Optical Tomography on Simple Riemannian Surfaces.

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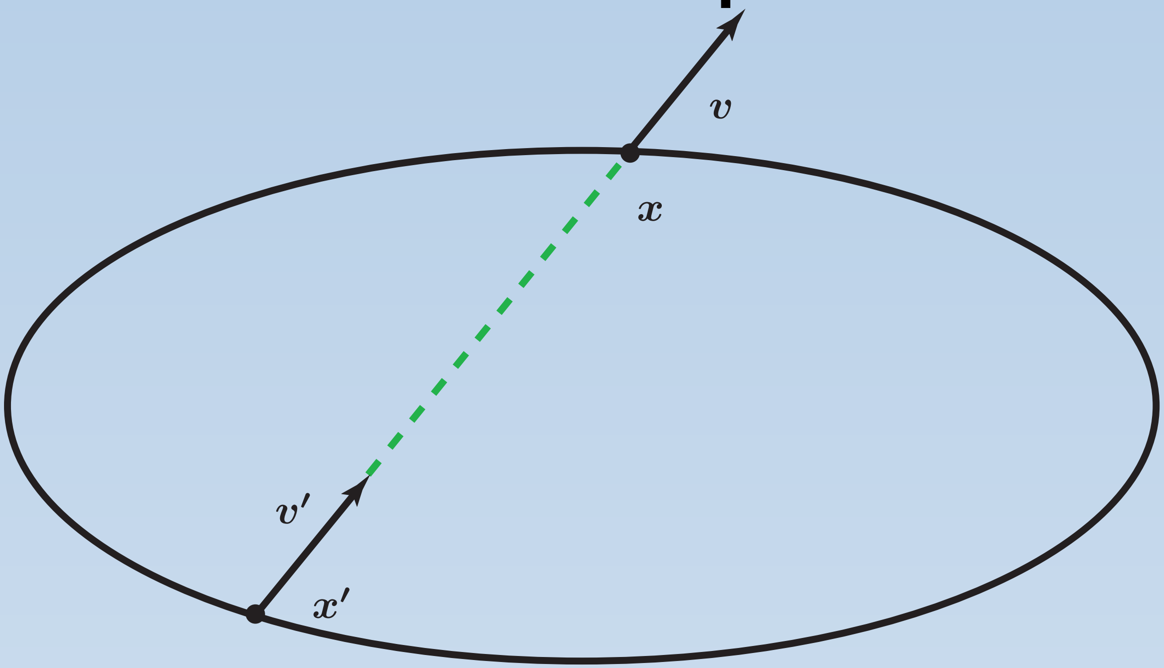
The forward problem.



- No interaction;
- Euclidean background (constant index of refraction).

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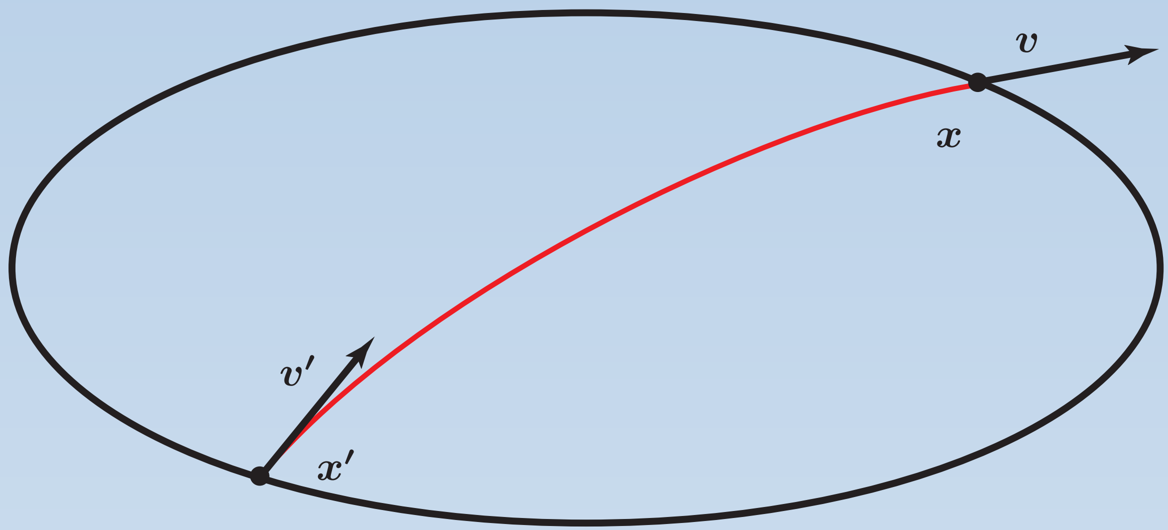

- No interaction;
- Background Riemannian metric (varying index of refraction).

The forward problem.

- Let M be a bounded domain with smooth boundary and trivial topology;
- let g be a Riemannian metric on M;
- denote by $\Omega_x M$ the unit tangent sphere at $x \in M$ and by ΩM be the unit tangent sphere bundle;

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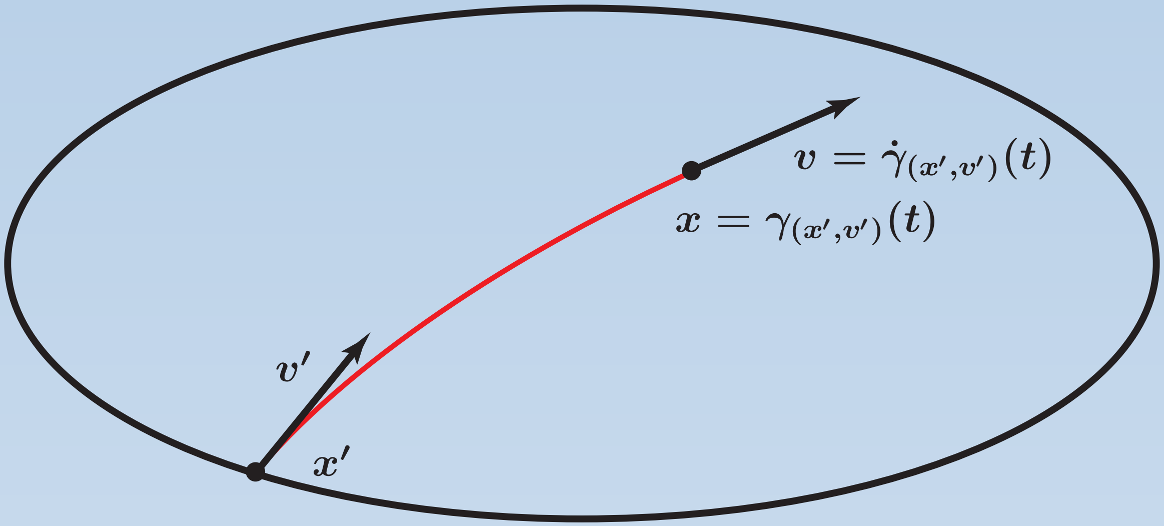
- Let M be a bounded domain with smooth boundary and trivial topology;
- let g be a Riemannian metric on M;
- denote by $\Omega_x M$ the unit tangent sphere at $x \in M$ and by ΩM be the unit tangent sphere bundle;
- define the sets of *incoming* (Γ_-) and *outgoing* (Γ_+) directions on the boundary

$$\Gamma_\pm = \{(x,v) \in \Omega M \mid x \in \partial M,\ \pm\langle v, \nu\rangle_{g_x} > 0\}$$

where ν is the outer unit normal to ∂M.

The forward problem.



- No interaction; a particle (or photon) enters M at $x' \in \partial M$ with velocity v';
- it follows the **geodesic** described in the sphere bundle by

$$\vec{\gamma}_{(x,v)}(t) = \left(\gamma_{(x',v')}(t), \dot{\gamma}_{(x',v')}(t)\right)$$

where $\gamma_{(x',v')}(0) = x',\ \dot{\gamma}_{(x',v')}(0) = v'$.

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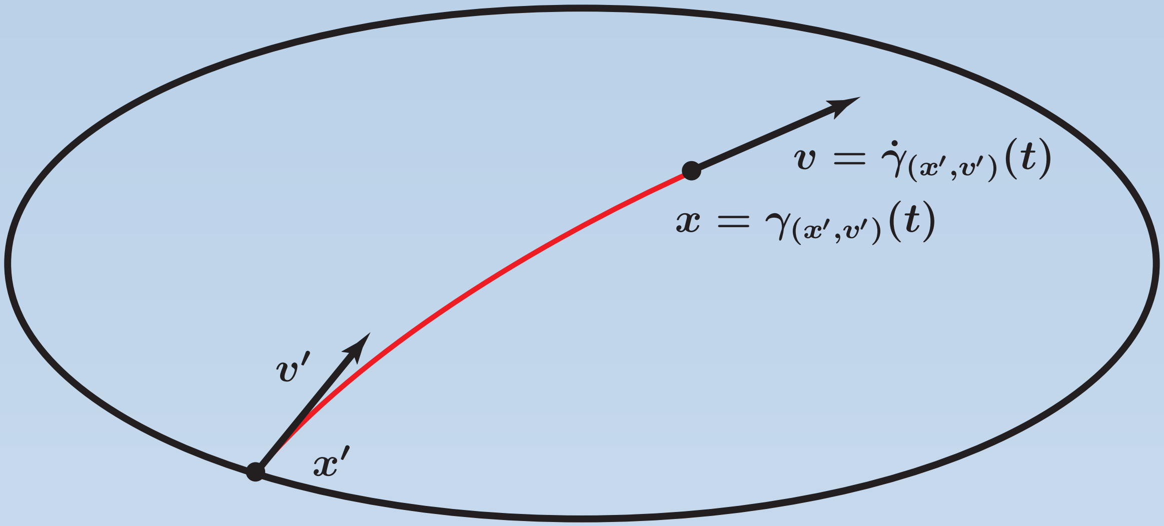

- No interaction; a particle (or photon) enters M at $x' \in \partial M$ with velocity v';
- If $f(x, v)$ is the density of particles at $(x, v) \in \Omega M$ then this free motion is described by

$$\mathcal{D}f(x,v) = \frac{d}{dt}\Big|_{t=0} f(\vec{\gamma}_{(x,v)}(t)) = 0.$$

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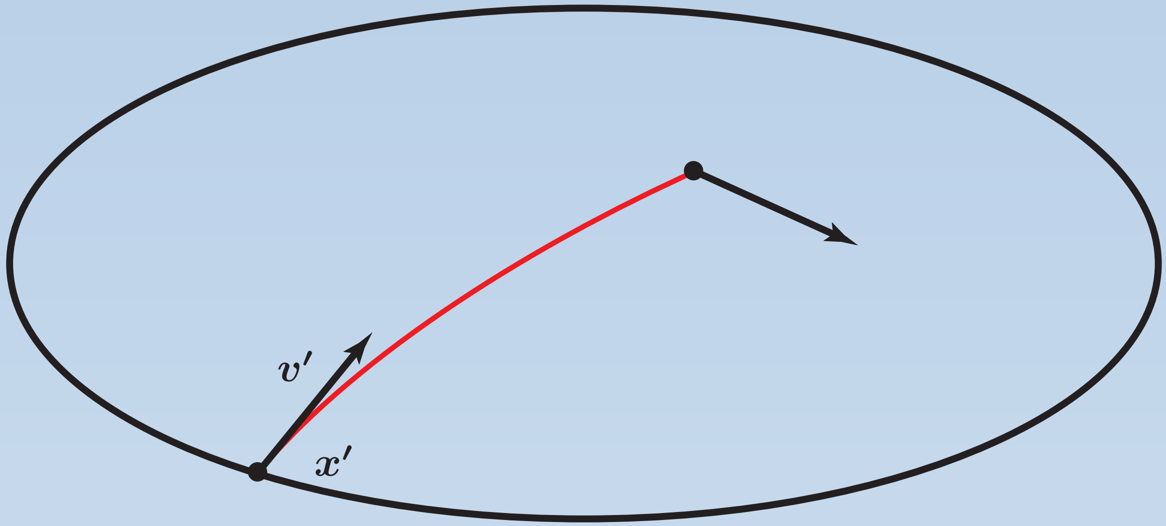

- At any given point, it may be absorbed or scatter to a new direction.

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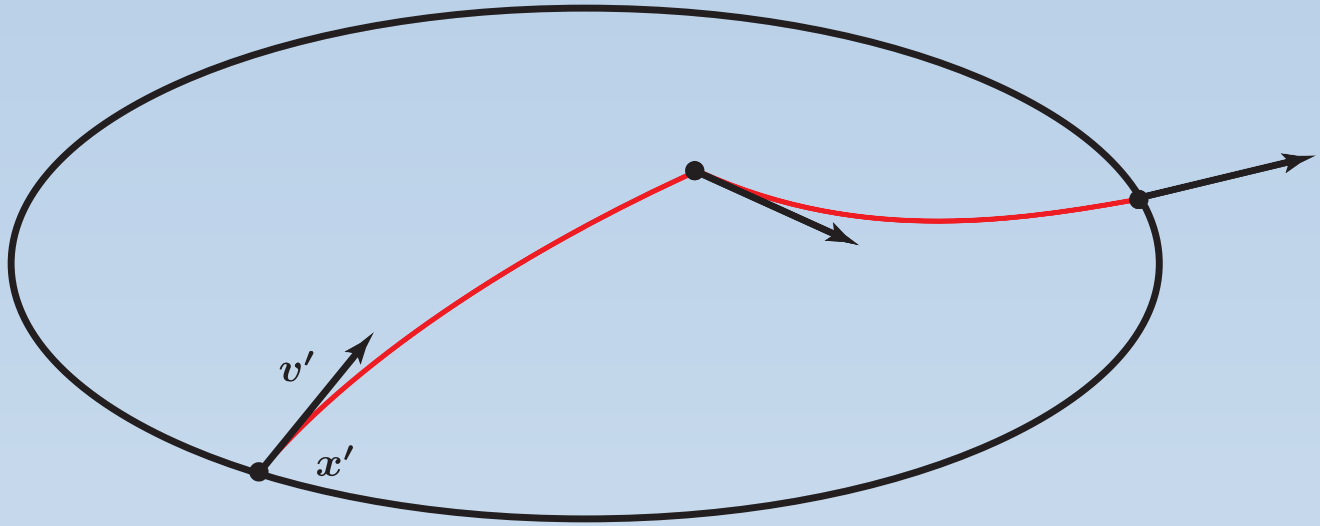

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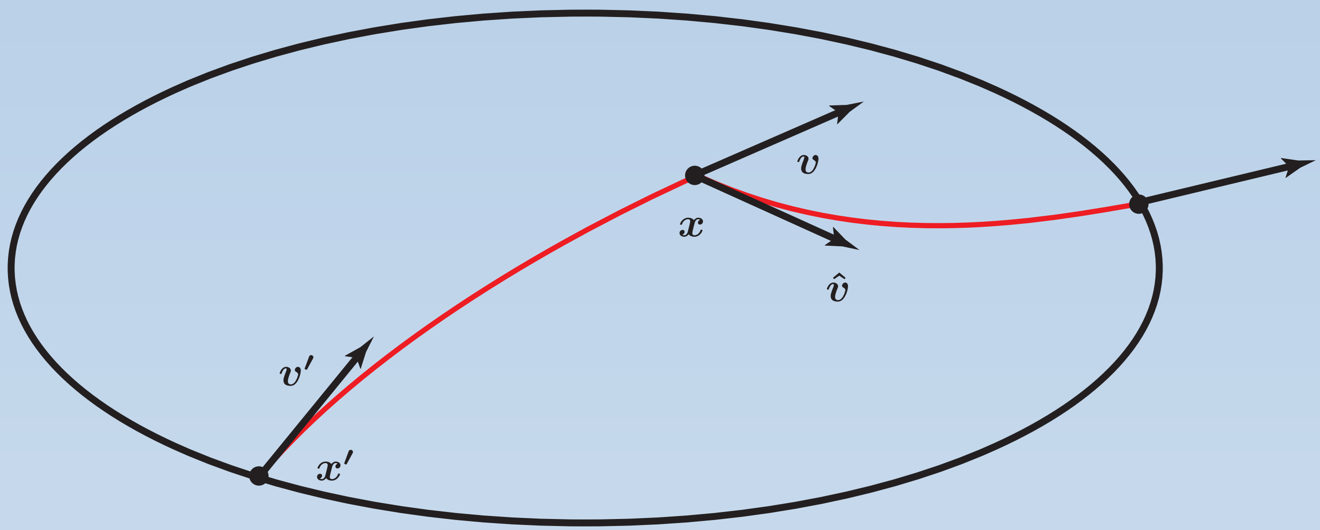

- Let $\sigma_a(x, v)$ describe the rate of absorption, and $k(x, v, \hat{v})$ describe the probability of a particle at x with velocity v scattering to velocity $\hat{v}$.

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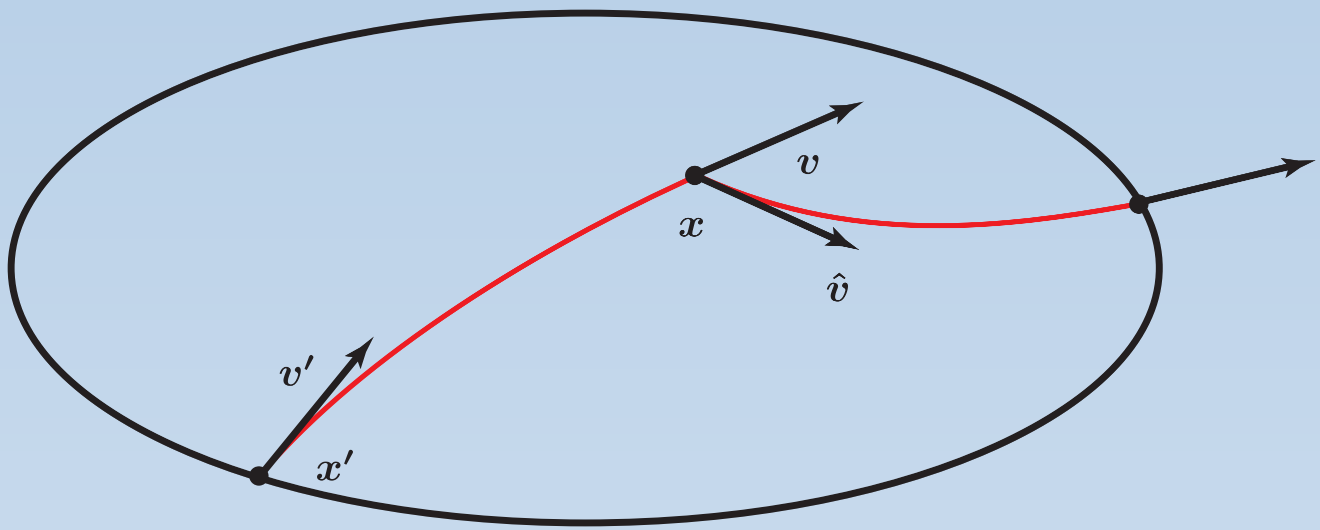

- Let $\sigma_a(x, v)$ describe the rate of absorption, and $k(x, v, \hat{v})$ describe the probability of a particle at x with velocity v scattering to velocity $\hat{v}$.

- If $f_-(x, v) \in C^\infty(\Gamma_-)$ describes the distribution of particles *entering* M, let $f(x, v)$ describe the resulting distribution of particles throughout M. Then f solves the linear stationary transport equation (TE)

$$-\mathcal{D}f(x, v) - \sigma_a(x, v)f(x, v) + \int_{\Omega_x M} k(x, v', v)f(x, v')dv' = 0,$$

$$f\big|_{\Gamma_-} = f_-.$$

The inverse problem.

Given $f_-(x,v) \in C^\infty(\Gamma_-)$, let f solve (TE) and define the *albedo map* $\mathcal{A}$ which maps f_- to the distribution of particles *leaving* M,

$$\mathcal{A}f_-(x,v) = f(x,v)\big|_{\Gamma_+}.$$

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Does $\mathcal{A}$ determine $\sigma_a(x, v)$ and $k(x, v', v)$ uniquely?

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Does $\mathcal{A}$ determine $\sigma_a(x,v)$ and $k(x,v',v)$ uniquely?

The answer is certainly “no” for general σ_a so we assume that $\sigma_a = \sigma_a(x)$ depends only on position x.

Inverse problem – the Euclidean case.

When g is Euclidean, Choulli and Stefanov [CS] (Osaka J. Math. 36 (1999)) proved (under mild assumptions on σ_a and k) that $\mathcal{A}$ uniquely determines

- $\sigma_a(x, |v|)$ and $k(x, v', v)$ in dimensions $n \geq 3$, and
- $\sigma_a(x, |v|)$ for dimension $n = 2$.

In dimension 2, Stefanov and Uhlmann [SU] proved that if M is convex and if k is assumed to be small relative to σ_a then $\mathcal{A}$ also uniquely determines $k(x, v', v)$. They also give an explicit constant describing the relative smallness and present a stability result.

Also in dimension 2, Tamasan proved uniqueness for a homogeneous kernel [T1] and for a “weakly anisotropic” kernel [T2]. Stability results are included in these, and further stability results are proven by Wang and Romanov.

Inverse Problem – the Riemannian case.

Following the ideas of [CS] it is proven (M, Pac. J. Math, 2004) that for a known “simple” metric g, $\mathcal{A}$ uniquely determines

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Here we follow the ideas of [SU] to prove that for a known simple metric g with no focal points, $\mathcal{A}$ uniquely determines $k(x, v', v)$ in dimension 2.

Assumptions on (M, g).

M1. (M, g) has no *focal points*: for every geodesic $\gamma : [a, b] \longrightarrow M$ and every non-zero Jacobi field $J(t)$ along γ satisfying $J(a) = 0$, $\|J(t)\|$ is a strictly increasing function on $[a, b]$.

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On the sphere of constant curvature $\kappa_0 > 0$ such Jacobi fields grow as $\|J(t)\| = \sin(\sqrt{\kappa_0}\, t)$ and so the diameter A of M is limited to a quarter of the circumference of the sphere.

If $\kappa \leq 0$ then $\|J(t)\|$ is always strictly increasing.

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M1. (M, g) has no *focal points*: for every geodesic $\gamma : [a, b] \longrightarrow M$ and every non-zero Jacobi field $J(t)$ along γ satisfying $J(a) = 0$, $\|J(t)\|$ is a strictly increasing function on $[a, b]$.

M2. ∂M is strictly convex: the second fundamental form of the boundary is positive definite at every $x \in \partial M$

It follows that (M, g) is "simple." In particular, for any $x \in \bar{M}$ the exponential map $\mathbf{Exp}_x : \mathbf{Exp}_x^{-1}(\bar{M}) \longrightarrow \bar{M}$ is a diffeomorphism.

M3. The sectional curvature of (M, g) is bounded above by κ_0.

M4. The diameter A of (M, g) satisfies $A < \pi/(2\sqrt{\kappa_0})$.

Assumptions on (σ_a, k).

A1. σ_a depends only on x.

A2. $\sigma_a \in L^\infty(M)$, $k \in L^\infty(\{(y, v', v) \in M \times \Omega_y M \times \Omega_y M\})$, and $\|k\|_{L^\infty} \leq (2\pi \,\mathrm{diam}(M))^{-1}$.

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Define the class

$$\mathcal{U}_{\Sigma,\varepsilon} = \Big\{ \big(\sigma_a(x), k(x, w', w)\big) \mid \|\sigma_a\|_{L^\infty} \leq \Sigma,\ \|k\|_{L^\infty} \leq \varepsilon, \text{ and } (\sigma_a, k) \text{ satisfy A1, A2} \Big\}.$$

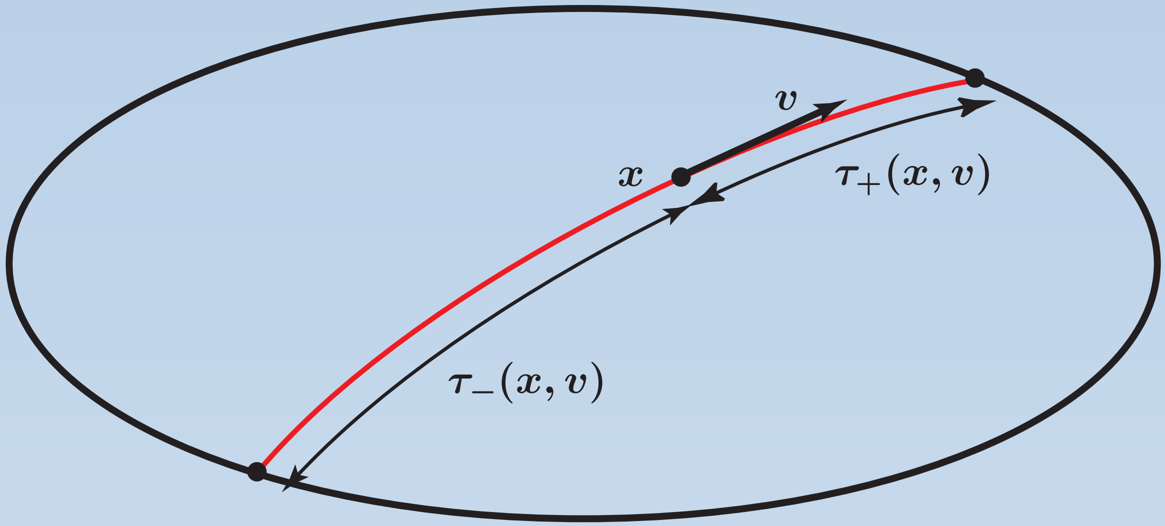
Result.

Theorem: (Comm. PDE., 2005)

Let (M, g) *satisfy assumptions M1–M4.*
(1) *If* (σ_a, k) *satisfy A1, A2 then the metric* g *is uniquely determined by the associated albedo operator* $\mathcal{A}$.
(2) *Given* $\Sigma > 0$ *there exists* $\varepsilon > 0$ *such that any pair* $(\sigma_a, k) \in \mathcal{U}_{\Sigma,\varepsilon}$ *is uniquely determined, within* $\mathcal{U}_{\Sigma,\varepsilon}$*, by the associated albedo operator* $\mathcal{A}$.

Furthermore, the ε *can be chosen to be* $\varepsilon = Ce^{-2A\Sigma}$ *where* C *depends only on* (M, g) *and* $A = \mathrm{diam}(M, g)$.

Preparations.



- Define the "*time to boundary*" functions $\tau_\pm : \Omega M \to \mathbb{R}^+$ by

$$\tau_\pm(x,v) = \min\{t \geq 0 \mid \gamma_{(x,v)}(\pm t) \in \partial M\},$$

and set $\tau = \tau_+ + \tau_-$.

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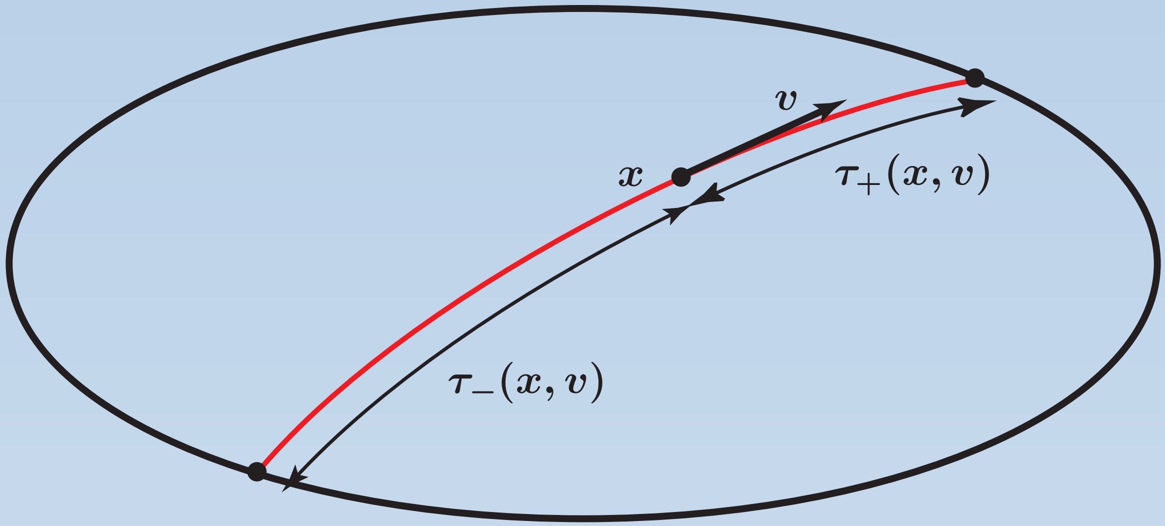

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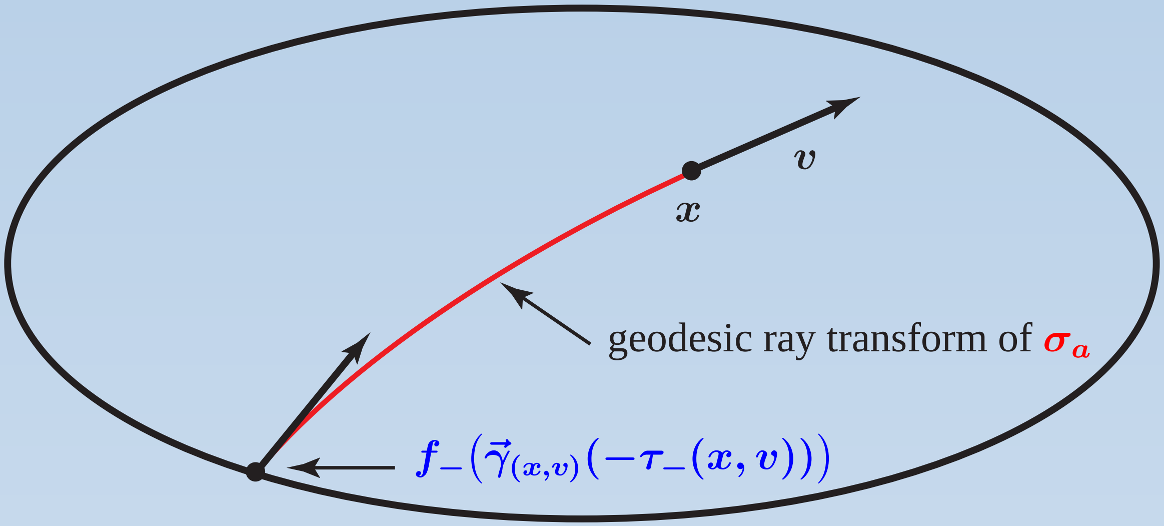
  and set $\tau = \tau_+ + \tau_-$.

- We define the measure $d\mu(x',v')$ on Γ_-: let $dv_x\, d\omega(x)$ be the *Liouville volume form* on ΩM (which is invariant under the geodesic flow) and then require that $d\mu(x',v')\, dt$ equals the pull–back of $dv_x\, d\omega(x)$ by the geodesic flow of Γ_- into ΩM.

An integral equation.



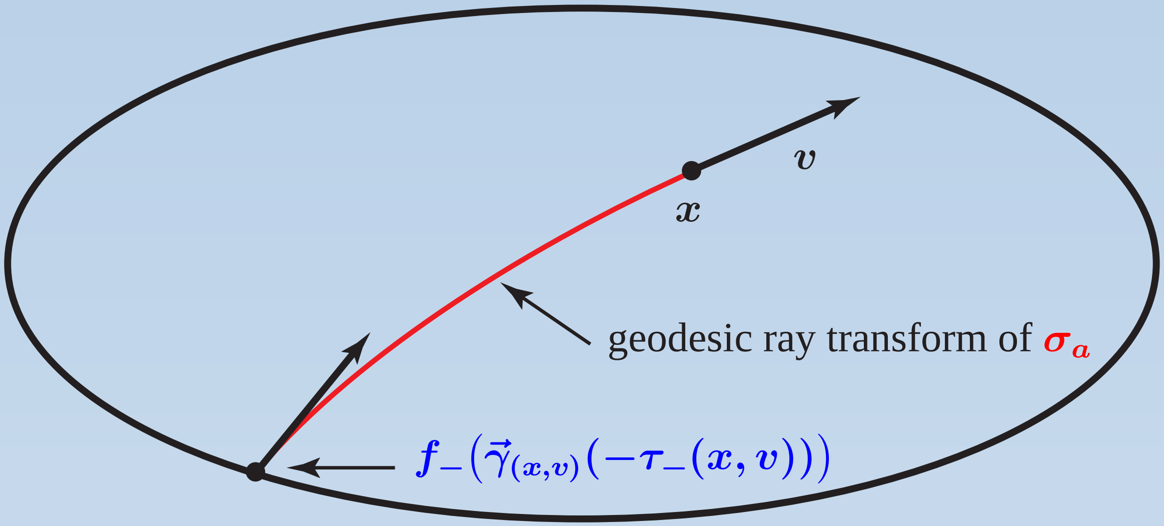
The solution to the homogeneous problem with no scattering

$$(\mathcal{D} + \sigma_a)f = 0, \qquad f|_{\Gamma_-} = f_-$$

is

$$Jf_-(x,v) = \exp\Big(-\int_0^{\tau_-(x,v)} \sigma_a(\gamma_{(x,v)}(s))\, ds\Big) f_-(\vec{\gamma}_{(x,v)}(-\tau_-(x,v)))$$

An integral equation.



If we define $E(x, v, t_1, t_2) = \exp\left(\int_{t_1}^{t_2} \sigma_a(\gamma_{(x,v)}(s))\, ds\right)$ then we re-write (TE) as an integral equation:

$$(\mathcal{D} + \sigma_a) = \int kf \qquad \Rightarrow \qquad \mathcal{D}(Ef) = E(\mathcal{D} + \sigma_a) = E \int kf$$

$$\Rightarrow \qquad f = E^{-1}\mathcal{D}^{-1}\left(E \int kf\right) =: -Kf$$

The albedo operator.

Taking into account the boundary condition, $(\boldsymbol{I} + \boldsymbol{K})\boldsymbol{f} = \boldsymbol{J}\boldsymbol{f}_-$.

The albedo operator.

Taking into account the boundary condition, $\big(I + K\big) f = J f_-$.

The bound on k guarantees $I + K$ is invertible on $L^\infty(\Omega M)$ and $f = (I + K)^{-1} J f_-$.

We find f has a well-defined trace and we may define the *albedo operator*

$$\mathcal{A} : L^1(\Gamma_-, d\mu) \longrightarrow L^1(\Gamma_+, d\mu), \qquad \mathcal{A} : f_- \mapsto f\big|_{\Gamma_+}.$$

Kernel Expansion

We calculate the **distribution kernel** of the solution operator for (TE):

$$\phi = \phi_0 + \phi_1 + \phi_2 = J\phi_- - KJ\phi_- + (I + K)^{-1}K^2 J\phi_-$$

where ϕ_- is a delta source on Γ_-.

The distribution kernel of $\mathcal{A}$ is then $\alpha = \phi|_{\Gamma_+ \times \Gamma_-}$.

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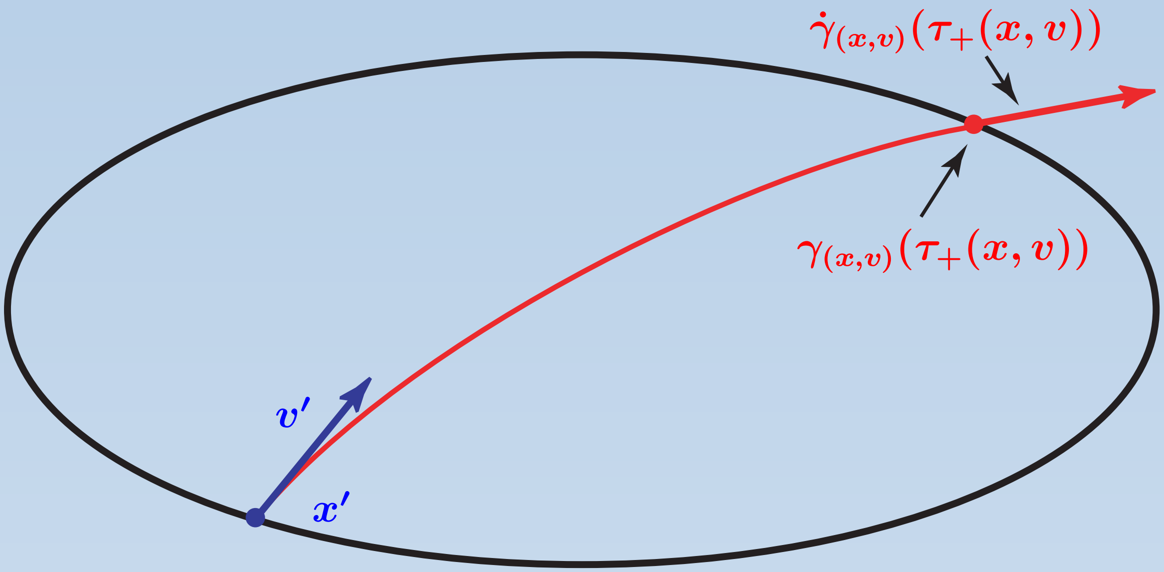
In [M], it is shown that

$$\alpha_0(x,v,x',v') = \int_0^{\tau_+(x',v')} E\big(x,v,0,-\tau_-(x,v)\big)\ \delta_{(x,v)}\big(\vec{\gamma}_{(x',v')}(t)\big)\ dt$$

(recall, $E\big(x,v,0,-\tau_-(x,v)\big)$ is the ray transform of the absorption σ_a).

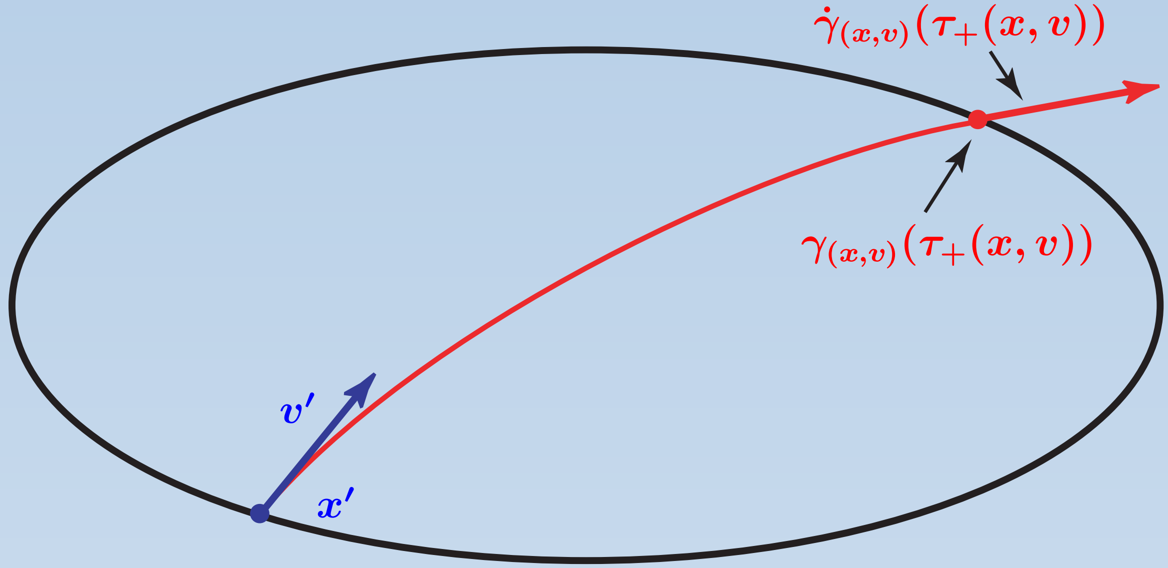
Determining the metric.



Thus, we can determine σ_a (and hence the function $E(\cdot)$), and the **scattering relation**

$$\mathcal{S} = \Big\{ \Big((x', v'), \big(\gamma_{(x',v')}(\tau_+(x',v')), \dot{\gamma}_{(x',v')}(\tau_+(x',v'))\big) \Big) \Big\}.$$

Determining the metric.



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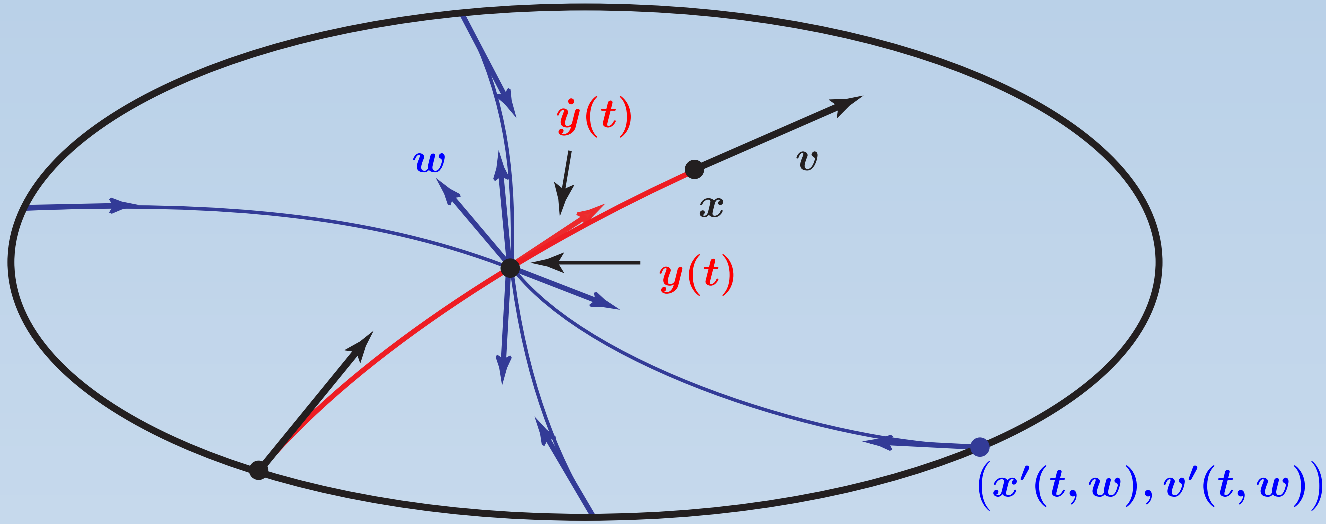
Pestov and Uhlmann have proven, in this setting, that $\mathcal{S}$ uniquely determines the metric g.

The term $\phi_1 = -KJ\phi_-$



For general f_-, $-KJf_-(x,v) =$

$$\int_0^{\tau_-(x,v)} E(\cdot) \int_{\Omega_{y(t)}M} k\big(y(t), w, \dot{y}(t)\big) E(\cdot) f_-\big(x'(t,w), v'(t,w)\big)\, dw\, dt$$

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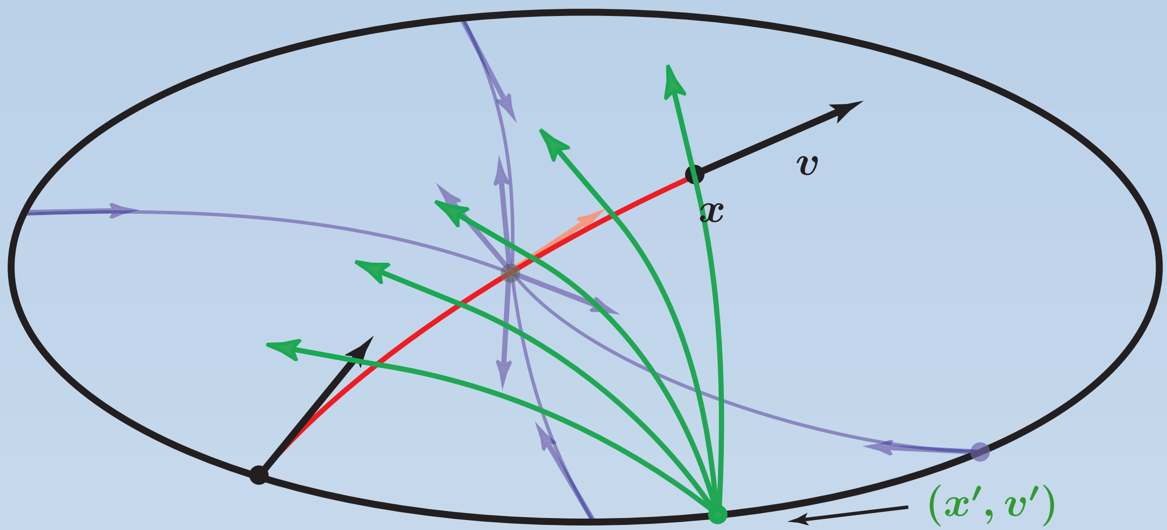

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To find the kernel, we need re-write this in terms integration over Γ_-; computing the change of volume element we obtain...

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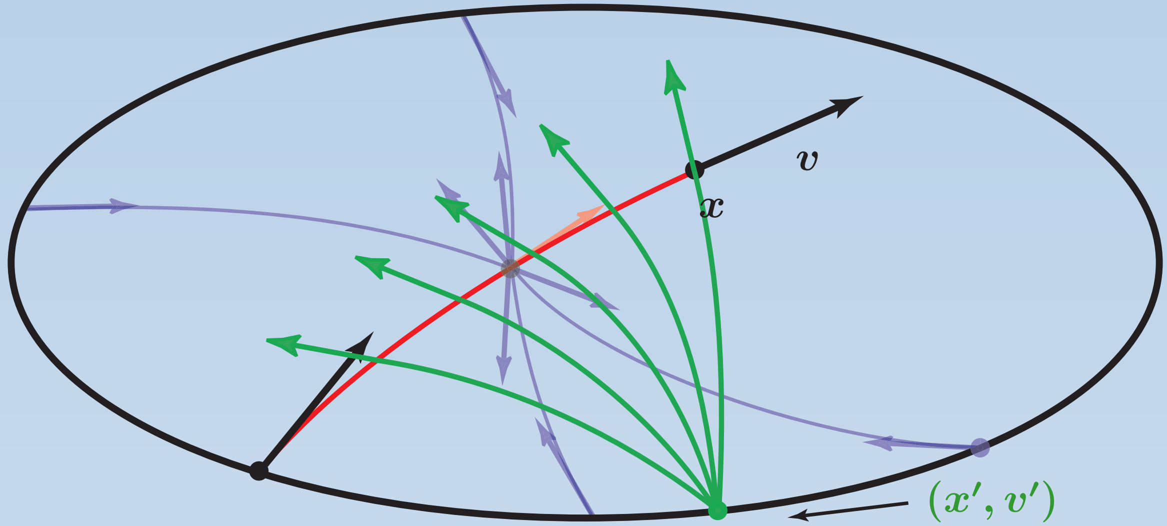

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$$= \int_{\Gamma_-} \chi\, E(\cdot)E(\cdot)k(\hat{x}, \hat{v}, \dot{y}) \frac{\mathcal{J}(x,v,x',v')}{|\sin(\psi(x,v,x',v'))|} f_-(x',v')\, d\mu(x',v')$$

The term $\phi_1 = -KJ\phi_-$

With $f_- = \phi_-$, the delta source,

$$\phi_1(x,v,x',v') = \chi(x,v,x',v')E(\cdot)E(\cdot)\mathcal{J}(\cdot)\frac{k(y,w,w')}{|\sin(\psi(x,v;x',v'))|}$$

where:

- $\chi(x,v,x',v') = 1$ if $\gamma_{(x,v)}$ and $\gamma_{(x',v')}$ intersect, and 0 otherwise,
- y is the point of intersection, and w, w' are the tangent vectors at y,
- and $\psi(x,v,x',v')$ is the angle of intersection.
- $\mathcal{J}(x,v,x',v')$ is uniformly bounded.

Outline of proof of theorem.

Suppose we have (M, g, σ_a, k) and $(M, g, \tilde{\sigma}_a, \tilde{k})$ with $\mathcal{A} = \tilde{\mathcal{A}}$.

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$$\alpha_2 - \tilde{\alpha}_2 = \tilde{\alpha}_1 - \alpha_1 = \chi E(\cdot) E(\cdot) \mathcal{J}(\cdot) \frac{(\tilde{k} - k)(y, w, w')}{|\sin(\psi(x, v; x', v'))|}$$

so that

$$\chi |\tilde{k} - k| \leq C_0 \chi |(\alpha_2 - \tilde{\alpha}_2) \sin \psi|.$$

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Analyzing $K^2 J \phi_-$ we will prove an estimate

$$\|(\alpha_2 - \tilde{\alpha}_2)\sin\psi\|_{L^\infty} \leq C\varepsilon \|k - \tilde{k}\|_{L^\infty}.$$

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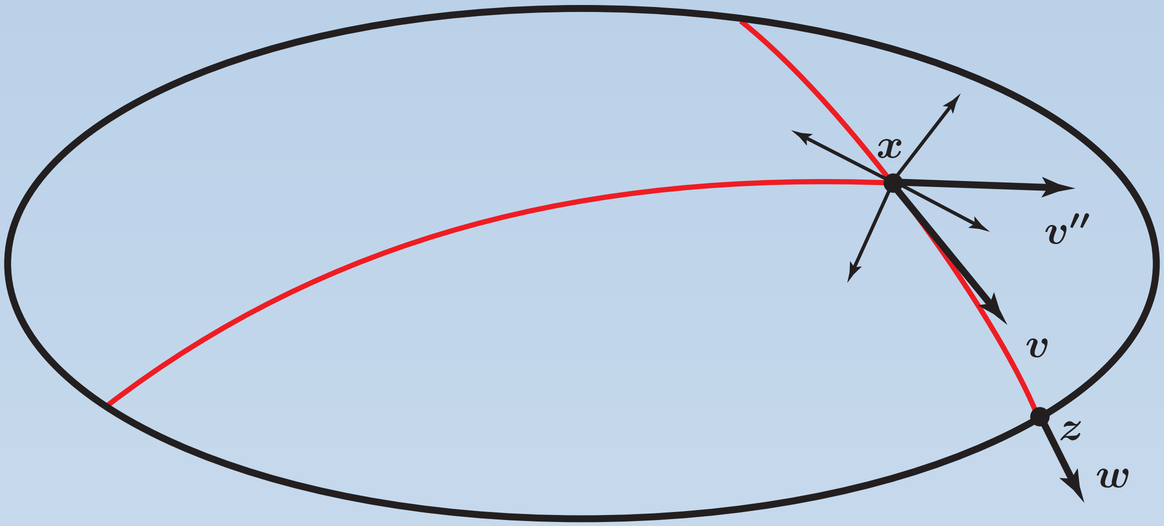
Combining these, $\|k - \tilde{k}\|_{L^\infty} \leq C\varepsilon\|k - \tilde{k}\|_{L^\infty}$ so that for sufficiently small ε, $k = \tilde{k}$.

Estimating $\phi_2 = K^2 J \phi_-$



$K^2 J \phi_-$ accounts for "double scattering"; for arbitrary f_-,

$$K^2 J f_-(z, w) = -\int_0^{\tau_-(z,w)} E(\cdot) \int_{\Omega_{x=\gamma(s)} M} k(x, v'', v) K J f_-(x, v'')\, dv''\, ds.$$

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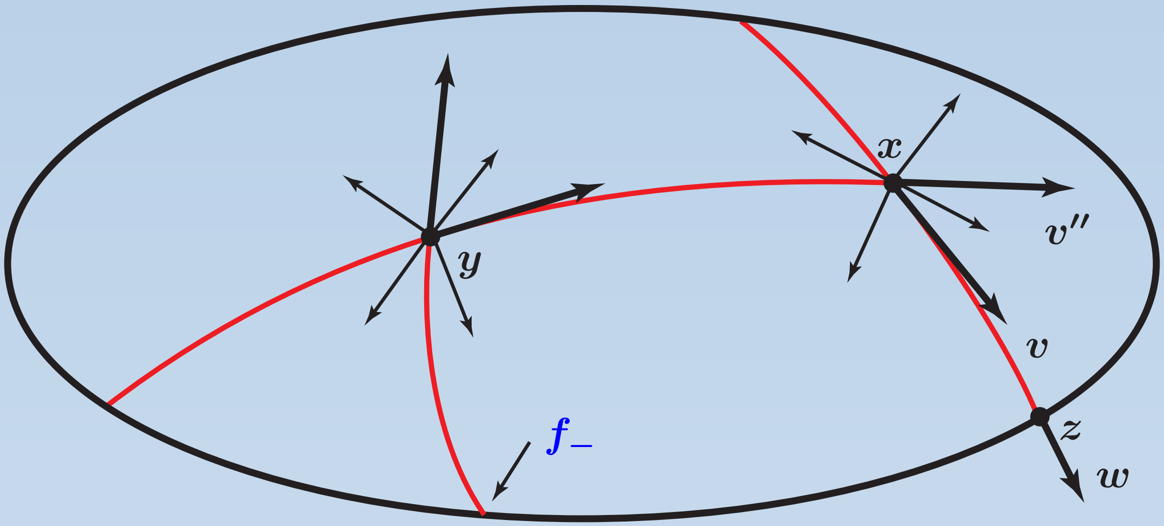

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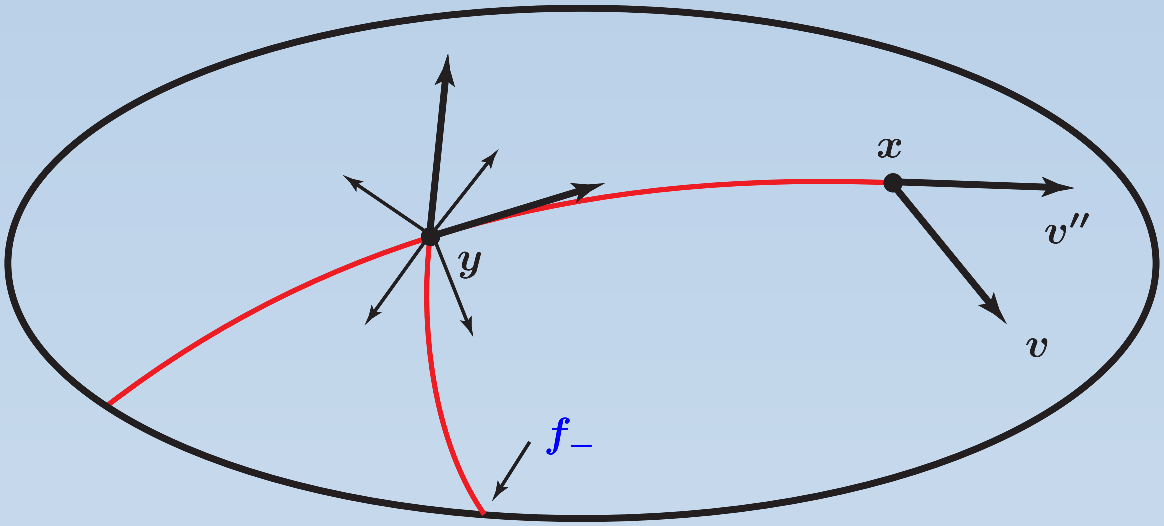

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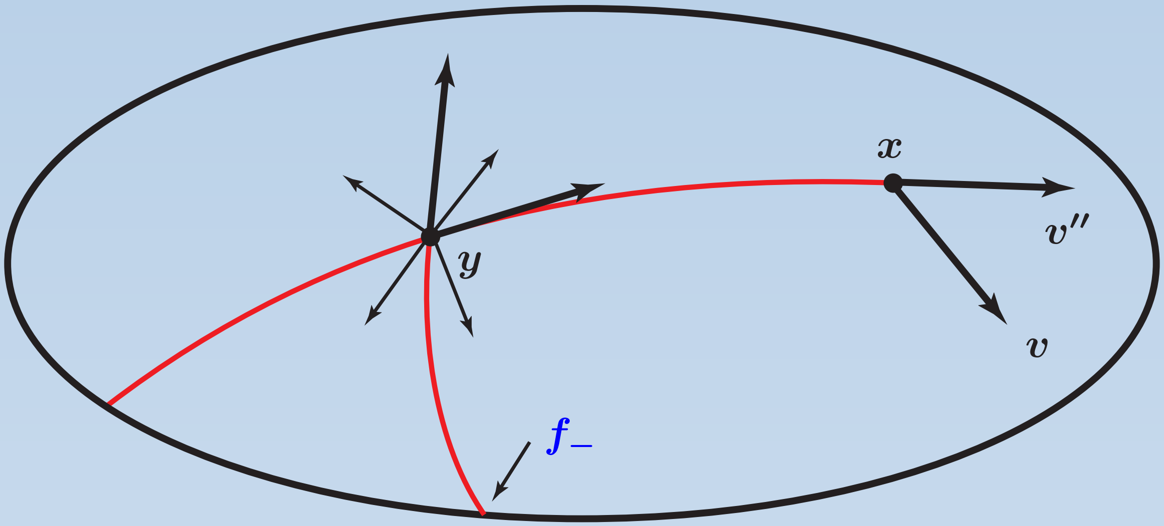


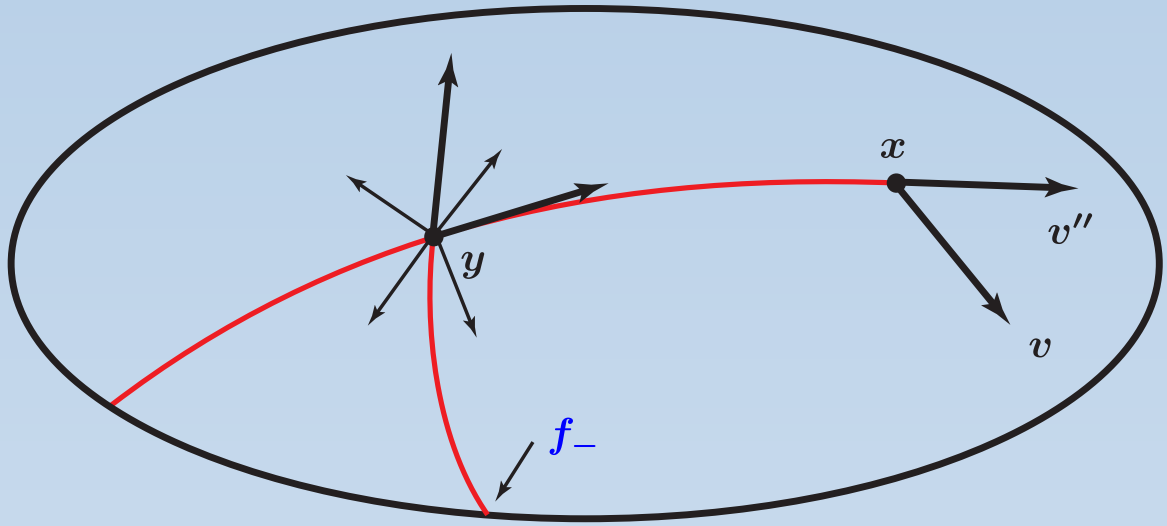
First,

$$(*) \quad \int_{\Omega_x M} k(x, v'', v) K J f_-(x, v'') dv' =$$

$$\int_{\Omega_x M} \int_0^{\tau_-(x,v'')} \int_{\Omega_y M} E(\cdot) k(\cdot) k(\cdot) J f_-(\cdot)\, d\tilde{v}\, dt\, dv''.$$

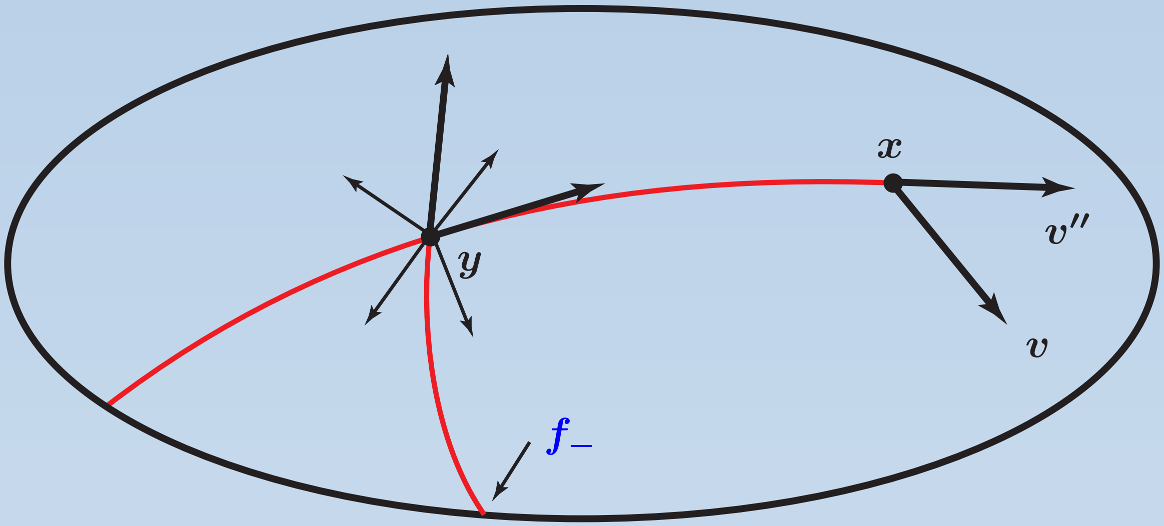
where $y = \gamma_{(x,v'')}(t - \tau_-(x, v''))$.

Estimating $\phi_2 = K^2 J\phi_-$



We change variables $(t, v'') \mapsto y$;

Estimating $\phi_2 = K^2 J \phi_-$



We change variables $(t, v'') \mapsto y$; the volume element is

$$dy = |Y_{(x,y)}(d(x,y))| \, dt \, dv''$$

where Y is the Jacobi field along $\gamma_{(x,v'')}$ which has $Y(0) = 0$ at x and $\dot{Y}(0)$ is perpendicular to v'' at x.

Estimating $\phi_2 = K^2 J \phi_-$

With the delta source supported at (x_0', v_0'), $(*)$ becomes

$$(*) = \int_0^{\tau_+(x_0', v_0')} E(\cdot)E(\cdot)k(\cdot)k(\cdot)\frac{dt}{|Y_{(x,y)}(d(x,y))|}.$$

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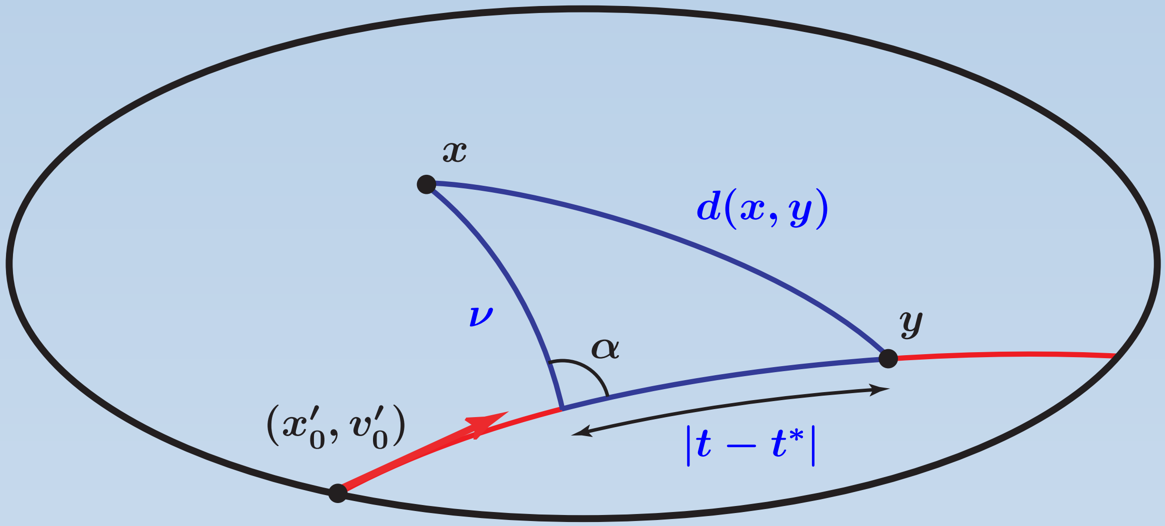
Now, we have $\kappa \leq \kappa_0$ so Rauch comparison gives

$$|Y_{(x,y)}(d(x,y))| \geq \frac{\sin\left(\sqrt{\kappa_0}\, d(x,y)\right)}{\sqrt{\kappa_o}}$$

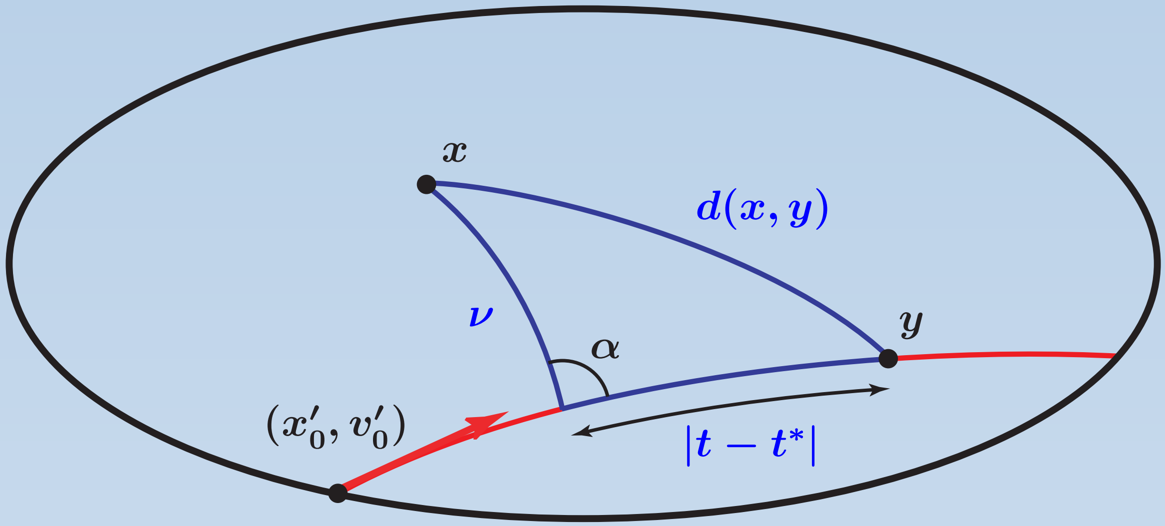
Estimating $\phi_2 = K^2 J\phi_-$



To estimate $d(x, y)$

- consider the geodesic triangle: x, y and the closest point to x on $\gamma_{(x_0', v_0')}$,
- and the equivalent comparison triangle (ν, α, $|t - t^*|$) on the sphere of constant curvature $\sqrt{\kappa_0}$.
- Then $d(x, y) \geq$ the distance on the sphere.

Estimating $\phi_2 = K^2 J \phi_-$

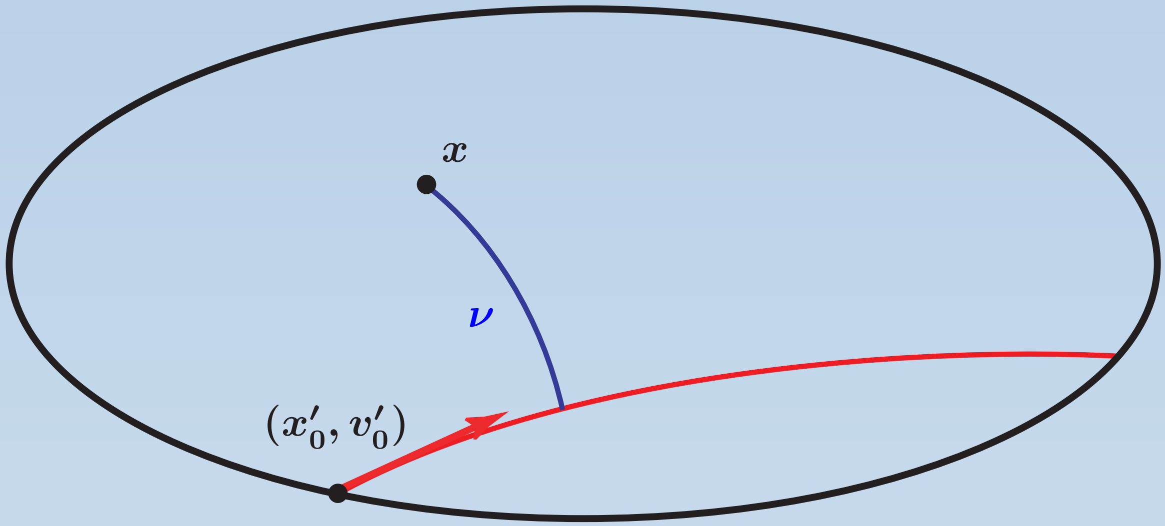


The law of cosines on the sphere then gives

$$\sin(\sqrt{\kappa_0}\, d(x,y)) \geq \frac{2\sqrt{\kappa_0}}{\pi}\sqrt{\nu^2 + |t - t^*|^2 \cos^2(\sqrt{\kappa_0} A)}$$

where $A = \mathrm{diam}\, M$.

Estimating $\phi_2 = K^2 J \phi_-$



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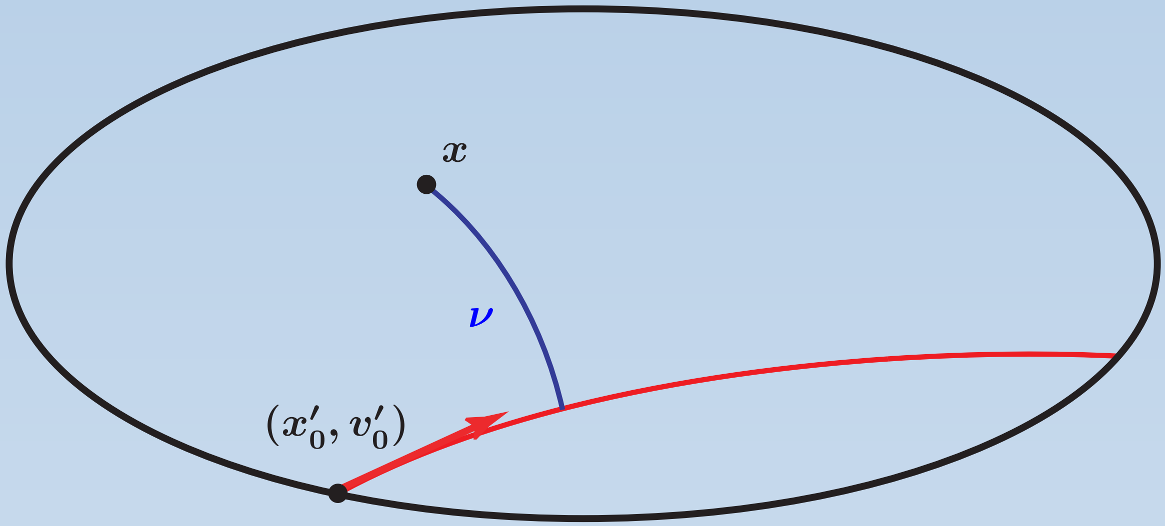
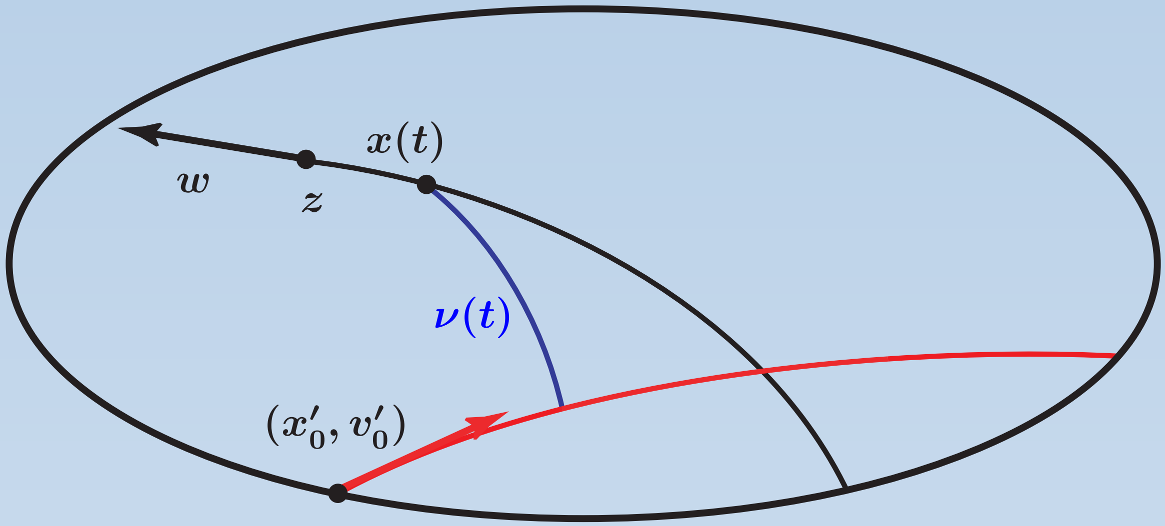


Integrating along $\gamma_{(x'_0,v'_0)}$,

$$(*) = \int_{\Omega_x M} k(x, v'', v) K J\phi_-(x, v'') dv' \leq \frac{\pi}{\cos(\sqrt{\kappa_0}A)} \|k\|^2 (C + \log \frac{A}{\nu})$$

where $\nu = \text{dist}(x, \gamma_{(x'_0,v'_0)})$ and $C = C(M, g)$. Notice the restriction on $\mathbf{diam}\, M$ comming in here.

Estimating $\phi_2 = K^2 J\phi_-$

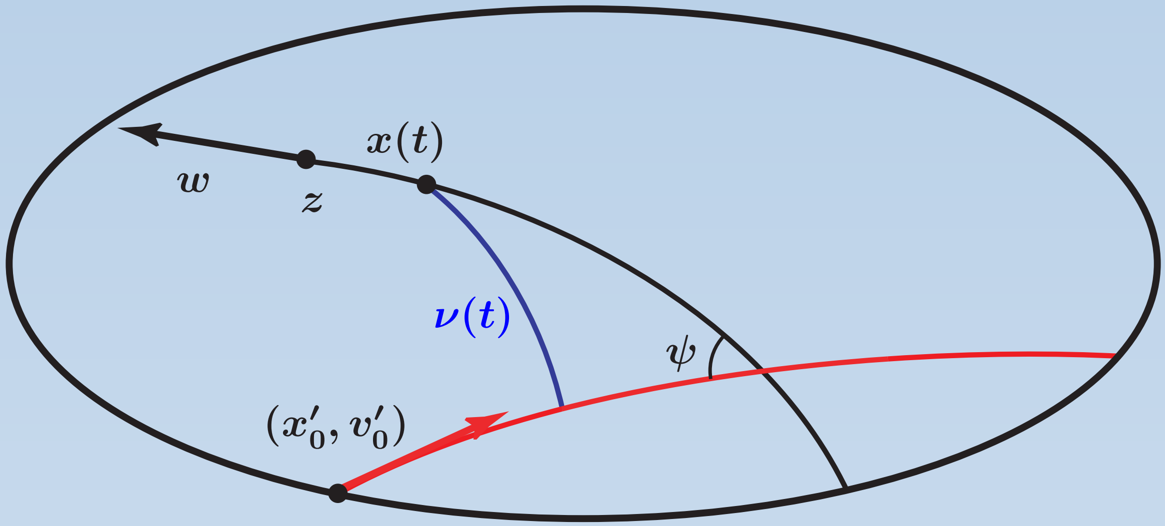


Finally $K^2 J\phi_-(z,w) = -\int_0^{\tau_-(z,w)} E(\cdot)(*)(\vec{\gamma}_{(z,w)}(t - \tau_-(z,w)))\, dt$, so we have

$$|K^2 J\phi_-| \le \frac{\pi A}{\cos(\sqrt{\kappa_0}A)} \|k\|^2 \int_0^{\tau_-(z,w)} \left(C' + \log\frac{A}{\nu(t)}\right) dt$$

where here $\nu(t)$ is the distance of $\gamma_{(z,w)}(t - \tau_-(z,w))$ from the geodesic $\gamma_{(x_0',v_0')}$.

Estimating $\phi_2 = K^2 J\phi_-$



Once again comparing to triangles on $S^2_{\kappa_0}$ we obtain

$$|K^2 J\phi_-| \leq \frac{\pi A}{\cos(\sqrt{\kappa_0}A)}\|k\|^2\left(1 + C' + \log\frac{\pi}{|\sin\psi|}\right).$$

Estimating $K^3 J\phi_-$

A similar but more involved analysis shows that

$$K^3 J\phi_- \in L^\infty(\Gamma_+ \times \Gamma_-)$$

with $\|K^3 J\phi_-\|^3_{L^\infty} \leq C(M, g)\|k\|^3_{L^\infty}$.

Putting it together.

Now, $\alpha_2 - \tilde{\alpha}_2 = (I+K)^{-1}(K^2 - \tilde{K}^2)J\phi_-$

$$+(I+\tilde{K})^{-1}(\tilde{K}-K)(I+K)^{-1}\tilde{K}^2 J\phi_-$$

and $K^2 - \tilde{K}^2 = K(K-\tilde{K}) + (K-\tilde{K})\tilde{K}$;

Putting it together.

Now, $\alpha_2 - \tilde{\alpha}_2 = (I + K)^{-1}(K^2 - \tilde{K}^2)J\phi_-$

$$+(I + \tilde{K})^{-1}(\tilde{K} - K)(I + K)^{-1}\tilde{K}^2 J\phi_-$$

and $K^2 - \tilde{K}^2 = K(K - \tilde{K}) + (K - \tilde{K})\tilde{K}$; applying

$$|K^2 J\phi_-| \leq \frac{\pi A}{\cos(\sqrt{\kappa_0}A)}\|k\|^2\big(1 + C' + \log\frac{\pi}{|\sin\psi|}\big)$$

to, for example, $K(K - \tilde{K})$ and multiplying by $|\sin\psi|$, the log term becomes bounded, $\|k\| \leq \varepsilon$ by assumption, and we can show that

$$\chi|(\alpha_2 - \tilde{\alpha}_2)\sin\psi| \leq C''\varepsilon\|k - \tilde{k}\|$$

almost everywhere.

Conclusion

Since we also know that $\|\boldsymbol{k} - \tilde{\boldsymbol{k}}\| \leq C_0|(\boldsymbol{\alpha_2} - \tilde{\alpha}_2)\,\mathbf{sin}\,\psi|$ almost everywhere, we now have

$$\|\boldsymbol{k} - \tilde{\boldsymbol{k}}\| \leq \boldsymbol{C\varepsilon}\|\boldsymbol{k} - \tilde{\boldsymbol{k}}\|.$$

For sufficiently small ε this implies that $\boldsymbol{k} = \tilde{\boldsymbol{k}}$.

The final constant involved implies that given the bound $\boldsymbol{\Sigma}$ for $\|\boldsymbol{\sigma_a}\|$ we may take $\boldsymbol{\varepsilon = C(M, g)e^{-2\Sigma A}}$.