Ricci Flow and the Geometrization Conjecture for 3–manifolds

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References

1 3-manifolds and the Geometrization conjecture


2 Hamilton’s Ricci Flow - a short selection


3 Survey articles (and a big book) on the Ricci Flow


4 Perelman’s papers and two relevant/related papers


5 Articles/Talks/Web-sites discussing Perelman’s work


Compact orientable Surfaces

- $S^2$ round metric $s_1^2 \to \mathbb{R}^3$
  - Constant curvature $= 1$

- $T^2$ flat metric $T^2 = \mathbb{R}^2 / \Gamma$
  - Constant curvature $= 0$

- hyperbolic metric
  - $\Sigma_g = \mathbb{H}^2 / \Gamma$
  - Constant curvature $= -1$

Uniformization Theorem: Every compact surface

(a) Admits a metric of constant Gauss curvature $+1$, $0$ or $-1$

(b) Is a quotient of $S^2$, $\mathbb{R}^2$ or $\mathbb{H}^2$ by a discrete group $\Gamma$ of isometries acting freely.
Geometric Structures on 3-manifolds

- A **homogeneous manifold** \((M, g)\) is a Riemannian manifold whose group of isometries acts transitively.

- A **locally homogeneous manifold** \((M, g)\) is one whose universal cover is homogeneous \((M, g)\) is locally isometric to a homogeneous manifold.

- \(M\) has a **geometric structure** if there is a metric \(g\) so that \((M, g)\) is a finite volume homogeneous manifold.

In dimension 3 there are 8 homogeneous geometries with finite volume quotients.
The 8 Geometries

I. The constant curvature geometries

1. $\mathbb{R}^3$ (Bianchi I) \( \text{Curv} = 0 \)
   - quotients are flat manifolds $\mathbb{T}^3/\Gamma$, $\Gamma$ finite

2. $S^3$ (Bianchi IX) \( \text{Curv.} = 1 \)
   - quotients include $\mathbb{RP}^3$, lens spaces

3. $H^3$ (Bianchi V) \( \text{Curv} = -1 \)
   - quotients are compact or finite volume hyperbolic manifolds

II. The product geometries

4. $H^2 \times \mathbb{R}$ (Bianchi III)
   - quotients are trivial $S^1$ bundles over hyperbolic surfaces

5. $S^2 \times \mathbb{R}$ (Kantursbi - Sachs)
   - quotients are $S^2 \times S^1$ or $\mathbb{RP}^3 \neq \mathbb{RP}^3$
III. The twisted product Geometries

6. Nil = left invariant metric on the (Bianchi II) nilpotent Heisenberg group
   \[
   \begin{pmatrix}
   1^* & \ast & \ast \\
   \ast & 1 & \ast \\
   0 & 0 & 1
   \end{pmatrix}
   \]
   quotients are non-trivial $S^1$ bundles over $T^2$, "nil-manifolds".

7. Sol = left invariant metric on the solvable group $\mathbb{R}^2 \ltimes \mathbb{R}^+$
   (Bianchi VIo) action of $t \in \mathbb{R}^+$ on $\mathbb{R}^2$ is given by
   \[
   \begin{pmatrix}
   e^t & 0 \\
   0 & e^{-t}
   \end{pmatrix}
   \]
   quotients are non-trivial $T^2$ bundles over $S^1$, "solu-manifolds".

8. $\widetilde{SL(2,\mathbb{R})}$ (Bianchi VIII)
   - left invariant metric on the universal covering group of $SL(2,\mathbb{R})$
   quotients are non-trivial $S^1$-bundles over $\Sigma_g$, a hyperbolic surface
   $g \geq 2$
Compact (orientable) 3-manifolds

I. Prime Decomposition (Kneser, Milnor)

M admits a finite connected sum decomposition

\[ M \cong \left( \Sigma_1/\Gamma_1 \# \ldots \# \Sigma_k/\Gamma_k \right) \# (S^2 \times S^1) \# \ldots \# (S^2 \times S^1) \]

\[ \# (K_1 \# \ldots \# K_p) \]

where

- each \( \Sigma_i \) = homotopy 3-sphere (possible counter-example to Poincare' conjecture)
- \( \Gamma_i \) acts freely on \( \Sigma_i \)
- each \( K_j \) = aspherical 3-manifold \( K(\pi,1) \)
  \( ( \tilde{K}_j \) is contractible \)

Poincare' & Spherical space-form conjectures

\[ \Sigma_i = S^3 \]

\[ \Gamma_i \subset SO(4) \]

What about the \( K_j \)'s?
II. Torus decomposition (Jaco-Shalen, Johannsen)

If $M$ is a closed, prime 3-manifold, then $\exists$ finite (possibly $\emptyset$) collection of disjoint, incompressible $T^2 \hookrightarrow M$ which separate $M$ into a finite collection of 3-manifolds with toral boundaries, each of which is torus irreducible or "Seifert fibered".

(M "Seifert fibered" has a locally free $S^1$ action; these are geometric $\triangleright$)

Thurston's Geometrization conjecture

$M$ closed, oriented 3-manifold. Then each component of the prime and torus decompositions admits a geometric structure.
• Assuming Poincare' & Spherical S-f Conj.
  each of the pieces, except the Kj's
  in the prime decomposition are "geometric".

  The Kj's need not be.

  Suppose \(_M\) is prime (irreducible).

  (a) A surface \(i: \Sigma^2 \hookrightarrow M\) is said to
  be incompressible if.

  \[ i_*: \pi_1(\Sigma^2) \hookrightarrow \pi_1(M) \text{ injective.} \]

  "essential" \(S^2\)'s and incompressible \(T^2\)'s
  are obstructions to the existence of
  a geometric structure on \(M\).

  (b) A 3-manifold \(N\) (possibly with
  boundary \(\partial N\)) is called
  torus irreducible if every embedded
  incompressible \(T^2\) may be
  deformed to a \(T^2 \hookrightarrow \partial N\).
• Thurston and others proved geometrization in a number of important cases
e.g. if \( M \) is Haken manifold
  \((M \text{ is prime and } \exists \text{ an incompressible} \Sigma_g \hookrightarrow M, \ g \geq 1)\)

• The geometrization conjecture implies both the Poincare' conjecture and the Spherical Space-form Conjecture

• Since different geometries do not fit together smoothly (along incompressible tori or otherwise) must incorporate degeneration of geometric structures on noncompact manifolds.

  \[
  \text{incompressible } T^2 \hookrightarrow \text{geometric singularities}
  \]

  cusps at \( \infty \)
Hamilton's Ricci Flow

Consider (*) \[ \frac{\partial g(x,t)}{\partial t} = -2 \text{Ric}(g(x,t)) \]

- Morally regard this as a geometric heat equation for the metric

In harmonic coordinates

\[ (x^k, 1 \leq k \leq n, \Delta g x^k = 0) \]

\[ -2 \text{Ric}(g) = \Delta g g + Q(g, \omega g) \]

- a cheat since this is not tensorial and must evolve the coordinates

\[ \Rightarrow \text{Expect:} \]

- **Smoothing**: heat equation regularizes initial data

- **A diffusion process**: tries to make the metric more homogeneous & isotropic
• Ricci flow is invariant under the diffeomorphism group
  • good for geometry (e.g. geometric manifolds remain geometric)
  • complicates the analysis
  - after taking this into account (DeTurck)
  can show (*) is a quasilinear parabolic equation
  \[ \Rightarrow \text{short time existence \& uniqueness} \]

• In general, expect **singularities** to occur in **finite time**!

• Stationary points of the volume-normalized Ricci flow are **Einstein metrics**. \[ n = 3 \text{ Einstein} \Rightarrow \text{Constant Curvature} \]

• Under \((*)\), the Ricci flow, the (scalar) curvature evolves by
  \[ \frac{\partial R}{\partial t} = \Delta R + 2 |Ric|^2 \]
  \[ \text{Diffusion Reaction} \]
  the reaction term sometimes wins.
Specific example

\((M, g(t))\) a sphere of radius \(r(t)\) = \((S^n, r^2(t) \, g_0)\)

\(\text{standard metric on } S^n \hookrightarrow \mathbb{R}^n\)

\(\text{Ric}(g(t)) = (n-1) \, g_0\)

(independent of \(r\))

Ricci flow becomes

\[
\frac{dr^2}{dt} = -2 (n-1)
\]

Solution: \(r^2(t) = r^2(0) - 2(n-1) \, t\)

Sphere collapses to a point in finite time \(t = \frac{r^2(0)}{2(n-1)}\)

"Pop!"
More generally expect "neck pinches"

⇒ Analyze blow up at a singularity
get "soliton" solutions

• Solutions which evolve via a one parameter family of diffeomorphisms

-need to understand these!
Hamilton's space-form Theorem

If $g(0)$ is a metric on a 3-manifold $M$ with $\text{Ric}(g(0)) > 0$, then the volume normalized Ricci flow exists for all time and converges to the round metric on $S^3/\Gamma$ where $\Gamma$ is a finite subgroup of $SO(4)$ acting freely on $S^3$.

- Hamilton also the case $\text{Ric}(g(0)) \geq 0$

- This established geometrization for 3-manifold admitting a metric with $\text{Ric}(g) \geq 0$.

In the following 20 years, Hamilton did a great deal of work in analysing other cases of the Ricci flow.

e.g. Geometrization for nonsingular solutions (Hamilton 1999)
A solution \((M, g(t))\) to the volume normalized Ricci flow is nonsingular if

- It exists for all time
- \(\sup_{M^3 \times [0, \infty)} |Ric| \leq C < \infty\)

\(\Rightarrow\) \(M\) is geometrizable

Proof uses Gromov-Hausdorff convergence and the notion of "collapse with bounded curvature" (Cheeger-Gromov F-structures).

Hamilton laid out a program to prove the full geometrization conjecture by using the Ricci flow with an arbitrary initial metric.

- A number of very difficult and important technical issues were left unresolved.
Perelman's work

Sketch of the Program.

1. Show that the singularities which occur are standard

   cup:

   neck:

   In particular Perelman rules out the "cigar soliton"

2. Locate where singularities are occurring (before they occur!) and perform surgery there
3. After each surgery:
   - discard the pieces with positive curvature (covered by Hamilton's theorem)
   - continue the flow on the others

4. Show that this process terminates

5. For \( t \gg 0 \) show that each remaining component splits along incompressible tori into
   \[ M_{\text{thick}} \cup M_{\text{thin}} \]

   - \( M_{\text{thick}} \rightarrow \) complete finite volume hyperbolic manifolds
   - \( M_{\text{thin}} \rightarrow \) collapse with bounded curvature

6. \( M_{\text{thin}} \) is a graph manifold (geometrizable)
One of the new ideas from Perelman

Guiding principle:
"Variational structures are good"

Problem: The Ricci flow is not the gradient flow for any functional.

Critical points must be Einstein metrics, only possibilities are:

Einstein-Hilbert functional with Cosmological constant $\Lambda$

$$S_{EH} = \int_{M} (R(g) - 2\Lambda) \, dvol$$

(gradient flow of $S_{EH}$ does not exist, since it implies a backwards heat eq'n for $R$)

Perelman: Consider the functional

$$E(g,f) = \int_{M} (R + \nabla f^2) \, e^{-f} \, dvol$$

on $\text{Met} \times C^\infty(M, \mathbb{R})$
Fix a smooth measure $dm$ on $M$ and require that $(g, f)$ satisfy the "Perelman coupling"

$$e^{-f} d\nu_g = dm$$

$\Rightarrow$ for any $g \in \text{Met}$, $\exists f$ (or $dm$) such that the gradient flow of

$$\Phi^m(g, f) = \int_M (R + |\nabla f|^2) dm$$

as a functional on $\text{Met}$, exists and is given by

$$\frac{\partial \tilde{g}}{\partial t} = -2 (\text{Ric} + \nabla^2 f)$$

$\text{with } \nabla^2 f = \text{Hess}_g(f)$

This is the Ricci flow up to diffeom.

$$(\nabla^2 f = \frac{d}{dt} (\phi_t^* \tilde{g}), \quad \frac{d}{dt} \phi_t = \nabla f)$$
This is a functional whose critical points are Einstein metrics and Ricci solitons!

—this is very useful.

Note: The functional $\mathcal{F}(g, f)$ arises in string theory (?)\textsuperscript{(!)}\textsuperscript{(!)}\textsuperscript{(!)}

"low energy effective action"

The function $f$ (scalar field) is called the "dilation".