Analysis 101:

Surfaces and Area

ABSTRACT

Here one will find a rigorous treatment of the simplest situation in Surface Area Theory, viz. the nonparametric case with domain the unit square in the plane.

SURFACES AND AREA

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§X. *THE FRÉCHET PROCESS*

Let (X,d) be a metric space and let $F:X \to [0, +\infty]$ be a lower semicontinuous function. Assume:

(A) For each $x \in X$, there is a sequence x_n $(n = 1,2,\ldots)$ in $X - \{x\}$ converging to x such that

$$\lim_{n \to \infty} F(x_n) = F(x).$$

Let $(\bar{X},\bar{d})$ be the completion of (X,d), the elements $\bar{x}$ of which being equivalence classes of Cauchy sequences in X. Extend F to a function $\bar{F}:\bar{X} \to [0, +\infty]$ by defining

$$\bar{F}(\bar{x}) = \inf_{\{x_n\}\in\bar{x}} \liminf_{n \to \infty} F(x_n),$$

where the infimum is taken over all Cauchy sequences in $\bar{x}$.

<u>1:</u> THEOREM $\bar{F}$ is an extension of F, i.e.,

$$\bar{F}|X = F.$$

Moreover $\bar{F}$ is lower semicontinuous and in addition is unique.

<u>2:</u> <u>N.B.</u> $\bar{F}$ has the following property:

(B) For each $\bar{x} \in \bar{X}$, there is a Cauchy sequence $\{x_n\} \in \bar{x}$ such that

$$\lim_{n \to \infty} F(x_n) = \bar{F}(\bar{x}).$$

To recapitulate:

<u>3:</u> SCHOLIUM Every nonnegative, extended real valued, lower semicontinuous function on a metric space X with property (A) can be extended to a unique lower

semicontinuous function on the completion $\bar{X}$ of X with property (B).

4: EXAMPLE Consider

$$\begin{cases} X =]0,1[& (d(x,y) = |x - y|) \\ \bar{X} = [0,1] & (\bar{d}(\bar{x},\bar{y}) = |\bar{x} - \bar{y}|) \end{cases}$$

and

$$\begin{cases} F = id_X \\ \bar{F} = id_{\bar{X}}. \end{cases}$$

§0. THE BEGINNING

Traditionally, a k-surface in n-space ($k \le n$) is an ordered pair $S = (A, \underline{f})$, where A is a subset of R^k with a nonempty interior (subject to certain restrictions) and $\underline{f}$ is a function from A to R^n, i.e., $\underline{f}: A \to R^n$, thus

$$\underline{f} = (f_1, \ldots, f_n).$$

1: N.B. If $k = n$, then $\underline{f}$ is said to be flat.

2: REMARK If $k = 1$ and $A = [a,b]$, then $\underline{f}$ is just a curve.

In this account, we shall take $k = 2$ and $n = 3$, thus

$$\underline{f}: \begin{cases} f_1: A \to R \\ f_2: A \to R \\ f_3: A \to R. \end{cases}$$

3: N.B. There are associated flat maps, viz.

$$\begin{cases} x = 0,\ y = f_2(u,v),\ z = f_3(u,v) \\ x = f_1(u,v),\ y = 0,\ z = f_3(u,v) \\ x = f_1(u,v),\ y = f_2(u,v),\ z = 0, \end{cases}$$

where $(u,v) \in A$.

In what follows, we do not intend to operate "in general" but instead will specialize matters to the so-called "nonparametric" situation.

Put

$$Q = [0,1] \times [0,1] \subset R^2 \quad (0 \le x \le 1,\ 0 \le y \le 1).$$

4: DEFINITION A nonparametric 2-surface in 3-space is an ordered pair $S_f = (Q,\underline{f})$, where

$$\underline{f}(x,y) = (x,y,f(x,y)),\ f:Q \to R$$

is a function, thus

$$\begin{cases} f_1(x,y) = x \\ f_2(x,y) = y \end{cases} \qquad f_3(x,y) = f(x,y).$$

5: REMARK Every function $f:Q \to R$ determines a nonparametric surface S_f. Because of this, the focus is on f, not on S_f.

Restricting matters to Q more or less eliminates the topological aspects of the theory, thus the discussion is "pure analysis", there being two aspects to the development, viz.

PART 1: The Continuous Case, $f \in C(Q)$.

PART 2: The Integrable Case, $f \in L^1(Q)$.

6: EXAMPLE Define $f:Q \to R$ by the prescription

$$\begin{cases} 0 \ (0 \le x \le \frac{1}{2}) \\ 1 \ (\frac{1}{2} < x \le 1). \end{cases}$$

Then f is not continuous but it is integrable.

§1. *QUASI LINEAR FUNCTIONS*

1: DEFINITION A quasi linear function is a continuous function $\Pi:Q \to R$ for which there exists a decomposition D of Q into a finite number of nonoverlapping triangles $T_1, T_2, \ldots, T_n$ such that Π is linear in each of these triangles, thus

$$\Pi(x,y) = a_i x + b_i y + c_i \quad ((x,y) \in T_i),$$

the a_i, b_i, c_i being real numbers.

2: EXAMPLE A constant function

$$f(x,y) = C((x,y) \in Q)$$

is quasi linear.

Suppose that $\Pi:Q \to R$ is quasi linear -- then Π maps each T_i into a triangle $\Delta_i \subset R^3$ (possibly a segment or a point).

3: NOTATION Let $|\Delta_i|$ stand for the area of Δ_i.

4: DEFINITION The elementary area of a quasi linear function $\Pi:Q \to R$ is the sum

$$a(\Pi) \equiv \Sigma \, |\Delta_i|,$$

where Σ is taken over the $T_i \in D$.

5: NOTATION Let $|T_i|$ stand for the area of T_i.

6: N.B. Let

$$(u_1,v_1), \ (u_2,v_2), \ (u_3,v_3)$$

be the vertices of T_i in Q -- then

$$|T_i| = \frac{1}{2} \left| \det \begin{bmatrix} u_1 & v_1 & 1 \\ u_2 & v_2 & 1 \\ u_3 & v_3 & 1 \end{bmatrix} \right| .$$

7: LEMMA

$$|\Delta_i| = |T_i| \ (1 + a_i^2 + b_i^2)^{1/2} .$$

Therefore

$$a(\Pi) = \sum_i |T_i| \ (1 + a_i^2 + b_i^2)^{1/2} .$$

8: SCHOLIUM

$$a(\Pi) = \int\int_Q [1 + (\partial\Pi/\partial x)^2 + (\partial\Pi/\partial y)^2]^{1/2} \ dxdy.$$

It follows from this that $a(\Pi)$ is independent of the subdivision D of Q into triangles of linearity for Π.

9: REMARK A quasi linear function $\Pi:Q \rightarrow R$ is Lipschitz continuous and

$$H^2(Gr_\Pi(Q)) = \int\int_Q [1 + (\partial\Pi/\partial x)^2 + (\partial\Pi/\partial y)^2]^{1/2} \ dxdy.$$

10: LEMMA Per uniform convergence, the elementary area is lower semi-continuous on the set of quasi linear functions.

§2. *LEBESGUE AREA*

Recall that

$$Q = [0,1] \times [0,1] \subset R^2 \quad (0 \leq x \leq 1,\ 0 \leq y \leq 1).$$

1: LEMMA Let $f:Q \to R$ be a continuous function — then there exists a sequence

$$\xi = \{\Pi_n : n = 1,2,\ldots\}$$

of quasi linear functions $\Pi_n:Q \to R$ such that $\Pi_n \to f$ uniformly $(n \to \infty)$.

2: NOTATION Given a continuous function $f:Q \to R$, denote by Ξ the collection of all sequences

$$\xi = \{\Pi_n : n = 1,2,\ldots\}$$

of quasi linear functions $\Pi_n:Q \to R$ such that $\Pi_n \to f$ uniformly $(n \to \infty)$.

3: N.B. The preceding lemma ensures that Ξ is nonempty.

4: DEFINITION The Lebesgue area $L_Q[f]$ of a continuous function $f:Q \to R$ is the entity

$$\inf_{\xi \in \Xi} \ \liminf_{n \to \infty} a(\Pi_n).$$

5: REMARK This definition and the considerations that follow are an instance of the Fréchet process: Take for X the quasi linear functions on Q, take for d the metric defined by the prescription

$$d(\Pi_1,\Pi_2) = \sup |\Pi_1(x,y) - \Pi_2(x,y)|,$$

and take for F the elementary area — then the completion $\bar{X}$ of X is C(Q), the set of continuous functions on Q, and the extension $\bar{F}$ of F assigns to each $f \in C(Q)$ its Lebesgue area:

$$\bar{F}(f) = L_Q[f].$$

<u>6:</u> CONSISTENCY PRINCIPLE The elementary area of a quasi linear function $\Pi:Q \to R$ equals its Lebesgue area.

<u>7:</u> LEMMA There is at least one $\xi \in \Xi$ such that

$$a(\Pi_n) \to L_Q[f] \quad (n \to \infty).$$

PROOF There are two possibilities:

$$L_Q[f] < +\infty \text{ or } L_Q[f] = +\infty.$$

Matters are manifest if $L_Q[f] = +\infty$, so assume that $L_Q[f] < +\infty$. Given any positive integer n, there exists a sequence $\{\Pi_m : m = 1,2,\ldots\}$ such that for $m \to \infty$, $\Pi_m \to f$ uniformly and

$$\liminf_{m \to \infty} a(\Pi_m) < L_Q[f] + \frac{1}{n},$$

thus there is an m such that

$$||\Pi_m - f||_\infty < \frac{1}{n}$$

and

$$a(\Pi_m) < L_Q[f] + \frac{1}{n}.$$

This m depends on n. Write $\Pi(n)$ in place of Π_m -- then

$$||\Pi(n) - f||_\infty < \frac{1}{n}$$

and

$$a(\Pi(n)) < L_Q[f] + \frac{1}{n}.$$

Let now $n \to \infty$ to conclude that

$$\Pi(n) \to f$$

uniformly and

$$\limsup_{n \to \infty} a(\Pi(n)) \leq L_Q[f].$$

On the other hand,

$$L_Q[f] \leq \liminf_{n \to \infty} a(\Pi(n)).$$

Hence the lemma.

8: N.B. This result is known as the proper sequential limit principle.

9: THEOREM Let $f:Q \to R$ be a continuous function. Suppose that $f_n:Q \to R$ $(n = 1,2,\ldots)$ is a sequence of continuous functions such that $f_n \to f$ uniformly -- then

$$L_Q[f] \leq \liminf_{n \to \infty} L[f_n].$$

PROOF Assume without loss of generality that

$$\liminf_{n \to \infty} L_Q[f_n] < +\infty \text{ and } L_Q[f_n] < +\infty \ (\forall\, n).$$

Given n, choose per supra a sequence $\{\Pi_{nm}:m = 1,2,\ldots\}$ of quasi linear functions uniformly convergent to f_n $(m \to \infty)$ with

$$a(\Pi_{nm}) \to L_Q[f_n] \quad (m \to \infty).$$

Accordingly

$$\delta_{nm} \equiv ||\Pi_{nm} - f_n||_\infty \to 0 \qquad (m \to \infty)$$

and for each n there exists an integer $m = m(n)$ such that

$$\delta_{nm} < \frac{1}{n} \text{ and } |a(\Pi_{nm}) - L_Q[f_n]| < \frac{1}{n}.$$

Next, $\forall\, w \in Q$,

$$|\Pi_{nm}(w) - f(w)| \leq ||\Pi_{nm} - f_n||_\infty + ||f_n - f||_\infty$$

$$\leq \delta_{nm} + ||f_n - f||_\infty$$

$$< \frac{1}{n} + ||f_n - f||_\infty$$

$$\to 0 \quad (n \to \infty).$$

Put

$$\Pi_n' = \Pi_{nm}$$

and let

$$\xi' = \{\Pi_n' : n = 1,2,\ldots\},$$

so $\xi' \in \Xi$. And

$$L_Q[f] \leq \liminf_{n \to \infty} a(\Pi_n')$$

$$= \liminf_{n \to \infty} (a(\Pi_n') - L_Q[f_n] + L_Q[f_n])$$

$$= \lim_{n \to \infty} (a(\Pi_n') - L_Q[f_n]) + \liminf_{n \to \infty} L_Q(f_n)$$

$$= 0 + \liminf_{n \to \infty} L_Q[f_n]$$

$$= \liminf_{n \to \infty} L_Q(f_n).$$

Therefore Lebesgue area is a lower semicontinuous functional in the class of continuous functions (the underlying convergence being uniform).

[Note: It can be shown that Lebesgue area is a lower semicontinuous functional in the class of continuous functions relative to pointwise convergence.]

Here is a simple application: If $\forall\, n$, $L_Q[f_n] \leq L_Q[f]$, then $L_Q[f_n] \to L_Q[f]$.

In fact,

$$\limsup_{n \to \infty} L_Q[f_n] \leq L_Q[f]$$

while on the other hand,

$$\liminf_{n \to \infty} L_Q[f_n] \geq L_Q[f].$$

10: LEMMA Let L* be a functional in the class of continuous functions which is lower semicontinuous per uniform convergence and has the property that for every quasi linear Π,

$$L^*[\Pi] = a(\Pi).$$

Then for every f,

$$L^*[f] \leq L_Q[f].$$

PROOF Choose $\xi \in \Xi$ such that

$$a(\Pi_n) \to L_Q[f] \qquad (n \to \infty)$$

and note that

$$L^*[f] \leq \liminf_{n \to \infty} L^*[\Pi_n]$$

$$= \liminf_{n \to \infty} a(\Pi_n)$$

$$\leq L_Q[f].$$

§3. GEÖCZE AREA

The setting for the notion of Lebesgue area is the unit square

$$Q = [0,1] \times [0,1].$$

However there is no difficulty in extending matters to oriented rectangles $R \subset Q$:

$$\begin{cases} a \leq x \leq b & (a < b) \\ c \leq y \leq d & (c < d) \end{cases}, \quad |R| = (b - a)\ (d - c).$$

The theory thus formulated applies to any real valued continuous function on R. In particular: Given a continuous function $f:Q \to \mathbf{R}$, let f_R be its restriction to R and denote its Lebesgue area per R by the symbol $L_Q[f_R]$.

Introduce

$$\begin{cases} G_X(f;R) = \int_a^b |f(x,d) - f(x,c)|dx \\ G_Y(f;R) = \int_c^d |f(b,y) - f(a,y)|dy \end{cases}$$

and put

$$\Gamma(f;R) = [(G_X(f;R))^2 + (G_Y(f;R)^2 + |R|^2]^{1/2}.$$

<u>1:</u> LEMMA

$$\Gamma(f;R) \leq L_Q[f_R].$$

Let D be a subdivision of Q into nonoverlapping oriented rectangles R (lines parallel to the coordinate axes).

<u>2:</u> DEFINITION The <u>sum of Geöcze</u> is the expression

$$G(f;D) = \Sigma\ \Gamma(f;R),$$

the summation being taken over the rectangles R in D.

So

$$G(f;D) \leq \Sigma\, L_Q[f_R].$$

And

$$\Sigma\, L_Q[f_R] \leq L_Q[f].$$

Therefore

$$G(f;D) \leq L_Q[f].$$

3: NOTATION Put

$$\Gamma_Q[f] = \sup_D G(f;D),$$

the Geöcze area of f.

Then $\forall$ D,

$$G(f;D) \leq L_Q[f]$$

=>

$$\Gamma_Q[f] \leq L_Q[f].$$

[Note: This inequality is trivial if $L_Q[f] = +\infty$, thus there is no loss of generality in assuming that $L_Q[f] < +\infty$.]

4: THEOREM

$$\Gamma_Q[f] = L_Q[f].$$

This assertion is nontrivial, the first step being to establish it when

$$\frac{\partial f}{\partial x} = p(x,y),\ \frac{\partial f}{\partial y} = q(x,y)$$

exist in Q and are continuous there.

- Write

$$G_X(f;R) = \int_a^b |f(x,d) - f(x,c)|dx$$

$$= (b-a)|f(\xi,d) - f(\xi,c)| \quad (a \le \xi \le b)$$

$$= (b-a)(d-c)|q(\xi,\eta)| \quad (c \le \eta \le d)$$

$$= |R|\ |q(\xi,\eta)|.$$

- Write

$$G_Y(f;R) = \int_c^d |f(b,y) - f(a,y)|dy$$

$$= (d-c)|f(b,\mu) - f(a,\mu)| \quad (c \le \mu \le d)$$

$$= (d-c)(b-a)|p(\nu,\mu)| \quad (a \le \nu \le b)$$

$$= |R|\,|p(\nu,\mu)|.$$

Consequently

$$\Gamma(f;R) = [1 + p(\nu,\mu)^2 + q(\xi,\eta)^2]^{1/2}|R|$$

$$= [1 + p(\xi,\eta)^2 + q(\xi,\eta)^2]^{1/2}|R| + \varepsilon_R|R|,$$

where ε_R tends to zero with the diameter of R.

Let again D be a subdivision of Q into nonoverlapping oriented rectangles R (lines parallel to the coordinate axes). Since $\Sigma\ |R| = |Q| = 1$, it follows that

$$G(f;D) = \Sigma\ \Gamma(f;R)$$

$$= \Sigma\ [1 + p(\xi,\eta)^2 + q(\xi,\eta)^2]^{1/2}|R| + \varepsilon.$$

Here $\varepsilon \to 0$ when $\delta \to 0$ (δ being the maximum diameter of the rectangles R in D).

Replace now D by a sequence $\{D_n\}$ and assume that $\delta_n \to 0$ $(n \to \infty)$ — then the sum

$$\Sigma\ [1 + p(\xi,\eta)^2 + q(\xi,\eta)^2]^{1/2}|R|$$

tends to the integral

$$\iint_Q (1 + p^2 + q^2)^{1/2} \, dxdy,$$

hence

$$\lim_{n \to \infty} G(f;D_n) = \iint_Q (1 + p^2 + q^2)^{1/2} \, dxdy$$

or still,

$$\Gamma_Q[f] \geq \iint_Q (1 + p^2 + q^2)^{1/2} \, dxdy$$

$$\equiv L_Q[f] \quad \text{(see below)}.$$

But, as has been noted above, it is always the case that

$$\Gamma_Q[f] \leq L_Q[f].$$

So in the end,

$$\Gamma_Q[f] = L_Q[f].$$

5: CONSTRUCTION There is a $\xi \in \Xi$ such that

$$a(\Pi_n)\,(n \to \infty) \to \iint_Q (1 + p^2 + q^2)^{1/2} \, dxdy.$$

6: LEMMA

$$L_Q[f] = \iint_Q (1 + p^2 + q^2)^{1/2} \, dxdy.$$

PROOF

$$\iint_Q (1 + p^2 + q^2)^{1/2} \, dxdy$$

$$\leq \Gamma_Q[f] \leq L_Q[f]$$

$$\leq \liminf_{n \to \infty} a(\Pi_n) = \lim_{n \to \infty} a(\Pi_n)$$

$$= \iint_Q (1 + p^2 + q^2)^{1/2} \, dxdy.$$

7: EXAMPLE Suppose that $f(x,y)$ is independent of y -- then $\frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial x} = f'(x)$, hence

$$\iint_Q (1 + p^2 + q^2)^{1/2} \, dxdy = \int_0^1 (1 + (f'(x))^2)^{1/2} \, dx.$$

It remains to establish that

$$\Gamma_Q[f] = L_Q[f]$$

in general. To this end, denote by $\underline{Q}$ a concentric square completely contained in the interior of Q, let $0 < h < \frac{1}{2}$, put

$$Q_h: \begin{cases} h \leq x \leq 1 - h \\ h \leq y \leq 1 - h, \end{cases}$$

and assume that for h sufficiently small, $\underline{Q} \subset Q_h$ -- then there exists a continuous function $f_h: Q_h \to R$ with the following properties.

(a) $\frac{\partial f_h}{\partial x}$, $\frac{\partial f_h}{\partial y}$ exist and are continuous functions in Q_h.

(b) $\Gamma_{\underline{Q}}[f_h] \leq \Gamma_Q[f]$.

(c) $f_h \to f$ $(h \to 0)$ uniformly in $\underline{Q}$.

Granted these points, on the basis of the earlier considerations, from (a),

$$\Gamma_{\underline{Q}}[f_h] = L_{\underline{Q}}[f_h],$$

thus by (b),

$$L_{\underline{Q}}[f_h] \leq \Gamma_Q[f] \leq L_Q[f]$$

=>

$$\limsup_{h \to 0} L_{\underline{Q}}[f_h] \leq \Gamma_Q[f].$$

But, thanks to (c),

$$L_{\underline{Q}}[f] \leq \liminf_{h \to 0} L_{\underline{Q}}[f_h].$$

And then

$$\begin{aligned} L_{\underline{Q}}[f] &\leq \liminf_{h \to 0} L_{\underline{Q}}[f_h] \\ &\leq \limsup_{h \to 0} L_{\underline{Q}}[f_h] \\ &\leq \Gamma_Q[f] \leq L_Q[f]. \end{aligned}$$

Suppose now that $\underline{Q}$ invades $Q : \underline{Q} \uparrow Q$, hence

$$L_{\underline{Q}}[f] \to L_Q[f]$$

$$\Rightarrow$$

$$L_Q[f] \leq \Gamma_Q[f] \leq L_Q[f]$$

$$\Rightarrow$$

$$\Gamma_Q[f] = L_Q[f].$$

§4. *APPROXIMATION THEORY*

To finish the proof that

$$\Gamma_Q[f] = L_Q[f],$$

we have yet to establish the validity of points (a), (b), (c) as formulated near the end of the preceding § and for this, it will be necessary to set up some machinery.

<u>1:</u> DEFINITION Let $f:Q \to R$ be a continuous function and let $0 < h < \frac{1}{2}$ -- then the function

$$f_h(x,y) = \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} f(x + \xi, y + \eta)d\xi d\eta$$

defined in the square

$$Q_h: \begin{cases} h \le x \le 1 - h \\ h \le y \le 1 - h \end{cases}$$

is called the <u>integral mean</u> of f.

<u>2:</u> LEMMA $f_h:Q_h \to R$ is a continuous function.

<u>3:</u> LEMMA $f_h \to f$ $(h \to 0)$ uniformly in $\underline{Q} \subset Q_h$.

<u>4:</u> LEMMA $\frac{\partial f_h}{\partial x}$, $\frac{\partial f_h}{\partial y}$ exist and are continuous functions on Q_h:

$$\begin{cases} \frac{\partial f_h}{\partial x} = \frac{1}{4h^2} \int_{-h}^{h} [f(x + h ,y + \eta) - f(x - h,y + \eta)]d\eta \\ \frac{\partial f_h}{\partial y} = \frac{1}{4h^2} \int_{-h}^{h} [f(x + \xi,y + h) - f(x + \xi,y - h)]d\xi . \end{cases}$$

5: N.B. Accordingly points (a) and (c) are settled.

The validity of point (b), i.e., the assertion that

$$\Gamma_{\underline{Q}}[f_h] \le \Gamma_Q[f]$$

is not so easy to prove.

Start by fixing an oriented rectangle $R \subset \underline{Q}$:

$$\left[\begin{array}{ll} a \le x \le b & (a < b) \\ c \le y \le d & (c < d) \end{array}\right., \quad |R| = (b - a)\ (c - d).$$

Then

$$|f_h(x,d) - f_h(x,c)|$$

$$\le \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} |f(x + \xi, d + \eta) - f(x + \xi, c + \eta)\ d\xi d\eta$$

$$\Rightarrow$$

$$G_X(f_h;R) = \int_a^b |f_h(x,d) - f_h(x,c)|dx$$

$$\le \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} d\xi d\eta \int_a^b |f(x + \xi, d + \eta) - f(x + \xi, c + \eta)|dx.$$

Let $R_{\xi\eta}$ be the rectangle obtained by subjecting R to the translation

$$\left[\begin{array}{l} \bar{x} = x + \xi \\ \bar{y} = y + \eta, \end{array}\right.$$

thus

$$G_X(f;R_{\xi\eta}) = \int_a^b |f(x + \xi, d + \eta) - f(x + \xi, c + \eta)|dx$$

and so

$$G_X(f_h;R) \le \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} G_X(f;R_{\xi\eta})d\xi d\eta.$$

Analogously

$$G_Y(f_h;R) \leq \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} G_Y(f;R_{\xi\eta})d\xi d\eta.$$

Finally

$$|R| = |R_{\xi\eta}| = \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} |R_{\xi\eta}|d\xi d\eta.$$

To summarize:

6: LEMMA

$$\Gamma(f_h;R) \leq [G_X(f_h;R)^2 + G_Y(f_h;R)^2 + |R|^2]^{1/2}$$

$$\leq \frac{1}{4h^2} [(\int_{-h}^{h} \int_{-h}^{h} G_X(f;R_{\xi\eta})d\xi d\eta)^2$$

$$+ (\int_{-h}^{h} \int_{-h}^{h} G_Y(f;R_{\xi\eta})d\xi d\eta)^2$$

$$+ (\int_{-h}^{h} \int_{-h}^{h} |R_{\xi\eta}|d\xi d\eta)^2]^{1/2}.$$

7: RAPPEL Under canonical assumptions,

$$((\int_X \phi_1)^2 + \cdots + (\int_X \phi_n)^2)^{1/2}$$

$$\leq \int_X (\phi_1^2 + \cdots + \phi_n^2)^{1/2}.$$

Therefore

$$\Gamma(f_h;R) \leq \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} (G_X(f;R_{\xi\eta})^2 + (G_Y(f;R_{\xi\eta})^2 + |R_{\xi\eta}|^2)^{1/2} d\xi d\bar{\eta}$$

$$= \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} \Gamma(f;R_{\xi\eta})d\xi d\eta.$$

Suppose now that $\underline{D}$ is a subdivision of $\underline{Q}$ into nonoverlapping rectangles R (lines parallel to the coordinate axes) -- then

$$G(f_h;\underline{D}) = \Sigma\, \Gamma(f_h;R)$$

$$\leq \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} \Sigma\, \Gamma(f;R_{\xi\eta})\, d\xi d\eta,$$

the sum under $\int_{-h}^{h} \int_{-h}^{h}$ being the sum of Geöcze (for f) relative to the division $\underline{D}_{\xi\eta}$ of $\underline{Q}_{\xi\eta} \subset Q$ into rectangles $R_{\xi\eta}$, thus a fortiori,

$$\Sigma\, \Gamma(f;R_{\xi\eta}) \leq \Gamma_Q[f]$$

=>

$$G(f_h;\underline{D}) \leq \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} \Gamma_Q[f]\, d\xi d\eta$$

$$= \frac{\Gamma_Q[f]}{4h^2} \int_{-h}^{h} \int_{-h}^{h} d\xi d\eta$$

$$= \Gamma_Q[f]$$

=>

$$\Gamma_{\underline{Q}}[f_h] = \sup_{\underline{D}} G(f_h;\underline{D})$$

$$\leq \Gamma_Q[f],$$

from which point (b).

<u>8:</u> LEMMA

$$L_{Q_h}[f_h] \leq L_Q[f]$$

and

$$L_Q[f] = \lim_{h \to 0} L_{Q_h}[f_h].$$

Since

$$L_{Q_h}[f_h] = \int\int_{Q_h} [1 + (\frac{\partial f_h}{\partial x})^2 + (\frac{\partial f_h}{\partial y})^2]^{1/2} dxdy,$$

it follows that

$$L_Q[f] = \lim_{h \to 0} \int_h^{1-h}\int_h^{1-h} [1 + (\frac{1}{4h^2} \int_{-h}^{h} (f(x+h,y+\eta) - f(x-h,y+\eta))d\eta)^2$$

$$+ (\frac{1}{4h^2} \int_{-h}^{h} (f(x+\xi,y+h) - f(x+\xi,y-h))d\xi)^2]^{1/2} dxdy.$$

* * * * * * * * * * * *

What follows will not be needed in the sequel but it is of independent interest.

9: DEFINITION Let $f \in L^1(Q)$ and let $0 < h < \frac{1}{2}$ -- then the function

$$f_h(x,y) = \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} f(x+\xi,y+\eta)d\xi d\eta$$

defined in the square

$$\begin{cases} h \leq x \leq 1 - h \\ h \leq y \leq 1 - h \end{cases}$$

is called the integral mean of f.

10: LEMMA $f_h:Q_h \to R$ is a continuous function, hence

$$\int\int_{Q_h} |f_h| < + \infty \Rightarrow f_h \in L^1(Q_h).$$

11: LEMMA $\forall f \in L^1(Q)$,

$$||f_h||_{L^1} \leq ||f||_{L^1}.$$

PROOF

$$\int\int_{Q_h} |f_h(x,y)|dxdy$$

$$= \int_h^{1-h}\int_h^{1-h}|f_h(x,y)|dxdy$$

$$\le \frac{1}{4h^2}\int_h^{1-h}\int_h^{1-h}\{\int_{-h}^{h}\int_{-h}^{h}|f(x+\xi,y+\eta)|d\xi d\eta\}dxdy$$

$$\le \frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h}\{\int_h^{1-h}\int_h^{1-h}|f(x+\xi,y+\eta)|dxdy\}d\xi d\eta$$

$$\le \frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h}\{\int_{h+\xi}^{1-h+\xi}\int_{h+\eta}^{1-h+\eta}|f(x,y)|dxdy\}d\xi d\eta$$

$$\le \frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h}\{\int_0^1\int_0^1|f(x,y)|dxdy\}d\xi d\eta$$

$$\le \frac{1}{4h^2}(2h)(2h)||f||_{L^1}$$

$$= ||f||_{L^1} < +\infty.$$

12: REMARK An analogous estimate obtains if $f \in L^p(Q)$ $(1 < p < +\infty)$:

$$||f_h||_{L^p} \le ||f||_{L^p}.$$

13: LEMMA As $h \to 0$, f_h converges almost everywhere to f.

14: LEMMA

$$\int\int_{Q_h} |f_h - f| \to 0 \quad (h \to 0).$$

PROOF Given $\varepsilon > 0$, write $f = \phi + \psi$, where ϕ is continuous in Q, ψ is integrable in Q, and $\int\int_Q |\psi| < \varepsilon$ -- then

$$\int\int_{Q_h} |f_h - f|$$

$$= \int\int_{Q_h} |(\phi_h + \psi_h) - (\phi + \psi)|$$

$$\leq \int\int_{Q_h} |\phi_h - \phi| + \int\int_{Q_h} |\psi_h - \psi|$$

$$\leq \int\int_{Q_h} |\phi_h - \phi| + \int\int_{Q_h} |\psi_h| + \int\int_{Q_h} |\psi|$$

$$\leq \int\int_{Q_h} |\phi_h - \phi| + \int\int_{Q} |\psi| + \int\int_{Q} |\psi|$$

$$\leq \int\int_{Q_h} |\phi_h - \phi| + 2 \int\int_{Q} |\psi|$$

$$\leq \int\int_{Q_h} |\phi_n - \phi| + 2\varepsilon.$$

Since ϕ is continuous in Q, it follows that in Q_h,

$$\phi_h \to \phi|Q_h \quad (h \to 0)$$

uniformly, hence

$$\int\int_{Q_h} |\phi_h - \phi| \to 0 \quad (h \to 0).$$

So for all sufficiently small h,

$$\int\int_{Q_h} |\phi_h - \phi| < \varepsilon$$

=>

$$\lim_{h \to 0} \int\int_{Q_h} |f_h - f| < 3\varepsilon.$$

15: REMARK An analogous statement obtains if $f \in L^p(Q)$ $(1 < p < +\infty)$:

$$\int\int_{Q_h} |f_h - f|^p \to 0$$

as $h \to 0$.

16: LEMMA If $f \in L^p(Q)$ $(1 \leq p < +\infty)$, then

$$\frac{\partial f_h}{\partial x} \ \& \ \frac{\partial f_h}{\partial y}$$

belong to $L^p(Q_h)$.

PROOF Take $p > 1$ and consider $\frac{\partial f_h}{\partial x}$, thus

$$\frac{\partial f_h}{\partial x} = \frac{1}{4h^2} \int_{y-h}^{y+h} [f(x+h,\eta) - f(x-h,\eta]d\eta$$

almost everywhere in Q_h, the claim being that the functions

$$\left[\begin{array}{l} \int_{y-h}^{y+h} f(x+h,\eta)d\eta \\ \\ \int_{y-h}^{y+h} f(x-h,\eta)d\eta \end{array}\right.$$

are in $L^p(Q_h)$. To discuss the first of these, write

$$\int_{y-h}^{y+h} f(x+h,\eta)d\eta = \int_{-h}^{h} f(x+h,y+\eta)d\eta.$$

Then

$$|\int_{-h}^{h} f(x+h,y+\eta)d\eta|^p$$

$$\leq (2h)^{p-1} \int_{-h}^{h} |f(x+h,y+\eta)|^p d\eta.$$

Since $f \in L^p(Q)$, $|f(x+h,y+\eta)|^p$ is integrable in

$$h \leq x \leq 1 - h, \; h \leq y \leq 1 - h, \; -h \leq \eta \leq h.$$

Therefore

$$\int_{-h}^{h} |f(x+h,y+\eta)|^p d\eta$$

is integrable in Q_h, hence

$$\int_{-h}^{h} |f(x+h,y+\eta)|^p d\eta$$

is in $L^p(Q_h)$.

§5. *TONELLI'S CHARACTERIZATION*

Let $f:Q \to R$ be a continuous function.

1: DEFINITION

$$\begin{cases} V_x(f;y) = T_{f(-,y)}[0,1] & (0 \leq y \leq 1) \\ V_y(f;x) = T_{f(x,-)}[0,1] & (0 \leq x \leq 1). \end{cases}$$

2: LEMMA

$$\begin{cases} V_x(f;-) \text{ is a lower semicontinuous function of } y \in [0,1] \\ V_y(f;-) \text{ is a lower semicontinuous function of } x \in [0,1]. \end{cases}$$

PROOF Consider the first assertion and suppose that $y_n \to y$ -- then

$$f(x,y_n) \to f(x,y)$$

=>

$$T_{f(-,y)}[0,1] \leq \liminf_{n \to \infty} T_{f(-,y_n)}[0,1].$$

I.e.:

$$V_x(f;y) \leq \liminf_{n \to \infty} V_x(f;y_n).$$

3: SCHOLIUM $V_x(f;-)$ and $V_y(f;-)$ are Lebesgue measurable.

4: DEFINITION (BVT) f is said to be of <u>bounded variation in the sense of Tonelli</u> if

$$\begin{cases} \int_0^1 V_x(f;y)\,dy < +\infty \\ \int_0^1 V_y(f;x)\,dx < +\infty. \end{cases}$$

5: NOTATION

$$V_T(f) = \int_0^1 V_x(f;y)\,dy + \int_0^1 V_y(f;x)\,dx.$$

6: N.B. Accordingly, if $V_T(f) < +\infty$, then

$$e_Y = \{y \in [0,1] : V_x(f;y) = +\infty\}$$

is of Lebesgue measure zero and

$$e_X = \{x \in [0,1] : V_y(f;x) = +\infty\}$$

is of Lebesgue measure zero.

7: LEMMA Suppose that $V_T(f) < +\infty$ -- then $f|Q^{\circ} \in BV(Q^{\circ})$ and

$$\begin{cases} f_x = \dfrac{\partial f}{\partial x} \text{ exists almost everywhere in } Q \\ f_y = \dfrac{\partial f}{\partial y} \text{ exists almost everywhere in } Q. \end{cases}$$

8: LEMMA Suppose that $V_T(f) < +\infty$ -- then

$$\begin{cases} \int\int_Q |f_x(x,y)|\,dxdy \le \int_0^1 V_x(f;y)\,dy < +\infty \\ \int\int_Q |f_y(x,y)|\,dxdy \le \int_0^1 V_y(f;x)\,dx < +\infty \end{cases}$$

=>

$$\left.\begin{matrix} f_x \\ f_y \end{matrix}\right\} \in L^1(Q)$$

=>

$$[1 + f_x^2 + f_y^2]^{1/2} \in L^1(Q).$$

9: THEOREM $L_Q[f]$ is finite iff f is of bounded variation in the sense of Tonelli.

Assume to begin with that $L_Q[f]$ is finite. Let D be the subdivision of Q specified by

$$\begin{cases} x_0 = 0 < x_1 < \cdots < x_j < \cdots < x_m = 1 \\ y_0 = 0 < y_1 < \cdots < y_k < \cdots < y_n = 1 \end{cases}$$

and introduce

$$\begin{cases} v_x(f;y;D) = \sum_{j=0}^{m-1} |f(x_{j+1},y) - f(x_j,y)| & (0 \leq y \leq 1) \\ v_y(f;x;D) = \sum_{k=0}^{n-1} |f(x,y_{k+1}) - f(x,y_k)| & (0 \leq x \leq 1). \end{cases}$$

Then

$$\begin{cases} \int_0^1 v_x(f;y;D)\,dy = \Sigma\, G_Y(f;R) \\ \int_0^1 v_y(f;x;D)\,dx = \Sigma\, G_X(f;R), \end{cases}$$

the summations being over the rectangles R in D. Next

$$\left.\begin{matrix} \Sigma\, G_Y(f;R) \\ \\ \Sigma\, G_X(f;R) \end{matrix}\right\} \leq G(f;D) \leq L_Q[f].$$

Therefore

$$\left.\begin{matrix} \int_0^1 v_x(f;y;D)\,dy \\ \\ \int_0^1 v_y(f;x;D)\,dx \end{matrix}\right\} \leq L_Q[f] < +\infty.$$

From the definitions,

$$\begin{cases} 0 \leq v_x(f;y;D) \leq V_x(f;y) \\ 0 \leq v_y(f;x;D) \leq V_y(f;x). \end{cases}$$

So, upon sending the maximum diameters of the rectangles R in D to zero sequentially, we conclude that

$$\begin{cases} \lim v_x(f;y;D) = V_x(f;y) \\ \lim v_y(f;x;D) = V_y(f;x) \end{cases}$$

=>

$$\begin{cases} \int_0^1 V_x(f;y)\,dy = \int_0^1 \lim v_x(f;y;D)\,dy \\ \int_0^1 V_y(f;x)\,dx = \int_0^1 \lim v_y(f;x;D)\,dx \end{cases}$$

or still,

$$\begin{cases} \leq \liminf \int_0^1 v_x(f;y;D)\,dy \\ \leq \liminf \int_0^1 v_y(f;x;D)\,dx \end{cases} \quad (\text{Fatou}) \leq L_Q[f] < +\infty.$$

Consequently, under the supposition that $L_Q[f]$ is finite, it follows that f is of bounded variation in the sense of Tonelli.

To reverse this, note first that for any D,

$$\begin{cases} v_x(f;y;D) \leq V_x(f;y) \\ v_y(f;x;D) \leq V_y(f;x) \end{cases}$$

=>

$$\left[\begin{array}{l} \Sigma\, G_Y(f;R) \le \int_0^1 V_x(f;y)\,dy \\ \\ \Sigma\, G_X(f;R) \le \int_0^1 V_y(f;x)\,dx. \end{array}\right.$$

And

$$\Gamma(f;R) \le G_X(f;R) + G_Y(f;R) + |R|$$

$\Rightarrow$

$$\begin{aligned} G(f;D) &= \Sigma\, \Gamma(f;R) \\ &\le \Sigma\, G_Y(f;R) + \Sigma\, G_X(f;R) + \Sigma\, |R| \\ &\le \int_0^1 V_x(f;y)\,dy + \int_0^1 V_y(f;x)\,dx + 1 \\ &= V_T(f) + 1. \end{aligned}$$

However

$$\Gamma_Q[f] = \sup_D G(f;D).$$

Therefore

$$\Gamma_Q[f] < +\infty$$

$\Rightarrow$

$$L_Q[f] < +\infty.$$

<u>10:</u> REMARK Individually

$$\int_0^1 V_x(f;y)\,dy,\ \int_0^1 V_y(f;x)\,dx,\ 1$$

are all $\le L_Q[f]$.

§6. *TONELLI'S ESTIMATE*

Let $f:Q \to R$ be a continuous function.

1: THEOREM Suppose that $L_Q[f]$ is finite -- then

$$L_Q[f] \geq \int\int_Q [1 + f_x^2 + f_y^2]^{1/2} \, dxdy.$$

Let $D = \{R_1, R_2, \ldots, R_n\}$ be a subdivision of Q, where

$$R_k = [a_k, b_k] \times [c_k, d_k] \quad (k = 1, 2, \ldots, n).$$

2: LEMMA Given $\varepsilon > 0$, there is a D such that

$$\left| \sum_{k=1}^{n} [(\int_{R_k}\int f_x \, dxdy)^2 + (\int_{R_k}\int f_y \, dxdy)^2 + |R_k|^2]^{1/2} - \int\int_Q [1 + f_x^2 + f_y^2]^{1/2} \, dxdy \right| < \varepsilon.$$

[Recall that

$$\left. \begin{array}{l} f_x \\ \\ f_y \end{array} \right] \in L^1(Q)$$

and use the Vitali covering lemma.]

Proceeding

$$\left| \sum_{k=1}^{n} [\ldots]^{1/2} - \int\int_Q \ldots \right| < \varepsilon$$

=>

$$\left| \int\int_Q - \sum_{k=1}^{n} [\ldots]^{1/2} \right| < \varepsilon$$

=>

$$\int_Q\int - \sum_{k=1}^{n} [\ldots]^{1/2} < \varepsilon$$

=>

$$\sum_{k=1}^{n} [\ldots]^{1/2} - \int_Q\int \ldots > - \varepsilon$$

=>

$$\sum_{k=1}^{n} [\ldots]^{1/2} > \int_Q\int \cdots - \varepsilon.$$

And

$$\Gamma_Q[f] \geq \sum_{k=1}^{n} [\ldots]^{1/2} > \int_Q\int \cdots - \varepsilon.$$

But

$$\Gamma_Q[f] = L_Q[f].$$

§7. *THE ROLE OF ABSOLUTE CONTINUITY*

Let $f:Q \to R$ be a continuous function.

1: DEFINITION (ACT) f is said to be absolutely continuous in the sense of Tonelli if it is of bounded variation in the sense of Tonelli and if

- For almost every $y \in [0,1]$, the function $x \to f(x,y)$ is absolutely continuous
- For almost every $x \in [0,1]$, the function $y \to f(x,y)$ is absolutely continuous.

2: REMARK Since f is BVT, the ordinary partial derivatives

$$\frac{\partial f}{\partial x} \ \& \ \frac{\partial f}{\partial y}$$

belong to $L^1(Q)$. So, thanks to ACL,

$$f \in W^{1,1}(Q^\circ).$$

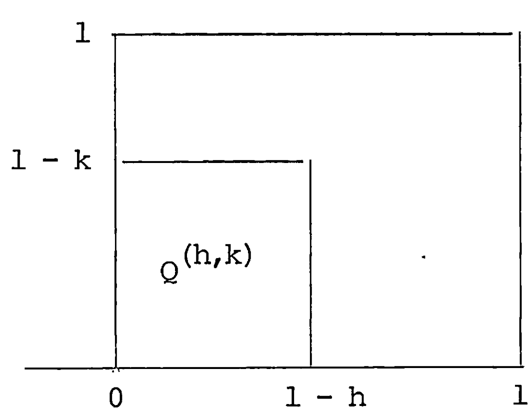
3: NOTATION Put

$$Q^{(h,k)} = [0, 1-h] \times [0, 1-k],$$

where

$$\begin{cases} 0 < h < 1 \\ 0 < k < 1. \end{cases}$$

4: PICTURE



5: NOTATION Given an ACT function f, put

$$f^{(h,k)}(x,y) = \frac{1}{hk}\int_x^{x+h}\int_y^{y+k} f(\xi,\eta)\, d\xi d\eta.$$

6: LEMMA

$$\int_0^{1-h}\int_0^{1-k} |f^{(h,k)}(x,y)|\, dxdy \le \int_0^1\int_0^1 |f(x,y)|\, dxdy.$$

7: LEMMA

$$\left[\begin{aligned} \frac{\partial f^{(h,k)}}{\partial x} &= \frac{1}{hk}\int_x^{x+h}\int_y^{y+k} \frac{\partial f}{\partial \xi}\, d\xi d\eta \\ \frac{\partial f^{(h,k)}}{\partial y} &= \frac{1}{hk}\int_x^{x+h}\int_y^{y+k} \frac{\partial f}{\partial \eta}\, d\xi d\eta. \end{aligned}\right.$$

[Note: It follows from these relations that $f^{(h,k)}$ is a C' function.]

Therefore

$$\int_0^{1-h}\int_0^{1-k} \sqrt{1 + [f_x^{(h,k)}]^2 + [f_y^{(h,k)}]^2}\, dxdy$$

$$= \int_0^{1-h}\int_0^{1-k}\{\sqrt{[\frac{1}{hk}\int_0^h\int_0^k d\xi d\eta]^2}$$

$$\overline{+ [\frac{1}{hk}\int_0^h\int_0^k f_\xi(x+\xi,\ y+\eta)\, d\xi d\eta]^2}$$

$$\overline{+ [\frac{1}{hk}\int_0^h\int_0^k f_\eta(x+\xi,\ y+\eta)\, d\xi d\eta]^2}\}\, dxdy$$

$$\leq \int_0^{1-h} \int_0^{1-k} \Big[\frac{1}{hk} \int_0^h \int_0^k \{\sqrt{1 + [f_\xi(x + \xi, y + \eta)]^2 + [f_\eta(x + \xi, y + \eta)]^2}\} d\xi d\eta\Big] dxdy$$

$$= \frac{1}{hk} \int_0^h \int_0^k \Big[\int_\xi^{1-h+\xi} \int_\eta^{1-k+\eta} \sqrt{1 + f_x^2 + f_y^2} \, dxdy\Big] d\xi d\eta$$

$$\leq \frac{1}{hk} \int_0^h \int_0^k \Big[\int_0^1 \int_0^1 \sqrt{1 + f_x^2 + f_y^2} \, dxdy\Big] d\xi d\eta$$

$$= \frac{1}{hk} \frac{hk}{1} \int_0^1 \int_0^1 \sqrt{1 + f_x^2 + f_y^2} \, dxdy$$

$$= \int\int_Q \sqrt{1 + f_x^2 + f_y^2} \, dxdy \leq L_Q[f].$$

8: RAPPEL During the course of establishing that

$$\Gamma_Q[f] = L_Q[f],$$

it was shown that if f was C', then

$$L_Q[f] = \int\int_Q \Big[1 + \Big(\frac{\partial f}{\partial x}\Big)^2 + \Big(\frac{\partial f}{\partial y}\Big)^2\Big]^{1/2} dxdy.$$

So, upon applying this to $f^{(h,k)}$, the upshot is that

$$L_{Q^{(h,k)}}[f^{(h,k)}]$$

$$= \int_0^{1-h} \int_0^{1-k} \sqrt{1 + [f_x^{(h,k)}]^2 + [f_y^{(h,k)}]^2} \, dxdy.$$

9: SCHOLIUM If f is absolutely continuous in the sense of Tonelli, then

$$L_Q[f] = \int\int_Q [1 + f_x^2 + f_y^2]^{1/2} dxdy.$$

[In fact,

$$L_Q[f] \leq \liminf_{\substack{h \to 0 \\ k \to 0}} L_{Q^{(h,k)}} [f^{(h,k)}]$$

$$\leq \limsup_{\substack{h \to 0 \\ k \to 0}} L_{Q^{(h,k)}} [f^{(h,k)}]$$

$$\leq \int\int_Q [1 + f_x^2 + f_y^2]^{1/2} \, dxdy$$

$$\leq L_Q[f].]$$

<u>10:</u> EXAMPLE Suppose that $f:R^2 \to R$ is a C' function. Put

$$Gr_f(Q) = \{(x,y), f(x,y):(x,y) \in Q\} .$$

Then

$$\mathcal{H}^2(Gr_f(Q)) = \int\int_Q [1 + f_x^2 + f_y^2]^{1/2} \, dxdy.$$

Consequently

$$\mathcal{H}^2(Gr_f(Q)) = L_Q[f].$$

Matters can be reversed, namely:

<u>11:</u> SCHOLIUM If f is of bounded variation in the sense of Tonelli and if

$$L_Q[f] = \int\int_Q [1 + f_x^2 + f_y^2]^{1/2} \, dxdy,$$

then f is absolutely continuous in the sense of Tonelli.

We shall sketch the proof.

12: LEMMA For every oriented rectangle $R \subset Q$,

$$L_R[f] = \int\int_R [1 + f_x^2 + f_y^2]^{1/2}\, dxdy.$$

Explicate $R \subset Q$:

$$\begin{cases} a \leq x \leq b \quad (a < b) \\ c \leq y \leq d \quad (c < d) \end{cases}, \quad |R| = (b - a)\ (c - d)$$

and introduce

$$\begin{cases} W_x(f;R) = \int_c^d V_x(f;y)\, dy \\ W_y(f;R) = \int_a^b V_y(f;x)\, dx. \end{cases}$$

13: LEMMA For every oriented rectangle $R \subset Q$,

$$\begin{cases} W_x(f;R) \leq L_R[f] \\ W_y(f;R) \leq L_R[f]. \end{cases}$$

Therefore

$$\begin{cases} W_x(f;R) \leq \int\int_R [1 + f_x^2 + f_y^2]^{1/2}\, dxdy \\ W_y(f;R) \leq \int\int_R [1 + f_x^2 + f_y^2]^{1/2}\, dxdy. \end{cases}$$

Denoting by $\mathcal{R}$ the set of oriented rectangles in Q, a rectangle function is a function $\phi:\mathcal{R} \rightarrow \mathbf{R}$. So, e.g., the assignments

$$\begin{cases} R \rightarrow W_x(f;R) \\ R \rightarrow W_y(f;R) \end{cases} \quad (R \in \mathcal{R})$$

are rectangle functions.

<u>14:</u> DEFINITION A rectangle function $R \to \phi(R)$ is said to be <u>absolutely continuous</u> if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\phi(R_1)| + \cdots + |\phi(R_n)| < \varepsilon$$

for every finite system of oriented rectangles $R_1, \ldots, R_n$ which satisfy the conditions

$$R_i^o \cap R_j^o = \emptyset \ (i \neq j) \text{ and } |R_1| + \cdots + |R_n| < \delta.$$

<u>15:</u> CRITERION If $\Phi \in L^1(Q)$ and if

$$\phi(R) = \int\int_R |\Phi| \, dxdy \quad (R \in \mathcal{R}),$$

then ϕ is absolutely continuous.

<u>16:</u> APPLICATION The rectangle functions

$$\left[\begin{array}{l} R \to W_x(f;R) \\ R \to W_y(f;R) \end{array}\right. \quad (R \in \mathcal{R})$$

are absolutely continuous.

[Note: Bear in mind that

$$[1 + f_x^2 + f_y^2]^{1/2} \in L^1(Q).]$$

Recall that the contention is that f is absolutely continuous in the sense of Tonelli, i.e.,

$$\left[\begin{array}{l} \text{For almost every } y \in [0,1], \text{ the function } x \to f(x,y) \text{ is absolutely continuous} \\ \text{For almost every } x \in [0,1], \text{ the function } y \to f(x,y) \text{ is absolutely continuous.} \end{array}\right.$$

Consider the first of these assertions. Using the absolute continuity of $W_x(f;R)$ to eliminate a potential singular term, we have

$$W_x(f;Q) = \int\int_Q |f_x(x,y)|\, dxdy.$$

On the other hand, by definition,

$$W_x(f;Q) = \int_0^1 V_x(f;y)\, dy.$$

Therefore

$$\int_0^1 [V_x(f;y) - \int_0^1 |f_x(x,y)|\, dx]\, dy = 0.$$

But

$$V_x(f;y) \geq \int_0^1 |f_x(x,y)|\, dx$$

for almost every y in [0,1]. Therefore

$$V_x(f;y) = \int_0^1 |f_x(x,y)|\, dx$$

for those $y \notin E$, where E is a certain subset of [0,1] of Lebesgue measure 0. And this implies that f(x,y) is absolutely continuous as a function of x for $y \notin E$.

<u>17:</u> <u>N.B.</u> In general, if f is of bounded variation in the sense of Tonelli, then

$$W_x(f;Q) = \int_0^1 V_x(f;y)\, dy$$

$$\geq \int_0^1 [\int_0^1 |f_x(x,y)|\, dx]\, dy$$

$$= \int\int_Q |f_x(x,y)|\, dxdy,$$

the inequality becoming an equality in the presence of the absolute continuity of $R \to W_x(f;R)$.

§8. *STEINER'S INEQUALITY*

Suppose that

$$\begin{cases} f_1 : Q \to R \\ f_2 : Q \to R \end{cases}$$

are continuous functions.

1: THEOREM

$$L_Q[(f_1 + f_2)/2] \leq \frac{L_Q[f_1] + L_Q[f_2]}{2} .$$

PROOF The assertion is trivial if

$$L_Q[f_1] = + \infty \text{ or } L_Q[f_2] = + \infty,$$

so it can be assumed that both are finite. Accordingly, given a subdivision D of Q, form the sums of Geöcze per f_1, f_2, and $(f_1 + f_2)/2$, hence

$$G((f_1 + f_2)/2;D) \leq \frac{G(f_1;D) + G(f_2;D)}{2}$$

=>

$$G((f_1 + f_2)/2;D) \leq \frac{L_Q[f_1] + L_Q[f_2]}{2}$$

=>

$$L_Q[(f_1 + f_2)/2] \leq \frac{L_Q[f_1] + L_Q[f_2]}{2} .$$

2: RAPPEL If $f:Q \to R$ is continuous, then $L_Q[f]$ is finite iff f is of bounded variation in the sense of Tonelli, there being the estimate

$$L_Q[f] \geq \int\int_Q [1 + f_x^2 + f_y^2]^{1/2} \, dxdy,$$

the inequality becoming an equality iff f is absolutely continuous in the sense of Tonelli.

Suppose that

$$\begin{cases} f_1 : Q \to R \\ f_2 : Q \to R \end{cases}$$

are absolutely continuous in the sense of Tonelli -- then the same is true of $(f_1 + f_2)/2$ and Steiner's inequality is the relation

$$\int\int_Q \{\frac{[1 + f_{1x}^2 + f_{1y}^2]^{1/2} + [1 + f_{2x}^2 + f_{2y}^2]^{1/2}}{2}$$

$$- [1 + (\frac{f_{1x} + f_{2x}}{2})^2 + (\frac{f_{1y} + f_{2y}}{2})^2]^{1/2}\} \, dxdy \geq 0$$

or still, that

$$\int\int_Q \{[(\frac{1}{2})^2 + (\frac{f_{1x}}{2})^2 + (\frac{f_{1y}}{2})^2]^{1/2} + [(\frac{1}{2})^2 + (\frac{f_{2x}}{2})^2 + (\frac{f_{2y}}{2})^2]^{1/2}$$

$$- [(\frac{1}{2} + \frac{1}{2}) + (\frac{f_{1x}}{2} + \frac{f_{2x}}{2})^2 + (\frac{f_{1y}}{2} + \frac{f_{2y}}{2})^2]^{1/2}\} \, dxdy \geq 0.$$

<u>3:</u> LEMMA

$$\sum_{i=1}^{k} (a_i^2 + b_i^2 + c_i^2)^{1/2} \geq [(\sum_{i=1}^{k} a_i)^2 + (\sum_{i=1}^{k} b_i)^2 + (\sum_{i=1}^{k} c_i)^2]^{1/2}.$$

To conclude that the foregoing integrand is nonnegative, take $k = 2$ and

$$\begin{cases} a_1 = \frac{1}{2},\ b_1 = \frac{f_{1x}}{2},\ c_1 = \frac{f_{1y}}{2} \\ a_2 = \frac{1}{2},\ b_2 = \frac{f_{2x}}{2},\ c_2 = \frac{f_{2y}}{2}. \end{cases}$$

Suppose that f_1 and f_2 are absolutely continuous in the sense of Tonelli and that equality obtains in Steiner — then the claim is that $f_1 - f_2$ is a constant. To establish this, observe first that

$$\int\int_Q \{\ldots\}\, dxdy = 0$$

and since the integrand is nonnegative, it must be equal to zero almost everywhere in Q. This implies that

$$f_{1x} = f_{2x},\ f_{1y} = f_{2y}$$

almost everywhere in Q or still, that

$$[(f_{1x} - f_{2x})^2 + (f_{1y} - f_{2y})^2]^{1/2} = 0$$

almost everywhere in Q.

<u>4:</u> NOTATION $E \subset Q$ is the set consisting of

(1) All lines $x = x_0$ such that $f_1(x_0,y)$, $f_2(x_0,y)$ are not both absolutely continuous in y.

(2) All lines $y = y_0$ such that $f_1(x,y_0)$, $f_2(x,y_0)$ are not both absolutely continuous in x.

(3) All points (x,y) such that

$$f_{1x}(x,y),\ f_{2x}(x,y),\ f_{1y}(x,y),\ f_{2y}(x,y)$$

are not all defined.

(4) All points (x,y) at which

$$[(f_{1x} - f_{2x})^2 + (f_{1y} - f_{2y})^2]^{1/2} \neq 0.$$

<u>5:</u> <u>N.B.</u> E has planer measure zero, hence for almost all points $(x_0,y_0) \in Q$ the lines $x = x_0$ and $y = y_0$ have in common with E at most a set of linear measure zero.

Fix one such point (x_0,y_0) and let (x,y) be any other point with the same property --- then

$$\begin{cases} f_1(x,y) - f_1(x_0,y_0) = \int_{x_0}^{x} f_{1x}(x,y_0)\, dx + \int_{y_0}^{y} f_{1y}(x,y)\, dy \\ f_2(x,y) - f_2(x_0,y_0) = \int_{x_0}^{x} f_{2x}(x,y_0)\, dx + \int_{y_0}^{y} f_{2y}(x,y)\, dy. \end{cases}$$

Since apart from a set of linear measure zero the integrands on the right are equal, it thus follows that

$$f_1(x,y) - f_2(x,y) = f_1(x_0,y_0) - f_2(x_0,y_0),$$

which is true for almost all (x,y) in Q, hence for all (x,y) in Q (f_1 and f_2 being continuous).

<u>6:</u> EXAMPLE It can happen that equality prevails in Steiner, yet neither f_1 nor f_2 is ACT.

[Let $\varphi(x)$ be a continuous monotonically increasing function such that $\varphi'(x) = 0$ almost everywhere and $\varphi(0) = 0$, $\varphi(1) = 1$. Working in $[0,2] \times [0,2]$, put

$$\begin{cases} f_1(x,y) = 0 & (0 \leq x \leq 1,\ 0 \leq y \leq 2) \\ f_1(x,y) = \varphi(x - 1) & (1 \leq x \leq 2,\ 0 \leq y \leq 2) \end{cases}$$

and

$$\begin{cases} f_2(x,y) = \varphi(x) & (0 \le x \le 1,\ 0 \le y \le 2) \\ f_2(x,y) = 1 & (1 \le x \le 2,\ 0 \le y \le 2). \end{cases}$$

Then

$$L_Q[(f_1 + f_2)/2] = 6$$

$$\begin{cases} L_Q[f_1] = 6 \\ L_Q[f_2] = 6 \end{cases}$$

=>

$$6 = \frac{6 + 6}{2} = \frac{12}{2} = 6.]$$

§9. *EXTENSION PRINCIPLES*

Let $\phi:\mathcal{R} \rightarrow R_{\geq 0}$ be a nonnegative rectangle function.

<u>1:</u> PROBLEM Determine conditions on ϕ which imply that ϕ can be extended to a measure on $\mathcal{B}(Q)$ (the σ-algebra of Borel subsets of Q).

<u>2:</u> DEFINITION ϕ satisfies <u>condition C</u> if for every choice of the systems

$$\left[\begin{array}{l} r_1,\ldots,r_k \\ \\ R_1,\ldots,R_n,\ldots \end{array}\right.$$

of oriented rectangles such that

$$r_i \cap r_j = \emptyset \quad (i \neq j)$$

and

$$r_1 \cup \ldots \cup r_k \subset R_1 \cup \ldots \cup R_n \cup \ldots \qquad \text{(finite or infinite)}$$

there follows

$$\phi(r_1) + \cdots + \phi(r_k) \leq \phi(R_1) + \cdots + \phi(R_n) + \cdots .$$

<u>3:</u> DEFINITION ϕ is <u>continuous</u> if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\phi(R) < \varepsilon$ for every oriented rectangle R such that $|R| < \delta$.

<u>4:</u> CRITERION If ϕ is finitely additive and continuous, then ϕ satisfies condition C.

<u>5:</u> <u>N.B.</u> Suppose that Φ is a Borel measure -- then the restriction $\phi \equiv \Phi|\mathcal{R}$ satisfies condition C.

[Put $B_1 = R_1$, $B_2 = R_2 \backslash R_1$,

$$\ldots B_n = R_n \backslash (R_1 \cup \cdots \cup R_{n-1}), \ldots .$$

Then

$$\phi(r_1) + \cdots + \phi(r_k)$$

$$= \Phi(r_1) + \cdots + \Phi(r_k)$$

$$= \Phi(r_1 \cup \cdots \cup r_k)$$

$$\leq \Phi(R_1 \cup \cdots \cup R_n \cup \cdots)$$

$$= \Phi(B_1 \cup \cdots \cup B_n \cup \cdots)$$

$$= \Phi(B_1) + \cdots + \Phi(B_n) + \cdots$$

$$\leq \Phi(R_1) + \cdots + \Phi(R_n) + \cdots$$

$$= \phi(R_1) + \cdots + \phi(R_n) + \cdots .$$

6: NOTATION Given a set $E \subset Q$, let

$$\Gamma^*(\phi;E) = \inf \Sigma\, \phi(R_n),$$

where the inf is taken over all rectangles $R_1, \ldots, R_n, \ldots$ (finite or infinite) of oriented rectangles in Q such that $E \subset \cup R_n$ (take $\Gamma(\emptyset,\phi) = 0$).

7: LEMMA Suppose that ϕ satisfies condition C -- then $\Gamma^*(\phi;—)$ is a metric outer measure.

8: NOTATION Put

$$\Gamma(\phi;—) = \Gamma^*(\phi;—) \,|\, \mathcal{B}(Q),$$

a measure on $\mathcal{B}(Q)$.

9: THEOREM ϕ extends to a measure on $\mathcal{B}(Q)$ iff ϕ satisfies condition C.

PROOF The necessity follows from #5 and the sufficiency follows from #7 (obviously, $\forall\ R \in \mathcal{R}$, $\Gamma(\phi;R) = \phi(R)$).

10: LEMMA If Φ and Ψ are Borel measures and if $\Phi(R) = \Psi(R)$ $(\forall\ R \in \mathcal{R})$, then $\Phi(E) = \Psi(E)$ $(\forall\ E \in \mathcal{B}(Q))$.

Suppose that $f:Q \to R$ is of bounded variation in the sense of Tonelli and recall that

$$\begin{cases} W_x(f;R) = \int_c^d V_x(f;y)\ dy \\ W_y(f;R) = \int_a^b V_y(f;x)\ dx. \end{cases}$$

It is clear that

$$\begin{cases} W_x(f;\text{—}) \\ W_y(f;\text{—}) \end{cases}$$

are finitely additive and it can be shown that they are continuous. Therefore

$$\begin{cases} W_x(f;\text{—}) \\ W_y(f;\text{—}) \end{cases}$$

satisfy condition C (cf. #4), thus they each admit a unique extension to a measure on $\mathcal{B}(Q)$, denoted

$$E \to \begin{cases} W_x(f;E) \\ \\ W_y(f;E) \end{cases} \qquad (E \in \mathcal{B}(Q)).$$

Accordingly there are Lebesgue decompositions

$$\begin{cases} W_x(f;E) = \int\int_E |f_x| \, dL^2 + W_x^0(f;E) \\ W_y(f;E) = \int\int_E |f_y| \, dL^2 + W_y^0(f;E), \end{cases}$$

where

$$\begin{cases} W_x^0(f;—) \\ W_y^0(f;—) \end{cases}$$

are singular.

§10. *ONE VARIABLE REVIEW*

In the Fréchet process, take for X the quasi linear functions Γ on [0,1], take for d the metric defined by the prescription

$$d(\Gamma_1,\Gamma_2) = \int_0^1 |\Gamma_1(x) - \Gamma_2(x)|dx,$$

and take for F the elementary length -- then lower semicontinuity is manifest, as is property (A). Here $\bar{X} = L^1[0,1]$ and property (B) is satisfied.

1: DEFINITION Put

$$\Psi[f] = \bar{F}(f) \quad (f \in L^1[0,1])$$

and call it the generalized variation of f.

2: DEFINITION (gBV) A function $f \in L^1[0,1]$ is of generalized bounded variation if

$$\Psi[f] < +\infty.$$

3: NOTATION gBV[0,1] is the set of functions of generalized bounded variation.

4: THEOREM Let $f \in L^1[0,1]$ -- then f is of generalized bounded variation iff there is a $g \in L^1[0,1]$ which is equal almost everywhere to f and $T_g[0,1] < +\infty$.

Therefore

$$BV[0,1] \subset gBV[0,1].$$

5: THEOREM Suppose that $f \in gBV[0,1]$ -- then

$$\Psi[f] = \inf\{T_g[0,1] : g = f \text{ almost everywhere}\}.$$

6: RAPPEL Given an $f \in L^1[0,1]$, $C_{ap}(f)$ is its set of points of approximate continuity.

7: N.B. $C_{ap}(f)$ is a subset of $[0,1]$ of full measure.

8: LEMMA If $f \in L^1[0,1]$, then

$$Ч[f] = \sup \sum_{i=1}^{n-1} |f(x_{i+1}) - f(x_i)|,$$

where the supremum is taken over all finite collections of points $x_i \in C_{ap}(f)$ subject to $x_i < x_{i+1}$.

[Note: If $E \subset C_{ap}(f)$ is a subset of full measure, then the supremum can be taken over the $x_i \in E$.]

9: RAPPEL If $f_n \to f$ in $L^1[0,1]$, then there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ almost everywhere.

10: LEMMA Ч is lower semicontinuous w.r.t. convergence almost everywhere, i.e., if $f_1, f_2, \ldots$ is a sequence in $L^1[0,1]$ that converges almost everywhere to $f \in L^1[0,1]$, then

$$Ч[f] \leq \liminf_{n \to \infty} Ч[f_n].$$

11: DEFINITION The essential derivative of f at a point x is the derivative of f computed at x after deleting a set of Lebesgue measure 0.

12: THEOREM Suppose that $Ч[f]$ is finite — then the essential derivative of f, denoted still by f', exists almost everywhere and

$$Ч[f] \geq \int_0^1 |f'(x)|\, dx.$$

Moreover equality obtains iff f is equivalent to an absolutely continuous function.

§11. *EXTENDED LEBESGUE AREA*

In the Fréchet process, take for X the quasi linear functions Π on $[0,1] \times [0,1]$ $(= Q)$, take for d the metric defined by the prescription

$$d(\Pi_1,\Pi_2) = \int\int_Q |\Pi_1(x,y) - \Pi_2(x,y)| \, dxdy,$$

and take for F the elementary area -- then lower semicontinuity is manifest, as is property (A). Here $\bar{X} = L^1(Q)$ and property (B) is satisfied.

1: DEFINITION Put

$$\Psi_Q[f] = \bar{F}(f) \quad (f \in L^1(Q))$$

and call it the generalized variation of f.

2: EXTENSION PRINCIPLE Suppose that $f:Q \to R$ is continuous -- then

$$\Psi_Q[f] = L_Q[f].$$

3: N.B. Therefore Ψ_Q can be viewed as an "area functional" on $L^1(Q)$, there being no a priori assumption of continuity, which justifies calling Ψ_Q extended Lebesgue area.

4: LEMMA Suppose that $f:Q \to R$ is continuous.

- If $L_Q[f] < +\infty$, then for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $g:Q \to R$ is continuous and

$$|f(x,y) - g(x,y)| < \delta$$

on a set of measure greater than $1 - \delta$, then

$$L_Q[g] > L_Q[f] - \varepsilon.$$

- If $L_Q[f] = +\infty$, then for every $M > 0$ there is a $\delta > 0$ such that if

$g:Q \to R$ is continuous and

$$|f(x,y) - g(x,y)| < \delta$$

on a set of measure greater than $1 - \delta$, then

$$L_Q[g] > M.$$

There are two possibilities:

$$L_Q[f] < +\infty \text{ or } L_Q[f] = +\infty.$$

For sake of argument, consider the first of these.

Since uniform convergence of $\{\Pi_n(x,y)\}$ to $f(x,y)$ implies that $d(\Pi_n,f)$ converges to zero, it follows that $Ч_Q[f] \leq L_Q[f]$. To go the other way, take $\varepsilon > 0$, let $\delta > 0$ be per supra, and choose a quasi linear function Π such that

$$\int_Q\int |f - \Pi| \, dL^2 < \delta^2.$$

Then

$$|f(x,y) - \Pi(x,y)| < \delta$$

on a set of measure greater than $1 - \delta$, hence

$$L_Q[\Pi] > L_Q[f] - \varepsilon$$

$$\Rightarrow$$

$$Ч_Q[f] \geq L_Q[f] - \varepsilon.$$

There is also a Geöcze version of these considerations.

5: DEFINITION Let $f \in L^1(Q)$ and let $R \subset Q$ be an oriented rectangle, thus in the usual notation,

$$\left[\begin{array}{ll} a \le x \le b & (a < b) \\ & \\ c \le y \le d & (c < d) \end{array}\right. , \quad |R| = (b - a)(d - c).$$

Then R is said to be <u>admissible</u> if $f(x,y)$ is approximately continuous in x for almost all y on the boundary lines of R parallel to the y axis and if $f(x,y)$ is approximately continuous in y for almost all x on the boundary lines of R parallel to the x axis.

[Note: A subdivision D of Q into nonoverlapping oriented rectangles R is admissible provided this is of the case of each of the R.]

Using this data, one can arrive at the extended Geöcze area, denoted

$$\text{И}_Q[f].$$

<u>6:</u> THEOREM

$$\text{И}_Q(f) = \text{Ч}_Q[f].$$

<u>7:</u> <u>N.B.</u> Recall that

$$\Gamma_Q[f] = L_Q[f] \qquad (f \in C(Q)),$$

i.e.,

Geöcze area = Lebesgue area.

§12. *THEORETICAL SUMMARY*

What is said below for the integrable case runs parallel to what has been said for the continuous case.

1: DEFINITION (gBVT) Let $f \in L^1(Q)$ -- then f is said to be of <u>generalized bounded variation in the sense of Tonelli</u> if

$$\begin{cases} \int_0^1 \Psi[f(\longleftarrow,y)]\ dy < +\infty \\ \int_0^1 \Psi[f(x,\longrightarrow)]\ dx < +\infty. \end{cases}$$

The gBVT-functions can be characterized.

2: THEOREM Let $f \in L^1(Q)$ -- then f is of generalized bounded variation in the sense of Tonelli iff there are functions g and h equal to f almost everywhere in Q such that

$$\begin{cases} \int_0^1 V_x(g;y)\ dy < +\infty \\ \int_0^1 V_y(g;x)\ dx < +\infty. \end{cases}$$

3: REMARK Suppose that f is gBVT -- then it can be shown that there is a function k equal to f almost everywhere in Q such that

$$\begin{cases} \int_0^1 V_x(k;y)\ dy < +\infty \\ \int_0^1 V_y(k;x)\ dx < +\infty. \end{cases}$$

4: <u>N.B.</u>

$$f \text{ BVT} \Rightarrow f \text{ gBVT}.$$

[Note: Recall that f BVT means, in particular, that $f \in C(Q)$, hence $f \in L^1(Q)$.]

5: THEOREM $\Psi_Q[f] < +\infty$ iff f is gBVT.

6: THEOREM Suppose that f is gBVT -- then the essential partial derivatives f_x and f_y exist almost everywhere, are integrable, and

$$\Psi_Q[f] \geq \int\int_Q [1 + f_x^2 + f_y^2]^{1/2} \, dxdy.$$

7: DEFINITION (gACT) Suppose that f is gBVT -- then f is said to be generalized absolutely continuous in the sense of Tonelli if f coincides almost everywhere with a function g which is absolutely continuous w.r.t. x for almost all y and absolutely continuous w.r.t. y for almost all x.

8: SCHOLIUM

- If f is gBVT and if

$$\Psi_Q[f] = \int\int_Q [1 + f_x^2 + f_y^2]^{1/2} \, dxdy,$$

then f is gACT.

- If f is gACT, then

$$\Psi_Q[f] = \int\int_Q [1 + f_x^2 + f_y^2]^{1/2} \, dxdy.$$

§13. *VARIANTS*

Up to this point, the discussion has taken

$$Q = [0,1] \times [0,1]$$

as the domain of discourse. Of course, matters can be extended with little change when Q is replaced by

$$[a,b] \times [c,d].$$

This done, the next step is to replace Q by a nonempty open subset $\Omega \subset R^2$.

<u>1:</u> RAPPEL A continuous function $f:]a,b[\rightarrow R$ is of bounded variation in a nonempty open interval $]a,b[\subset R$ provided

$$T_f]a,b[< +\infty.$$

<u>2:</u> DEFINITION A continuous function $f:\Omega \rightarrow R$ is of bounded variation in a nonempty open subset $\Omega \subset R$ provided

$$T_f \Omega < +\infty,$$

where

$$T_f \Omega = \sum_n T_f]a_n, b_n[,$$

the nonempty open intervals $]a_n, b_n[$ running through the connected components of Ω (admit $\pm\infty$).

<u>3:</u> NOTATION Let Ω be a nonempty open subset of R^2.

- For any real number $\bar{x}$, let $\Omega(\bar{x})$ denote the open linear set which is the intersection of Ω with the straight line $x = \bar{x}$.

- For any real number $\bar{y}$, let $\Omega(\bar{y})$ denote the open linear set which is the intersection of Ω with the straight line $y = \bar{y}$.

Given a continuous function $f:\Omega \to R$, introduce

$$\begin{cases} V_x(f;\bar{y};\Omega) = T_f\Omega(\bar{y}) \\ V_y(f;\bar{x};\Omega) = T_f\Omega(\bar{x}). \end{cases}$$

[Note: Take

$$\begin{cases} V_x = 0 \text{ if } \Omega(\bar{y}) = \emptyset \\ V_y = 0 \text{ if } \Omega(\bar{x}) = \emptyset. \end{cases}]$$

4: LEMMA

$$\begin{cases} V_x(f;\bar{y};\Omega) \text{ is a lower semicontinuous function of } \bar{y} \\ V_y(f;\bar{x};\Omega) \text{ is a lower semicontinuous function of } \bar{x} \end{cases} \text{ in }]-\infty,+\infty[.$$

5: DEFINITION (BVT) f is said to be of bounded variation in the sense of Tonelli if

$$\begin{cases} \int_{-\infty}^{+\infty} V_x(f;\bar{y};\Omega)\,d\bar{y} < +\infty \\ \int_{-\infty}^{+\infty} V_y(f;\bar{x};\Omega)\,d\bar{x} < +\infty. \end{cases}$$

6: LEMMA Suppose that $f:\Omega \to R$ is of bounded variation in the sense of Tonelli — then

$$\begin{cases} f_x = \dfrac{\partial f}{\partial x} \\ f_y = \dfrac{\partial f}{\partial y} \end{cases} \text{ exists almost everywhere in } \Omega$$

and

$$\left[\begin{array}{l} \int\int_{\Omega} |f_x(x,y)|\ dxdy \le \int_{-\infty}^{+\infty} V_x(f;\bar{y};\Omega)\ d\bar{y} < +\infty \\ \\ \int\int_{\Omega} |f_y(x,y)|\ dxdy \le \int_{-\infty}^{+\infty} V_y(f;\bar{x};\Omega)\ d\bar{x} < +\infty \end{array}\right.$$

$$\Rightarrow$$

$$\left[\begin{array}{l} f_x \\ \\ f_y \end{array}\right. \in L^1(\Omega).$$

Another setting for the theory is a nonempty open subset $\Omega \subset R^2$, $L^1(Q)$ then being replaced by $L^1(\Omega)$, the analog of a gBVT function now being an element of $BVL^1\Omega$.

<u>7:</u> DEFINITION Let $f \in L^1(\Omega)$ -- then f is a <u>function of bounded variation</u> in Ω if the distributional partial derivatives of f are finite signed Radon measures

$$\left[\begin{array}{ll} \mu_x & \int_{\Omega} f \frac{\partial \phi}{\partial x}\, dx = - \int_{\Omega} \phi\, d\mu_x \\ \quad : & \qquad\qquad\qquad\qquad \forall\ \phi \in C_c^{\infty}(\Omega) \\ \mu_y & \int_{\Omega} f \frac{\partial \phi}{\partial y}\, dy = - \int_{\Omega} \phi\, d\mu_y \end{array}\right.$$

of finite total variation.

<u>8:</u> NOTATION $BVL^1\Omega$ is the set of functions of bounded variation in Ω.

Given $g \in L^1(\Omega)$, put

$$V_T(g;\Omega) = \int_{-\infty}^{+\infty} V_x(g;\bar{y};\Omega)\ d\bar{y} + \int_{-\infty}^{+\infty} V_y(g;\bar{x};\Omega)\ d\bar{x}.$$

<u>9:</u> THEOREM Let $f \in L^1(\Omega)$ -- then $f \in BVL^1\Omega$ iff

$$\inf\{V_T(g;\Omega) : g = f \text{ almost everywhere}\} < +\infty.$$