A function and its Fourier transform cannot both be sharply localized.


**Notation:**

\[ |E| = \text{Lebesgue measure of } E, \quad \int = \int_{-\infty}^{\infty} \]
The Fourier transform (on $\mathbb{R}$):

$$\hat{f}(\xi) = \int e^{-2\pi i \xi x} f(x) \, dx$$

$$f(x) = \int e^{2\pi i x \xi} \hat{f}(\xi) \, d\xi.$$ 

Basic facts:

$$\|\hat{f}\|_\infty \leq \|f\|_1, \quad \|\hat{f}\|_2 = \|f\|_2,$$

$$\|\hat{f}\|_q \leq \|f\|_p \quad (1 \leq p \leq 2, \ p^{-1} + q^{-1} = 1),$$

$$\hat{f}'(\xi) = 2\pi i \xi \hat{f}(\xi).$$

A probability distribution function (PDF) on $\mathbb{R}$ is a function $\rho \geq 0$ with $\int \rho(x) \, dx = 1$. If $\rho$ is a PDF, its variance is

$$\text{var}(\rho) = \inf_{a \in \mathbb{R}} \int (x - a)^2 \rho(x) \, dx.$$
Interpretations:

1. Classical: $f(t)$ is the amplitude of a signal at time $t$. 
   
   $f(t) = \int e^{2\pi i \omega t} \hat{f}(\omega) \, d\omega$ expresses $f$ as a superposition of sine waves of different frequencies.

2. Quantum: Suppose $||f||_2 = 1$. Then $|f|^2$ and $|\hat{f}|^2$ are both PDFs. $f$ is the “wave function” of a quantum particle moving on the line, $|f|^2$ is the PDF of its position, and $|\hat{f}|^2$ is the PDF of its momentum (taking Planck’s constant $= 1$).
**Heisenberg’s Inequality.** If $f \in L^2(\mathbb{R})$, then for all $a, \alpha \in \mathbb{R}$,

$$
\int (x - a)^2 |f(x)|^2 \, dx \int (\xi - \alpha)^2 |\hat{f}(\xi)|^2 \, d\xi \geq \frac{||f||^4}{16\pi^2}. \quad (1)
$$

In particular, if $||f||^2 = 1$,

$$
\text{var}(|f|^2) \text{var}(|\hat{f}|^2) \geq 1/16\pi^2.
$$

Equality holds in (1) for a given $a$ and $\alpha \iff f(x) = ce^{2\pi i \alpha x}e^{-b(x-a)^2}$ for some $b > 0$, $c \in \mathbb{C}$. 

**Proof:** By considering $g(x) = e^{-2\pi i \alpha x} f(x + a)$, reduce to the case $a = \alpha = 0$. WLOG, assume that
\[
\int x^2 |f(x)|^2 \, dx < \infty \quad \text{and} \quad \int \xi^2 |\hat{f}(\xi)|^2 \, d\xi < \infty.
\]
Note that
\[
\int \xi^2 |\hat{f}(\xi)|^2 \, d\xi = \frac{1}{4\pi^2} \int |\hat{f}'(\xi)|^2 \, d\xi = \frac{1}{4\pi^2} \|f'\|_2^2, \quad (2)
\]
so $f' \in L^2$. Since
\[
\frac{d}{dx} (x|f(x)|^2) = |f(x)|^2 + 2 \text{Re} \, x f(x) \overline{f'(x)},
\]
\[
-2 \text{Re} \int_c^d x f(x) \overline{f'(x)} \, dx = -x|f(x)|^2 \bigg|_c^d + \int_c^d |f(x)|^2 \, dx.
\]
Let $c \to -\infty$, $d \to \infty$ to get
\[
-2 \text{Re} \int x f(x) \overline{f'(x)} \, dx = \|f\|_2^2,
\]
then use Cauchy-Schwarz and (2):
\[
\|f\|_2^2 \leq 2\|xf\|_2 \|f'\|_2 = 4\pi \|xf\|_2 \|\xi f'\|_2.
\]
Equality holds $\iff f'(x) = bxf(x)$ for some $b \in \mathbb{R}$, which gives $f(x) = ce^{-bx^2}$.
There are generalizations involving other $L^p$ norms. For example, if $1 \leq p \leq 2$,
\[
\|f\|_2^2 \leq 2\|xf\|_p\|f'\|_{p'} \leq 2\|xf\|_p\|\hat{f}'\|_p = 4\pi\|xf\|_p\|xf'\|_p.
\]
Cowling and Price have obtained general results relating to norms of the form $\|\ |x|^\alpha f\|_p$.

**Local Uncertainty Inequality (Faris-Price).**

Suppose $0 < \alpha < \frac{1}{2}$. There exists $C_\alpha > 0$ such that for all $f \in L^2(\mathbb{R})$ and all measurable $E \subset \mathbb{R}$,
\[
\int_E |\hat{f}|^2 \leq C_\alpha |E|^{2\alpha} \|\ |x|^\alpha f\|_2^2.
\]
Proof: Let $\chi_r(x) = 1$ if $|x| < r$, $\chi_r(x) = 0$ otherwise, and $\chi'_r = 1 - \chi_r$.

$$
\|\hat{f}\chi_E\|_2 \leq \|f\chi_r\chi_E\|_2 + \|f\chi'_r\chi_E\|_2
\leq |E|^{1/2}\|f\chi_r\|_\infty + \|f\chi'_r\|_2.
$$

Now

$$
\|f\chi_r\|_\infty \leq \|f\chi_r\|_1
\leq \left( \int_{-r}^r |x|^{-2\alpha} \, dx \right)^{1/2} \left( \int_{-r}^r |x|^{2\alpha} |f(x)|^2 \, dx \right)^{1/2}
\leq C_\alpha r^{(1/2)-\alpha} \|x^\alpha f\|_2,
$$

and

$$
\|f\chi'_r\|_2 \leq r^{-\alpha} \|x^\alpha f\|_2,
$$

so

$$
\|\chi_E\hat{f}\|_2 \leq (C_\alpha |E|^{1/2}r^{(1/2)-\alpha} + r^{-\alpha})\|x^\alpha f\|_2.
$$

Choose $r$ to minimize this quantity.
Qualitative Uncertainty Principles:

Let $\Sigma(f) = \{x : f(x) \neq 0\}$.

1. $\Sigma(f)$ bounded $\Rightarrow \hat{f}$ entire $\Rightarrow \mathbb{R} \setminus \Sigma(\hat{f})$ countable or $f = 0$.

2. If $f \in L^2$ and $f \neq 0$, $|\Sigma(f)| \|\Sigma(\hat{f})\| \geq 1$.

$$
\int |\hat{f}|^2 \leq |\Sigma(f)| \|\hat{f}\|_\infty^2 \leq |\Sigma(\hat{f})| \|f\|_1^2 \\
\leq |\Sigma(\hat{f})| |\Sigma(f)| \|f\|_2^2.
$$

3. (Benedicks) If $f \in L^p$ for some $p \geq 1$ and $|\Sigma(f)| \|\Sigma(\hat{f})\| < \infty$, then $f = 0$.

4. (Hardy) Suppose

$$
|f(x)| \leq Ce^{-a\pi x^2}, \quad |\hat{f}(\xi)| \leq Ce^{-b\pi \xi^2}
$$

for some $a, b, C > 0$. If $ab = 1$ then $f(x) = ce^{-ax^2}$; if $ab > 1$ then $f = 0$. 
**Definition:** A function $f \in L^2(\mathbb{R})$ is $\epsilon$-concentrated on a set $A \subset \mathbb{R}$ if $\|f(1 - \chi_A)\|_2 \leq \epsilon \|f\|_2$, or equivalently $\|f\chi_A\|_2 \geq \sqrt{1 - \epsilon^2} \|f\|_2$.

For $A, B \subset \mathbb{R}$, let

$$P_A f = \chi_A f, \quad (Q_B f)^\wedge = \chi_B \hat{f}.$$

**Theorem (Donoho-Stark).**

a. $\|P_A Q_B f\|_2 \leq |A|^{1/2} |B|^{1/2} \|f\|_2$.

b. If there exists $f \neq 0$ such that $f$ is $\epsilon$-concentrated on $A$ and $\hat{f}$ is $\delta$-concentrated on $B$, then

$$|A|^{1/2} |B|^{1/2} \geq 1 - \epsilon - \delta.$$
Proof: For (a),

\[ \left| P_A Q_B f(x) \right| = \chi_A(x) \left| \int \hat{\chi}_B(x - y) f(y) \, dy \right| \]

\[ \leq \chi_A(x) \| f \|_2 \| \hat{\chi}_B \|_2 = \chi_A(x) |B|^{1/2} \| f \|_2. \]

Take \( L^2 \) norms of both sides.

For (b), assume \( \| f \|_2 = 1 \). Then

\[ \| P_A (1 - Q_B) f \|_2 \leq \| (1 - Q_B) f \|_2 \leq \delta, \]

so

\[ 1 - \epsilon - \delta \leq \| f \|_2 - \| f - P_A f \|_2 - \| P_A (1 - Q_B) f \|_2 \]

\[ \leq \| P_A Q_B f \|_2 \leq |A|^{1/2} |B|^{1/2}. \]
Landau-Pollak-Slepian Theory:

Take \( A = (-\frac{1}{2}T, \frac{1}{2}T) \) and \( B = (-\Omega, \Omega) \) and consider

\[
S = Q_B P_A Q_B = (P_A Q_B)^*(P_A Q_B).
\]

This is a compact self-adjoint operator on \( L^2 \), so it has an orthonormal eigenbasis \( \{\phi_n\} \) with eigenvalues \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq 0 \).

If \( Sf = \lambda f \) then

\[
\|P_A f\|_2^2 = \|P_A Q_B f\|_2^2 = \langle Q_B P_A Q_B f, f \rangle = \lambda \|f\|_2^2,
\]

so \( f \) is \( \sqrt{1 - \lambda} \)-concentrated on \( A \) and \( \hat{f} \) is 0-concentrated on \( B \).

**Theorem:** \( \lambda_n \approx 1 \) for \( n \ll 2\Omega T \), \( \lambda_n \approx 0 \) for \( n \gg 2\Omega T \), and the transition interval has width \( O(\log \Omega T) \). Eigenfunctions \( \phi_n \) are “prolate spheroidal wave functions.”
If $\rho$ is a PDF on $\mathbb{R}$, its entropy $E(\rho)$ is

$$E(\rho) = - \int \rho(x) \log \rho(x) \, dx.$$  

**Proposition:** If $\operatorname{var}(\rho) < \infty$ then

$$E(\rho) \leq \frac{1}{2} \log(2\pi e \operatorname{var}(\rho)).$$

**Proof:** By composing with translations and dilations, reduce to the case where $\rho$ has mean 0 and variance $\operatorname{var}(\rho) = \int x^2 \, d\rho(x) = 1$. Let

$$\phi(x) = \sqrt{2\pi} e^{x^2/2} \rho(x), \quad d\gamma(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx,$$

so $\int \phi \, d\gamma = \int \rho \, dx = 1$. By Jensen’s inequality,

$$0 = \left( \int \phi \, d\gamma \right) \log \left( \int \phi \, d\gamma \right) \leq \int \phi \log \phi \, d\gamma$$

$$= \int \rho(x) \left[ \frac{1}{2} \log 2\pi + \frac{1}{2} x^2 + \log \rho(x) \right] \, dx$$

$$= \frac{1}{2} \log 2\pi + \frac{1}{2} - E(\rho).$$
**Theorem:** If \( \|f\|_2 = 1 \) then
\[
E(|f|^2) + E(|\hat{f}|^2) \geq 1 - \log 2.
\]

**Lemma:** If \( \phi(t) \leq \psi(t) \) for \( t \geq a \) and \( \phi(a) = \psi(a) \), then \( \phi'(a) \leq \psi'(a) \).

Applying this to the Hausdorff-Young inequality \( \|\hat{f}\|_q \leq \|f\|_p \) (\( q \geq 2, p = q/(q - 1), a = 2 \)), we get
\[
E(|f|^2) + E(|\hat{f}|^2) \geq 0.
\]

But using Beckner’s sharp Hausdorff-Young inequality
\[
\|\hat{f}\|_q \leq p^{1/2p}q^{-1/2q}\|f\|_p,
\]
we get the theorem.

**Corollary:** Heisenberg’s inequality.

**Corollary:** Suppose \( \int |g|^2 \, d\gamma = 1 \). Let \( Tg(x) = 2^{1/4}e^{-\pi x^2}g(2\sqrt{\pi} \, x) \), and \( \tilde{g} = T^{-1}(\hat{Tg}) \). Then
\[
\int |g|^2 \log |g| \, d\gamma + \int |\tilde{g}|^2 \log |\tilde{g}| \, d\gamma \leq \int |\nabla g|^2 \, d\gamma.
\]

Without the second term on the left, this is Gross’s logarithmic Sobolev inequality.