lambda h_1'(z_0)$$

$$\Rightarrow$$

$$h_1'(z_0) = 0.$$

But

$$h_1(z) = (z-a_1)h_0(z) - 2\lambda h_0'(z)$$

=>

$$h_1'(z) = h_0(z) + (z-a_1)h_0'(z) - 2\lambda h_0''(z)$$

=>

$$0 = h_1'(z_0) = h_0(z_0) + (z_0-a_1)h_0'(z_0) - 2\lambda h_0''(z_0)$$

$$= - 2\lambda h_0''(z_0)$$

=>

$$h_0''(z_0) = 0.$$

ETC. However the claim leads to a contradiction: $h_0 = e^{-\lambda D^2}g$ has at most K nonreal zeros in the open upper half-plane.

N.B. The condition on K is obviously fulfilled if the number of real zeros of f is infinite.

APPENDIX

Here a proof of 36.32 will be sketched. So take an $f \in * - S - L - P$ -- then the claim is that

$$(\Gamma_\lambda^2 + \beta)f \qquad (\Gamma_\lambda^2 \equiv \Gamma_{\lambda,0}^2)$$

remains within $* - S - L - P$ and for this, it can be assumed that f has infinitely many real zeros.

SETUP Write

$$f(z) = e^{az^2+bz}\, Q(z) \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) e^{z/\lambda_n},$$

where a is real and ≤ 0, b is real, $Q(z)$ is a real polynomial, the λ_n are real and distinct with

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \frac{1}{4\beta} \qquad \text{(cf. 10.19)}.$$

Choose a positive constant B such that $|t| \geq B$

$$\Rightarrow Q(t) \neq 0,\ \frac{d}{dt}\, \frac{Q'(t)}{Q(t)} < 0, \text{ and } \left| \frac{b}{t} + \frac{Q'(t)}{tQ(t)} \right| < \frac{1}{4\lambda}\,.$$

Assume further that the zeros of $f(z)$ that lie in $|z| \geq B$ are real and simple.

NOTATION For $R > 0$, put

$$f_R(z) = e^{az^2+bz}\, Q(z) \prod_{|\lambda_n|<R} \left(1 - \frac{z}{\lambda_n}\right) e^{z/\lambda_n}.$$

N.B.

$$(\Gamma_\lambda^2 + \beta) f_R \in * - L - P$$

and

$$(\Gamma_\lambda^2 + \beta) f_R \to (\Gamma_\lambda^2 + \beta) f \qquad (R \to \infty)$$

uniformly on compact subsets of $\mathbf{C}$.

LEMMA

$$\frac{\Gamma_\lambda f_R(z)}{f_R(z)}$$

$$= (1 - 4\lambda a)z - 2\lambda b - 2\lambda \frac{Q'(z)}{Q(z)}$$

$$- 2\lambda \sum_{|\lambda_n|<R} \frac{z}{\lambda_n(z-\lambda_n)} \, .$$

APPLICATION If λ', λ'' are two consecutive real zeros of $f_R(z)$ such that $\lambda' < \lambda'' \leq -B$ or $B \leq \lambda' < \lambda''$, then

$$\frac{\Gamma_\lambda f_R(z)}{f_R(z)}$$

has exactly one real zero between λ' and λ''.

[In fact,

$$\lim_{t \downarrow \lambda'} \frac{\Gamma_\lambda f_R(t)}{f_R(t)} = -\infty, \quad \lim_{t \uparrow \lambda''} \frac{\Gamma_\lambda f_R(t)}{f_R(t)} = \infty$$

and

$$\frac{\Gamma_\lambda f_R(t)}{f_R(t)}$$

is strictly increasing in the interval $]\lambda', \lambda''[$.]

LEMMA Suppose that

$$\frac{\Gamma_\lambda f_R(r_0)}{f_R(r_0)} = 0 \qquad (r_0 \in R, \ |r_0| \geq B).$$

Then the real numbers

$$f_R(r_0) \text{ and } (\Gamma_\lambda^2 + \beta) f_R(r_0)$$

are of opposite sign.

PROOF Trivially,

$$r_0 = \frac{2\lambda f'_R(r_0)}{f_R(r_0)} .$$

Therefore

$$\frac{r_0}{2\lambda} = 2ar_0 + b + \frac{Q'(r_0)}{Q(r_0)} + \sum_{|\lambda_n|<R} \frac{r_0}{\lambda_n(r_0-\lambda_n)}$$

=>

$$\frac{1}{2\lambda} = 2a + (\frac{b}{r_0} + \frac{Q'(r_0)}{r_0 Q(r_0)}) + \sum_{|\lambda_n|<R} \frac{1}{\lambda_n(r_0-\lambda_n)}$$

$$\leq (\frac{b}{r_0} + \frac{Q'(r_0)}{r_0 Q(r_0)}) + \sum_{|\lambda_n|<R} \frac{1}{\lambda_n(r_0-\lambda_n)}$$

$$\leq \left| \frac{b}{r_0} + \frac{Q'(r_0)}{r_0 Q(r_0)} \right| + \left| \sum_{|\lambda_n|<R} \frac{1}{\lambda_n(r_0-\lambda_n)} \right|$$

$$< \frac{1}{4\lambda} + \left| \sum_{|\lambda_n|<R} \frac{1}{\lambda_n(r_0-\lambda_n)} \right|$$

=>

$$\frac{1}{4\lambda} < \left| \sum_{|\lambda_n|<R} \frac{1}{\lambda_n(r_0-\lambda_n)} \right|$$

$$\leq \sum_{|\lambda_n|<R} \frac{1}{|\lambda_n|\,|r_0-\lambda_n|}$$

$$\leq \Big(\sum_{|\lambda_n|<R} \frac{1}{\lambda_n^2} \Big)^{1/2} \Big(\sum_{|\lambda_n|<R} \frac{1}{(r_0-\lambda_n)^2} \Big)^{1/2}$$

$$< \frac{1}{2\sqrt{\beta}} \Big(\sum_{|\lambda_n|<R} \frac{1}{(r_0-\lambda_n)^2} \Big)^{1/2}$$

=>

$$\sum_{|\lambda_n|<R} \frac{1}{(r_0-\lambda_n)^2} > \Big(\frac{1}{4\lambda}\Big)^2 \ (2\ \sqrt{\beta})^2$$

$$= \frac{\beta}{4\lambda^2} .$$

Moving on,

$$\frac{(\Gamma_\lambda^2 + \beta)f_R(r_0)}{f_R(r_0)} = \beta - 2\lambda \ + 4\lambda^2 \ \frac{f_R''(r_0)f_R(r_0) \ - f'(r_0)^2}{f_R(r_0)^2}$$

$$= \beta - 2\lambda + 4\lambda^2 \frac{d}{dt} \Big(\frac{f_R'(t)}{f_R(t)}\Big) \Big|_{t = r_0}$$

$$= \beta - 2\lambda + 4\lambda^2 \Big(2a + \frac{d}{dt} \Big(\frac{Q'(t)}{Q(t)}\Big) \Big|_{t = r_0} - \sum_{|\lambda_n|<R} \frac{1}{(r_0-\lambda_n)^2}\Big)$$

$$< \beta + 4\lambda^2 \Big(- \sum_{|\lambda_n|<R} \frac{1}{(r_0-\lambda_n)^2}\Big).$$

But

$$\sum_{|\lambda_n|<R} \frac{1}{(r_0-\lambda_n)^2} > \frac{\beta}{4\lambda^2} ,$$

so

$$\frac{(\Gamma_\lambda^2 + \beta)f_R(r_0)}{f_R(r_0)} < \beta - \beta = 0.$$

APPLICATION If λ', λ'', λ''' are three consecutive real zeros of $f_R(z)$

such that $\lambda' < \lambda'' < \lambda''' \leq -B$ or $B \leq \lambda' < \lambda'' < \lambda'''$ and if r_1 and r_2 are real zeros of $\frac{\Gamma_\lambda f_R(z)}{f_R(z)}$ such that $\lambda' < r_1 < \lambda'' < r_2 < \lambda'''$, then $(\Gamma_\lambda^2 + \beta) f_R(z)$ has a real zero between r_1 and r_2.

[As a part of the overall setup, the zeros of $f_R(z)$ are real and simple.]

NOTATION Given an entire function $F(z)$ and a subset S of $\mathbb{C}$, let

$$N(F(z);S)$$

denote the number (counting multiplicity) of zeros of $F(z)$ that lie in S.

EXAMPLE

$$N((\Gamma_\lambda^2 + \beta)f_R(z);\mathbb{C}) = N(f_R(z);\mathbb{C}) + 2.$$

EXAMPLE

$$N((\Gamma_\lambda^2 + \beta)f_R(z);\ ]-\infty, -B] \cup [B,\infty[)$$

$$\geq N(f_R(z);\ ]-\infty, -B] \cup [B,\infty[) - 4.$$

LEMMA We have

$$N((\Gamma_\lambda^2 + \beta)f_R(z);\text{Im } z \neq 0)$$

$$\leq N(f(z);\text{Im } z \neq 0) + N(f(z);\ ]-B,B[) + 6.$$

PROOF Rewrite the first term as

$$N((\Gamma_\lambda^2 + \beta)f_R(z);\mathbb{C}) - N((\Gamma_\lambda^2 + \beta)f_R(z);\mathbb{R})$$

and then bound it by

$$N(f_R(z);C) + 2 - N((\Gamma_\lambda^2 + \beta)f_R(z);\ ]-\infty, -B] \cup [B,\infty[)$$

or still, by

$$N(f_R(z);C) - N(f_R(z);\ ]-\infty, -B] \cup [B,\infty[) + 6$$

or still, by

$$N(f_R(z);\text{Im } z \neq 0) + N(f_R(z);\ ]-B,B[) + 6$$

or still, by

$$N(f(z);\text{Im } z \neq 0) + N(f(z);\ ]-B,B[) + 6.$$

Accordingly,

$$(\Gamma_\lambda^2 + \beta)f \in * - L - P$$

but there remains the possibility that it might have infinitely many multiple zeros. However, if this were the case, then we would have

$$\lim_{A \to \infty} (N((\Gamma_\lambda^2 + \beta)f(z);\ ]-A,A[) - N(f(z);\ ]-A,A[)) = \infty.$$

And:

LEMMA Take $A > B$ –– then $\exists\ R_0 > A$ such that

$$N((\Gamma_\lambda^2 + \beta)f(z);|\text{Re } z| < A)$$

$$\leq N((\Gamma_\lambda^2 + \beta)f_{R_0}(z);|\text{Re } z| < A).$$

On the other hand,

$$N((\Gamma_\lambda^2 + \beta)f(z);\ ]-A,A[)$$

$$\leq N((\Gamma_\lambda^2 + \beta)f(z);|\text{Re } z| < A)$$

$$\leq N((\Gamma_\lambda^2 + \beta)f_{R_0}(z);|\mathrm{Re}\ z| < A)$$

$$= N((\Gamma_\lambda^2 + \beta)f_{R_0}(z);C) - N((\Gamma_\lambda^2 + \beta)f_{R_0}(z);|\mathrm{Re}\ z| \geq A)$$

$$\leq N((\Gamma_\lambda^2 + \beta)f_{R_0}(z);C) - N((\Gamma_\lambda^2 + \beta)f_{R_0}(z);\ ]-\infty,-A] \cup [A,\infty[)$$

$$\leq N(f_{R_0}(z);C) + 2 - N(f_{R_0}(z);\ ]-R_0,-A] \cup [A,R_0[) + 4$$

$$= N(f_{R_0}(z);\mathrm{Im}\ z \neq 0) + N(f_{R_0}(z);\ ]-A,A[) + 6$$

$$\leq N(f(z);\mathrm{Im}\ z \neq 0) + N(f(z);\ ]-A,A[) + 6$$

=>

$$N((\Gamma_\lambda^2 + \beta)f(z);\ ]-A,A[) - N(f(z);\ ]-A,A[)$$

$$\leq N(f(z);\mathrm{Im}\ z) + 6,$$

from which a contradiction (send A to ∞).

§37. THE $\mathcal{F}_0$ - CLASS

Let F be a real entire function such that

$$\log M(r;F) = O(r^4) \qquad (r \to \infty)$$

and

$$\int_{-\infty}^{\infty} |F(\sqrt{-1}\, t)|dt < \infty.$$

[Note: Since F is real, $\overline{F(z)} = F(\bar{z})$, hence if $G(t) = F(\sqrt{-1}\, t)$, then

$$g(-t) = F(\sqrt{-1}\,(-t)) = F((-\sqrt{-1})t)$$

$$= F(\overline{\sqrt{-1}}\, t) = F(\overline{\sqrt{-1}\, t}) = \overline{F(\sqrt{-1}\, t)} = \overline{G(t)}.]$$

37.1 DEFINITION $F \in \mathcal{F}_0$ provided all its zeros are real and

$$\sum_n \frac{1}{\lambda_n^4} < \infty \quad (F(\lambda_n) = 0,\ \lambda_n \neq 0).$$

[Note: The sum is finite or infinite.]

37.2 THEOREM Suppose that $F \in \mathcal{F}_0$ and

$$f(z) \equiv \int_{-\infty}^{\infty} F(\sqrt{-1}\, t)e^{\sqrt{-1}\, zt}dt.$$

Then $f \in L - P$.

[Note: While not quite obvious, the assumptions on F imply that f is entire (see below). Moreover f is real:

$$\overline{f(x)} = \int_{-\infty}^{\infty} \overline{F(\sqrt{-1}\, t)}e^{-\sqrt{-1}\, xt}dt$$

$$= \int_{-\infty}^{\infty} F(-\sqrt{-1}\, t)e^{-\sqrt{-1}\, xt}dt$$

$$= \int_{-\infty}^{\infty} F(\sqrt{-1}\, t) e^{\sqrt{-1}\, xt} dt = f(x).]$$

37.3 RAPPEL If $f_n \in L - P$ $(n = 1,2,\ldots)$ and if $f_n \to f$ uniformly on compact subsets of $\mathbf{C}$, then $f \in L - P$.

The proof of 37.2 falls into two cases, according to whether the number of zeros of F is finite or infinite.

So suppose first that F has finitely many zeros -- then there exists a real polynomial P and real constants $\alpha,\beta,\gamma,\delta$ such that P has only real zeros, α is nonnegative, $\max(\alpha,\gamma)$ is positive, and

$$F(z) = P(z)\exp(- \alpha^2 z^4 - \beta^3 z^3 + \gamma z^2 + \delta z).$$

Choose a positive integer N:

$$2n\alpha + \frac{3}{2} n\beta^2 + \gamma > 0 \qquad (n \geq N).$$

Then define $F_n(z)$ $(n \geq N)$ by

$$F_n(z) = P(z)\left((1 - \frac{\alpha z^2}{n})\exp(\frac{\alpha z^2}{n})\right)^{2n^2}$$

$$\times \left((1 - \frac{\beta z}{n})\exp(\frac{\beta z}{n} + \frac{\beta^2 z^2}{2n^2})\right)^{3n^3} e^{\gamma z^2 + \delta z}$$

and set

$$f_n(z) = \int_{-\infty}^{\infty} F_n(\sqrt{-1}\, t) e^{\sqrt{-1}\, zt} dt.$$

37.4 LEMMA $f_n \to f$ uniformly on compact subsets of $\mathbf{C}$.

PROOF In fact,

$$\left((1 - \frac{\alpha z^2}{n})\exp(\frac{\alpha z^2}{n})\right)^{2n^2} \to e^{-\alpha^2 z^4}$$

and

$$((1 - \frac{\beta z}{n})\exp(\frac{\beta z}{n} + \frac{\beta^2 z^2}{2n^2}))^{3n^3} \to e^{-\beta^3 z^3}$$

uniformly on compact subsets of C. On the other hand,

$$\left| (1 - \frac{\beta\sqrt{-1}\,t}{n})\exp(\frac{\beta\sqrt{-1}\,t}{n} + \frac{\beta^2(\sqrt{-1}\,t)^2}{2n^2}) \right| \le 1 \quad (t \in R).$$

In addition, there are positive constants C, t_0 such that

$$((1 + \frac{\alpha t^2}{n})\exp(- \frac{\alpha t^2}{n}))^{2n^2} e^{-\gamma t^2} \le e^{-Ct^2} \quad (n \ge N, \quad |t| \ge t_0).$$

And this sets the stage for dominated convergence.

37.5 LEMMA $\forall\, n \ge N$, $f_n \in L - P$.

PROOF We have

$$F_n(z) = P(z)(1 - \frac{\alpha z^2}{n})^{2n^2} (1 - \frac{\beta z}{n})^{3n^3}$$

$$\times \exp((2n\alpha + \frac{3}{2} n\beta^2 + \gamma)z^2 + (3n^2\beta + \delta)z).$$

But

$$2n\alpha + \frac{3}{2} n\beta^2 + \gamma > 0$$

and replacing z by $\sqrt{-1}\,t$ leads to

$$- (2n\alpha + \frac{3}{2} n\beta^2 + \gamma)t^2,$$

thus an application of 12.37 completes the proof.

Taking into account 37.3, it then follows from 37.4 and 37.5 that $f \in L - P$.

Suppose now that F has infinitely many zeros (by hypothesis real) and write

$$F(z) = Mz^m \exp(A_4 z^4 + A_3 z^3 + A_2 z^2 + A_1 z)$$

$$\times \prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n}) \exp(\frac{z}{\lambda_n} + \frac{z^2}{2\lambda_n^2} + \frac{z^3}{3\lambda_n^3}),$$

where $M \neq 0$ is real, m is a nonnegative integer, A_1, A_2, A_3, A_4 are real constants, the λ_n are real with $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^4} < \infty$ -- then $\forall\ t \in R$,

$$|F(\sqrt{-1}\ t)| = |M|\ |t|^m e^{A_4 t^4 - A_2 t^2} \prod_{n=1}^{\infty} (1 + \frac{t^2}{\lambda_n^2})^{1/2} \exp(- \frac{t^2}{2\lambda_n^2}).$$

37.6 LEMMA There exists a positive integer N with the property that

$$\max(- A_4, A_2 + \sum_{k=1}^{n} \frac{1}{\lambda_k^2}) > 0 \quad (n \geq N).$$

PROOF Since

$$\int_{-\infty}^{\infty} |F(\sqrt{-1}\ t)| dt < \infty,$$

A_4 must be ≤ 0, thus matters are obvious if A_4 is < 0. Assume, therefore, that $A_4 = 0$ -- then

$$|F(\sqrt{-1}\ t)| \geq |M|\ |t|^m e^{-A_2 t^2} \prod_{n=1}^{\infty} \exp(- \frac{t^2}{2\lambda_n^2})$$

$$= |M|\ |t|^m e^{-A_2 t^2} \exp((- \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2}) t^2),$$

so if

$$\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty,$$

the condition on A_2 is that

$$-A_2 - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2} < 0$$

or still,

$$A_2 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2} > 0$$

=>

$$A_2 + \sum_{n=1}^{\infty}\frac{1}{\lambda_n^2} > 0$$

=>

$$A_2 + \sum_{k=1}^{n}\frac{1}{\lambda_k^2} > 0 \qquad (n >> 0).$$

However, in the event that

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2} = \infty,$$

then it is automatic that

$$\max(0, A_2 + \sum_{k=1}^{n}\frac{1}{\lambda_k^2}) > 0$$

$\forall\, n >> 0$, there being in this case no condition on A_2.

Define $F_n(z)$ $(n \geq N)$ by

$$F_n(z) = Mz^m \exp(A_4z^4 + A_3z^3 + A_2z^2 + A_1z)$$

$$\times \prod_{k=1}^{n} (1 - \frac{z}{\lambda_k})\exp(\frac{z}{\lambda_k} + \frac{z^2}{2\lambda_k^2} + \frac{z^3}{3\lambda_k^3})$$

$$\equiv P_n(z)\exp(A_4 z^4 + A_{3,n} z^3 + A_{2,n} z^2 + A_{1,n} z),$$

where

$$P_n(z) = Mz^m \prod_{k=1}^{n} (1 - \frac{z}{\lambda_k})$$

and

$$A_{j,n} = A_j + \frac{1}{j} \sum_{k=1}^{n} \frac{1}{\lambda_k^j} \quad (j = 1,2,3),$$

and set

$$f_n(z) = \int_{-\infty}^{\infty} F_n(\sqrt{-1}\, t) e^{\sqrt{-1}\, zt} dt.$$

37.7 LEMMA $\forall\, n \geq N$, $f_n \in \mathcal{L} - \mathcal{P}$.

PROOF From the definitions, $F_n \in \mathcal{F}_0$. But F_n has finitely many zeros, hence by the earlier work, $f_n \in \mathcal{L} - \mathcal{P}$.

37.8 LEMMA $F_n \to F$ uniformly on compact subsets of $\mathbf{C}$.

37.9 LEMMA $\forall\, n \geq N$,

$$|F_n(\sqrt{-1}\, t)| \leq |F_N(\sqrt{-1}\, t)| \qquad (t \in \mathbf{R}).$$

PROOF This is because

$$\left|(1 - \frac{\sqrt{-1}\, t}{\lambda_n})\exp(\frac{\sqrt{-1}\, t}{\lambda_n} + \frac{(\sqrt{-1}\, t)^2}{2\lambda_n^2} + \frac{(\sqrt{-1}\, t)^3}{3\lambda_n^3})\right| \leq 1$$

for all n and for all t.

Consequently, $f_n \to f$ uniformly on compact subsets of $\mathbf{C}$, thus 37.3 can be invoked to conclude that $f \in \mathcal{L} - \mathcal{P}$, thereby finishing the proof of 37.2.

37.10 LEMMA If $F \in \mathcal{F}_0$, then $\forall\ \lambda > 0$, the function

$$e^{\lambda z^2} F(z)$$

is in $\mathcal{F}_0$, hence the function

$$\int_{-\infty}^{\infty} F(\sqrt{-1}\ t) e^{-\lambda t^2} e^{\sqrt{-1}\ zt} dt$$

is in $L - P$ (cf. 37.2).

[Note:

$$\begin{aligned} \text{Re}(-\ \lambda t^2 + \sqrt{-1}\ zt) &= -\ \lambda t^2 - t\ \text{Im}\ z \\ &\leq -\ \lambda t^2 + |t|\ |\text{Im}\ z| \\ &\leq -\ \lambda t^2 + |t|\ |z|. \end{aligned}$$

As a function of t, the max of

$$-\ \lambda t^2 + |t|\ |z|$$

is at $|t| = \frac{|z|}{2\lambda}$ and the maximum value is

$$-\ \lambda \frac{|z|^2}{4\lambda^2} + \frac{|z|}{2\lambda} |z| = \frac{|z|^2}{4\lambda}.$$

And then

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} F(\sqrt{-1}\ t) e^{-\lambda t^2} e^{\sqrt{-1}\ zt} dt \right| \\ &\qquad \leq \left(\int_{-\infty}^{\infty} |F(\sqrt{-1}\ t)| dt\right) \exp\left(\frac{|z|^2}{4\lambda}\right).] \end{aligned}$$

The foregoing considerations can, in a certain sense, be reversed.

37.11 THEOREM[†] Let μ be an even, finite, absolutely continuous Borel measure on the real line. Suppose that $\forall\ \lambda < 0$, the function

$$\int_{-\infty}^{\infty} e^{\lambda t^2}\, e^{\sqrt{-1}\, zt} d\mu(t)$$

has real zeros only -- then

$$d\mu(t) = F(\sqrt{-1}\, t)dt$$

for some $F \in \mathcal{F}_0$.

N.B. In this situation, $F(\sqrt{-1}\, t)$ is nonnegative, even, and admits the decomposition

$$F(\sqrt{-1}\, t) = Mt^{2m} \exp(-\alpha t^4 - \beta t^2) \prod_j (1 + \frac{t^2}{a_j^2})\exp(-\frac{t^2}{a_j^2}),$$

where $M > 0$, $m = 0,1,\ldots$, $a_j > 0$, $\sum_j \frac{1}{a_j^4} < \infty$, $\alpha > 0$ and β real or $\alpha = 0$ and $\beta + \sum_j \frac{1}{a_j^2} > 0$.]

[Note: The product is over a set of j which may be empty, finite, or infinite and the condition $\beta + \sum_j \frac{1}{a_j^2} > 0$ is considered to be satisfied if $\sum_j \frac{1}{a_j^2} = \infty$.]

37.12 SUBLEMMA $\forall\ x \in R$,

$$(1 + x^2)\exp(-x^2) \geq \exp(-x^4/2).$$

PROOF $\forall\ y \geq 0$,

$$\log(1 + y) \geq y - \frac{y^2}{2}.$$

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† C. Newman, *Proc. Amer. Math. Soc.* 61 (1976), pp. 245-251.

Therefore

$$1 + y \geq \exp(y - \frac{y^2}{2})$$

$\Rightarrow$

$$(1 + y)\exp(- y) \geq \exp(- \frac{y^2}{2}).$$

Now take $y = x^2$.

37.13 APPLICATION We have

$$F(\sqrt{-1}\ t) \geq Mt^{2m} \exp(-(\alpha + \sum_j \frac{1}{2a_j^4})t^4 - \beta t^2).$$

Let $\Phi \in L^1(-\infty,\infty)$ be real analytic, positive and even. Assume:

$$\Phi(t) = O(\exp(A|t|^a - Be^{C|t|^c})) \qquad (|t| \to \infty)$$

for positive constants A, $a \geq 1$, B, C, $c \geq 1$.

N.B. Therefore Φ is of regular growth (cf. 35.14).

Given any real λ, put

$$\Xi_\lambda(z) = \int_{-\infty}^{\infty} \Phi(t)e^{\lambda t^2}\ e^{\sqrt{-1}\ zt}dt.$$

37.14 THEOREM If the zeros of Ξ_0 lie in the strip $\{z: |\text{Im } z| \leq \Delta\}$, then the zeros of Ξ_λ $(\lambda > 0)$ are real provided $\frac{\Delta^2}{2} \leq \lambda$ and simple provided $\frac{\Delta^2}{2} < \lambda$ (cf. 36.20).

37.15 LEMMA There does not exist an $F \in \mathcal{F}_0$ such that $\Phi(t) = F(\sqrt{-1}\ t)$.

PROOF For if this were the case, then

$$\Phi(t) \geq Mt^{2m} \exp(-(\alpha + \sum_j \frac{1}{2a_j^4})t^4 - \beta t^2) \qquad \text{(cf. 37.13)},$$

so

$$Mt^{2m} \exp(-(\alpha + \sum_j \frac{1}{2a_j^4})t^4 - \beta t^2)$$

$$= O(\exp(A|t| - Be^{C|t|})).$$

Setting $T = |t|$, it thus follows that

$$\log M + 2m \log T - (\alpha + \sum_j \frac{1}{2a_j^4})T^4 - \beta T^2 - AT + Be^{CT}$$

stays bounded as $T \to \infty$, an absurdity.]

Supposing still that the zeros of Ξ_0 lie in the strip $\{z:|\text{Im } z| \leq \Delta\}$, there must exist a negative λ_0 such that Ξ_{λ_0} has a nonreal zero (otherwise, taking $d\mu(t) = \Phi(t)dt$ in 37.11 forces $\Phi(t) = F(\sqrt{-1}\, t)$ for some $F \in \mathcal{F}_0$ contradicting 37.15).

37.16 LEMMA $\forall \lambda < \lambda_0$, Ξ_λ has a nonreal zero.

PROOF In fact, if all the zeros of Ξ_λ were real, then all the zeros of Ξ_{λ_0} would also be real (cf. 36.8).

Let L be the set of λ such that Ξ_λ has a nonreal zero and let R be the set of λ such that all the zeros of Ξ_λ are real -- then

$$\lambda_1 \in L,\ \lambda_2 \in R \Rightarrow \lambda_1 < \lambda_2.$$

Therefore the pair (L,R) defines a Dedekind cut and we shall denote its cut point by Λ_0, hence

$$\begin{cases} \lambda < \Lambda_0 \Rightarrow \lambda \in L \\ \lambda > \Lambda_0 \Rightarrow \lambda \in R. \end{cases}$$

N.B. A priori,

$$\Lambda_0 \leq \frac{\Delta^2}{2} \quad \text{(cf. 37.14).}$$

37.17 LEMMA

$$\Lambda_0 \in \mathbf{R}.$$

PROOF Put $\lambda_n = \Lambda_0 + \frac{1}{n}$ $(n = 1,2,\ldots)$ -- then $\Xi_{\lambda_n} \to \Xi_{\Lambda_0}$ uniformly on compact subsets of $\mathbf{C}$ (the assumptions serve to ensure that the Ξ_{λ_n} constitute a normal family). But the zeros of Ξ_{λ_n} are real and a zero of Ξ_{Λ_0} is either a zero of Ξ_{λ_n} for all sufficiently large values of n or else is a limit point of the set of zeros of the Ξ_{λ_n}. And this means that the zeros of Ξ_{Λ_0} are real, i.e., $\Lambda_0 \in \mathbf{R}$.

N.B. Therefore L consists of all λ such that $\lambda < \Lambda_0$ and R consists of all λ such that $\Lambda_0 \leq \lambda$.

37.18 THEOREM If $\lambda < \Lambda_0$, then Ξ_λ has a nonreal zero and if $\Lambda_0 \leq \lambda$, then all the zeros of Ξ_λ are real.

[This is a statement of recapitulation.]

37.19 THEOREM Suppose that Ξ_λ has a multiple real zero x_0 -- then $\lambda \leq \Lambda_0$.

PROOF Take $x_0 = 0$ and in 36.19, take $f(z) = \Xi_\lambda(z)$ -- then for all $\delta > 0$ and sufficiently small, $e^{\delta D^2}\Xi_\lambda(z)$ has a nonreal zero. But

$$e^{\delta D^2}\Xi_\lambda(z) = e^{\delta D^2}\, e^{-\lambda D^2}\Xi_0(z) \quad \text{(cf. 36.12)}$$

$$= e^{(\delta-\lambda)D^2}\ \Xi_0(z) \quad (\text{cf. } 36.16)$$

$$= \Xi_{\lambda-\delta}(z) \quad (\text{cf. } 36.12),$$

so

$$\lambda - \delta < \Lambda_0 \Rightarrow \lim_{\delta\to 0} (\lambda-\delta) \le \Lambda_0 \Rightarrow \lambda \le \Lambda_0.$$

37.20 SCHOLIUM If $\lambda > \Lambda_0$, then all the zeros of Ξ_λ are real and simple.

37.21 APPLICATION If Ξ_0 has a multiple real zero, then $0 \le \Lambda_0$.

[Note: If Ξ_0 has a nonreal zero, then $\Lambda_0 > 0$.]

37.22 CRITERION Suppose that there exists a $\lambda_0 < \Lambda_0$ with the property that $\forall\ \varepsilon > 0$, all but a finite number of zeros of Ξ_{λ_0} lie in the strip $|\text{Im } z| \le \varepsilon$ -- then $\forall\ \lambda \in \,]\lambda_0, \Lambda_0[$, $\Xi_\lambda \in * - S - L - P$.

[By definition,

$$\Xi_{\lambda_0}(z) = \int_{-\infty}^{\infty} \Phi(t) e^{\lambda_0 t^2}\ e^{\sqrt{-1}\ zt} dt.$$

Put

$$\phi(t) = \Phi(t) e^{\lambda_0 t^2},$$

so that

$$\Xi_{\lambda_0}(z) = \int_{-\infty}^{\infty} \phi(t) e^{\sqrt{-1}\ zt} dt$$

$$= f_\infty(z).$$

Pass now to

$$f_\infty(z;\lambda-\lambda_0) = \int_{-\infty}^{\infty} \phi(t)e^{(\lambda-\lambda_0)t^2} e^{\sqrt{-1}\,zt}dt,$$

a function in $* - S - L - P$ (cf. 36.33). But

$$f_\infty(z;\lambda-\lambda_0) = \int_{-\infty}^{\infty} \Phi(t)e^{\lambda t^2} e^{\sqrt{-1}\,zt}dt$$

$$= \Xi_\lambda(z).]$$

§38. ζ, ξ, AND Ξ

If $\zeta(s)$ is the Riemann zeta function and if

$$\xi(s) = \frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma(\frac{1}{2})\zeta(s)$$

is the completed Riemann zeta function, then

$$\xi(s) = \xi(1-s).$$

38.1 NOTATION Put

$$\Xi(z) = \xi(\frac{1}{2} + \sqrt{-1}\ z).$$

Then Ξ is even, i.e., $\Xi(z) = \Xi(-z)$.

38.2 LEMMA Ξ is a real entire function of order 1 and of maximal type.

38.3 LEMMA The zeros of Ξ lie in the strip $\{z:|\text{Im } z| < \frac{1}{2}\}$.

[Note: Recall that $\zeta(s)$ is zero free on the lines Re s = 1, Re s = 0.]

38.4 LEMMA If $\rho = \alpha + \sqrt{-1}\ \beta$ is a zero of Ξ, then

$$\bar{\rho} = \alpha - \sqrt{-1}\ \beta, -\rho = -\alpha - \sqrt{-1}\ \beta, -\bar{\rho} = -\alpha + \sqrt{-1}\ \beta$$

are also zeros of Ξ.

38.5 LEMMA Ξ has an infinity of zeros.

If $\rho_1, \rho_2, \ldots$ are the zeros of Ξ and if $r_n = |\rho_n|$, and if

$$0 < r_1 \leq r_2 \leq \cdots \quad (r_n \to \infty),$$

then $\forall\ \varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{r_n^{1+\varepsilon}} < \infty$$

but

$$\sum_{n=1}^{\infty} \frac{1}{r_n} < \infty.$$

[Note: Therefore the convergence exponent of the zeros of Ξ is equal to 1.]

38.6 LEMMA gen $\Xi = 1$ and

$$\Xi(z) = \Xi(0) \prod_{n=1}^{\infty} (1 - \frac{z}{\rho_n})e^{z/\rho_n}.$$

[Note: $\forall\ \rho$,

$$(1 - \frac{z}{\rho})e^{z/\rho} \cdot (1 + \frac{z}{\rho})e^{-z/\rho} = (1 - \frac{z^2}{\rho^2}).]$$

Therefore

$$\Xi \in \frac{1}{2} - L - P.$$

38.7 DEFINITION The Riemann Hypothesis (RH) is the statement that all the zeros of Ξ are real.

38.8 LEMMA RH holds iff

$$\Xi \in L - P.$$

[Note: Since $L - P$ is closed under differentiation, if the Riemann Hypothesis obtains, then $\forall$ n,

$$\Xi^{(n)}(z) = \frac{d^n}{dz^n} \Xi \in L - P.]$$

38.9 THEOREM Ξ has an infinity of real zeros.

[There are a number of proofs of this result, one of which is delineated below.]

38.10 NOTATION Put

$$\Phi(t) = \sum_{n=1}^{\infty} (4\pi^2 n^4 e^{\frac{9}{2}t} - 6\pi n^2 e^{\frac{5}{2}t}) \exp(-\pi n^2 e^{2t}).$$

38.11 THEOREM Ξ and Φ are connected by the relation

$$\Xi(z) = \int_{-\infty}^{\infty} \Phi(t) e^{\sqrt{-1}\, zt} dt.$$

38.12 RAPPEL The theta function is defined by

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 z} \quad (\mathrm{Re}\, z > 0).$$

38.13 LEMMA Φ and θ are connected by the relation

$$\Phi(t) = \frac{1}{2}\left(\frac{d^2}{dt^2} - \frac{1}{4}\right)(e^{\frac{t}{2}} \theta(e^{2t})).$$

38.14 LEMMA Φ is an even function of $t: \Phi(t) = \Phi(-t)$.

PROOF In the functional equation

$$\theta(x) = \left(\frac{1}{x}\right)^{1/2} \theta\left(\frac{1}{x}\right),$$

take $x = e^{2t}$, hence

$$e^{\frac{t}{2}} \theta(e^{2t}) = e^{-\frac{t}{2}} \theta(e^{-2t}).$$

38.15 LEMMA Φ is a positive function of $t: \Phi(t) > 0$.

[Note: In particular,

$$\Xi(0) = \int_{-\infty}^{\infty} \Phi(t) dt$$

$$= 2 \int_0^{\infty} \Phi(t) dt > 0.]$$

38.16 LEMMA We have

$$\Phi(t) = O(\exp(\frac{9}{2}|t| - \pi e^{2|t|})) \text{ as } |t| \to \infty.$$

38.17 LEMMA $\Phi(t)$ admits an analytic continuation into the strip $|\text{Im } z| < \frac{\pi}{4}$ and $\forall\ n = 0,1,2,\ldots,$

$$\lim_{t \to \frac{\pi}{4}} \Phi^{(n)}(\sqrt{-1}\ t) = 0.$$

[Note: Φ cannot be extended to an entire function.]

N.B. Therefore Φ is real analytic.

38.18 REMARK The data above thus fits within the framework of §37, viz. $\Phi \in L^1(-\infty,\infty)$ is real analytic, positive and even, the growth constants being $A = \frac{9}{2}$, $a = 1$, $B = \pi$, $C = 2$, $c = 1$.

[Note: This theme is pursued in §39.]

Here is Polya's proof of 38.9. To begin with, Fourier inversion is clearly possible, hence

$$\Phi(t) = \frac{1}{\pi}\int_0^\infty \Xi(x)\cos tx\, dx,$$

from which

$$\Phi^{(2n)}(t) = \frac{(-1)^n}{\pi}\int_0^\infty \Xi(x)x^{2n}\cos tx\, dx.$$

Write

$$\Phi(\sqrt{-1}\ t) = c_0 + c_1 t^2 + c_2 t^4 + \cdots \quad (|t| < \frac{\pi}{4}),$$

so

$$(2n)!\ c_n = (-1)^n \Phi^{(2n)}(0) = \frac{1}{\pi} \int_0^\infty \Xi(x) x^{2n} dx.$$

To get a contradiction, suppose now that the sign of $\Xi(x)$ is eventually constant, say $\Xi(x) > 0$ for $x > X$ -- then

$$\int_0^\infty \Xi(x) x^{2n} dx > \int_{X+1}^{X+2} \Xi(x) x^{2n} dx - \int_0^X |\Xi(x)| x^{2n} dx$$

$$> (X+1)^{2n} \int_{X+1}^{X+2} \Xi(x) dx - X^{2n} \int_0^X |\Xi(x)| dx$$

$$> 0 \quad (n >> 0)$$

=>

$$c_n > 0 \quad (n >> 0).$$

Therefore $\Phi^{(2n)}(\sqrt{-1}\, t)$ increases monotonically in t for $n >> 0$, whereas

$$\Phi^{(2n)}(\sqrt{-1}\, t) \to 0$$

for $t \to 0$, $t \to \frac{\pi}{4}$ (cf. 38.17).

38.19 LEMMA If $t > 0$, then $\Phi'(t) < 0$.

[This is a brute force computation (see the Appendix to §42 for the "how to").]

38.20 LEMMA Φ is a strictly decreasing function of t on $[0,\infty[$.

39. *THE de BRUIJN-NEWMAN CONSTANT*

Take Ξ and Φ as in §38, hence

$$\Xi(z) = \int_{-\infty}^{\infty} \Phi(t)e^{\sqrt{-1}\, zt}dt \quad (\text{cf. } 38.11),$$

and Φ meets the growth requirements per §37 (cf. 38.18). Since the zeros of Ξ lie in the strip $\{z:|\text{Im } z| < \frac{1}{2}\}$ (cf. 38.3),

$$\Delta = \frac{1}{2} \Rightarrow \frac{\Delta^2}{2} = \frac{1}{8}.$$

Given a real λ, set

$$\Xi_\lambda(z) = \int_{-\infty}^{\infty} \Phi(t)e^{\lambda t^2}\, e^{\sqrt{-1}\, zt}dt \quad (\Xi_0 = \Xi).$$

Then the zeros of Ξ_λ $(\lambda > 0)$ are real provided $\frac{1}{8} \leq \lambda$ and simple provided $\frac{1}{8} < \lambda$ (cf. 37.14). Now introduce Λ_0 and recall: If $\lambda < \Lambda_0$, then Ξ_λ has a nonreal zero and if $\Lambda_0 \leq \lambda$, then all the zeros of Ξ_λ are real (cf. 37.18).

N.B. It is automatic that

$$\Lambda_0 \leq \frac{1}{8}.$$

39.1 DEFINITION Λ_0 is called the de Bruijn-Newman constant.

[Note: Some authorities reserve this term for $4\Lambda_0$.]

39.2 LEMMA RH holds iff $\Lambda_0 \leq 0$.

N.B. The Newman Conjecture is the statement that $\Lambda_0 \geq 0$, "a quantitative version of the dictum that the Riemann Hypothesis, if true, is only barely so".

[Note: The Newman Conjecture would be resolved in the affirmative if Ξ had a multiple real zero (cf. 37.21).]

39.3 REMARK[†] It can be shown that

$$4\Lambda_0 > -1 . 14541 \times 10^{-11}.$$

[Note: It is true but not obvious that $\Lambda_0 < \frac{1}{8}$ (cf. 39.10).]

39.4 LEMMA If f is an entire function order < 2, then the order of

$$e^{\lambda D^2} f(z)$$

is < 2 (cf. 36.15) and, in fact, the orders of $f(z)$ and $e^{\lambda D^2} f(z)$ are equal.

39.5 APPLICATION Ξ_λ is a real entire function of order 1.

[Thanks to 36.12,

$$\Xi_\lambda(z) = e^{-\lambda D^2} \Xi(z).$$

39.6 LEMMA Ξ_λ is of maximal type.

PROOF If Ξ_λ were of finite type, then Ξ_λ would be of exponential type but this is ruled out by the Paley-Wiener theorem (cf. 22.7).

On general grounds, Ξ_λ has an infinity of zeros but more is true: Ξ_λ has an infinity of real zeros (argue as in 38.9).

[†] Y. Saouter et al., *Math. Compu.* 80 (2011), pp. 2281-2287.

39.7 LEMMA[†] Take $\lambda > 0$ -- then $\forall\ \varepsilon > 0$, all but a finite number of zeros of $\Xi_\lambda(z)$ lie in the strip $|\text{Im } z| \leq \varepsilon$.

39.8 APPLICATION $\forall\ \lambda > 0$, all but a finite number of zeros of Ξ_λ are real and simple (cf. 36.35).

39.9 LEMMA Suppose that $0 < \lambda < \frac{1}{8}$ -- then the zeros of Ξ_λ lie in the strip

$$\{z:|\text{Im } z| \leq A_\lambda\}$$

for some $A_\lambda < (\frac{1}{4} - 2\lambda)^{1/2}$.

PROOF Choose $\lambda_0 : 0 < \lambda_0 < \lambda$ and put $A_0 = (\frac{1}{4} - 2\lambda_0)^{1/2}$. Since the zeros of Ξ_0 $(= \Xi)$ are confined to the strip $\{z:|\text{Im } z| \leq \frac{1}{2}\}$ and since $\Xi_{\lambda_0} = e^{-\lambda_0 D^2}\Xi_0$, it follows from 36.5 (and subsequent comment) that the zeros of Ξ_{λ_0} are confined to the strip $\{z:|\text{Im } z| \leq A_0\}$ (the A^2 there is $(\frac{1}{2})^2$ here $(f_\infty = \Xi_0)$). On the other hand, the number of nonreal zeros of Ξ_{λ_0} is finite (cf. 39.8) and Ξ_{λ_0} has an infinity of real zeros. Observing now that

$$2(\lambda - \lambda_0) < A_0^2 = \frac{1}{4} - 2\lambda_0,$$

on the basis of 36.37, the zeros of

$$\Xi_\lambda = e^{-\lambda D^2}\Xi_0$$

[†] H. Ki et al., *Advances in Math.* 222 (2009), pp. 281-306.

$$= e^{-(\lambda+\lambda_0-\lambda_0)D^2}\Xi_0$$

$$= e^{-(\lambda-\lambda_0)D^2} e^{-\lambda_0 D^2}\Xi_0 \quad \text{(cf. 36.16)}$$

$$= e^{-(\lambda-\lambda_0)D^2}\Xi_{\lambda_0}$$

lie in the strip

$$\{z: |\mathrm{Im}\ z| \leq A_\lambda\}$$

for some

$$A_\lambda < (A_0^2 - 2(\lambda-\lambda_0))^{1/2} = (\frac{1}{4} - 2\lambda)^{1/2}.$$

39.10 THEOREM The de Bruijn-Newman constant Λ_0 is $< \frac{1}{8}$.

PROOF Fix $\lambda: 0 < \lambda < \frac{1}{8}$ and then choose λ_0 subject to

$$A_\lambda^2 < 2\lambda_0 < \frac{1}{4} - 2\lambda,$$

hence

$$2\lambda + 2\lambda_0 < \frac{1}{4} \Rightarrow \lambda + \lambda_0 < \frac{1}{8}.$$

Now take in 36.22 $f = \Xi_\lambda$, $A = A_\lambda$ and conclude that the zeros of

$$e^{-\lambda_0 D^2}\Xi_\lambda$$

are real. But

$$e^{-\lambda_0 D^2}\Xi_\lambda = e^{-\lambda_0 D^2} e^{-\lambda D^2}\Xi_0$$

$$= e^{-(\lambda+\lambda_0)D^2}\Xi_0 \quad \text{(cf. 36.16)}$$

$$= \Xi_{\lambda+\lambda_0}.$$

And this implies that

$$\Lambda_0 \leq \lambda + \lambda_0 < \frac{1}{8}.$$

39.11 REMARK Consider $\Xi_{1/8}$ -- then its zeros are real and simple (cf. 37.20).

Per

$$\Xi^{(n)}(z) = \frac{d^n}{dz^n}\Xi,$$

one has the analog of Λ_0, call it $\Lambda_0^{(n)}$ $(\Lambda_0 \equiv \Lambda_0^{(0)})$.

N.B.

$$\Xi_\lambda^{(n)}(z) = e^{-\lambda D^2}\Xi^{(n)}(z).$$

39.12 THEOREM The sequence $\{\Lambda^{(n)}\}$ is decreasing and its limit is ≤ 0.

PROOF By definition, $\Lambda^{(n)}$ is the infimum of the set of λ such that $\Xi_\lambda^{(n)}$ has real zeros only. But if $\Xi_\lambda^{(n)}$ has real zeros only, then the same is true of $\Xi_\lambda^{(n+1)}$, hence $\Lambda^{(n+1)} \leq \Lambda^{(n)}$. Next, $\forall\ \lambda > 0$, Ξ_λ has at most a finite number of nonreal zeros (cf. 39.8), thus $\Xi_\lambda \in * - L - P$, so $\exists\ n : \Xi_\lambda^{(n)}$ is in $L - P$ (cf. 11.9) from which $\Lambda^{(n)} \leq \lambda$. Now send λ to 0 and conclude that

$$\lim_{n \to \infty} \Lambda^{(n)} \leq 0.$$

§40. *TOTAL POSITIVITY*

A sequence $\{c_n : n \geq 0\}$ $(c_0 \neq 0)$ of real numbers is said to be <u>totally positive</u> if all the minors of all orders of the infinite lower triangular matrix

$$\mathfrak{C}: \begin{bmatrix} c_0 & 0 & 0 & 0 & 0 & \cdots \\ c_1 & c_0 & 0 & 0 & 0 & \cdots \\ c_2 & c_1 & c_0 & 0 & 0 & \cdots \\ c_3 & c_2 & c_1 & c_0 & 0 & \cdots \\ & & \cdots & & & \end{bmatrix}$$

are nonnegative.

[Note: Therefore the c_n are nonnegative.]

40.1 LEMMA If for some n, $c_n = 0$, then $\forall\ k = 1,2,\ldots,\ c_{n+k} = 0$.

PROOF The minor

$$\begin{vmatrix} c_n & c_0 \\ c_{n+k} & c_k \end{vmatrix} = -\ c_0 c_{n+k}$$

is nonnegative. But c_0 is > 0 and c_{n+k} is ≥ 0, hence $c_{n+k} = 0$.

With the understanding that $c_n = 0$ if $n < 0$, put

$$D(n,r) = \begin{vmatrix} c_n & c_{n-1} & \cdots & c_{n-r+1} \\ c_{n+1} & c_n & \cdots & c_{n-r+2} \\ \vdots & \vdots & & \vdots \\ c_{n+r-1} & c_{n+r-2} & & c_n \end{vmatrix} .$$

Here n = 0,1,2,..., while r = 1,2,3,... .

40.2 EXAMPLE Take r = 1 -- then

$$D(n,1) = c_n.$$

40.3 EXAMPLE Take r = 2 -- then

$$D(n,2) = \begin{vmatrix} c_n & c_{n-1} \\ c_{n+1} & c_n \end{vmatrix}.$$

In particular:

$$D(0,2) = \begin{vmatrix} c_0 & 0 \\ c_1 & c_0 \end{vmatrix}.$$

40.4 EXAMPLE Take r = 3 -- then

$$D(n,3) = \begin{vmatrix} c_n & c_{n-1} & c_{n-2} \\ c_{n+1} & c_n & c_{n-1} \\ c_{n+2} & c_{n+1} & c_n \end{vmatrix}.$$

In particular:

$$D(0,3) = \begin{vmatrix} c_0 & 0 & 0 \\ c_1 & c_0 & 0 \\ c_2 & c_1 & c_0 \end{vmatrix}, \quad D(1,3) = \begin{vmatrix} c_1 & c_0 & 0 \\ c_2 & c_1 & c_0 \\ c_3 & c_2 & c_1 \end{vmatrix}.$$

40.5 FEKETE CRITERION A sequence $\{c_n : n \geq 0\}$ $(c_0 \neq 0)$ of nonnegative real numbers is totally positive if

$$\forall\, n,\ \forall\, r,\ D(n,r) > 0.$$

40.6 THEOREM[†] Suppose that

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

is a real entire function with $f(0) > 0$ -- then the sequence $c_0, c_1, c_2, \ldots$ is totally positive iff f has a representation of the form

$$f(z) = f(0)e^{az} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right),$$

where a is real and ≥ 0, the λ_n are real and < 0 with $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$.

40.7 EXAMPLE Take $f(z) = e^z$ -- then the sequence $\frac{1}{0!}, \frac{1}{1!}, \frac{1}{2!}, \ldots$ is totally positive.

40.8 EXAMPLE Take $f(z) = (1+z)^n$ -- then the sequence $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots$ is totally positive.

40.9 RAPPEL (cf. 10.11) Let $f \not\equiv 0$ be a real entire function -- then $f \in \text{ent}(]-\infty,0])$ iff f has a representation of the form

$$f(z) = Cz^m e^{az} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right),$$

---

† M. Aissen et al., *Proc. Nat. Acad. Sci. U.S.A.* 37 (1951), pp. 303-307.

where $C \neq 0$ is real, m is a nonnegative integer, a is real and ≥ 0, the λ_n are real and < 0 with $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$.

40.10 NOTATION Denote by

$$\text{ent}_+(]-\infty,0])$$

the subset of ent(]- ∞,0]) (cf. 10.26) consisting of those f such that

$$f(z) = f(0)e^{az} \prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n})$$

with $f(0) > 0$.

40.11 SCHOLIUM If

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

is a real entire function with $f(0) > 0$, then the sequence $c_0, c_1, c_2, \ldots$ is totally positive iff

$$f \in \text{ent}_+(]-\infty,0]).$$

40.12 NOTATION Write

$$\mathfrak{C}: [c_{i-j}]_{i=1,\ j=1}^{\infty}.$$

So, e.g.,

$$c_{1-1} = c_0,\ c_{1-2} = 0,\ c_{2-1} = c_1,\ c_{2-2} = c_0,\ c_{2-3} = 0 \text{ etc.}$$

40.13 NOTATION Given a positive integer n, let

$$\begin{cases} 1 \leq i_1 < i_2 < \cdots < i_n \\ 1 \leq j_1 < j_2 < \cdots < j_n \end{cases}$$

be positive integers and let

$$\mathfrak{C}(i_1, i_2, \ldots, i_n \mid j_1, j_2, \ldots, j_n)$$

denote the $n \times n$ minor obtained from $\mathfrak{C}$ by deleting all the rows and columns except those labeled $i_1, i_2, \ldots, i_n$ and $j_1, j_2, \ldots, j_n$ respectively.

40.14 THEOREM† Let

$$f \in \mathrm{ent}_+(]-\infty, 0]).$$

Assume: a is equal to 0, the c_n are greater than 0, and the product

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right)$$

is infinite -- then the minor

$$\mathfrak{C}(i_1, i_2, \ldots, i_n \mid j_1, j_2, \ldots, j_n)$$

is positive if $j_1 \leq i_1$, $j_2 \leq i_2, \ldots, j_n \leq i_n$.

40.15 APPLICATION For $n = 0,1,2,\ldots$ and $r = 1,2,3,\ldots$,

$$D(n,r) = \mathfrak{C}(n+1,\ n+2,\ \ldots,\ n+r \mid 1,2,\ldots r),$$

so $D(n,r)$ is positive.

40.16 EXAMPLE

$$D(n,2) = \begin{vmatrix} c_n & c_{n-1} \\ c_{n+1} & c_n \end{vmatrix}$$

† S. Karlin, *Total Positivity*, Stanford University Press, 1968, pp. 427-432.

$$= c_n^2 - c_{n-1}c_{n+1}$$

$$= \mathfrak{C}(n+1,\ n+2 \mid 1,2) > 0.$$

[Note:

$$D(n,1) = c_n = \mathfrak{C}(n+1|1) > 0.]$$

40.17 LEMMA Suppose that

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

is a real entire function with $f(0) > 0$ and $\forall\ n$, $c_n \geq 0$. Assume: $f \in L - P$ -- then

$$f \in \text{ent}_+(]-\infty,0]).$$

40.18 EXAMPLE Take

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{e^{n^2}} z^n.$$

Then

$$f \in \text{ent}_+(]-\infty,0]).$$

[The Jensen polynomials

$$J_n(f;z) = \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{e^{k^2}} z^k$$

associated with f have real zeros only, thus $f \in L - P$ (cf. 12.14).]

§41. CHANGE OF VARIABLE

Continuing the discussion initiated in §38, from the definitions

$$\Xi\left(\frac{z}{2}\right) = \int_{-\infty}^{\infty} \Phi(t) e^{\sqrt{-1}\,\frac{z}{2}\,t}\, dt$$

$$= 2 \int_0^{\infty} \Phi(t) \cos z \frac{t}{2}\, dt$$

$$= 4 \int_0^{\infty} \Phi(2t) \cos zt\, dt$$

$$= 8 \int_0^{\infty} \Phi(t) \cos zt\, dt,$$

where, in a flagrant abuse of notation, the "new" $\Phi(t)$ is

$$\Phi(t) = \sum_{n=1}^{\infty} (2\pi^2 n^4 e^{9t} - 3\pi n^2 e^{5t}) \exp(-\pi n^2 e^{4t}).$$

Expand now the cosine and integrate term by term to get the representation

$$Ш(z) \equiv \frac{1}{8} \Xi\left(\frac{z}{2}\right)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} b_k z^{2k}.$$

Here

$$b_k = \int_0^{\infty} t^{2k} \Phi(t)\, dt.$$

41.1 NOTATION Put

$$F_\zeta(z) = \sum_{k=0}^{\infty} \frac{b_k}{(2k)!} z^k$$

and set

$$c_k = \frac{b_k}{(2k)!}.$$

Accordingly,

$$Ш(z) = F_\zeta(- z^2).$$

Therefore if z_0 is a zero of $Ш(z)$, then $- z_0^2$ is a zero of $F_\zeta(z)$.

41.2 LEMMA F_ζ is a real entire function of order $\frac{1}{2}$ and of maximal type.

41.3 LEMMA $\forall\ k \geq 0$, C_k is positive (cf. 38.15).

N.B. In particular:

$$F_\zeta(0) = C_0 > 0.$$

41.4 SCHOLIUM RH is equivalent to the statement that all the zeros of F_ζ are real and negative.

41.5 SCHOLIUM RH is equivalent to the statement that

$$F_\zeta \in \text{ent}_+(]- \infty,0]).$$

41.6 THEOREM If RH obtains, then

$$\forall\ n,\ \forall\ r,\ D(n,r) > 0.$$

PROOF In fact,

$$\text{RH} \Rightarrow F_\zeta \in \text{ent}_+(]- \infty,0]).$$

But if

$$F_\zeta \in \text{ent}_+(]- \infty,0]),$$

then

$$F_\zeta(z) = F_\zeta(0) \prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n})$$

and, as there is no exponential term, in view of 40.15,

$$\forall\, n,\ \forall\, r,\ D(n,r) > 0.$$

41.7 THEOREM If

$$\forall\, n,\ \forall\, r,\ D(n,r) > 0,$$

then RH obtains.

PROOF The assumption implies that the sequence $C_0, C_1, C_2, \ldots$ is totally positive (cf. 40.5), hence

$$F_\zeta \in \mathrm{ent}_+(]-\infty,0]) \qquad (\text{cf. } 40.11),$$

from which RH.

41.8 SCHOLIUM RH is equivalent to the statement that

$$\forall\, n, \forall\, r,\ D(n,r) > 0.$$

<u>N.B.</u> Trivially,

$$D(n,1) = C_n > 0.$$

§42. D(n,2)

Here it will be shown that D(n,2) is positive (cf. 41.8).

N.B. We have

$$D(0,2) = \begin{vmatrix} C_0 & 0 \\ C_1 & C_0 \end{vmatrix} = C_0^2 > 0,$$

so it can be assumed that $n \geq 1$.

42.1 LEMMA[†] $\forall\ t > 0$,

$$\frac{d}{dt}\left(\frac{\Phi'(t)}{t\Phi(t)}\right) < 0.$$

42.2 THEOREM $\forall\ n \geq 1$,

$$C_n^2 - (1 + \frac{1}{n})C_{n-1}C_{n+1} \geq 0.$$

PROOF Write

$$C_n^2 - (1 + \frac{1}{n})C_{n-1}C_{n+1}$$

$$= \frac{b_n^2}{(2n!)^2} - \frac{n+1}{n}\,\frac{1}{(2n-2)!}\,\frac{1}{(2n+2)!}\,b_{n-1}b_{n+1}$$

$$= \frac{1}{(2n!)^2}\,(b_n^2 - \frac{n+1}{n}\,\frac{(2n)!}{(2n-2)!}\,\frac{(2n)!}{(2n+2)!}\,b_{n-1}b_{n+1})$$

$$= \frac{1}{(2n!)^2}\,(b_n^2 - \frac{n+1}{n}\,\frac{2n(2n-1)}{1}\,\frac{1}{2(n+1)(2n+1)}\,b_{n-1}b_{n+1})$$

[†] G. Csordas and R. Varga, *Constr. Approx.* 4 (1988), pp. 175-198.

$$= \frac{1}{(2n!)^2} (b_n^2 - \frac{2n-1}{2n+1} b_{n-1}b_n).$$

Put

$$\Delta_n = b_n^2 - \frac{2n-1}{2n+1} b_{n-1}b_n$$

and then make the claim that $\Delta_n \geq 0$. First

$$b_n = \int_0^\infty t^{2n}\Phi(t)dt$$

=>

$$b_n = - \frac{1}{2n+1} \int_0^\infty t^{2n+1}\Phi'(t)dt.$$

Therefore

$$\int_0^\infty \int_0^\infty u^{2n}v^{2n}\Phi(u)\Phi(v)(v^2-u^2)$$

$$(\int_u^v - \frac{d}{dt} (\frac{\Phi'(t)}{t\Phi(t)})dt)dudv$$

$$= \int_0^\infty \int_0^\infty u^{2n-1}v^{2n-1}(v^2-u^2)$$

$$(v\Phi(v)\Phi'(u) - u\Phi(u)\Phi'(v))dudv$$

$$= - (2n-1)b_{n-1} \int_0^\infty v^{2n+2}\Phi(v)dv$$

$$+ (2n+1)b_n \int_0^\infty v^{2n}\Phi(v)dv$$

$$+ (2n+1)b_n \int_0^\infty u^{2n}\Phi(u)du$$

$$- (2n-1)b_{n-1} \int_0^\infty u^{2n+2}\Phi(u)du$$

$$= -(2n-1)b_{n-1}b_{n+1} + (2n+1)b_n^2$$

$$+ (2n+1)b_n^2 - (2n-1)b_{n-1}b_{n+1}$$

$$= 2(2n+1)b_n^2 - 2(2n-1)b_{n-1}b_{n+1}$$

$$= 2(2n+1)\left(b_n^2 - \frac{2(2n-1)}{2(2n+1)} b_{n-1}b_{n+1}\right)$$

$$= 2(2n+1)\Delta_n.$$

But $\forall\, t > 0$,

$$-\frac{d}{dt}\left(\frac{\Phi'(t)}{t\Phi(t)}\right) > 0 \quad \text{(cf. 41.9)}.$$

Consequently,

$$(v^2-u^2)\left(\int_u^v -\frac{d}{dt}\left(\frac{\Phi'(t)}{t\Phi(t)}\right)dt\right)du\,dv$$

is nonnegative for all $0 \leq u, v < \infty$, hence Δ_n is ≥ 0, as claimed.

42.13 APPLICATION $\forall\, n \geq 1$,

$$C_n^2 \geq \left(1 + \frac{1}{n}\right)C_{n-1}C_{n+1} > C_{n-1}C_{n+1}$$

=>

$$C_n^2 > C_{n-1}C_{n+1}$$

=>

$$D(n,2) = \begin{vmatrix} C_n & C_{n-1} \\ C_{n+1} & C_n \end{vmatrix}$$

$$= C_n^2 - C_{n-1}C_{n+1} > 0.$$

42.14 REMARK Put

$$\Gamma_n = F_\zeta^{(n)}(0) \quad (\Rightarrow C_n = \frac{\Gamma_n}{n!}).$$

Then

$$\Gamma_n^2 - \Gamma_{n-1}\Gamma_{n+1} \geq 0.$$

I.e.:

$$(F_\zeta^{(n)}(0))^2 - F_\zeta^{(n-1)}(0)F_\zeta^{(n+1)}(0) \geq 0.$$

Take now n = 1 and, in the notation of 13.6, ask: Is it true that for ALL real t,

$$L_1(F_\zeta)(t) = (F_\zeta'(t))^2 - F_\zeta(t)F_\zeta''(t) \geq 0?$$

The answer is unknown (although the inequality does hold in a finite interval containing the origin...).

[Note: If $\forall$ t,

$$L_1(F_\zeta)(t) > 0,$$

then it would follow that all the real zeros of F_ζ are simple.]

There is another proof of the positivity of D(n,2) that is based on a different set of ideas, these being important for their associated methodology.

42.5 LEMMA $\forall$ t > 0,

$$-\begin{vmatrix} \Phi(t) & \Phi'(t) \\ \Phi'(t) & \Phi''(t) \end{vmatrix} > 0.$$

PROOF Owing to 42.1, $\forall\, t > 0$,

$$\frac{d}{dt}\left(\frac{\Phi'(t)}{t\Phi(t)}\right) < 0$$

which, when written out, is equivalent to the inequality

$$t((\Phi'(t))^2 - \Phi(t)\Phi''(t)) + \Phi(t)\Phi'(t)$$

$$> 0$$

or still,

$$t((\Phi'(t))^2 - \Phi(t)\Phi''(t)) > - \Phi(t)\Phi'(t).$$

But $\Phi(t)$ is positive (cf. 38.15) and $\Phi'(t)$ is negative (cf. 38.19). Therefore

$$- \Phi(t)\Phi'(t) > 0$$

$$\Rightarrow$$

$$(\Phi'(t))^2 - \Phi(t)\Phi''(t)$$

$$= - \begin{vmatrix} \Phi(t) & \Phi'(t) \\ \Phi'(t) & \Phi''(t) \end{vmatrix} > 0.$$

[Note:

$$\frac{d^2}{dt^2} \log \Phi(t)$$

$$= \frac{d}{dt}\left(\frac{\Phi'(t)}{\Phi(t)}\right)$$

$$= \frac{\Phi(t)\Phi''(t) - (\Phi'(t))^2}{\Phi(t)^2}$$

$$< 0.]$$

N.B. It is to be emphasized that it is possible to give a proof of 42.5 which is independent of 42.1 (see the Appendix to this §).]

[Note: It is shown there that the inequality persists to $t = 0$ (or directly:

$$((\Phi'(t))^2 - \Phi(t)\Phi''(t))\Big|_{t=0}$$

$$= 0^2 - \Phi(0)\Phi''(0) > 0,$$

$\Phi(0)$ being positive and $\Phi''(0)$ being negative.]

42.6 SUBLEMMA Let $f_1(t)$, $f_2(t)$, $g_1(t)$, $g_2(t)$ be continuous and absolutely integrable on $[0,\infty[$. Assume: $f_i(t)g_j(t)$ $(1 \leq i, j \leq 2)$ and $f_1(t)f_2(t)g_1(t)g_2(t)$ are also absolutely integrable on $[0,\infty[$ -- then

$$\det \begin{bmatrix} \int_0^\infty f_1(t)g_1(t)dt & \int_0^\infty f_1(t)g_2(t)dt \\ \int_0^\infty f_2(t)g_1(t)dt & \int_0^\infty f_2(t)g_2(t)dt \end{bmatrix}$$

$$= \iint_{0<u<v<\infty} \det \begin{bmatrix} f_1(u) & f_1(v) \\ f_2(u) & f_2(v) \end{bmatrix} \cdot \det \begin{bmatrix} g_1(u) & g_1(v) \\ g_2(u) & g_2(v) \end{bmatrix} dudv.$$

42.7 NOTATION Given nonempty subsets X and Y of R and a real valued function f on $X \times Y$, put

$$f \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \det \begin{bmatrix} f(x_1,y_1) & f(x_1,y_2) \\ f(x_2,y_1) & f(x_2,y_2) \end{bmatrix}.$$

Put

$$\phi(v,t) = \frac{v^{t-1}}{\Gamma(t)} \ (v > 0,\ t > 0).$$

42.8 LEMMA $\forall\ t > 0,\ \forall\ s > 0,$

$$\phi(v,t+s) = \int_0^v \phi(u,t)\phi(v-u,s)du.$$

PROOF Start with the RHS:

$$\int_0^v \frac{u^{t-1}}{\Gamma(t)} \frac{(v-u)^{s-1}}{\Gamma(s)}\ du$$

$$= \frac{1}{\Gamma(t)} \frac{1}{\Gamma(s)} \int_0^v u^{t-1}(v-u)^{s-1}du$$

$$= \frac{1}{\Gamma(t)} \frac{1}{\Gamma(s)} v^{s-1} \int_0^v u^{t-1}(1 - \frac{u}{v})^{s-1}du$$

$$= \frac{1}{\Gamma(t)} \frac{1}{\Gamma(s)} v^{s-1} \int_0^1 (vw)^{t-1}(1-w)^{s-1}vdw$$

$$= \frac{1}{\Gamma(t)} \frac{1}{\Gamma(s)} v^{t+s-1}B(t,s)$$

$$= \frac{1}{\Gamma(t)} \frac{1}{\Gamma(s)} v^{t+s-1} \frac{\Gamma(t)\Gamma(s)}{\Gamma(t+s)}$$

$$= \frac{v^{t+s-1}}{\Gamma(t+s)} = \phi(v,t+s).$$

Put

$$\lambda(t) = \int_0^\infty \Phi(v)\phi(v,t)dv \quad (t > 0).$$

Then

$$\lambda(2n+1) = \int_0^\infty \Phi(v)\phi(v,2n+1)dv$$

$$= \int_0^\infty \Phi(v) \frac{v^{2n+1-1}}{\Gamma(2n+1)} dv$$

$$= \int_0^\infty \Phi(v) \frac{v^{2n}}{(2n)!} dv$$

$$= \frac{1}{(2n)!} \int_0^\infty \Phi(v) v^{2n} dv = \frac{b_n}{(2n)!} = c_n.$$

42.9 LEMMA $\forall\ t > 0,\ \forall\ s > 0,$

$$\Lambda(s,t) \equiv \lambda(s+t) = \int_0^\infty \Phi(v)\phi(v,s+t)dv$$

$$= \int_0^\infty \phi(u,s) (\int_0^\infty \Phi(u+v)\phi(v,t)dv)du.$$

PROOF In the double integral, let

$$\begin{cases} x = u \\ y = u + v. \end{cases}$$

Then the Jacobian equals 1, so there is no $J(x,y)$ factor and since u and v are nonnegative, if x is varied first, it goes from 0 to y. This said, upon inverting, thus

$$\begin{cases} u = x \\ v = y - x, \end{cases}$$

we arrive at

$$\int_{y=0}^\infty \int_{x=0}^y \phi(x,s)\phi(y-x,t)\Phi(y)dxdy$$

or still,

$$\int_{y=0}^\infty \Phi(y) (\int_{x=0}^y \phi(x,s)\phi(y-x,t)dx)dy$$

or still,

$$\int_{y=0}^\infty \Phi(y)\phi(y,s+t)dy \qquad \text{(cf. 42.8)}$$

or still,

$$\int_0^\infty \Phi(v)\phi(v,s+t)dv.$$

42.10 LEMMA If $0 < v_1 < v_2$ and if $0 < t_1 < t_2$, then

$$\phi\begin{bmatrix} v_1 & v_2 \\ t_1 & t_2 \end{bmatrix} > 0.$$

PROOF In fact,

$$\det\begin{bmatrix} \phi(v_1,t_1) & \phi(v_1,t_2) \\ \phi(v_2,t_1) & \phi(v_2,t_2) \end{bmatrix}$$

$$= \phi(v_1,t_1)\phi(v_2,t_2) - \phi(v_1,t_2)\phi(v_2,t_2)$$

$$= \frac{v_1^{t_1}}{v_1\Gamma(t_1)}\ \frac{v_2^{t_2}}{v_2\Gamma(t_2)} - \frac{v_1^{t_2}}{v_1\Gamma(t_2)}\ \frac{v_2^{t_1}}{v_2\Gamma(t_1)}$$

$$= \frac{1}{\Gamma(t_1)\Gamma(t_2)}\left[\frac{v_1^{t_1}v_2^{t_2}}{v_1v_2} - \frac{v_1^{t_2}v_2^{t_1}}{v_1v_2}\right]$$

$$= \frac{1}{\Gamma(t_1)\Gamma(t_2)}\left[v_1^{t_1-1}v_2^{t_1-1+t_2-t_1} - v_1^{t_1-1+t_2-t_1}v_2^{t_1-1}\right]$$

$$= \frac{v_1^{t_1-1}\ v_2^{t_1-2}}{\Gamma(t_1)\Gamma(t_2)}\left[v_2^{t_2-t_1} - v_1^{t_2-t_1}\right]$$

$$> 0.$$

42.11 SUBLEMMA Let I be an open interval (bounded or unbounded). Suppose that f is twice continuously differentiable on I and

$$\frac{d^2}{dt^2} f(t) < 0 \quad (t \in I).$$

Then for any four points a,b,c,d in I with a < c < d < b,

$$\frac{f(c) - f(a)}{c - a} > \frac{f(b) - f(d)}{b - d}.$$

PROOF By the mean value theorem,

$$\left[\begin{array}{l} \frac{f(c) - f(a)}{c - a} = f'(x) \quad (\exists\ x \in\ ]a,c[) \\ \\ \frac{f(b) - f(d)}{b - d} = f'(y) \quad (\exists\ y \in\ ]d,b[). \end{array}\right.$$

But the assumption on f implies that f' is strictly decreasing on I, hence

$$x < y \Rightarrow f'(x) > f'(y).$$

[Note: If c - a = b - d, then

$$f(c) + f(d) > f(a) + f(b).]$$

<u>N.B.</u> In the applications (as below), it can happen that during the course of a "labeling procedure", one has "c = d", so

$$\left[\begin{array}{l} \frac{f(c) - f(a)}{c - a} = f'(x) \quad (\exists\ x \in\ ]a,c[) \\ \\ \frac{f(b) - f(c)}{b - c} = f'(y) \quad (\exists\ y \in\ ]c,b[), \end{array}\right.$$

thus if c - a = b - c, then

$$f(c) + f(c) > f(a) + f(b).]$$

Put

$$K(u,v) = \Phi(u+v) \quad (u > 0,\ v > 0).$$

42.12 LEMMA If $0 < u_1 < u_2$ and if $0 < v_1 < v_2$, then

$$K\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} < 0.$$

PROOF In 42.11, take

$$f(t) = \log \Phi(t) \qquad (\text{cf. } 42.5).$$

Define a,b,c,d as follows:

$$a = u_1 + v_1,\ b = u_2 + v_2,\ c = u_2 + v_1,\ d = u_1 + v_2.$$

Therefore

$$a < c < b,\ a < d < b, \text{ and } c - a = b - d.$$

Now, while the setup in 42.11 called for $c < d$, if $d < c$, then their roles can be interchanged and the possibility that $c = d$ is not excluded (cf. supra). Consequently,

$$\log \Phi(c) + \log \Phi(d) > \log \Phi(a) + \log \Phi(b)$$

$$\Rightarrow$$

$$\Phi(c)\Phi(d) > \Phi(a)\Phi(b)$$

$$\Rightarrow$$

$$\Phi(u_2+v_1)\Phi(u_1+v_2) > \Phi(u_1+v_1)\Phi(u_2+v_2)$$

or still,

$$\Phi(u_1+v_1)\Phi(u_2+v_2) - \Phi(u_1+v_2)\Phi(u_2+v_1) < 0.$$

And

$$K\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} = \det\begin{bmatrix} K(u_1,v_1) & K(u_1,v_2) \\ K(u_2,v_1) & K(u_2,v_2) \end{bmatrix}$$

$$= \det\begin{bmatrix} \Phi(u_1+v_1) & \Phi(u_1+v_2) \\ \Phi(u_2+v_1) & \Phi(u_2+v_2) \end{bmatrix}$$

$$< 0.$$

Put

$$L(u,t) = \int_0^\infty K(u,v)\phi(v,t)dv.$$

42.13 LEMMA If $0 < u_1 < u_2$ and if $0 < t_1 < t_2$, then

$$L\begin{bmatrix} u_1 & u_2 \\ t_1 & t_2 \end{bmatrix} < 0.$$

PROOF Using 42.6, write

$$L\begin{bmatrix} u_1 & u_2 \\ t_1 & t_2 \end{bmatrix}$$

$$= \iint_{0<u<v<\infty} K\begin{bmatrix} u_1 & u_2 \\ u & v \end{bmatrix} \phi\begin{bmatrix} u & v \\ t_1 & t_2 \end{bmatrix} dudv.$$

In this connection, it is necessary to observe that

$$\det \begin{bmatrix} \phi(u,t_1) & \phi(v,t_1) \\ \phi(u,t_2) & \phi(v,t_2) \end{bmatrix}$$

$$= \det \begin{bmatrix} \phi(u,t_1) & \phi(u,t_2) \\ \phi(v,t_1) & \phi(v,t_2) \end{bmatrix}$$

$$= \phi\begin{bmatrix} u & v \\ t_1 & t_2 \end{bmatrix}.$$

But

$$K\begin{bmatrix} u_1 & u_2 \\ u & v \end{bmatrix} < 0 \quad \text{(cf. 42.12)}$$

and

$$\phi\begin{bmatrix} u & v \\ t_1 & t_2 \end{bmatrix} > 0 \quad \text{(cf. 42.10)}.$$

Therefore

$$L\begin{bmatrix} u_1 & u_2 \\ & \\ t_1 & t_2 \end{bmatrix} < 0.$$

Using the notation of 42.9, we have

$$\begin{aligned}\Lambda(s,t) \equiv \lambda(s+t) &= \int_0^\infty \phi(u,s)\,(\int_0^\infty \Phi(u+v)\,\phi(v,t)\,dv)\,du \\ &= \int_0^\infty \phi(u,s)\,(\int_0^\infty K(u,v)\,\phi(v,t)\,dv)\,du \\ &= \int_0^\infty \phi(u,s)\,L(u,t)\,du.\end{aligned}$$

42.14 LEMMA If $0 < s_1 < s_2$ and if $0 < t_1 < t_2$, then

$$\Lambda\begin{bmatrix} s_1 & s_2 \\ & \\ t_1 & t_2 \end{bmatrix} < 0.$$

PROOF Appealing once again to 42.6, write

$$\Lambda\begin{bmatrix} s_1 & s_2 \\ & \\ t_1 & t_2 \end{bmatrix} = \iint_{0<u<v<\infty} \phi\begin{bmatrix} u & v \\ & \\ s_1 & s_2 \end{bmatrix} L\begin{bmatrix} u & v \\ & \\ t_1 & t_2 \end{bmatrix} dudv$$

and then apply 42.10 and 42.13.

42.15 SCHOLIUM If $0 < s_1 < s_2$ and if $0 < t_1 < t_2$, then

$$\begin{vmatrix} \lambda(s_1+t_1) & \lambda(s_1+t_2) \\ \lambda(s_2+t_1) & \lambda(s_2+t_2) \end{vmatrix} < 0.$$

Consider now the determinant

$$\begin{vmatrix} C_{n-1} & C_n \\ C_n & C_{n+1} \end{vmatrix} \quad (n \geq 1),$$

hence

$$C_{n-1} = \lambda(2n-1),\ C_n = \lambda(2n+1),\ C_{n+1} = \lambda(2n+3).$$

In 42.15, let

$$s_1 = t_1 = n - \frac{1}{2},\ s_2 = t_2 = n + \frac{3}{2}.$$

Then

$$s_1+t_1 = 2n-1,\ s_1+t_2 = 2n+1,\ s_2+t_1 = 2n+1,\ s_2+t_2 = 2n+3.$$

Therefore

$$\begin{vmatrix} \lambda(2n-1) & \lambda(2n+1) \\ \lambda(2n+1) & \lambda(2n+3) \end{vmatrix} < 0.$$

I.e.:

$$\begin{vmatrix} C_{n-1} & C_n \\ C_n & C_{n+1} \end{vmatrix} < 0$$

or still,

$$C_{n-1}C_{n+1} - C_n^2 < 0$$

or still,

$$D(n,2) = C_n^2 - C_{n-1}C_{n+1} > 0.$$

42.16 REMARK The condition

$$C_n^2 - C_{n-1}C_{n+1} > 0$$

is weaker than the condition

$$C_n^2 - (1 + \frac{1}{n})C_{n-1}C_{n+1} \geq 0$$

and this is because less was used in its derivation (viz. 42.5 as opposed to 42.1).

A similar but more complicated analysis serves to establish that $D(n,3)$ is positive (for this and additional information, see Nuttall†).

APPENDIX

THEOREM $\forall\, t \geq 0$,

$$(\Phi'(t))^2 - \Phi(t)\Phi''(t) > 0.$$

We shall proceed via a list of lemmas.

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† arXiv:1111.1128 [math. NT]; also *Constr. Approx.* 38 (2013), pp. 193-212.

Write

$$\Phi(t) = \sum_{n=1}^{\infty} a_n(t),$$

where

$$a_n(t) = (2\pi^2 n^4 e^{9t} - 3\pi n^2 e^{5t})\exp(-\pi n^2 e^{4t}),$$

and put

$$a(t) = a_1(t), \ \Psi(t) = \sum_{n=2}^{\infty} a_n(t),$$

thus

$$\Phi(t) = a(t) + \Psi(t)$$

and so

$$(\Phi'(t))^2 - \Phi(t)\Phi''(t)$$

$$= (a'(t) + \Psi'(t))^2 - (a(t) + \Psi(t))(a''(t) + \Psi''(t))$$

$$= V(t) + U(t) + (\Psi'(t))^2.$$

Here, by definition,

$$V(t) = (a'(t))^2 - a(t)a''(t)$$

and

$$U(t) = 2a'(t)\Psi'(t) - a''(t)\Psi(t) - \Phi(t)\Psi''(t).$$

NOTATION Let

$$y = \pi e^{4t} (t \geq 0) \Rightarrow y \geq \pi.$$

LEMMA 1 $\forall\, t \geq 0$,

$$0 < \Psi(t) \leq 64 e^{t} y^2 e^{-4y}.$$

PROOF

$$0 < \Psi(t) = \sum_{n=2}^{\infty} (2\pi^2 n^4 e^{9t} - 3\pi n^2 e^{5t}) \exp(-\pi n^2 e^{4t})$$

$$\leq 2e^{t} \sum_{n=2}^{\infty} n^4 \pi^2 e^{8t} \exp(-\pi n^2 e^{4t})$$

$$= 2e^{t}(16y^2 e^{-4y} + \sum_{n=1}^{\infty} y^2 n^4 e^{-n^2 y}).$$

And

$$\sum_{n=3}^{\infty} y^2 n^4 e^{-n^2 y} \leq \int_2^{\infty} y^2 x^4 e^{-yx^2} dx$$

$$< \int_2^{\infty} y^2 x^5 e^{-tx^2} dx$$

$$= \frac{1}{y} e^{-4y}(1 + 4y + 8y^2)$$

$$< 16y^2 e^{-4y}.$$

Therefore

$$\Psi(t) \leq 2e^{t}(16y^2 e^{-4y} + 16y^2 e^{-4y})$$

$$= 64e^{t} y^2 e^{-4y}.$$

LEMMA 2 $\forall\ t \geq 0$,

$$|\Psi'(t)| \leq 565e^{t} y^3 e^{-4y}.$$

PROOF

$$|\Psi'(t)| = |\sum_{n=2}^{\infty} \pi n^2 (8\pi^2 n^4 e^{8t} - 30\pi n^2 e^{4t} + 15) \exp(5t - \pi n^2 e^{4t})|$$

or still, if $x = e^t$,

$$|\Psi'(t)| = 8\pi^3 x^5 \left|\sum_{n=2}^{\infty} n^6 \left(x^8 - \frac{15}{4\pi n^2} x^4 + \frac{15}{8\pi^2 n^4}\right) \exp(-\pi n^2 x^4)\right|.$$

To examine $\left|\sum_{n=2} \ldots\right|$, first pull out x^8:

$$x^8 \left|\sum_{n=2}^{\infty} n^6 \left(1 - \frac{15}{4\pi n^2}\frac{1}{x^4} + \frac{15}{8\pi^2 n^4}\frac{1}{x^8}\right) \exp(-\pi n^2 x^4)\right|$$

and consider

$$-\frac{15}{4\pi n^2}\frac{1}{x^4} + \frac{15}{8\pi^2 n^4}\frac{1}{x^8},$$

which we claim is strictly trapped between -1 and 0.

- $$\frac{1}{2\pi n^2} < x^4 \Rightarrow \frac{1}{2\pi n^2}\frac{1}{x^4} < 1$$

  $\Rightarrow$

  $$-1 + \frac{1}{2\pi n^2}\frac{1}{x^4} < 0$$

  $\Rightarrow$

  $$-15 + \frac{15}{2\pi n^2}\frac{1}{x^4} < 0$$

  $\Rightarrow$

  $$-\frac{15}{4\pi n^2}\frac{1}{x^4} + \frac{15}{8\pi^2 n^4}\frac{1}{x^8} < 0.$$

- $$\frac{4\pi n^2}{15} > \frac{1}{x^4}$$

=>

$$\frac{1}{2\pi n^2}\frac{1}{x^8} + \frac{4\pi n^2}{15} > \frac{1}{x^4}$$

=>

$$-\frac{1}{x^4} + \frac{1}{2\pi n^2}\frac{1}{x^8} > -\frac{4\pi n^2}{15}$$

=>

$$-\frac{1}{4\pi n^2}\frac{1}{x^4} + \frac{1}{8\pi^2 n^4}\frac{1}{x^8} > -\frac{1}{15}$$

=>

$$-\frac{15}{4\pi n^2}\frac{1}{x^4} + \frac{15}{8\pi^2 n^4}\frac{1}{x^8} > -1.$$

Accordingly, if

$$C_{x,n} = -\frac{15}{4\pi n^2}\frac{1}{x^4} + \frac{15}{8\pi^2 n^4}\frac{1}{x^8},$$

then

$$-1 < C_{x,n} < 0$$

=>

$$0 < 1 + C_{x,n} < 1$$

=>

$$|1 + C_{x,n}| = 1 + C_{x,n} < 1$$

=>

$$\left|\sum_{n=2}^{\infty} n^6\left(1 - \frac{15}{4\pi n^2}\frac{1}{x^4} + \frac{15}{8\pi^2 n^4}\frac{1}{x^8}\right)\exp(-\pi n^2 x^4)\right|$$

$$= \left|\sum_{n=2}^{\infty} n^6 (1 + C_{x,n})\exp(-\pi n^2 x^4)\right|$$

$$\leq \sum_{n=2}^{\infty} n^6 |1 + C_{x,n}| \exp(-\pi n^2 x^4)$$

$$< \sum_{n=2}^{\infty} n^6 \exp(-\pi n^2 x^4)$$

=>

$$|\Psi'(t)| < \frac{8y^{13/4}}{\pi^{1/4}} \sum_{n=2}^{\infty} n^6 e^{-n^2 y} \quad (y = \pi x^4 \geq \pi).$$

And

$$\sum_{n=2}^{\infty} n^6 e^{-n^2 y} < 64e^{-4y} + \int_2^{\infty} s^6 e^{-s^2 y} ds$$

$$< 64e^{-4y} + \frac{e^{-4y}}{2y^{7/2}} \Big((4y)^{5/2} + \frac{5}{2}(4y)^{3/2}$$

$$+ \frac{15}{4}(4y)^{1/2} + \frac{15e^{4y}}{8} \int_{4y}^{\infty} \frac{e^{-u}}{\sqrt{u}} du\Big).$$

But $\frac{1}{\sqrt{u}} < 1$ for $u \geq 4y \geq 4\pi$, hence

$$e^{4y} \int_{4y}^{\infty} \frac{e^{-u}}{\sqrt{u}} du < 1,$$

so

$$\sum_{n=2}^{\infty} n^6 e^{-n^2 y}$$

is bounded above by

$$64e^{-4y} \Big(1 + \frac{1}{4y} + \frac{5}{32y^2} + \frac{15}{256y^3} + \frac{15}{1024y^{7/2}}\Big) \quad (y \geq \pi).$$

The expression in parentheses is strictly decreasing, thus is majorized by its value at $y = \pi$ and it follows that

$$\sum_{n=2}^{\infty} n^6 e^{-n^2 y} < 64e^{-4y} \left(1 + \frac{13}{40\pi}\right).$$

Therefore

$$|\Psi'(t)| < \frac{8y^{13/4}}{\pi^{1/4}} \left(64e^{-4y}\left(1 + \frac{13}{40\pi}\right)\right)$$

$$= 512\left(1 + \frac{13}{40\pi}\right)\pi^3 \exp(13t - 4\pi e^{4t})$$

$$< 565\pi^3 \exp(13t - 4\pi e^{4t})$$

$$= 565e^t y^3 e^{-4y}.$$

LEMMA 3 $\forall\, t \geq 0$,

$$|\Psi''(t)| \leq (1.031)2^{13} e^t y^4 e^{-4y}.$$

PROOF Let

$$p(x) = 32x^3 - 224x^2 + 330x - 75.$$

Then $p(x)$ has three distinct positive roots

$$0 < x_1 < x_2 < x_3 = 5.049720\ldots\ .$$

Therefore

$$x > x_3 \Rightarrow p(x) > 0.$$

On the other hand,

$$x > x_3 \Rightarrow 0 < p(x) < 32x^3.$$

These points made, from the definitions

$$\Psi''(t) = \sum_{n=2}^{\infty} \pi n^2 p(\pi n^2 e^{4t}) \exp(5t - \pi n^2 e^{4t}).$$

But

$$\pi n^2 e^{4t} \geq 4\pi > x_3$$

=>

$$|\Psi''(t)| \leq 32 \sum_{n=2}^{\infty} \pi n^2 (\pi n^2 e^{4t})^3 \exp(5t - \pi n^2 e^{4t})$$

$$= 32\pi^4 e^{17t} \sum_{n=2}^{\infty} \frac{n^8}{\exp(\pi n^2 e^{4t})}$$

$$= 32\pi^4 e^{17t} \sum_{n=2}^{\infty} \frac{n^8}{\exp(n^2 y)}$$

$$= 32\pi^4 e^{17t} \sum_{n=2}^{\infty} \frac{1}{\exp(n^2 y - 8\log n)}$$

$$\leq 32\pi^4 e^{17t} \sum_{n=2}^{\infty} \frac{1}{K(y)^n}$$

$$= 32\pi^4 e^{17t} \frac{1}{K(y)^2 (1 - \frac{1}{K(y)})}$$

if

$$K(y) = \frac{e^{2y}}{16}$$

as then

$$n^2 y - 8\log n \geq n \log K(y).$$

But

$$\frac{1}{K(y)^2(1 - \frac{1}{K(y)})} = \frac{2^8 e^{-4y}}{1 - \frac{16}{e^{2y}}}$$

$$\leq \frac{2^8 e^{-4y}}{1 - \frac{16}{e^{2\pi}}} \quad (y \geq \pi).$$

And

$$\frac{1}{1 - \frac{16}{e^{2\pi}}} < 1.031,$$

leaving

$$< (1.031) 2^8 e^{-4y}.$$

Finally

$$\pi^4 e^{17t} = e^t \pi^4 e^{16t}$$

$$= e^t y^4.$$

LEMMA 4 $\forall\, t \geq 0$,

$$0 < \Phi(t) < \frac{203}{202}\, a(t).$$

PROOF

$$\Psi(t) < 64\pi^2 \exp(9t - 4\pi e^{4t})$$

$$< \frac{1}{202}\, a(t)$$

=>

$$\Phi(t) = a(t) + \Psi(t)$$

$$< a(t) + \frac{1}{202} a(t)$$

$$= \frac{203}{202} a(t).$$

NOTATION Put

$$E(y) = e^{2t} e^{-2y} y^3.$$

LEMMA 5 $\forall\, t \geq 0$,

$$V(t) \geq 256 e^{2t} e^{-2y} y^3 \equiv 256 E(y).$$

PROOF

$$V(t) = 16 \exp(-2\pi e^{4t} + 14t) \pi^3 (15 - 12\pi e^{4t} + 4\pi^2 e^{8t})$$

$$= 16 e^{14t} e^{-2y} \pi^3 (15 - 12y + 4y^2)$$

$$= 16 e^{2t} e^{-2y} y^3 (15 - 12y + 4y^2).$$

But

$$15 - 12y + 4y^2 = 4(y - \tfrac{3}{2})^2 + 6$$

is an increasing function of $y \geq \pi$, so

$$4(y - \tfrac{3}{2})^2 + 6 \geq 4(\pi - \tfrac{3}{2})^2 + 6$$

$$\geq 16.$$

Therefore

$$V(t) \geq 256 e^{2t} e^{-2y} y^3 \equiv 256 E(y).$$

NOTATION Write

$$\begin{cases} a(t) = e^t e^{-y} y(2y-3) \\ a'(t) = -e^t e^{-y} y(15 - 30y + 8y^2) \\ a''(t) = e^t e^{-y} y(-75 + 330y - 224y^2 + 32y^3). \end{cases}$$

LEMMA 6 $\forall\, t \geq 0$,

$$|U(t)| \leq 56{,}424 E(y) e^{-3y} y^3.$$

PROOF Start from the inequality

$$|U(t)| \leq |2a'(t)\Psi'(t)| + |a''(t)\Psi(t)| + |\Phi(t)\Phi''(t)|$$

and estimate separately each of the three summands.

-

$$|2a'(t)\Psi'(t)|$$

$$\leq |2(-e^t e^{-y} y(15 - 30y + 8y^2)| \cdot |565 e^t y^3 e^{-4y}|$$

$$\leq E(y)A(y),$$

where

$$A(y) = 1{,}130 e^{-3y}(15y + 30y^2 + 8y^3).$$

-

$$|a''(t)\Psi(t)|$$

$$\leq |e^t e^{-y} y(-75 + 330y - 224y^2 + 32y^3)| \cdot |64 e^t y^2 e^{-4y}|$$

$$\leq E(y)B(y),$$

where

$$B(y) = 64 e^{-3y}(75 + 330y + 224y^2 + 32y^3).$$

$$|\Phi(t)\Psi''(t)|$$

$$\leq \left|\frac{203}{202}e^{t}e^{-y}y(2y-3)\right| \cdot \left|(1.031)2^{13}e^{t}y^{4}e^{-4y}\right|$$

$$\leq E(y)C(y),$$

where

$$C(y) = 8{,}562e^{-3y}(2y^3 + 3y^2).$$

Combining these estimates then gives

$$|U(t)| \leq E(y)(A(y) + B(y) + C(y))$$

$$\leq E(y)2e^{-3y}(2{,}400 + 19{,}035y + 36{,}961y^2 + 14{,}206y^3)$$

$$\leq E(y)2e^{-3y}(14{,}206y^3) \cdot \frac{2{,}400 + 19{,}035y + 36{,}961y^2 + 14{,}206y^3}{14{,}206y^3}$$

$$\leq E(y)2e^{-3y}(14{,}206y^3)(1.97)$$

$$\leq 56{,}424E(y)e^{-3y}y^3.$$

Recall now the statement of the theorem: $\forall\ t \geq 0$,

$$(\Phi'(t))^2 - \Phi(t)\Phi''(t) > 0.$$

Proof: In fact,

$$V(t) + U(t) \geq V(t) - |U(t)|$$

$$\geq 256E(y) - 56{,}424E(y)e^{-3y}y^3$$

$$\geq E(y)(256 - 56{,}424e^{-3\pi}\pi^3)$$

$$> 114E(y) > 0.$$

§43. *POSITIVE QUADRATIC FORMS*

Let $p \not\equiv 0$ be a real polynomial of degree $n \geq 1$:

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n \quad (a_0 \neq 0).$$

Let $z_1, \ldots, z_n$ be its zeros and put

$$S_0 = n, \; S_k = z_1^k + z_2^k + \cdots + z_n^k \quad (k = 1,2,\ldots).$$

43.1 LEMMA There is an expansion

$$z\,\frac{p'(z)}{p(z)} = \sum_{k=0}^{\infty} S_k z^{-k} = S_0 + \frac{S_1}{z} + \cdots .$$

In addition,

$$\sum_{k=0}^{m} a_{n-k} S_{m-k} = (n-m) a_{n-m}$$

if $m < n$ but vanishes if $m \geq n$.

43.2 BORCHARDT-HERMITE CRITERION The zeros of p are real iff the determinants

$$\Delta_k = \begin{vmatrix} S_0 & S_1 & \cdots & S_{k-1} \\ S_1 & S_2 & \cdots & S_k \\ \vdots & \vdots & & \vdots \\ S_{k-1} & S_k & \cdots & S_{2k-2} \end{vmatrix} \quad (k = 1,2,\ldots,n)$$

are nonnegative. Moreover, the number of distinct zeros of p is equal to the index k of the last $\Delta_k \neq 0$ in the above sequence.

[Note: Spelled out

$$\Delta_1 = S_0 \;, \; \Delta_2 = \begin{vmatrix} S_0 & S_1 \\ S_1 & S_2 \end{vmatrix} , \ldots .]$$

<u>N.B.</u> If $\Delta_{k+1} = 0$, then $\Delta_{k+2} = \cdots = \Delta_n = 0$.

43.3 EXAMPLE Take $n = 2$ and consider $p(z) = z^2 - 1$ -- then $s_0 = 2$, $s_1 = 1 + (-1) = 0$, $s_2 = 1^2 + (-1)^2 = 2$, hence

$$\Delta_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4.$$

43.4 EXAMPLE Take $n = 2$ and consider $p(z) = z^2 + 1$ -- then $s_0 = 2$, $s_1 = \sqrt{-1} + (-\sqrt{-1}) = 0$, $s_2 = (\sqrt{-1})^2 + (-\sqrt{-1})^2 = 1 - 1 = -2$, hence

$$\Delta_2 = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = 4.$$

43.5 EXAMPLE Take $n = 2$ and consider $p(z) = (z-1)^2$ -- then $s_0 = 2$, $s_1 = 1 + 1$, $s_2 = 1^2 + 1^2 = 2$, hence

$$\Delta_2 = \begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} = 0.$$

43.6 RAPPEL Let $A = [a_{ij}]$ be a real symmetric matrix of degree n -- then the quadratic form $\underline{A}$ associated with A is the function of n real variables $x_1, \ldots, x_n$ defined by

$$\underline{A}(\underline{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j.$$

- $\underline{A}$ is positive if $\forall\ \underline{x} \neq \underline{0}$,

$$\underline{A}(\underline{x}) > 0.$$

FACT $\underline{A}$ is positive iff all successive principal minors of A are positive, i.e.,

$$a_{11} > 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \ldots, \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} > 0.$$

43.7 SCHOLIUM The zeros of p are real and simple iff the quadratic form

$$\sum_{i,j=0}^{n-1} s_{i+j} x_i x_j$$

is positive.

Put

$$s_k = \frac{1}{z_1^k} + \frac{1}{z_2^k} + \cdots + \frac{1}{z_n^k} \quad (k = 1,2,\ldots).$$

43.8 LEMMA There is an expansion

$$-\frac{p'(z)}{p(z)} = s_1 + s_2 z + s_3 z^2 + \cdots .$$

N.B. This is the point of departure for the ensuing extension of the theory.

[Note: By way of reconciliation, observe that

$$\frac{p(z)}{a_0} = (1 - \frac{z}{z_1}) \cdots (1 - \frac{z}{z_n})$$

$$= e^{-s_1 z} \prod_{k=1}^{n} (1 - \frac{z}{z_k}) e^{a/z_k},$$

so the "b" below is, in fact, $- s_1$.]

Let $f \not\equiv 0$ be a transcendental real entire function with an infinity of zeros such that $f(0) \neq 0$:

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

$$= \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} z^n \qquad (\gamma_n = f^{(n)}(0)).$$

Assume further that $f \in L - P$ -- then in view of 10.19, f has a representation of the form

$$f(z) = Ce^{az^2+bz} \prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n}) e^{z/\lambda_n},$$

where $C \neq 0$ is real, a is real and ≤ 0, b is real, the λ_n are real with $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty$.

Consider now the expansion

$$-\frac{f'(z)}{f(z)} = -2az - b + \sum_{n=1}^{\infty} (\frac{1}{\lambda_n - z} - \frac{1}{\lambda_n})$$

$$= -b - 2az + \sum_{n=1}^{\infty} (\frac{z}{\lambda_n^2} + \frac{z^2}{\lambda_n^3} + \cdots)$$

$$= s_1 + s_2 z + s_3 z^2 + \cdots ,$$

thus

$$s_1 = -b, \ s_2 = -2a + \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2}$$

and

$$s_k = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^k} \ (k \geq 3).$$

43.9 THEOREM $\forall\, r \geq 0$, the quadratic form

$$\sum_{i,j=0}^{r} s_{2+i+j} x_i x_j$$

is positive.

PROOF Inserting the data, consider

$$- 2ax_0^2 + \sum_{n=1}^{\infty} \Big(\sum_{i,j=0}^{r} \frac{x_i x_j}{\lambda_n^{2+i+j}} \Big)$$

or still,

$$- 2ax_0^2 + \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \Big(x_0 + \frac{x_1}{\lambda_n} + \cdots + \frac{x_r}{\lambda_n^r}\Big)^2,$$

an expression in which each term is manifestly nonnegative. Suppose that $\exists$ $x_0^{(0)}, x_1^{(0)}, \ldots, x_r^{(0)}$ such that

$$\sum_{i,j=0}^{r} s_{2+k+j} x_i^{(0)} x_j^{(0)} = 0.$$

Let

$$P_r(x) = x_0^{(0)} + x_1^{(0)} x + \cdots + x_r^{(0)} x^r.$$

Then

$$P_r\Big(\frac{1}{\lambda_n}\Big) = 0 \quad (n = 1,2,\ldots).$$

But the number of distinct $\frac{1}{\lambda_n}$ is infinite implying, therefore, that $P_r \equiv 0$, hence $x_0^{(0)} = 0,\ x_1^{(0)} = 0, \ldots,\ x_r^{(0)} = 0$.

43.10 SCHOLIUM if $f \not\equiv 0$ is a transcendental real entire function with an infinity of zeros such that $f(0) \neq 0$ and if $f \in \mathcal{L} - \mathcal{P}$, then the determinants

$$D_r \equiv \begin{vmatrix} s_2 & s_3 & \cdots & s_{2+r} \\ s_3 & s_4 & \cdots & s_{2+r+1} \\ \vdots & \vdots & & \vdots \\ s_{2+r} & s_{2+r+1} & \cdots & s_{2+r+r} \end{vmatrix} \quad (r \geq 0)$$

are positive.

43.11 EXAMPLE Take $r = 0$ -- then

$$D_0 = s_2 = -2a + \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} > 0.$$

[Note: Assume that $c_0 = 1$ -- then from the theory

$$-2a = c_1^2 - 2c_2 - \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2}$$

or still,

$$-2a + \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} = c_1^2 - 2c_2$$

or still,

$$s_2 = c_1^2 - 2c_2 \quad \text{(cf. 43.13).]}$$

43.12 EXAMPLE Take $r = 1$ -- then

$$D_1 = \begin{vmatrix} s_2 & s_3 \\ s_3 & s_4 \end{vmatrix} > 0.$$

43.13 LEMMA We have

$$\begin{aligned}
&c_0s_1 + c_1 = 0\\
&c_0s_2 + c_1s_1 + 2c_2 = 0\\
&c_0s_3 + c_1s_2 + c_2s_1 + 3c_3 = 0\\
&c_0s_4 + c_1s_3 + c_2s_2 + c_3s_1 + 4c_4 = 0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

43.14 APPLICATION Suppose that c_0 is positive and f is even -- then $c_1 = 0$, $c_3 = 0,\ldots$ and $s_1 = 0$, $s_3 = 0$, ... Therefore

$$s_2 = -\frac{2c_2}{c_0} > 0 \quad (\Rightarrow c_2 < 0)$$

while

$$c_0s_4 + c_2\ (-\frac{2c_2}{c_0}) + 4c_4 = 0$$

=>

$$c_0s_4 = \frac{2c_2^2}{c_0} - 4a_4 \Rightarrow \frac{c_2^2}{c_0} - 2c_4 > 0.$$

43.15 EXAMPLE In the notation of §41, take

$$f(z) = \text{Ш}(z) = \frac{1}{8}\,\Xi\,(\frac{z}{2})$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} b_k z^{2k}.$$

Then Ш is even and under RH, Ш $\in L - P$, thus the positivity of the D_r $(r \geq 0)$

provides a countable set of necessary conditions for its validity. To illustrate, in the case at hand

$$c_0 = b_0,\ c_1 = 0,\ c_2 = -\frac{1}{2!}b_1,\ c_3 = 0,\ c_4 = \frac{1}{4!}b_2.$$

Accordingly,

$$\begin{aligned}\frac{c_2^2}{c_0} - 2c_4 &= \frac{1}{b_0}\left(-\frac{1}{2}b_1\right)^2 - \frac{2}{24}b_2 \\ &= \frac{1}{4}\frac{b_1^2}{b_0} - \frac{1}{12}b_2 \\ &= \frac{1}{4b_0}\left(b_1^2 - \frac{1}{3}b_0b_2\right).\end{aligned}$$

And

$$\begin{aligned}&b_1^2 - \frac{1}{3}b_0b_2 \\ &\quad = 3.\ 588\ 449\ 148\ldots > 0.\end{aligned}$$

The central conclusion thus far is 43.9: If $f \in L - P$, then $\forall\ r \geq 0$, the quadratic form

$$\sum_{i,j=0}^{r} s_{2+i+j}x_ix_j$$

is positive. But this can be turned around.

43.16 THEOREM[†] Suppose that

$$f(z) = Ce^{az^2+b}\prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right)e^{z/z_n}$$

[†] J. Grommer, *J. Reine Angew. Math.* 144 (1914), pp. 114-166; see also N. Kritikos, *Math. Annalen* 81 (1920), pp. 97-118.

is in $A - L - P$ (cf. 10.31). Assume: $\forall\ r \geq 0$, the quadratic form

$$\sum_{i,j=0}^{r} s_{2+i+j} x_i x_j$$

is positive -- then $f \in L - P$.

Since

$$Ш \in 1 - L - P,$$

one approach to RH is potentially through 43.16.

§44. *ONE EQUIVALENCE*

There are a number of statements which are equivalent to the Riemann Hypothesis. What follows is one of them (of a semi-trivial nature...).

Per §41,

$$\text{Ш}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} b_k z^k,$$

where

$$b_k = \int_0^{\infty} t^{2k}\Phi(t)dt \quad (k = 0,1,\ldots).$$

In particular:

$$b_0 = \int_0^{\infty} \Phi(t)dt, \; b_1 = \int_0^{\infty} t^2\Phi(t)dt.$$

Let $0 < x_1 \leq x_2 \leq \ldots$ be the positive real zeros of Ш.

Let $S = \{\rho\}$ be the set of nonreal zeros of Ш whose imaginary part is positive:

$$\rho = \alpha + \sqrt{-1}\,\beta \;(0 < \beta < 1).$$

[Note: A sum over the empty set is 0 and a product over the empty set is 1.]

44.1 LEMMA

$$\text{Ш}(z) = \text{Ш}(0) \prod_{n=1}^{\infty} (1 - \frac{z^2}{x_n}) \prod_{\rho\in S} (1 - \frac{z^2}{\rho^2}).$$

44.2 LEMMA

$$\frac{d}{dz} (\frac{\text{Ш}'(z)}{\text{Ш}(z)}) = - \sum_{n=1}^{\infty} (\frac{1}{(z-x_n)^2} + \frac{1}{(z+x_n)^2})$$

$$- \sum_{\rho \in S} \left(\frac{1}{(z-\rho)^2} + \frac{1}{(z+\rho)^2}\right).$$

Now evaluate the left hand side of 44.2 at $z = 0$:

$$\frac{d}{dz}\left(\frac{\text{Ш}'(z)}{\text{Ш}(z)}\right)\Big|_{z=0} = \left(\frac{\text{Ш}'}{\text{Ш}}\right)'(0)$$

$$= \frac{\text{Ш}(0)\text{Ш}''(0) - \text{Ш}'(0)^2}{\text{Ш}(0)^2}$$

$$= \frac{\text{Ш}''(0)}{\text{Ш}(0)} .$$

And

$$\begin{cases} b_0 = \text{Ш}(0) \\ b_1 = - \text{Ш}''(0). \end{cases}$$

[Note: $\text{Ш}'(0) = 0$ (Ш being even).]

On the other hand, the right hand side of 44.2 evaluated at $z = 0$ is

$$- 2 \sum_{n=1}^{\infty} x_n^2 - 2 \sum_{\rho \in S} \frac{1}{\rho^2} .$$

And

$$\frac{1}{\rho^2} = \frac{1}{\alpha^2 - \beta^2 + 2\sqrt{-1}\,\alpha\beta}$$

$$= \frac{\alpha^2 - \beta^2 - 2\sqrt{-1}\,\alpha\beta}{(\alpha^2 - \beta^2)^2 + 4\alpha^2\beta^2}$$

$$= \frac{\alpha^2 - \beta^2 - 2\sqrt{-1}\,\alpha\beta}{\alpha^4 + 2\alpha^2\beta^2 + \beta^4} .$$

[Note: Working instead with $-\bar{\rho} = -\alpha + \sqrt{-1}\,\beta$ leads to

$$\frac{\alpha^2-\beta^2 + 2\sqrt{-1}\,\alpha\beta}{\alpha^4+2\alpha^2\beta^2 + \beta^4},$$

hence when summed the imaginary parts cancel out.]

Therefore

$$\frac{b_1}{2b_0} = \sum_{n=1}^{\infty} \frac{1}{x_n^2} + \sum_{\rho\in S} \frac{\alpha^2-\beta^2}{\alpha^4+2\alpha^2\beta^2 + \beta^4}.$$

N.B. $\forall\ \rho \in S$:

$$\left[\begin{array}{l} 1 < |\alpha| \\ 0 < \beta < 1 \end{array}\right. \Rightarrow \alpha^2-\beta^2 > 0.$$

44.3 THEOREM RH holds iff

$$\sum_{n=1}^{\infty} \frac{1}{x_n^2} = \frac{b_1}{2b_0}.$$

[The point is that if S is not empty, then $\forall\ \rho \in S$, $\alpha^2-\beta^2 > 0$.]

§45. *SUGGESTED READING*

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