Time fractional equations and probabilistic representation

Zhen-Qing Chen

March 5, 2017

Abstract

In this paper, we study the existence and uniqueness of solutions for general fractional-time parabolic equations of mixture type, and their probabilistic representations in terms of the corresponding inverse subordinators with or without drifts. An explicit relation between occupation measure for Markov processes time-changed by inverse subordinator in open sets and that of the original Markov process in the open set is also given.

AMS 2010 Mathematics Subject Classification: Primary 26A33, 60H30; Secondary: 34K37

Keywords and phrases: fractional-time derivative, subordinator, inverse subordinator, Lévy measure, occupation measure

1 Introduction

Fractional calculus has attracted lots of attentions in several fields including mathematics, physics, chemistry, engineering, hydrology and even finance and social sciences (see [9, 19, 21, 20]). The classical heat equation $\partial_t u = \Delta u$ describes heat propagation in homogeneous medium. The time-fractional diffusion equation $\partial_t^\beta u = \Delta u$ with $0 < \beta < 1$ has been widely used to model the anomalous diffusions exhibiting subdiffusive behavior, due to particle sticking and trapping phenomena (see e.g. [19, 22]). Here the fractional-time derivative ∂_t^β is the Caputo derivative of order $\beta \in (0, 1)$, which can be defined by

$$\partial_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-s)^{-\beta} \left(f(s) - f(0) \right) ds, \tag{1.1}$$

where $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1} e^{-t} dt$ is the Gamma function. The above definition says that the fractional derivative of f at time t depends on the whole history of $f(s)$ on $(0, t)$ with the nearest past affecting the present more. An interesting probabilistic representation is derived by Baeumer and Meerschaert [2] for the solution $u = u(t, x)$ of $\partial_t^\beta u = \Delta u$ with $u(0, x) = f(x)$:

$$u(t, x) = \mathbb{E}_x[f(X_{E_t})], \quad x \in \mathbb{R}^d,$$

where X is Brownian motion on $\mathbb{R}^d$ with infinitesimal generator Δ and E_t is an inverse β-subordinator that is independent of X. In fact, the above representation holds for any operator $\mathcal{L}$ in place of Δ that generates a strong Markov process X. This representation connects probability theory to time fractional equations. The scaling property of the β-stable subordinator is used in a crucial way in their derivation.

In applications and numerical approximations [8], there is a need to consider generalized fractional-time derivatives where its value at time t may depend only on the finite range of the past from $t-\delta$ to t, for example, $\frac{d}{dt}\int_{(t-\delta)^+}^t (t-s)^{-\beta}\,(f(s)-f(0))\,ds$. Here for $a\in\mathbb{R}$, $a^+ := \max\{a,0\}$. Motivated by this, for a given function $w:(0,\infty)\to[0,\infty)$ that is locally integrable on $[0,\infty)$, we introduce a generalized fractional-time derivative

$$\partial_t^w f(t) = \frac{d}{dt}\int_0^t w(t-s)\,(f(s)-f(0))\,ds, \tag{1.2}$$

whenever it is well defined. Typically $w(t)$ is a non-negative decreasing function on $(0,\infty)$ that blows up at $t=0$. Clearly, when $w(s)=\frac{1}{\Gamma(1-\beta)}s^{-\beta}$ for $\beta\in(0,1)$, $\partial_t^w f$ is just the Caputo derivative of order β defined by (1.1).

Let $X=\{X_t, t\geq 0; \mathbb{P}_x, x\in E\}$ be a strong Markov process on a separable locally compact Hausdorff space E whose transition semigroup $\{\mathbb{P}_t, t\geq 0\}$ is a uniformly bounded strong continuous semigroup in some Banach space $(\mathbb{B}, \|\cdot\|)$. For example, $\mathbb{B}=L^p(E;m)$ for some measure m on E and $p\geq 1$ or $\mathbb{B}=C_\infty(E)$, the space of continuous functions on E that vanish at infinity equipped with uniform norm. Let $(\mathcal{L},\mathcal{D}(\mathcal{L}))$ be the infinitesimal generator of $\{\mathbb{P}_t, t\geq 0\}$ in $\mathbb{B}$. We are interested in the existence and uniqueness of solution $u=u(t,x)$ for

$$\kappa\frac{\partial u}{\partial t} + \partial_t^w u = \mathcal{L}u \quad \text{with } u(0,x)=f(x)$$

and its probabilistic representation, where $\kappa\geq 0$ is a positive constant.

Given a constant $\kappa\geq 0$ and an unbounded right continuous non-increasing function $w(x)$ on $(0,\infty)$ with $\lim_{x\to\infty} w(x)=0$ and $\int_0^\infty (1\wedge x)(-dw(x))<\infty$, there is a unique non-negative valued Lévy process $\{S_t; t\geq 0\}$ with $S_0=0$ (called subordinator) associated with it in the following way. Here for $a,b\in\mathbb{R}$, $a\wedge b := \min\{a,b\}$. Let μ be the measure on $(0,\infty)$ so that $w(x)=\mu(x,\infty)$. Clearly

$$\mu(0,\infty)=\infty \quad \text{and} \quad \int_0^\infty (1\wedge x)\mu(dx)<\infty.$$

It is well-known (cf. [3]) that there is subordinator $\{S_t; t\geq 0\}$ with Laplace exponent ϕ:

$$\mathbb{E}\left[e^{-\lambda S_t}\right] = e^{-t\phi(\lambda)}, \quad \lambda>0, \tag{1.3}$$

so that

$$\phi(\lambda) = \kappa\lambda + \int_0^\infty (1-e^{-\lambda x})\mu(dx). \tag{1.4}$$

The measure μ is called the Lévy measure of the subordinator.

Conversely, given a subordinator subordinator $\{S_t; t\geq 0\}$, there is a unique constant $\kappa\geq 0$ and a Lévy measure μ on $(0,\infty)$ satisfying $\int_0^\infty (1\wedge x)\mu(dx)<\infty$ so that (1.3) and (1.4) hold. Throughout this paper, $\{S_t; t\geq 0\}$ is such a general subordinator with infinite Lévy measure μ and possibly with drift $\kappa\geq 0$. When $\kappa=0$, we say the subordinator is driftless or with no drift. Define for $t>0$, $E_t=\inf\{s>0: S_s>t\}$, the inverse subordinator. The assumption that the Lévy measure μ is infinite (which is equivalent to $w(x):=\mu(x,\infty)$ being unbounded) excludes compounded Poisson processes. Under this assumption, almost surely, $t \mapsto S_t$ is strictly increasing and hence $t \mapsto E_t$ is continuous.

The main purpose of this paper is to establish the following.

Theorem 1.1 *Under the above setting, let $w(x) = \mu(x, \infty)$, which is an unbounded right continuous non-increasing function on $(0, \infty)$. The function $u(t, x) := \mathbb{E}_x[f(X_{E_t})]$ is the unique solution in $\mathbb{B}$ to the time fractional equation*

$$(\kappa \partial_t + \partial_t^w)\, u = \mathcal{L} u \quad \textit{with } u(0, x) = f(x) \tag{1.5}$$

for every $f \in \mathcal{D}(\mathcal{L})$. Here ∂_t is the time derivative $\frac{\partial}{\partial t}$.

Our method of proof to the above theorem is different from that of [2] which is for stable subordinators, as there is no scaling property for a general subordinator S_t. Our approach is quite robust and direct that works for any subordinator with infinite Lévy measure and for a wide class of infinitesimal generators. One feature of this paper is that possible mixture of the standard time derivative ∂_t and the general fractional time derivative ∂_t^w is covered and treated in a unified way. In fact, we will establish a more general result for $\mathcal{L}$ being the infinitesimal generator of any uniformly bounded strongly continuous semigroup in general Banach spaces; see Theorem 2.3 for a precise statement. Our Theorem 2.3 not only gives the existence but also the uniqueness of solutions to the time fractional equation. The generalized Caputo derivative defined by (1.2) with $w(x) = \mu(x, \infty)$ extends the distributed order fractional derivative defined in [17] where S_t is a mixture of β-subordinators. An important application of these more general time fractional derivatives is to model "ultraslow diffusion" where a plume spreads at a logarithmic rate; see [17].

In Section 3 of this paper, we will study the relation between occupation measure for the time-changed process $X^* := X_{E_t}$ by inverse subordinator in an open set $D \subset E$ with that of X in D.

2 General time fractional equations

Recall that $\{S_t; t \geq 0\}$ is a general subordinator with infinite Lévy measure μ and drift $\kappa \geq 0$, whose Laplace exponent $\phi(\lambda)$ is given by (1.4). Define $w(x) = \mu(x, \infty)$ for $x > 0$ and $\phi_0(\lambda) := \int_0^\infty \left(1 - e^{-\lambda x}\right) \mu(dx)$. Note that $\phi_0(\lambda)$ is the Laplace exponent of the driftless subordinator $\{\bar{S}_t := S_t - \kappa t, t \geq 0\}$ having Lévy measure μ. Clearly

$$\phi(\lambda) = \kappa \lambda + \phi_0(\lambda) \quad \text{ and } \quad S_t = \kappa t + \bar{S}_t. \tag{2.1}$$

Since $\mu(0, \infty) = \infty$, almost surely, $t \mapsto \bar{S}_t$ is strictly increasing.

For every $a > 0$, by Fubini theorem,

$$\int_0^a w(x) dx = \int_0^a \left(\int_{(x,\infty)} \mu(d\xi) \right) dx = \int_0^\infty \left(\int_0^{\xi \wedge a} dx \right) \mu(d\xi) = \int_a^\infty (\xi \wedge a) \mu(d\xi) < \infty. \tag{2.2}$$

The Laplace transform of w is

$$\begin{aligned} \int_0^\infty e^{-\lambda x} w(x) dx &= \int_0^\infty e^{-\lambda x} \int_{(x,\infty)} \mu(d\xi) dx = \int_0^\infty \left(\int_0^\xi e^{-\lambda x} dx \right) \mu(d\xi) \\ &= \frac{1}{\lambda} \int_0^\infty \left(1 - e^{-\lambda \xi}\right) \mu(d\xi) = \frac{\phi_0(\lambda)}{\lambda}. \end{aligned} \tag{2.3}$$

Lemma 2.1 *There is a Borel set $\mathcal{N} \subset (0,\infty)$ having zero Lebesgue measure so that*

$$\mathbb{P}(\bar{S}_s \geq t) = \int_0^s \mathbb{E}\left[w(t-\bar{S}_r) 1_{\{t \geq \bar{S}_r\}} \right] dr \quad \textit{for every } s>0 \textit{ and } t \in (0,\infty) \setminus \mathcal{N}.$$

Consequently, for every $t \in (0,\infty) \setminus \mathcal{N}$, $s \mapsto \mathbb{P}(\bar{S}_s \geq t)$ is continuous and $\mathbb{P}(\bar{S}_s = t) = 0$ for every $s > 0$.

Proof. Note that since $r \mapsto \bar{S}_r$ is strictly increasing a.s., by Fubini theorem,

$$\int_0^s \mathbb{E}\left[w(t-\bar{S}_r) 1_{\{t \geq \bar{S}_r\}} \right] dr = \int_0^s \mathbb{E}\left[w(t-\bar{S}_r) 1_{\{t > \bar{S}_r\}} \right] dr.$$

For each fixed $s > 0$, the Laplace transform of $t \mapsto \mathbb{P}(\bar{S}_s \geq t)$ is

$$\begin{aligned} \int_0^\infty e^{-\lambda t} \mathbb{P}(\bar{S}_s \geq t) dt &= \int_0^\infty e^{-\lambda t} \mathbb{P}(\bar{S}_s > t) dt \\ &= -\frac{1}{\lambda} \int_0^\infty \mathbb{P}(\bar{S}_s > t) d e^{-\lambda t} = \frac{1}{\lambda} + \frac{1}{\lambda} \int_0^\infty e^{-\lambda t} d_t \mathbb{P}(\bar{S}_s > t) \\ &= \frac{1}{\lambda} - \frac{1}{\lambda} \mathbb{E}\left[e^{-\lambda \bar{S}_s} \right] = \frac{1 - e^{-s\phi_0(\lambda)}}{\lambda}. \end{aligned}$$

By Fubini theorem and (2.3), the Laplace transform of $t \mapsto \int_0^s \mathbb{E}\left[w(t-\bar{S}_r) 1_{\{t \geq \bar{S}_r\}} \right] dr$ is

$$\begin{aligned} \int_0^\infty e^{-\lambda t} \left(\int_0^s \mathbb{E}\left[w(t-\bar{S}_r) 1_{\{t \geq \bar{S}_r\}} \right] dr \right) dt &= \int_0^s \mathbb{E}\left[\int_0^\infty e^{-\lambda t} w(t-\bar{S}_r) 1_{\{t > \bar{S}_r\}} dt \right] dr \\ &= \int_0^s \mathbb{E}\left[e^{-\lambda \bar{S}_r} \int_0^\infty e^{-\lambda x} w(x) dx \right] dr \\ &= \frac{\phi(\lambda)}{\lambda} \int_0^s e^{-r\phi_0(\lambda)} dr = \frac{1 - e^{-s\phi_0(\lambda)}}{\lambda}, \end{aligned}$$

which is the same as the Laplace transform of $t \mapsto \mathbb{P}(\bar{S}_s > t)$. By the uniqueness of the Laplace transform that for each fixed $s > 0$,

$$\mathbb{P}(\bar{S}_s \geq t) = \int_0^s \mathbb{E}\left[w(t-\bar{S}_r) 1_{\{t \geq \bar{S}_r\}} \right] dr \tag{2.4}$$

for a.e. $t > 0$. Hence there is a Borel subset $\mathcal{N} \subset (0,\infty)$ having zero Lebesgue measure so that (2.4) holds for every $t \in (0,\infty) \setminus \mathcal{N}$ and for every rational $s > 0$. Note that for each fixed $t > 0$, $s \mapsto \mathbb{P}(\bar{S}_s \geq t)$ is right-continuous. On the other hand, for each fixed $t > 0$, $s \mapsto \int_0^s \mathbb{E}\left[w(t-\bar{S}_r) 1_{\{t \geq \bar{S}_r\}} \right] dr$ is continuous. It follows that (2.4) holds for every $t \in (0,\infty) \setminus \mathcal{N}$ and every $s > 0$. Consequently, for every $t \in (0,\infty) \setminus \mathcal{N}$, $s \mapsto \mathbb{P}(\bar{S}_s \geq t)$ is continuous. Since the subordinator $t \mapsto \bar{S}_t$ is strictly increasing a.s. and is stochastically continuous in the sense that $\mathbb{P}(\bar{S}_r = \bar{S}_{r-}) = 1$ for all $r > 0$, we have

$$\mathbb{P}(\bar{S}_s \geq t) = \lim_{r \uparrow s} \mathbb{P}(\bar{S}_r \geq t) = \mathbb{P}(\bar{S}_s > t) \quad \text{for every } s > 0.$$

In other words, $\mathbb{P}(\bar{S}_s = t) = 0$ for every $t \in (0,\infty) \setminus \mathcal{N}$ and all $s > 0$. □

Define $G(0)=0$ and $G(x)=\int_0^x w(t)dt$ for $x>0$. Then by (2.2), $G(x)$ is a continuous function on $[0,\infty)$ with $G'(x)=w(x)$ on $(0,\infty)$. By the integration by parts formula, for every $t>0$,

$$\begin{aligned}\int_0^t w(t-r)\mathbb{P}(S_s>r)dr &= -\int_0^t \mathbb{P}(S_s>r)d_rG(t-r)\\ &= G(t)+\int_0^t G(t-r)d_r\mathbb{P}(S_s>r)\\ &= G(t)-\int_0^t G(t-r)d_r\mathbb{P}(S_s\le r)\\ &= G(t)-\mathbb{E}\left[G(t-S_s)1_{\{t\ge S_s\}}\right]. \end{aligned}\tag{2.5}$$

In particular,

$$\mathbb{E}\left[G(t-S_s)1_{\{t\ge S_s\}}\right]\le G(t)\quad \text{for every } t>0.$$

For each fixed $t>0$, by (2.2) and dominated convergence theorem,

$$s\mapsto \int_0^t w(t-r)\mathbb{P}(S_s>r)dr=\int_0^t w(t-r)\mathbb{P}(S_s\ge r)dr$$

is a right continuous increasing function. Hence by (2.5), $s\mapsto \mathbb{E}\left[G(t-S_s)1_{\{t\ge S_s\}}\right]$ is a right continuous decreasing function on $[0,\infty)$.

Corollary 2.2 *Let $\mathcal{N}\subset(0,\infty)$ be the set in Lemma 2.1, which has zero Lebesgue measure.*

(i) $\int_0^\infty \mathbb{E}\left[w(t-\bar S_r)1_{\{t\ge \bar S_r\}}\right]dr=1$ *for every* $t\in(0,\infty)\setminus\mathcal{N}$.

(ii) $\int_0^\infty \mathbb{E}\left[G(t-\bar S_r)1_{\{t\ge \bar S_r\}}\right]dr=t$ *for every* $t>0$.

(iii) $\int_0^\infty \mathbb{E}\left[G(t-\ S_r)1_{\{t\ge S_r\}}\right]dr\le t$ *for every* $t>0$.

Proof. (i) just follows from Lemma 2.1 by taking $s\to\infty$.

(ii) For $t>0$, we have by (i) and Fubini theorem that

$$\begin{aligned} t &= \int_0^t\left(\int_0^\infty \mathbb{E}\left[w(s-\bar S_r)1_{\{s\ge S_r\}}\right]dr\right)ds\\ &= \int_0^\infty \mathbb{E}\left[\int_0^t w(s-\bar S_r)1_{\{s\ge S_r\}}ds\right]dr\\ &= \int_0^\infty \mathbb{E}\left[G(t-\bar S_r)1_{\{t\ge \bar S_r\}}\right]dr. \end{aligned}$$

(iii) Since $G(x)$ is an increasing function in x, we have by (ii)

$$\int_0^\infty \mathbb{E}\left[G(t-S_r)1_{\{t\ge S_r\}}\right]dr\le \int_0^\infty \mathbb{E}\left[G(t-\bar S_r)1_{\{t\ge \bar S_r\}}\right]dr\le t.$$

This proves the corollary. □

We define the generalized Caputo derivative ∂_t^w by

$$\partial_t^w f(t) := \frac{d}{dt} \int_0^t w(t-s)(f(s)-f(0))ds, \tag{2.6}$$

whenever it is well-defined in some function space of f.

Suppose that $\{T_t; t \geq 0\}$ is a strongly continuous semigroup with infinitesimal generator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ in some Banach space $(\mathbb{B}, \|\cdot\|)$ with the property that $\sup_{t>0} \|T_t\| < \infty$. Here $\|T_t\|$ denotes the operator norm of the linear map $T_t : \mathbb{B} \to \mathbb{B}$. Note that by the uniform boundedness principle, $\sup_{t>0} \|T_t\| < \infty$ is equivalent to $\sup_{t>0} \|T_t f\| < \infty$ for every $f \in \mathbb{B}$. Typical examples of such uniformly bounded strongly continuous semigroups are:

(i) Transition semigroup $\{P_t; t \geq 0\}$ of a strong Markov process $X = \{X_t, t \geq 0; \mathbb{P}_x, x \in E\}$ on a Lusin space E that has a weak dual with respect to some reference measure m on E. Then for every $p \geq 1$, $\{P_t; t \geq 0\}$ is a strongly continuous semigroup in $\mathbb{B} := L^p(E; m)$ with $\sup_{t>0} \|P_t\|_{p\to p} \leq 1$. The infinitesimal generator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ of $\{P_t; t \geq 0\}$ in $L^p(E; m)$ is called the L^p generator of the Markov process X.

(ii) Transition semigroup $\{P_t; t \geq 0\}$ of a Feller process $X = \{X_t, t \geq 0; \mathbb{P}_x, x \in E\}$ on a locally compact separable Hausdorff space E. In this case, $\{P_t; t \geq 0\}$ is a strongly continuous semigroup in the space $(C_\infty(E), \|\cdot\|_\infty)$ of continuous functions on E that vanish at infinity equipped with uniform norm. The infinitesimal generator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ of $\{P_t; t \geq 0\}$ in $\mathbb{B} := (C_\infty(E), \|\cdot\|_\infty)$ is called the Feller generator of X.

(iii) Certain Feynman-Kac semigroups (can be non-local Feynman-Kac semigroups or even generalized Feynman-Kac semigroups) in L^p-space or in $C_\infty(E)$ of a Hunt process X; cf. [4, 6].

For $\alpha > 0$, let $G_\alpha := \int_0^\infty e^{-\alpha t} T_t dt$ be the resolvent of the semigroup $\{T_t; t \geq 0\}$ on Banach space $\mathbb{B}$. Then by the resolvent equation, $\mathcal{D}(\mathcal{L}) = G_\alpha(\mathbb{B}) = G_1(\mathbb{B})$, which is dense in the Banach space $(\mathbb{B}, \|\cdot\|)$.

Let $E_t := \inf\{s > 0 : S_s > t\}$, $t \geq 0$, be the inverse subordinator. Define

$$u(t,x) = \mathbb{E}\left[T_{E_t} f(x)\right] = \int_0^\infty T_s f(x) d_s \mathbb{P}(E_t \leq s) = \int_0^\infty T_s f(x) d_s \mathbb{P}(S_s \geq t). \tag{2.7}$$

The following is the main result of this paper, which gives the existence and uniqueness of solutions to time fractional equation (2.8). Theorem 1.1 is its particular case, where T_t is the transition semigroup of a strong Markov process X given by $T_t f(x) = \mathbb{E}_x[f(X_t)]$.

Theorem 2.3 *Suppose that $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is the infinitesimal generator of a uniformly bounded strongly continuous semigroup $\{T_t; t \geq 0\}$ in a Banach space $(\mathbb{B}, \|\cdot\|)$. For every $f \in \mathcal{D}(\mathcal{L})$, $u(t,x) := \mathbb{E}\left[T_{E_t} f(x)\right]$ is a solution in $(\mathbb{B}, \|\cdot\|)$ to*

$$\left(\kappa \partial_t + \partial_t^w\right) u(t,x) = \mathcal{L} u(t,x) \quad \text{with } u(0,x) = f(x) \tag{2.8}$$

in the following sense:

(i) $\sup_{t>0} \|u(t,\cdot)\| < \infty$, $x \mapsto u(t,x)$ *is in* $\mathcal{D}(\mathcal{L})$ *for each* $t \geq 0$ *with* $\sup_{t\geq 0} \|\mathcal{L}u(t,\cdot)\| < \infty$, *and both* $t \mapsto u(t,\cdot)$ *and* $t \mapsto \mathcal{L}u(t,\cdot)$ *are continuous in* $(\mathbb{B}, \|\cdot\|)$*;*

(ii) *for every* $t > 0$, $I_t^w(u) := \int_0^t w(t-s)(u(s,x) - f(x))ds$ *is absolutely convergent in* $(\mathbb{B}, \|\cdot\|)$ *and*

$$\lim_{\delta\to 0} \frac{1}{\delta} \left(u(t+\delta,\cdot) - u(t,\cdot) + I_{t+\delta}^w(u) - I_t^w(u) \right) = \mathcal{L}u(t,x) \quad \text{in } (\mathbb{B}, \|\cdot\|).$$

Conversely, if $u(t,x)$ *is a solution to* (2.8) *in the sense of* (i) *and* (ii) *above with* $f \in \mathcal{D}(\mathcal{L})$*, then* $u(t,x) = \mathbb{E}\left[T_{E_t} f(x)\right]$ *in* $\mathbb{B}$ *for every* $t \geq 0$.

Proof. (a) (Existence) Clearly for $f \in \mathcal{D}(\mathcal{L})$,

$$\sup_{t>0} \|u(t,\cdot)\| \leq \sup_{t>0} \mathbb{E}\left[\|T_{E_t} f\|\right] \leq \sup_{r>0} \|T_r f\| < \infty.$$

By the same reason, $\sup_{t>0} \mathbb{E}\left[\|T_{E_t} \mathcal{L} f\|\right] \leq \sup_{r>0} \|T_r \mathcal{L} f\| < \infty$. It follows from the closed graph theorem for the generator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ that $u(t,\cdot) \in \mathcal{D}(\mathcal{L})$ and $\mathcal{L}u(t,\cdot) = \mathbb{E}\left[T_{E_t}(\mathcal{L}f)\right]$. Since $\{T_t; t \geq 0\}$ is a strongly continuous semigroup on $\mathbb{B}$ with $\sup_{t\geq 0} \|T_t\| < \infty$ and $t \mapsto E_t$ is continuous a.s., we have by bounded convergence theorem that both $t \mapsto u(t,\cdot) = \mathbb{E}\left[T_{E_t} f\right]$ and $t \mapsto \mathcal{L}u(t,\cdot) = \mathbb{E}\left[T_{E_t}(\mathcal{L}f)\right]$ are continuous in $(\mathbb{B}, \|\cdot\|)$.

It follows from (2.7), (2.5), and the integration by parts formula that for every $t > 0$,

$$\begin{aligned}
& \int_0^t w(t-r)(u(r,x) - u(0,x))dr \\
= & \int_0^t w(t-r) \left(\int_0^\infty (T_s f(x) - f(x)) d_s \mathbb{P}(S_s \geq r) \right) dr \\
= & \int_0^\infty (T_s f(x) - f(x)) d_s \left(\int_0^t w(t-r) \mathbb{P}(S_s > r) dr \right) \\
= & -\int_0^\infty (T_s f(x) - f(x)) d_s \mathbb{E}\left[G(t - S_s) 1_{\{t \geq S_s\}} \right] \\
= & \int_0^\infty \mathbb{E}\left[G(t - S_s) 1_{\{t \geq S_s\}} \right] \mathcal{L} T_s f(x) ds.
\end{aligned}$$

Note that since $\sup_{s>0} \|T_s f\| < \infty$ and $\sup_{s>0} \|\mathcal{L} T_s f\| = \sup_{s>0} \|T_s \mathcal{L} f\| < \infty$, by (2.2) and Corollary 2.2, all the integrals in above display are absolutely convergent in the Banach space $(\mathbb{B}, \|\cdot\|)$, while the second inequality is justified by the Riemann sum approximation of Stieltjes integrals, Fubini theorem and the dominated convergence theorem. On the other hand, $\mathbb{P}(S_r \geq s) = 1$ when $s \leq \kappa r$, while for a.e. $s \in (\kappa r, \infty)$, we have by Lemma 2.1 that

$$\mathbb{P}(S_r \geq s) = \mathbb{P}(\bar{S}_r \geq s - \kappa r) = \int_0^r \mathbb{E}\left[w(s - \kappa r - \bar{S}_y) 1_{\{s - \kappa r > \bar{S}_y\}} \right] dy. \tag{2.9}$$

So for every $t > 0$,

$$\begin{aligned}
\int_0^t \mathbb{P}(S_r \geq s) ds &= (\kappa r) \wedge t + \mathbb{E} \int_0^r \left(\int_{(\kappa r)\wedge t}^t w(s - \kappa r - \bar{S}_y) 1_{\{s - \kappa r > \bar{S}_y\}} ds \right) dy \\
&= (\kappa r) \wedge t + 1_{\{\kappa r < t\}} \mathbb{E} \int_0^r G(t - \kappa r - \bar{S}_y) 1_{\{t - \kappa r > \bar{S}_y\}} dy. \qquad (2.10)
\end{aligned}$$

Since

$$\begin{aligned}
\mathcal{L}u(s,x) &= \mathcal{L}\mathbb{E}\left[T_{E_s}f(x)\right] = \mathbb{E}\left[T_{E_s}\mathcal{L}f(x)\right] \\
&= \int_0^\infty T_r\mathcal{L}f(x)d_r\mathbb{P}(E_s \le r) = \int_0^\infty T_r\mathcal{L}f(x)d_r\mathbb{P}(S_r \ge s),
\end{aligned}$$

we have by (2.9) and (2.10) that

$$\begin{aligned}
&\int_0^t \mathcal{L}u(s,x)ds \\
&= \int_0^t \left(\int_0^\infty T_r\mathcal{L}f(x)d_r\mathbb{P}(S_r \ge s)\right)ds \\
&= \int_0^\infty T_r\mathcal{L}f(x)d_r\left(\int_0^t \mathbb{P}(S_r \ge s)ds\right) \\
&= \mathbb{E}\int_0^{t/\kappa} T_r\mathcal{L}f(x)\left(\kappa + G(t-\kappa r-\bar{S}_r)1_{\{t-\kappa r>\bar{S}_r\}} - \kappa\int_0^r w(t-\kappa r-\bar{S}_y)1_{\{t-\kappa r>\bar{S}_y\}}dy\right)dr \\
&= \int_0^\infty T_r\mathcal{L}f(x)\mathbb{E}\left[G(t-S_r)1_{\{t\ge S_r\}}\right]dr + \kappa\int_0^{t/\kappa} T_r\mathcal{L}f(x)\left(1-\mathbb{P}(S_r \ge t)\right)dr \\
&= \int_0^\infty T_r\mathcal{L}f(x)\mathbb{E}\left[G(t-S_r)1_{\{t\ge S_r\}}\right]dr + \kappa\int_0^\infty \mathbb{P}(S_r < t)d_r\left(T_rf(x)-f(x)\right) \\
&= \int_0^\infty T_r\mathcal{L}f(x)\mathbb{E}\left[G(t-S_r)1_{\{t\ge S_r\}}\right]dr + \kappa\int_0^{t/\kappa} \mathbb{P}(E_t > r)d_r\left(T_rf(x)-f(x)\right) \\
&= \int_0^\infty T_r\mathcal{L}f(x)\mathbb{E}\left[G(t-S_r)1_{\{t\ge S_r\}}\right]dr + \kappa\int_0^\infty \left(T_rf(x)-f(x)\right)d_r\mathbb{P}(E_t \le r) \\
&= \int_0^\infty T_r\mathcal{L}f(x)\mathbb{E}\left[G(t-S_r)1_{\{t\ge S_r\}}\right]dr + \kappa\mathbb{E}\left[T_{E_t}f(x)-f(x)\right] \\
&= \int_0^\infty T_r\mathcal{L}f(x)\mathbb{E}\left[G(t-S_r)1_{\{t\ge S_r\}}\right]dr + \kappa(u(t,x)-u(0,x)).
\end{aligned}$$

Thus we have for every $t > 0$,

$$\kappa(u(t,x)-u(0,x)) + \int_0^t w(t-r)(u(r,x)-u(0,x))dr = \int_0^t \mathcal{L}u(s,x)ds.$$

Consequently, $\left(\kappa\partial_t + \partial_t^w\right)u(t,x) = \mathcal{L}u(t,x)$ in $\mathbb{B}$ as $t \mapsto \mathcal{L}u(t,\cdot)$ is continuous in $(\mathbb{B}, \|\cdot\|)$.

(b) (Uniqueness) Suppose that $u(t,x)$ is a solution to (2.8) in the sense of (i) and (ii) with $f \in \mathcal{D}(\mathcal{L})$. Then $v(t,x) := u(t,x) - \mathbb{E}\left[T_{E_t}f(x)\right]$ is a solution to (2.8) with $v(0,x) = 0$. Hence we have for every $t > 0$,

$$\kappa v(t,x) + \int_0^t w(t-r)v(r,x)dr = \int_0^t \mathcal{L}v(s,x)ds. \tag{2.11}$$

Let $V(\lambda,x) := \int_0^\infty e^{-\lambda t}v(t,x)dt$, $\lambda > 0$, be the Laplace transform of $t \mapsto v(t,x)$. Clearly for every $\lambda > 0$, $V(\lambda,\cdot) \in \mathbb{B}$ with $\|V(\lambda,\cdot)\| \le \lambda^{-1}\sup_{t>0}\|v(t,\cdot)\|$. By the closed graph theorem, for each $\lambda > 0$, $V(\lambda,\cdot) \in \mathcal{D}(\mathcal{L})$ with

$$\mathcal{L}V(\lambda,\cdot) = \int_0^\infty e^{-\lambda t}\mathcal{L}v(t,\cdot)dt \quad \text{and} \quad \|\mathcal{L}V(\lambda,\cdot)\| \le \int_0^\infty e^{-\lambda t}\|\mathcal{L}v(t,\cdot)\|dt \le \frac{1}{\lambda}\sup_{t>0}\|\mathcal{L}v(t,\cdot)\|.$$

Taking Laplace transform in t on both sides of (2.11) yields

$$V(\lambda, x)\left(\kappa + \int_0^\infty e^{-\lambda x} w(x)dx\right) = \frac{1}{\lambda}\int_0^\infty e^{-\lambda t}\mathcal{L}v(t,x)dt = \frac{\mathcal{L}V(\lambda, x)}{\lambda}.$$

Thus by (2.1) and (2.3), $\mathcal{L}V(\lambda, x) = (\kappa\lambda + \phi_0(\lambda))\, V(\lambda, x) = \phi(\lambda)V(\lambda, x)$. In other words,

$$(\phi(\lambda) - \mathcal{L})V(\lambda, x) = 0 \quad \text{for every } \lambda > 0.$$

Since $\mathcal{L}$ is the infinitesimal generator of a uniformly bounded strongly continuous semigroup $\{T_t, t \geq 0\}$ in Banach space $\mathbb{B}$, for every $\alpha > 0$, the resolvent $G_\alpha = \int_0^\infty e^{-\alpha t} T_t dt$ is well defined and is the inverse to $\alpha - \mathcal{L}$. Hence we have from the last display that $V(\lambda, \cdot) = 0$ in $\mathbb{B}$ for every $\lambda > 0$. By the uniqueness of Laplace transform, we have $v(t, \cdot) = 0$ in $\mathbb{B}$ for every $t > 0$. This establishes that $u(t,x) = \mathbb{E}\left[T_{E_t} f(x)\right]$ in $\mathbb{B}$ for every $t \geq 0$. □

Remark 2.4 (i) The assumption that $f \in \mathcal{D}(\mathcal{L})$ in Theorem 2.3 is to ensure that all the integrals involved in the proof of Theorem 2.3 are absolutely convergent in the Banach space $\mathbb{B}$. This condition can be relaxed if we formulate the equation (2.8) in the weak sense when the uniformly bounded strongly continuous semigroup $\{T_t; t \geq 0\}$ is symmetric in a Hilbert space $L^2(E; m)$ and so its quadratic form can be used to formulate weak solutions. This will be carried out in the ongoing joint work [5] with Kim, Kumagai and Wang. It in particular applies to the case where $\{T_t; t \geq 0\}$ is the transition semigroup of any m-symmetric Markov process on a Lusin space E, which is a strongly continuous contraction symmetric semigroup in $L^2(E; m)$.

(ii) There are two closely related work [18, 12]. Suppose that $X = \{X_t, t \geq 0; \mathbb{P}_x, x \in \mathbb{R}^d\}$ is a Lévy process on $\mathbb{R}^d$ and generator $\mathcal{L}$, and $\{S_t; t \geq 0\}$ is a driftless subordinator with Laplace exponent ϕ and Lévy measure μ. Let $E_t := \inf\{s > 0 : S_s > t\}$ be the inverse subordinator. Under the assumption that $\kappa = 0$, $\mu(0, \infty) = \infty$, $\int_0^1 x|\log x|\mu(dx) < \infty$ and that the Lévy process X has a transition density function, it is shown in [18, Theorem 4.1] that $u(t,x) := \mathbb{E}_x\left[f(X_{E_t})\right]$ is a mild solution of the following pseudo-differential equation

$$\phi(\partial_t)u(t,x) = \mathcal{L}u(t,x) + f(x)\mu(t, \infty).$$

Here $\phi(\partial_t)$ is a pseudo-differential operator in time variable t formulated using Fourier multiplier.

Under the assumption that the Lévy measure μ of the subordinator S_t satisfying condition $\mu(d\xi) \geq \xi^{1+\beta} d\xi$ on $(0, \varepsilon)$ for some $\varepsilon > 0$ and $\beta > 0$, and $\{T_t; t \geq 0\}$ is the transition semigroup of a Feller process $X = \{X_t, t \geq 0; \mathbb{P}_x, x \in \mathbb{R}^d\}$ on $\mathbb{R}^d$ whose domain of infinitesimal generator contains $C^2(\mathbb{R}^d) \cap C_\infty(\mathbb{R}^d)$, [12, Theorem 8.4.2] asserts that for every $f \in C^2(\mathbb{R}^d) \cap C_\infty(\mathbb{R}^d)$, $u(t,x) := \mathbb{E}_x\left[f(X_{E_t})\right]$ satisfies

$$A_t^* u(t,x) = \mathcal{L}u(t,x) + f(x)A^*(1_{(0,\infty)})(t) \quad \text{with } u(0,x) = f(x),$$

where A^* is the dual of the infinitesimal generator of the subordinator S_t and notation $A_t^* u(t,x)$ means that the operator A^* is applied to the function $t \mapsto u(t,x)$. Here $C^2(\mathbb{R}^d)$ is the space of C^2-smooth functions on $\mathbb{R}^d$ and $C_\infty(\mathbb{R}^d)$ is the space of continuous functions on $\mathbb{R}^d$ that vanish at infinity. In [12, Theorem 8.4.2] , the subordinator S_t may have drift $\kappa \geq 0$.

(iii) Cauchy problems with distributed order time fractional derivatives (where $\kappa = 0$) were also studied in [16] for uniformly elliptic generators of divergence form in bounded $C^{1,\gamma}$ domains with Dirichlet boundary condition, under certain regularity conditions of the diffusion matrices. We also mention [14, Theorem 2] where $\{S_t; \geq 0\}$ is a subordinator without drift and $\{T_t; t \geq 0\}$ is the transition semigroup of a one-dimensional diffusion killed at certain rate via Feynman-Kac transform.

(iv) There are limited results in literature on the uniqueness for the time fractional equations (2.8); see [10, 11, 15] for cases of $\partial_t^\beta u = \mathcal{L}u$ and [13] for distributed order time fractional equation $\partial_t^w u = \mathcal{L}u$ where $\mathcal{L}$ is a one-dimensional differential operator in a bounded interval. We mention that Remark 3.1 of a recent preprint [1] contains a uniqueness result for solutions to $\partial_t^\beta u = \mathcal{L}u$, where $\mathcal{L}$ is the Feller generator of a doubly Feller process killed upon leaving a bounded regular domain, proved also by using Laplace transform similar to our uniqueness proof for Theorem 2.3 in this paper.

(v) When the uniformly bounded strongly continuous semigroup $\{T_t; t \geq 0\}$ in Theorem 2.3 has an integral kernel $p(t, x, y)$ with respect to some measure $m(dx)$, then there is a kernel $q(t, x, y)$ so that

$$u(t,x) := \mathbb{E}\left[T_{E_t} f(x)\right] = \int_E q(t,x,y) f(y) m(dy);$$

in other words,

$$q(t,x,y) := \mathbb{E}\left[p(E_t, x, y)\right] = \int_0^\infty p(s,x,y) d_s \mathbb{P}(E_t \leq s)$$

is the fundamental solution to the time fractional equation $(\kappa\partial_t + \partial_t^w)\, u = \mathcal{L}u$ under the setting of this paper. In [5], two-sided estimates on $q(t, x, y)$ are obtained when $\kappa = 0$ and $\{T_t; t \geq 0\}$ is the transition semigroup of a diffusion process that satisfies two-sided Gaussian-type estimates or of a stable-like process on metric measure spaces.

Example 2.5 (i) When $\{S_t; t \geq 0\}$ is a β-subordinator with $0 < \beta < 1$ with Laplace exponent $\phi(\lambda) = \lambda^\beta$, it is easy to check that S_t has no drift (i.e. $\kappa = 0$) and its Lévy measure is $\mu(dx) = \frac{\beta}{\Gamma(1-\beta)} x^{-(1+\beta)} dx$. Hence

$$w(x) := \mu(x, \infty) = \int_x^\infty \frac{\beta}{\Gamma(1-\beta)} y^{-(1+\beta)} dy = \frac{x^{-\beta}}{\Gamma(1-\beta)}.$$

Thus the fractional derivative $\partial_t^w f$ defined by (1.2) is exactly the Caputo derivative of order β defined by (1.1). In this case, Theorem 2.3 recovers the main result of [2].

(ii) We call a subordinator $\{S_t; t \geq 0\}$ truncated β-subordinator if it is driftless and its Lévy measure is

$$\mu_\delta(dx) = \frac{\beta}{\Gamma(1-\beta)}\, x^{-(1+\beta)} 1_{(0,\delta]}(x) dx$$

for some $\delta > 0$. In this case,

$$w_\delta(x) := \mu_\delta(x, \infty) = 1_{\{0<x\leq\delta\}} \int_x^\delta \frac{\beta}{\Gamma(1-\beta)} y^{-(1+\beta)} dy = \frac{1}{\Gamma(1-\beta)} \left(x^{-\beta} - \delta^{-\beta}\right) 1_{(0,\delta]}(x).$$

So the corresponding the fractional derivative of (1.2) is

$$\partial_t^{w_\delta} f(t) := \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_{(t-\delta)^+}^t \left((t-s)^{-\beta} - \delta^{-\beta} \right) (f(s) - f(0)) ds.$$

This is the fractional-time derivative whose value at time t depends only on the δ-range of the past of f as mentioned in the Introduction. Theorem 2.3 says that the corresponding time fractional equation (1.5) can be solved by using the inverse of truncated β-subordinator. Clearly, as $\lim_{\delta\to\infty} w_\delta(x) = w(x) := \frac{1}{\Gamma(1-\beta)} x^{-\beta}$. Consequently, the fractional derivative $\partial_t^{w_n} f(t) \to \partial_t^{w} f(t))$, the Caputo derivative of f of order β, in the distributional sense. Using the probabilistic representation in Theorem 2.3, one can deduce that as $\delta \to \infty$, the solution to the equation $\partial_t^{w_\delta} u = \mathcal{L}u$ with $u(0,x) = f(x)$ converges to the solution of $\partial_t^\beta u = \mathcal{L}u$ with $u(0,x) = f(x)$.

If we define

$$\eta_\delta(r) = \frac{\Gamma(2-\beta)\,\delta^{\beta-1}}{\beta} w_\delta(r) = (1-\beta)\delta^{\beta-1} \left(x^{-\beta} - \delta^{-\beta} \right) 1_{(0,\delta]}(x),$$

then $\eta_\delta(r)$ converges weakly to the Dirac measure concentrated at 0 as $\delta \to 0$. So the fractional derivative $\partial_t^{\eta_\delta} f(t)$ converges to $f'(t)$ for every differentiable f. It can be shown that the subordinator corresponding to η_δ, that is, subordinator with Lévy measure

$$\nu_\delta(dx) := \frac{(1-\beta)\,\delta^{\beta-1}}{\beta}\, x^{-(1+\beta)} 1_{(0,\delta]}(x) dx,$$

converges as $\delta \to 0$ to deterministic motion t moving at constant speed 1. Using Theorem 2.3, one can show that the solution to the equation $\partial_t^{\eta_\delta} u(t,x) = \mathcal{L}u(t,x)$ with $u(0,x) = f(x)$ converges to the solution of the heat equation $\partial_t u = \mathcal{L}u$ with $u(0,x) = f(x)$. $\square$

3 Occupation measure for processes time-changed by inverse subordinator

Suppose $X = \{X_t, t \geq 0; \mathbb{P}_x, x \in E\}$ is a general strong Markov process on state space E and $\{S_t; t \geq 0\}$ is a subordinator independent of X whose Lévy measure μ satisfies $\mu(0,\infty) = \infty$. Let ϕ be the Laplace exponent of X; that is, $\mathbb{E}e^{-\lambda S_t} = e^{-t\phi(\lambda)}$. Note that $\mathbb{E}[S_t] = t\phi'(0)$ so in particular $\phi'(0) = \mathbb{E}[S_1]$. Let $E_t := \inf\{s > 0 : S_s > t\}$ be the inverse subordinator, and $X_t^* := X_{E_t}$. Suppose D is an open subset of E and define $\tau_D := \{t > 0 : X_t \notin D\}$ to be the first exit time from D by the process X. In general, the time-changed process X^* is not a Markov process but we can still define its first exit time from D by

$$\tau_D^* := \inf\{t > 0 : X_t^* \notin D\}.$$

Let ∂ be a cemetery point. The process $X^{*,D}$ defined by $X_t^{*,D} := X_t^*$ when $t < \tau_D^*$ and $X_t^* := \partial$ for $t \geq \tau_D^*$ is called the part process of X^* in D. The part process X^D of X in D is defined in an analogous way. We use $\mathbb{E}_x$ to denote mathematical expectation taken with respect to the probability law $\mathbb{P}_x$, under which the Markov process X starts from $x \in E$. For every $x \in D$, the occupation measures for X^D and $X^{*,D}$ are defined by

$$\nu_x^D(A) = \mathbb{E}_x\left[\int_0^{\tau_D} 1_A(X_s)ds\right] \quad \text{and} \quad \nu_x^{*,D}(A) = \mathbb{E}_x\left[\int_0^{\tau_D^*} 1_A(X_s^*)ds\right], \qquad A \subset D.$$

Occupation measures describe the average amount of time spent by the processes in subsets of the state space.

The next theorem says that the occupation measure for the part process $X^{*,D}$ of X^* in D is proportional to that of the part process X^D of X in D when $\phi'(0) < \infty$, that is, when the subordinator S_t has finite mean. When the subordination S_t has infinite mean, the occupation measure for the part process $X^{*,D}$ of X^* in D is always infinite.

Theorem 3.1 *For every measurable function $f \geq 0$ on D and $x \in D$,*

$$\mathbb{E}_x\left[\int_0^{\tau_D^*} f(X_t^*)dt\right] = \phi'(0)\,\mathbb{E}_x\left[\int_0^{\tau_D} f(X_t)dt\right] = \phi'(0)G_D f(x).$$

In other words, $\nu_x^{*,D} = \phi'(0)\nu_x^D$ for every open set $D \subset E$ and every $x \in D$.

Proof. First note that

$$\begin{aligned}
\tau_D^* &= \inf\{t > 0 : X_{E_t} \notin D\} = \inf\{t > 0 : E_t = \tau_D\} \\
&= \inf\{t > 0 : S_{\tau_D} > t\} = S_{\tau_D}.
\end{aligned}$$

For any $f \geq 0$ on D, we have

$$\begin{aligned}
\mathbb{E}_x\left[\int_0^{\tau_D^*} f(X_t^*)dt\right] &= \mathbb{E}_x\left[\int_0^{S_{\tau_D}} f(X_{E_t})dt\right] = \mathbb{E}_x\left[\int_0^{\tau_D} f(X_r)dS_r\right] \\
&= \mathbb{E}_x\left[\int_0^{S_{\tau_D}} f(X_{E_t})dt\right] = \mathbb{E}_x\mathbb{E}_x\left[\int_0^{\tau_D} f(X_r)dS_r\Big|X\right] \\
&= \mathbb{E}_x\left[\int_0^{\tau_D} f(X_r)d(\mathbb{E}S_r)\right] = \mathbb{E}_x\left[\int_0^{\tau_D} f(X_r)dr\right]\phi'(0) \\
&= \phi'(0)G_D f(x).
\end{aligned}$$

□

Remark 3.2 Taking $f = 1$ in Theorem 3.1 in particular yields the following relation on mean exit times:

$$\mathbb{E}_x[\tau_D^*] = \phi'(0)\,\mathbb{E}_x[\tau_D] \quad \text{for every } x \in D. \tag{3.1}$$

When X is either a diffusion process determined by a stochastic differential equation driven by Brownian motion or a rotationally symmetric α-stable process on $\mathbb{R}^d$, and $\{S_t; t \geq 0\}$ is a tempered β-subordinator having Laplace exponent $\phi(\lambda) = (\lambda + m)^\beta - m^\beta$ for some $m > 0$ and $0 < \beta < 1$, (3.1) recovers the main result of [7], derived there using a PDE method.

Acknowledgement. The author thanks M. M. Meerschaert for the invitation to the Workshop "Future Directions in Fractional Calculus Research and Applications" held at Michigan State University, East Lansing, from October 17-21, 2016, and for helpful comments. He also thanks T. Kumagai and J. Wang for helpful comments.

References

[1] B. Baeumer, T. Luks and M. M. Meerschaert, Space-time fractional Dirichlet problems. Preprint.

[2] B. Baeumer and M. M. Meerschaert, Stochastic solutions for fractional Cauchy problems. *Fract. Calc. Appl. Anal.* **4** (2001), 481-500.

[3] J. Bertoin, *Lévy Processes*. Cambridge Univ. Press, 1998.

[4] Z.-Q. Chen, P. J. Fitzsimmons, K. Kuwae and T.-S. Zhang, Perturbation of symmetric Markov processes. *Probab. Theory Relat. Fields* **140** (2008), 239-275.

[5] Z.-Q. Chen, P. Kim, T. Kumagai and J. Wang, Heat kernel estimates for solutions of general time fractional equations. In preparation.

[6] Z.-Q. Chen and R. Song, Conditional gauge theorem for non-local Feynman-Kac transforms. *Probab. Theory Relat. Fields,* **125** (2003), 45-72.

[7] W. Deng, X. Wu and W. Wang, Mean exit time and escape probability for the anomalous processes with the tempered power-law waiting times. To appear in *Europhysics Letters*.

[8] Q. Du, J. Yang and Z. Zhou, Analysis of a nonlocal-in-time parabolic equation. To appear in *Disc. Cont. Dyn. Sys. B*, 2016.

[9] R. Herrmann, *Fractional Calculus: An Introduction for Pysicists, 2nd Edition*. Singapore: World Scientific, 2011.

[10] A. N. Kochubei, The Cauchy problem for evolution equations of fractional order. (Russian); translated from *Differentsial'nye Uravneniya* **25** (1989), 1359-1368, 1468 *Differential Equations* **25** (1989), 967-974.

[11] A. N. Kochubei, Diffusion of fractional order. (Russian); translated from *Differentsial'nye Uravneniya* **26** (1990), 660-670, 733-734 *Differential Equations 26* (1990), 485-492.

[12] V. Kolokoltsov, *Markov Processes: Semigroups and Generators*. De Gruyter, Berlin/New York, 2011.

[13] Y. Luchko, Boundary value problems for the generalized time-fractional diffusion equation of distributed order. *Fract. Calc. Appl. Anal.* **12** (2009), 409?422.

[14] M. Magdziarz and R. L. Schilling, Asymptotic properties of Brownian motion delayed by inverse subordinators. *Proc. Amer. Math. Soc.* **143** (2015), 4485-4501.

[15] M. M. Meerschaert, E. Nane and P. Vellaisamy, Fractional Cauchy problems on bounded domains. *Ann. Probab.* **37** (2009), 979-1007.

[16] M. M. Meerschaert, E. Nane and P. Vellaisamy, Distributed-order fractional Cauchy problems on bounded domains. *J. Math. Anal. Appl.* **379** (2011), 216-228.

[17] M. M. Meerschaert and H. P. Scheffler, Stochastic model for ultraslow diffusion. *Stochastic Process Appl.* **116** (2006), 1215-1235.

[18] M. M. Meerschaert and H. P. Scheffler, Triangular array limits for continuous time random walks. *Stochastic Process Appl.* **118** (2008), 1606-1633.

[19] M. M. Meerschaert and A. Sikorskii, *Stochastic Models for Fractional Calculus*, De Gruyter Studies in Mathematics **43**. Walter de Gruyter, Berlin/Boston, 2012.

[20] R. Metzler and J. Klafter, The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics. *J. Pyhsics A* **37** (2004), R161–R208.

[21] R. Metzeler and J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys. Peps.* **339(1)** (2000), 1-77.

[22] M. Shlesinger, J. Klafter and Y. M. Wong, Random walks with infinite spatial and temporal moments. *J. Statist. Phys.* **27** (1982), 499–512.

Zhen-Qing Chen
Department of Mathematics, University of Washington, Seattle, WA 98195, USA
E-mail: zqchen@uw.edu