Random Sorting Networks
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with: Omer Angel  Dan Romik
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1 \times 2
2 \times 1
3 - 3
4 - 4
1 2×2 4
2 1 3 3
3×3 1 2
4 4 4 1
1 \times 2 - 2 - 2 \times 3 \quad 4
2 \times 1 - 3 - 3 \times 2 \quad 3
3 - 3 \times 1 - 4 - 4 \quad 2
4 - 4 - 4 \times 1 - 1 \quad 1
1 \times 2 - 2 \times 2 - 2 \times 3 - 3 \times 4
2 - 1 \times 3 - 3 \times 2 - 4 \times 3
3 - 3 \times 1 - 4 - 4 \times 2 - 2
4 - 4 - 4 \times 1 - 1 - 1 - 1 - 1
To get from $1 \cdots n$ to $n \cdots 1$ requires

$$N := \binom{n}{2}$$

nearest-neighbour swaps
E.g. \( n=4 \):

\[
\begin{array}{cccccc}
1 & 2 & 2 & 2 & 3 & 3 & 4 \\
2 & 1 & 3 & 3 & 2 & 4 & 3 \\
3 & 3 & 1 & 4 & 4 & 2 & 2 \\
4 & 4 & 4 & 1 & 1 & 1 & 1 \\
\end{array}
\]

A Sorting Network =

any route from 1\ldots n to n\ldots 1

in exactly

\[ N := \binom{n}{2} \]

nearest-neighbour swaps
Theorem (Stanley 1984). 
\# of n-particle sorting networks = \[ \frac{n!}{1^{n-1}3^{n-2}5^{n-3}\cdots(2n-3)^1} \]

Uniform Sorting Network (USN): choose an n-particle sorting network uniformly at random.

E.g. n=3:

\[ P(\begin{array}{c} x \\ x \\ x \end{array}) = P(\begin{array}{c} x \\ x \\ x \end{array}) = \frac{1}{2} \]
swap locations
1 2 2 2 3 3 3 4
2 1 3 3 2 4 4 3
3 3 1 4 4 2 2
4 4 4 1 1 1 1

swap locations

particle trajectory
Efficient simulation algorithm for USN...
Swap locations, n=100
Swap locations, n=2000
Theorem (Angel, H, Romik, Virag, 2007) For USN:

1. **Sequence of swap locations**
   
   $(s_1, \ldots, s_N)$ is stationary \( \forall n \)

2. **Scaled first swap location**

   \[
   \frac{s_1}{n} \xrightarrow{\text{dist}} \text{semicircle random variable} \quad \text{as } n \to \infty
   \]

3. **Scaled swap process**

   \[
   \xrightarrow{\text{dist}} \text{semicircle } \times \text{Lebesgue} \quad \text{as } n \to \infty
   \]

(Note: not true for all sorting networks, e.g. bubble sort)
Proof of stationarity:

\[
\begin{array}{ccccccc}
1 & \times & 2 & - & 2 & - & 2 & \times & 3 & - & 3 & \times & 4 \\
2 & \times & 1 & \times & 3 & - & 3 & \times & 2 & \times & 4 & \times & 3 \\
3 & - & 3 & \times & 1 & - & 4 & - & 4 & - & 2 & - & 2 \\
4 & - & 4 & - & 4 & \times & 1 & - & 1 & - & 1 & - & 1
\end{array}
\]
Proof of stationarity:
Proof of stationarity:

1 — 1 — 1 × 3 — 3 × 4
2 × 3 — 3 × 1 × 4 × 3
3 × 2 × 4 — 4 × 1 — 1
4 — 4 × 2 — 2 — 2 — 2
Proof of stationarity:

1 — 1 — 1 × 3 — 3 × 4 ···
2 × 3 — 3 × 1 × 4 × 3 ···
3 × 2 × 4 — 4 × 1 — 1
4 — 4 × 2 — 2 — 2 — 2 ×
Proof of stationarity:

\[
\begin{array}{cccc}
1 & 1 & 1 & 3 & 3 & 4 & 4 \\
2 & 3 & 3 & 1 & 4 & 3 & 3 \\
3 & 2 & 4 & 4 & 1 & 1 & 2 \\
4 & 4 & 2 & 2 & 2 & 2 & 1 \\
\end{array}
\]
Proof of stationarity:

\[(s_1,...,s_N) \mapsto (s_2,...,s_N,n-s_1)\] is a bijection from \{sorting networks\} to itself.

So for USN:

\[(s_2, \ldots, s_N) \overset{d}{=} (s_1, \ldots, s_{N-1})\]
Selected trajectories, n=2000
Scaled trajectory of particle $i$: $T_i: [0,1] \rightarrow [-1,1]$
**Conjecture (AHRV)**

trajectories → random Sine curves:

\[
\max_{i,t} |T_i(t) - A_i^n \sin(\pi t + \Theta_i^n)| \xrightarrow{\text{Prob}} 0
\]

(random) as \( n \to \infty \)

**Theorem (AHRV)**

scaled trajectories have subsequential limits which are Hölder(\( \frac{1}{2} \)) with prob 1

as \( n \to \infty \)
Half-time permutation matrix, $n=2000$
Conjecture (AHRV)

Scaled permutation matrix at time $N/2 \xrightarrow{d} \text{Archimedes measure}$

Projection of surface area measure on sphere $S^2 \subset \mathbb{R}^3$ onto $\mathbb{R}^2$

(unique circularly symmetric measure with uniform linear projections; $dx \, dy \over 2\pi \sqrt{1-x^2-y^2}$ on $x^2+y^2<1$)

Scaled permutation matrix at time $tN \xrightarrow{d} \begin{pmatrix} 1 & 0 \\ \cos \pi t & \sin \pi t \end{pmatrix} \circ \text{Arch. meas.}$
Theorem (AHRV)
scaled permutation matrix at time $tN$
is supported within a certain octagon
with prob $\rightarrow 1$
as $n \rightarrow \infty$

$(1 - \frac{1}{2} \sqrt{3 - \varepsilon})n$
Tools in proofs:

1. Bijection (Edelman-Greene 1987)
   \{\text{sorting networks}\} \leftrightarrow \{\text{standard staircase Young tableaux}\}

   (jeu de taquin algorithm)

2. New result for limiting profile of random staircase Young tableau
   (from similar result for square tableaux, Pittel-Romik)
Why do we believe the conjectures?

The permutahedron: embedding of Cayley graph \((S_n, \text{n.n. swaps})\) in \(\mathbb{R}^n\):

\[\sigma \mapsto \sigma^{-1} = (\sigma^{-1}(1), \ldots, \sigma^{-1}(n)) \in \mathbb{R}^n\]

\(n=4\): embeds in \((n-2)\)-sphere

1...n and n...1 are antipodal

\(n=5\)
Conjecture (AHRV)
USN lies close to some great circle on the permutahedron with prob $\to 1$
as $n \to \infty$

e.g. $o(n)$ in $| |_\infty$

In fact simulations suggest more like $O(\sqrt{n})$!

(Again, not true for every sorting network, e.g. bubble sort)
Analagous (much easier) fact:

random shortest route

$1^{st}$ St & $1^{st}$ Ave to $n^{th}$ St & $n^{th}$ Ave

$\approx$ straight line as $n \to \infty$
**Theorem (AHRV)** If a (non-random) sorting network lies close to some great circle, then:

(o(n) in $| |_{\infty}$)

1. Trajectories $\approx$ Sine curves

2. Half-time permutation $\approx$ Archimedes measure

3. Swap process $\approx$ semicircle x Lebesgue

**Simulation**
Proof of Theorem:

close to great circle $\Rightarrow$

$\approx$ Sine trajectories (up to a time change)

$\Leftrightarrow$ $\approx$ rotating disc picture

projections uniform $\Rightarrow$ $\approx$ Archimedes

swap rate uniform $\Rightarrow$ rotation uniform

$\Rightarrow$ no time change

calculation $\Rightarrow$ semicircle law
Geometric Sorting Networks
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Goodman, Pollack (1980):
- all 4-item sorting networks are geometric
- but not all 5-item ones:
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- all 4-item sorting networks are geometric
- but not all 5-item ones:
Great circle conjecture says:
USN is \( \approx \) geometric as \( n \to \infty \)

but:

**Theorem** (Angel, H, Gorin, in prep)
\[
P(\text{USN is geometric}) \to 0 \quad \text{as} \quad n \to \infty
\]

**Proof**: in fact:
\[
P(\text{USN contains fixed swap pattern}) > 1 - e^{-cn}
\]

\[\text{e.g. Goodman-Pollack counterexample}\]
Subnetworks

1 2 3 4 5
Subnetworks
Subnetworks
Subnetworks
Subnetworks
Random Subnetworks

Take an $n$-item USN. Choose $m$ out of the $n$ items uniformly at random, indep. of USN.

Great circle conjecture $\Rightarrow$

$m$ fixed, $n \to \infty$:

random m-out-of-n network $\xrightarrow{d}$ geom. network of $m$ indep. points from Archimedes distn.
Conjecture (Warrington, 2009)

\[ P \left( \begin{array}{c} \text{random 4-out-of-} \normalsize n \\ \text{network} \end{array} \in \left\{ \text{geom. networks with 1 point in} \right. \right. \]

\[ \left. \left. \text{hull(other 3)} \right\} \right) = \frac{1}{4} \]

for all \( n \)!
Theorem (Angel, H 2009)
Warrington’s conjecture is true.

Moreover, $\forall j < m \leq n,$

$$E(\text{# swaps in location } j \text{ in random } m\text{-out-of-}n \text{ network})$$

does not depend on $n$

and

$$= \frac{(j - \frac{1}{2}) \cdots \frac{7}{2} \frac{5}{2} \frac{3}{2} \cdots \frac{7}{2} \frac{5}{2} \frac{3}{2} \cdots}{(j - 1)! \times (m - j - 1)!}$$

consistent with Archimedeses distribution conjecture about $n \to \infty$ limit
Ingredients of proof

\[ P(s_1 = k) = P(k-1 \text{ white balls added in first } n-2 \text{ in Polya urn}) \]

1\textsuperscript{st} swap location in USN

Stationarity of USN

**Exchangeability** of Polya urn

\[ P(wwwbb) = P(wbwbw) \]

Compute

\[ P(\text{given space-time point in USN} \Rightarrow \text{swap at location } j \text{ in subnetwork}) \]
Uniform swap model...

Angel, H, Romik 2008
Amir, Angel, Valko
N.B. Not every sorting network lies close to a great circle! E.g. typical network through

(But this permutation is very unlikely).
Staircase Young diagram:

$\begin{array}{cccc}
\text{N cells} \\
\hline
\text{(E.g. } n=5) \\
\end{array}$
Standard staircase Young tableau:

```
1 2 4 8
3 5 6
7 10
9
```

Fill with 1, \ldots, N so each row/col increasing
Edelman-Greene algorithm:

1. Remove largest entry
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Edelman-Greene algorithm:

2. Replace with larger of neighbours
Edelman-Greene algorithm:

2. Replace with larger of neighbours \(\uparrow \leftrightarrow\)
Edelman-Greene algorithm:

2. Replace with larger of neighbours \[\uparrow \leftarrow\] ...repeat
Edelman-Greene algorithm:

2. Replace with larger of neighbours \( \uparrow \leftarrow \) ...repeat
Edelman-Greene algorithm:

3. Add 0 in top corner
Edelman-Greene algorithm:

4. Increment
Edelman-Greene algorithm:

5. Repeat everything...
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5. Repeat everything...
Edelman-Greene algorithm:

5. Repeat everything...
Edelman-Greene algorithm:
Edelman-Greene algorithm:

\[ \text{etc} \]
Edelman-Greene Theorem:

After $N$ steps,
Edelman-Greene Theorem:

After $N$ steps,
get swap process
of a sorting network!
Edelman-Greene Theorem:

After $N$ steps,
get swap process
of a sorting network

And this is a bijection!

And can explicitly describe inverse!
Theorem (Pittel-Romik): For a uniform random $n \times n$ square tableau, there exists a limiting shape with contours.

$$h_\alpha(u) = \frac{2}{\pi} [u \tan^{-1}(u/R) + \tan^{-1} R]$$

where $R = \frac{\sqrt{\alpha(2 - \alpha) - u^2}}{1 - \alpha}$

Corollary (AHRV): For uniform random staircase tableau, limiting shape is half of this. (Proof uses Greene-Nijenhuis-Wilf Hook Walk)
Proof of LLN (swap process $\Rightarrow$ semic. x Leb.)

Swaps in space-time window $[an, bn] \times [0, \varepsilon N]$ come from entries $>(1-\varepsilon)N$ in tableau:

$\# \approx \text{area under contour} \approx \text{semicircle}$
Proof of octagon and Holder bounds

Inverse Edelman-Greene bijection
(≈ RSK algorithm) ⇒

# entries <k in 1st row

≥ longest ↦ subseq. of swaps
  by time k

≥ furthest any particle moves up
  by time k

So can bound this using
limit shape.
Angel, H, Virag (in preparation):

Process of first $k$ swaps in positions $cn \ldots cn+k$\text{→} random limit as $n \to \infty$

not depending on $c \in (0,1)$