

## Abstract: Analytic Capacity, Bilipschitz Maps, Cantor Sets and Menger Curvature

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Let  $E \subset \mathbb{C}$  be a compact plane set. The **analytic capacity** of  $E$  is

$$\gamma(E) = \sup\{|f'(\infty)| : f \text{ is analytic on } \mathbb{C} \setminus E \text{ and } \sup_{\mathbb{C} \setminus E} |f(z)| \leq 1\}$$

where

$$f'(\infty) = \lim_{z \rightarrow \infty} z(f(z) - f(\infty)).$$

Thus  $\gamma(E) > 0$  if and only if  $\mathbb{C} \setminus E$  supports a non-constant bounded analytic function. The general problem is to find a geometric characterization of sets of positive analytic capacity.

Classical theorems of Riemann say that  $\gamma(E) = 0$  if  $E$  is finite and  $\gamma(E) > 0$  if  $E$  is infinite and connected. It is also known that  $\gamma(E) = 0$  if  $E$  has zero one-dimensional measure and that  $\gamma(E) > 0$  if the Hausdorff dimension  $\dim(E) > 1$ .

Recent exciting work of Tolsa, using ideas of Melnikov, Verdera, David and many others, shows that  $\gamma(E) > 0$  if and only if  $E$  supports a positive measure  $\mu$  with  $\mu(B(z, r)) \leq r$  for every ball  $B(z, r)$  that has finite **Menger curvature**

$$\int \int \int \frac{1}{R(z, w, \zeta)^2} d\mu(w) d\mu(\zeta) d\mu(z) < \infty$$

where  $R(z, w, \zeta)$  denotes the radius of the circle through  $z, w$  and  $\zeta$ .

A homeomorphism  $T : E \rightarrow T(E)$  is **bilipschitz** if  $T$  and  $T^{-1}$  satisfy Lipschitz conditions

$$\frac{1}{K}|z - w| \leq |T(z) - T(w)| \leq K|z - w|$$

for all  $z, w \in E$ .

**Conjecture.** *If  $T$  is bilipschitz, then  $\gamma(T(E)) \leq C(K)\gamma(E)$ , where  $C(K)$  depends only on the constant  $K$ .*

The talk will give an introduction to analytic capacity and its connections with harmonic analysis and describe a proof with Verdera of the bilipschitz conjecture for planar Cantor sets.