An Inversion Analysis for Optical Tomography

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Diffusion Approximation

Under the assumption that scattering dominates absorption in a region Ω of interest,

Governing equation: the Boltzman transport equation

$\Longrightarrow$ diffusion approximation (DA): ($\Phi(x,t)$ — photon density)

$$\frac{1}{c}\frac{\partial \Phi}{\partial t}(x,t) - \nabla \cdot \kappa \nabla \Phi(x,t) + \mu_a \Phi(x,t) = q_0(x,t),\ x \in \Omega \quad (1)$$

$+$ Robin type boundary condition:

$$\Phi(x,t) + 2\kappa\theta \frac{\partial \Phi(x,t)}{\partial \nu} = 0,\ x \in \partial\Omega, \quad (2)$$

- $\Omega \subset \mathbb{R}^n (n = 2, 3)$ — a domain with C^∞ boundary $\partial\Omega$,
- ν — a unit normal vector of $\partial\Omega$ directed outside Ω,
- c — speed of light,
- $q_0(x, t)$ — a source term,
- $$\kappa = \frac{1}{3(\mu_s' + \mu_a)}$$

  with $C^\infty(\bar{\Omega})$ functions $\mu_s' > 0, \mu_a > 0$
- μ_a — absorption coefficient,
- μ_s' — reduced scattering coefficient given by

  $$\mu_s' = (1 - g_1)\mu_s$$

  with the mean cosine g_1 of the scattering angle,
- $\theta = \frac{1+R}{1-R}$, $R \sim -1.4399\, n_{in}^{-2} + 0.7099\, n_i n{-}1 + 0.6681 + 0.0636\, n_{in}$ with the refractive index n_{in}.

Light Source

- The light source in the optical tomography experiment is generated by an inward directed laser beam or optical fiber which is perpendicular to $\partial\Omega$ on a source cite $\varepsilon_k \subset \partial\Omega$.
- The diffuse boundary source (DS) model is the most common approximation which is used in the DA to model these anisotropic laser beam.

Diffuse Boundary Source (DS) Model

DS model: the source is absorbed into the boundary condition.

That is the Robin type boundary condition with the diffuse source model becomes

$$\Phi(x,t) + 2\kappa\theta \frac{\partial \Phi(x,t)}{\partial \nu} = g = \begin{cases} -4\Gamma_s w_k, & x \in \varepsilon_k \\ 0, & \text{elsewhere} \end{cases} \tag{3}$$

where the minus sign is due to the fact that the source current is inward directed, Γ_s is the diffuse boundary current and the indicator function w_k on ε_k is either 1 or 0 depending if the source is on or off.

Homogeneous Diffusion Equation

Needless to say, when the source term is absorbed into the Robin type boundary condition,

we have the homogeneous diffusion equation:

$$\frac{1}{c}\frac{\partial \Phi}{\partial t}(x,t) - \nabla \cdot \kappa \nabla \Phi(x,t) + \mu_a \Phi(x,t) = 0 \ (x \in \Omega,\ \ t > 0). \tag{4}$$

At $t = 0$, we assume that there is not any photon in Ω. That is

$$\Phi(x,0) = 0, \quad x \in \Omega. \tag{5}$$

Measured Quantity

The measured quantity of the trans-illuminate light is

$$J(r, t) \text{ — photon current}$$

By the Fick's law, we have

$$J(x, t) = -\kappa \nabla \Phi(x, t) \tag{6}$$

and then the measured quantity Γ can be written as

$$\Gamma = -\kappa \frac{\partial \Phi(x, t)}{\partial \nu} \quad x \in \partial \Omega. \tag{7}$$

Our Problem

Our problem: recover μ_a, μ_s' from the data set $\{g, \Gamma\}$ or many of them.

We even idealize the data set to the data set with infinitely many inputs. The mapping $g \mapsto \Gamma$ is called the Robin-to-Neumann map.

$\Longrightarrow$ our problem: reconstruct μ_a, μ_s' from the Robin-to-Neumann map.

we are especially interested in giving a scheme for the reconstruction.

Remark.

The use of infinitely many data set looks unrealistic. However, we want to remark that our scheme gives an approximate reconstruction of μ_a, μ_s' from the finitely many data sets.

Inverse Problem

We usually can assume that θ is known. For simplicity, we assume that $\kappa|_{\partial\Omega}, \frac{\partial\kappa}{\partial\nu}|_{\partial\Omega}$ are known.

Define $u := \kappa^{\frac{1}{2}}\Phi$, then we have

$$\rho\frac{\partial u}{\partial t} - \triangle u + Vu = 0, \tag{8}$$

where

$$\rho = \frac{1}{c\kappa}, \qquad V = \frac{\triangle\kappa^{\frac{1}{2}}}{\kappa^{\frac{1}{2}}} + \frac{\mu_a}{\kappa}.$$

It is easy to see that we only need to reconstruct ρ, V for obtaining μ_a, μ_s'.

Inverse Problem: reconstruct ρ and V from the Robin-to-Neumann map.

Reduction to a Harmonic Moment Problem

Let $\eta > 0$ be a large constant such that for any $w \in H^1(\Omega)$, we have

$$\int_\Omega \{|\nabla w|^2 + (V+\rho\eta)|w|^2\} + \int_{\partial\Omega} (2\kappa)^{-1}(1-\partial_\nu\kappa)|w|^2 \geq \delta \parallel w \parallel_{H^1(\Omega)}^2 \quad (9)$$

with some constant $\delta > 0$ independent of w.

We also let $v := e^{-\eta t} u$.

For the input in (3), we take

$$g = e^{\eta t} \kappa^{-\frac{1}{2}} \chi(t)(Bp)|_{\partial\Omega},$$

where

$$B := (1 - \theta\partial_\nu\kappa) + 2\kappa\theta\partial_\nu,$$

$p \in H^2(\Omega)$ is a harmonic function in Ω (i.e. $\triangle p = 0$ in Ω) and $\chi \in C_0^\infty((0,\infty))$ such that

$$0 \leq \chi \leq 1, \int_0^\infty \chi(t) > 0.$$

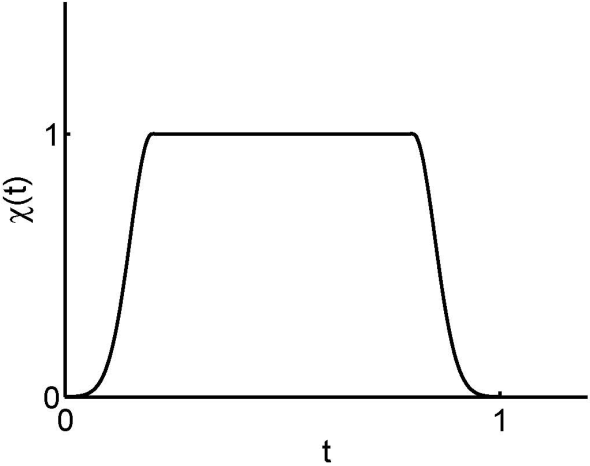

Figure: The graph of $\chi(t)$

Then, in terms of v, the mixed problem becomes

$$\begin{cases} \rho\partial_t v - \triangle v + (V + \rho\eta)v = 0 \quad \text{in} \quad \Omega \times (0,\infty) \\ Bv = \chi(t)Bp \quad \text{on} \quad \partial\Omega \times (0,\infty) \\ v(x,0) = 0 \quad (x \in \Omega) \end{cases} \tag{10}$$

and the measured data Γ becomes

$$\Gamma(x,t) = \frac{1}{2}e^{\eta t}(2\kappa^{\frac{1}{2}}\partial_\nu v - \kappa^{-\frac{1}{2}}\partial_\nu \kappa v) \quad \text{on} \quad \partial\Omega \times (0,\infty) \tag{11}$$

By (9) and the standard argument for showing the well-posedness of diffusion equation,

there exists a unique solution $v \in H^{2,1}(\Omega \times (0,\infty))$ to (10),

where $H^{\ell,m}(\Omega \times (0,\infty))$ is the Sobolev space defined in terms of the L^2 derivatives such that the indices $\ell \in \mathbb{N}$ and $m \in \mathbb{N}$ denote the L^2 derivatives in x and t up to ℓ and m, respectively.

The Solution v to (10)

In terms of the solution v to (10), we define the "Robin to Neumann map" R as a map which maps each Bv to $Nv := (-\partial_\nu \kappa v + 2\kappa \partial_\nu v)|_{\partial\Omega}$. Here, $\chi(t)$ is fixed and $p \in H^2(\Omega)$ can be any harmonic function in Ω.

For the asymptotic behavior of the solution v to (10) as $t \to \infty$, we have the following lemma.

Lemma 2.1.

There exist positive constants C, λ independent of p such that

$$\| v(\cdot,t) - \chi(t)p \| \leq Ce^{-\lambda t} \qquad (t \geq 0). \tag{12}$$

If Ω is bounded, then λ can be taken as the first eigenvalue of $-\rho^{-1}\triangle + \rho^{-1}(V + \rho\eta)$ with Robin type boundary condition on $\partial\Omega$ and $\| \cdot \| := \| \cdot \|_{L^2(\Omega)}$.

Now for the solution $v \in H^{2,1}(\Omega \times (0,\infty))$ to (10), we have by the Green formula

$$\int_\Omega v(\cdot,t)\bar{q}\rho - \int_{\partial\Omega}\int_0^t (\partial_\nu v\bar{q} - v\partial_\nu \bar{q}) + \int_\Omega \int_0^t (V+\rho\eta)v\bar{q} = 0 \tag{13}$$

for any harmonic function $q \in H^2(\Omega)$ in Ω.

Here, we have by the definition of B and R,

$$\begin{cases} \partial_\nu v = (2\kappa)^{-1}R(\chi(t)(Bp)|_{\partial\Omega}) + (2\kappa)^{-1}\partial_\nu \kappa v, \\ v = Bv - R(\chi(t)(Bp)|_{\partial\Omega}) \end{cases} \tag{14}$$

on $\partial\Omega$.

Hence, substituting (14) into (13), we have by a direct computation

$$\begin{aligned} &\int_{\partial\Omega}\int_0^t [(2\kappa)^{-1}R(\chi(t)(Bp)|_{\partial\Omega})\bar{q} + \{Bv - R(\chi(t)(Bp)_{\partial\Omega})\}\{(2\kappa)^{-1}\partial_\nu\kappa\bar{q} - \partial_\nu\bar{q}\}] \\ &= \int_\Omega v(\cdot,t)\bar{q}\rho + \int_\Omega\int_0^t (V+\rho\eta)v\bar{q}. \end{aligned} \tag{15}$$

Harmonic Moment Problem

By Lemma 2.1, the right hand side of (15) converges exponentially to

$$(\int_0^\infty \chi(t))M(p,q) \text{ with } M(p,q) := \int_\Omega (V+\rho\eta)p\bar{q} \tag{16}$$

as $t \to \infty$. Since the left hand side of (15) is known for any $t > 0$, we can know $M(p,q)$ for any harmonic function $p, q \in H^2(\Omega)$ in Ω.

So, the problem is to reconstruct $V+\rho\eta$ by knowing $M(p,q)$ for any harmonic functions $p, q \in H^2(\Omega)$ in Ω, which is a moment problem.

- We call this moment problem harmonic moment problem.
- Thus, we have reduced our inverse problem to harmonic moment problem.

Theoretical Method

First of all we give a theoretical method to solve the harmonic problem and reconstruct V and ρ.

- For an arbitrarily fixed $k \in \mathbb{R}^3 \setminus \{0\}$,
- let $\gamma_1, \gamma_2 \in \mathbb{R}^3$ be unit vectors such that k, γ_1, γ_2 are mutually orthogonal,
- let $\zeta := \gamma_1 + i\gamma_2$, $\xi := \zeta + g(k)\gamma_1$ with $g(k) := \frac{|k|^2}{2(2+\sqrt{4+|k|^2})}$.

Then, it is easy to see that

- $p := e^{x\cdot\xi_1}$ and $q := e^{x\cdot\xi_2}$ with $\xi_1 := \frac{i}{2}\,k + \xi$ and $\xi_2 := -\frac{i}{2}\,k - \overline{\xi}$ are harmonic functions,
- $p\overline{q} = e^{ik\cdot x}$.

These p, q are examples of the so called complex geometric optic solutions (see [4]).

Hence, we can obtain the Fourier transform of $V + \rho\eta\chi_\Omega$ where χ_Ω is the characteristic function of Ω.

Fourier Transform

- Therefore, by the given harmonic functions p and q, the moment problem becomes how to obtain $V + \rho\,\eta$ from its Fourier transform.
- Numerically, the problem is to compute the discrete inverse Fourier transform.
- We choose finitely many different $\{k_i\}_1^N$ and they generate $\{M_i\}_1^N$ by calculating the left side of (17), then $V + \rho\,\eta$ can be obtained easily from $\{M_i\}_1^N$ by Fast Fourier Transform.

Boundary Input

Remark.

We can use a CGO sol. with nonlinear phase to essentially restrict the input source cite.

It seems that we have solved our inverse problem. Unfortunately, the input used in the optical tomography is a pulse like input concentrated at several points of $\partial\Omega$ or the superpositions of such kind of inputs, and the input generated by the complex geometric optic solution is not that kind of input.

By a proper choice of the linear combinations of the boundary element C^2 functions approximating the Robin boundary value of complex geometric optic solutions, we can approximate $e^{ix\cdot k}$ by the $L^2(\Omega)$ basis generated from these linear combinations. Hence, by sampling $k \in \mathbb{R}^3$, we can apply the fast Fourier transformation to compute $V + \eta\rho$ numerically.

Procedure of Reconstruction Scheme

This scheme is given after the transformation $v = \kappa^{\frac{1}{2}} e^{-\eta t} \Phi$ has been already made. It consists of step (i) to (vi).

(i) Choose a constant $\eta > 0$ large enough such that (9) is satisfied.

(ii) Take the Robin input data in the form $\chi(t)Bp$, where $B = (1 - \theta \partial_\nu \kappa) + 2\kappa\theta\partial_\nu$, $p \in H^2(\Omega)$ is a harmonic function in Ω, and $\chi(t) \in C_0^\infty(\mathbb{R})$ is a cutoff function defined as in Figure 1.

(iii) Compute the Neumann output data $Nv = (\partial_\nu \kappa v + 2\kappa \partial_\nu v)|_{\partial\Omega}$, where v is the solution to (10).

Procedure of Reconstruction Scheme

(iv) Compute the left hand side of

$$\lim_{t\to\infty}\int_{\partial\Omega}\int_0^t[(2\kappa)^{-1}(Nv)\bar{q}+\{\chi(\cdot)(Bp)|_{\partial\Omega}-Nv\}\{(2\kappa^{-1})\partial_\nu\kappa\bar{q}-\partial_\nu\bar{q}\}]$$
$$=(\int_0^\infty\chi)M(p,q), \tag{17}$$

where $q \in H^2(\Omega)$ is a harmonic function in Ω and

$$M(p,q)=\int_\Omega(V+\rho\eta)p\,\bar{q}\,dx. \tag{18}$$

(v) Solve the harmonic moment problem (18) and obtain $H := V+\rho\eta$.

(vi) Take two different values η_1,η_2 for η and denote the associated H by H_1,H_2. Solve V and ρ from $H_j = V+\rho\eta\,(j=1,2)$.

An Application to Mammography

Let Ω be a slab domain bounded by two parallel planes $x_3 = 0$ and $x_3 = h$ with a constant $h > 0$.

We only put source cites on the plane $x_3 = h$.

Then, the harmonic input source p must satisfy

$$Bp = 0 \quad \text{on} \quad x_3 = 0. \tag{19}$$

An Application to Mammography

Since (9) holds for the slab domain Ω, we can reduce our inverse problem to the harmonic moment problem.

That is the problem to reconstruct $V + \rho\eta$ by knowing

$$M(p, q) := \int_\Omega (V + \rho\eta) p\overline{q} \tag{20}$$

for any harmonic functions p, $q \in H^2(\Omega)$ in Ω with the condition (19) only for p.

An Application to Mammography

We assume that $\alpha := (2\kappa\theta)^{-1}(1-\theta\partial_\nu\kappa)|_{x_3=0}$ is constant on $x_3 = 0$.

Here we note that if it is natural to assume that p decays locally in $x' = (x_1, x_2)$ away from $x_3 = 0$, then we have to assume $\alpha > 0$.

An Application to Mammography

To solve this harmonic moment problem, we need to modify the choice of p, q in the previous numerical scheme for solving the harmonic moment problem.

That is we need to modify the choice of ξ_1, ξ_2 for defining $p = e^{x\cdot\xi_1}$, $q = e^{x\cdot\xi_2}$ as follows.

Let $k = (k', k_3) = (k_1, k_2, k_3) \in \mathbb{R}^3$, $k' = (k_1, k_2) \neq 0$.

We define μ, β by

$$\mu = \frac{1}{2}|k'|^{-2}(|k|^2 - 2k_3\alpha), \quad \beta = \sqrt{\gamma^2 + |k'|^{-2}\alpha^2}. \tag{21}$$

An Application to Mammography

ξ_1, ξ_2 are given by

$$\begin{cases} \xi_1 = ((1-\gamma)k_1 + i\beta k_2, (1-\gamma)k_2 - i\beta k_1, k_3 - \alpha), \\ \xi_2 = (\gamma k_1 + i\beta k_2, \gamma k_2 - i\beta k_1, \alpha). \end{cases} \tag{22}$$

Having these p, q we can repeat the procedure of the reconstruction scheme given in Section 4.

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