On the uniqueness of the inverse source problem for linear particle transport theory

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The inverse source problem in radiative transfer and in linear transport theory consists of determining an internal source from a set of internal and/or external measurements. The problem of interest in practical applications and, more recently, for home security surveillance, is when the measurements are only external. More precisely, given the transport equation

$$\begin{array}{cc} (\mathbf{\Omega}\cdot\mathbf{\nabla}+\Sigma-H)\psi=S, & x\in X,\\ \psi=\psi_-, & x\in\Gamma_-, \end{array}$$

where $x=(\mathbf{r},E,\mathbf{\Omega})$ is a point in phase space $X=\{x,\mathbf{r}\in D,E\in\mathcal{E},\mathbf{\Omega}\in\mathcal{S}_2\}$, $\mathcal{E}$ is the energy domain, $\mathcal{S}_2$ the unit sphere, $\Gamma_-=\{x,\mathbf{r}\in\partial D,E\in\mathcal{E},\mathbf{\Omega}\in\mathcal{S}_2,\mathbf{\Omega}\cdot\mathbf{n}_+<0\}$ is the incoming boundary of X, S and ψ_- are, respectively, the internal source and the incoming angular flux and H is the scattering operator

$$(H\psi)(x)=\int dE'\int d\mathbf{\Omega}'\Sigma_s(\mathbf{r},E'\rightarrow E,\mathbf{\Omega}'\cdot\mathbf{\Omega})\psi(\mathbf{r},E',\mathbf{\Omega}'),$$

obtain the source $S(x)$ in terms of the boundary fluxes, assuming that the properties of the medium, Σ and Σ_s, are known.

Counter examples show that this 'noninvasive' inverse problem has a non unique solution. However, all the counterexamples are based on finding equivalent anisotropic sources and one may suspect that if one constraints the sources to be isotropic then the solution might be unique.

By recasting the problem in the form of an attenuated Radon transform (AtRT) and then using the proof of uniqueness for the inverse of the AtRT, we prove that for isotropic sources the noninvasive source problem has a unique solution in three-dimensional domains with energy dependence *when* the scattering is isotropic. This could provide a regularization if any source has an equivalent isotropic source, where by 'equivalent' we understand producing the same exiting flux. Unfortunately, this is not true as we prove by constructing a family of anisotropic sources that are not equivalent to an isotropic source.

We also investigate the relation of the noninvasive source problem with the 'canonical' inverse problem, which consists of finding the flux entering a domain without internal sources from the flux exiting the domain and discuss related inverse boundary problems.

We end with a discussion of how to construct inverse source methods using the duality between the direct and the adjoint transport equations. An application, based on Case's singular eigenfunctions, is given for the simplified one-velocity problem in a slab.

References

[1] Bal G., 2004. On the attenuated Radon transform with full and partial measurements, *Inverse Problems* **20**, 399–418.

[2] Bal G. and Tamasan A., 2006. Inverse source problems in transport equations, submitted to *Inverse Problems*.

[3] Beylkin G., 1984. The inversion problem and applications of the generalized Radon transform, *Commun. Pure Appl. Math.* **37**, 579–99.

[4] Bleistein N. and Cohen J. K., 1977. Nonuniqueness in the inverse source problem in acoustics and electromagnetics, *J. Math. Phys.* **18**, 2, 194–201.

[5] Case K. M. and Zweifel P. F., 1967. *Linear Transport Theory*, Addison-Wesley, Reading, Massachusetts.

[6] Dautray R. and Lions J. L., 1993. *Mathematical analysis and numerical methods for science and technology*, Vol. 6, Springer-Verlag, Berlin.

[7] Finch D. V., 1986. Uniqueness for the attenuated x-ray transform in the physical range, *Inverse Problems* **2** 197–203.

[8] Hoenders B. J., 1978. The uniqueness of inverse problems, pp. 41-82 in *Inverse Source Problems in Optics*, H.P. Baltes, Editor, Springer-Verlag, Berlin.

[9] Hoenders B. J., 1997. Existence of invisible nonscattering objects and nonradiating sources, *J. Opt. Soc. Am. A* **14**, 262–266.

[10] Larsen E. W., 1975. The inverse source problem in radiative transfer, *J. Quant. Spec. Radiat. Transf.* **15**, 1–5.

[11] McCormick N. J., 1992. Inverse radiative transfer problems: a review, *Nucl. Sci. Eng.* **112**, 185–198.

[12] Natterer F., 2001. Inversion of the attenuated Radon transform, *Inverse Problems* **17**, 113–119.

[13] Natterer F. and Wubbeling F., 2001. *Mathematical Methods in Image Reconstruction*, Siam, Philadelphia.

[14] Novikov R. G., 2002. An inversion formula for the attenuated X-ray transformation, *Ark. Math.* **40**, 145–167 (Rapport de Recherche 00/05-3, Université de Nantes, Laboratoire de Mathématiques).

[15] Novikov R. G., 2002a. On the range characterization for the two-dimensional attenuated x-ray transformation, *Inverse Problems* **18**, 677–700.

[16] Zweifel P. F., 1999. The canonical inverse problem, *Trans. Th. Statist. Phys.* **28**, 171–179.