On the uniqueness of the inverse source problem for linear transport theory

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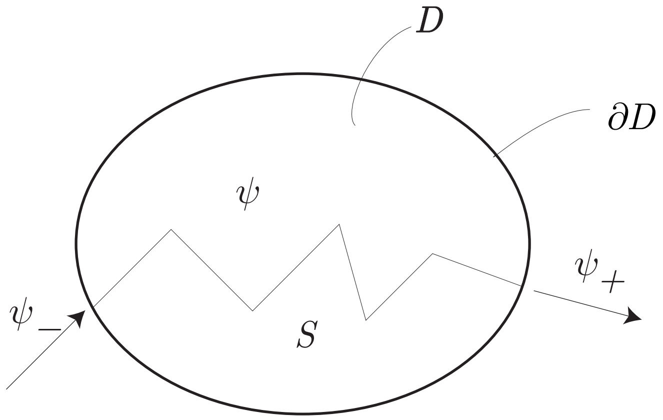
outline

direct transport problem

direct problem



$$\left.\begin{array}{ll} B\psi = S, & x \in X \\ \psi = \psi_- + \beta\psi, & x \in \Gamma_- \end{array}\right\}$$

$$\psi(x), S(x)$$

phase space $X = \{x = (\mathbf{r}, E, \mathbf{\Omega}) \in D \times \mathcal{E} \times \mathcal{S}^2\}$

boundaries $\Gamma_\pm = \{x \in \partial D \times \mathcal{E} \times \mathcal{S}^2, \pm\mathbf{\Omega}\cdot\mathbf{n}_+ > 0\}$

transport operator $B = \mathbf{\Omega}\cdot\mathbf{\nabla} + \Sigma - H$

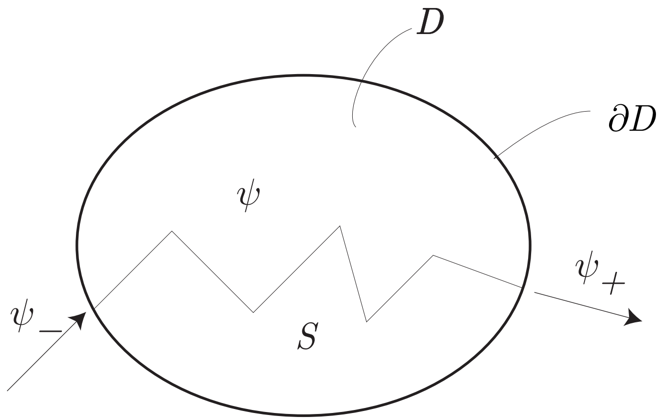
$$\Sigma(x), \quad (Hf)(x) = \int \Sigma_s(\mathbf{r}, E' \to E, \mathbf{\Omega}' \to \mathbf{\Omega}) f(\mathbf{r}, E', \mathbf{\Omega}') dE' d\mathbf{\Omega}'$$

isotropic media : $\Sigma(\mathbf{r}, E), \quad \Sigma_s(\mathbf{r}, E' \to E, \mathbf{\Omega}'\cdot\mathbf{\Omega})$

subcriticality $\left.\begin{array}{ll} B\psi = 0, & x \in X \\ \psi = \beta\psi, & x \in \Gamma_- \end{array}\right\} \Rightarrow \psi = 0$

inverse transport problems

$$\left.\begin{array}{ll} B\psi = S, & x \in X \\ \psi = \psi_- + \beta\psi, & x \in \Gamma_- \end{array}\right\}$$



$\psi \to S$

$\psi_+ \to S$

measurements
$$\begin{cases} \textit{invasive}: & \psi(x),\ x \in X \\ \textit{noninvasive}: & \psi_+(x),\ x \in \Gamma_+ \end{cases}$$

data to be reconstructed
$$\begin{cases} \textit{cross sections}: & \Sigma(x), \Sigma_s(\mathbf{r}, E' \to E, \mathbf{\Omega}' \to \mathbf{\Omega}),\ x \in X \\ \textit{albedo}: & \beta(x' \to x),\ x' \in \Gamma_+,\ x \in \Gamma_- \\ \textit{source}: & S(x),\ x \in X \end{cases}$$

in this talk we analyze uniqueness for the invasive and noninvasive inverse source problems (ψ_-, Σ, Σ_s *assumed known)*

invasive inverse source problem

source problem

$$\left.\begin{array}{ll} B\psi = S, & x \in X \\ \psi = \psi_-, & x \in \Gamma_- \end{array}\right\} \to \psi \qquad\qquad \psi(x), \forall x \in X \to S(x)$$

$$L^p = L^p(X, dx),\ W^p = W^p(X) = \{f, \mathbf{\Omega}\cdot\mathbf{\nabla} f \in L^p\},\ L^p_\pm = L^p_\pm(\Gamma_\pm, d_b x),$$

$$dx = d\mathbf{r}\, dE d\mathbf{\Omega},\ d_b x = |\mathbf{\Omega}\cdot\mathbf{n}(\mathbf{r})| dS dE d\mathbf{\Omega}$$

then the mapping $\pi : L^p \times L^p_- \to W^p$ *is an isomorphism (Dautray,1993)*

$$(S, \psi_-) \mapsto \psi$$

$\to$ *the invasive inverse source has a unique solution*

(very ill-conditioned) inverse method : $S = B\psi$

noninvasive inverse source problem (NISP)

source problem :

$$\left.\begin{aligned} B\psi = S, &\quad x \in X \\ \psi = 0, &\quad x \in \Gamma_- \end{aligned}\right\} \to \psi_+ \qquad\qquad \psi_+ = \psi(x)\Big|_{\Gamma_+} \to S(x)$$

the mapping $\pi_+ : L^p \to L^p_+$ *is a continuous morphism (non necessarily onto)*

$$S \mapsto \psi_+$$

$\ker(\pi_+) = \pi_+^{-1}(0) = \{S_{nr}\} =$ *set of nonradiating sources :* $S_{nr} \to \psi\Big|_{\Gamma} = 0$

thus, the general solution of the NISP is : $S + \{S_{nr}\}$

uniqueness $\Leftrightarrow \{S_{nr}\} = \varnothing$

examples of nonradiating sources

acoustics & electromagnetics (Blestein & Cohen,1977;Hoenders,1978; Arridge,1988)

(i) exponential transformation : $f(\mathbf{r}) : |\boldsymbol{\nabla} f| < \infty, \; f|_{\Gamma} = 0$

$$\left.\begin{aligned} B\psi &= S, \quad x \in X \\ \psi &= \psi_-, \quad x \in \Gamma_- \end{aligned}\right\} \quad \psi = e^{f(\mathbf{r})}\psi' \rightarrow \quad \left.\begin{aligned} B\psi' &= S', \quad x \in X \\ \psi' &= \psi_-, \quad x \in \Gamma_- \end{aligned}\right\}$$

$$S' = e^{-f(\mathbf{r})}(S - \psi \boldsymbol{\Omega} \cdot \boldsymbol{\nabla} f) \rightarrow \psi'\big|_{\Gamma} = \psi|_{\Gamma}$$

thus, $S - S'$ *is a nonradiating source.*

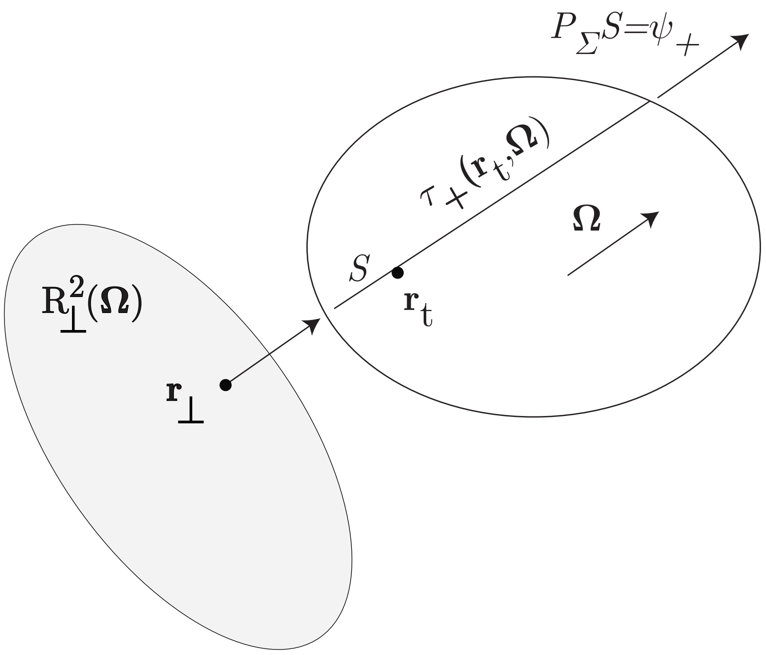
(ii) let $D_{nr} \subseteq D, \quad \psi_{nr} \in W^p(X_{nr}), \quad \psi_{nr}|_{\Gamma_{nr}} = 0$

then $S_{nr} = B\psi_{nr}$ *is a nonradiating source :* $S_{nr} \rightarrow \psi_{nr}|_{\Gamma} = 0$

attenuated Radon transform (AtRT)

one-group direct problem with no scattering :

$$\left.\begin{aligned} L\psi = S, &\quad x \in X \\ \psi = 0, &\quad x \in \Gamma_- \end{aligned}\right\}$$



$$L = \mathbf{\Omega} \cdot \mathbf{\nabla} + \Sigma, \quad \psi(\mathbf{r}, \mathbf{\Omega}),\, S(\mathbf{r}),\, \Sigma(\mathbf{r})$$

AtRT :

$$(P_\Sigma S)(\mathbf{r}_\perp, \mathbf{\Omega}) = \int_{-\infty}^{\infty} e^{-\tau_+(\mathbf{r}_t, \mathbf{\Omega})} S(\mathbf{r}_t)\,dt, \quad \mathbf{r}_\perp \in \mathbb{R}^2_\perp(\mathbf{\Omega}),\ \ \mathbf{\Omega} \in \mathcal{S}^2$$

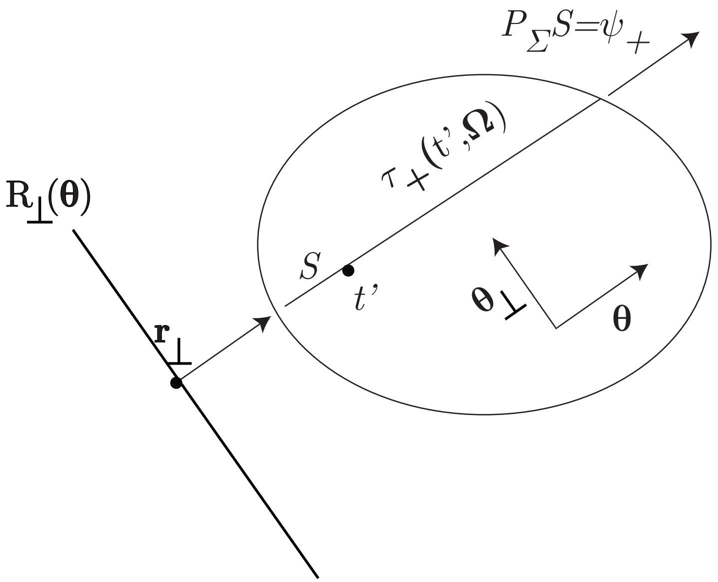
$$\mathbf{r}_t = \mathbf{r}_\perp - t\mathbf{\Omega}, \quad \tau_+(\mathbf{r}, \mathbf{\Omega}) = \int_0^{\infty} \Sigma(\mathbf{r} + t\,\mathbf{\Omega})\,dt$$

inverse of the 2D AtRT (Novikov,2002)

$$\left.\begin{array}{ll}(\boldsymbol{\theta}\cdot\nabla+\Sigma)\psi=S, & \mathbf{r}\in D, \theta\in \mathcal{S}^1 \\ \psi(\mathbf{r},\theta)=0 & \mathbf{r}\in\partial D, \boldsymbol{\theta}\cdot\mathbf{n}_+<0\end{array}\right\}$$

$$g(t,\boldsymbol{\theta})=P_\Sigma S\rightarrow S=\frac{1}{4\pi}\mathrm{div}P^*_{-\Sigma}(\boldsymbol{\theta}_\perp e^{-h}He^h g)$$

$$t\in\mathbb{R}^1, \boldsymbol{\theta}\in S^1$$



$$h=\frac{1}{2}(1+iH)\tau_{max}$$

$$\tau_{max}(t,\vec{\theta})=\int_{-\infty}^{\infty}\Sigma(t\boldsymbol{\theta}_\perp+t'\boldsymbol{\theta})dt'$$

$$(Hf)(t)=\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{f(t')}{t-t'}dt'$$

data redundancy : $(P_\Sigma S)(-t,-\boldsymbol{\theta})=e^{-\tau_{max}(t,\boldsymbol{\theta})}(P_{-\Sigma}S)(t,\boldsymbol{\theta})$

examples of nonradiating sources

(iii) (Bal,2004) : from the redundancy of data for the 2D AtRT one can get two independent equations to reconstruct, from $\psi_+(x)\ \forall x \in \Gamma_+$, *an anisotropic source of the form*

$$S_\omega(\mathbf{r},\varphi) = S_0(\mathbf{r}) + S_1(\mathbf{r})\cos(\varphi+\omega),\ \mathbf{r} \in \mathbb{R}^2,\ \varphi \in [0,2\pi[$$

for $\omega \in [0,2\pi[$ *given.*

then $S_\omega - S_{\omega'}$ *is a nonradiating source.*

$\rightarrow$ *(iii) can be generalized to an anisotropic source of the form*

$$S_f(\mathbf{r},\varphi) = S_0(\mathbf{r}) + S_1(\mathbf{r}) f(\varphi)$$

for given, weakly anisotropic $f(\varphi) = \sum_{k=1}^{\infty}(a_k \cos\varphi + b_k \sin\varphi)$

Bal (2004) : $\mathbf{r} \rightarrow z = x + iy, \lambda = e^{i\theta}$ *so that* $(\boldsymbol{\theta}\cdot\boldsymbol{\nabla} + \Sigma)\psi = S \rightarrow (\lambda\partial_z + \lambda^{-1}\partial_{\bar{z}} + \Sigma)\psi = S$ *with* $\partial_{\bar{z}} = (1/2)(\partial_x + i\partial_y)$ *and write a Riemann-Hilbert problem for* $\psi(z,\lambda)$ *as a sectionally analytic function* $\phi(z,\lambda)$ *for* $\lambda \in \mathbb{C} \setminus (T \cup \{0\})$ *, where the jump at the section* $T = \{\lambda, |\lambda| = 1\}$ *can be written in terms of the boundary data.*

(a) for a source $S(\mathbf{r},\varphi) = \sum_{k=-N}^{k=N} S_k(\mathbf{r})e^{ik\theta},\ S_{-k} = \bar{S}_k$ *one has :*

$$\sum_{k=-N}^{k=N} (\mathcal{H}_m S_{n-m} - \bar{\mathcal{H}}_m S_{n+m})(z) = \varphi_n(z),\ n = 0{,}1$$

where $(\mathcal{H}_1 f)(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(z)}{z - z'} dz'$ *and* $\sum_{k=-N}^{k=N} (\partial_{\bar{z}}\mathcal{H}_{k+2} + \Sigma\mathcal{H}_{k+1} + \delta_{k>0}\partial_z\mathcal{H}_k) = 0,\ k \geq 0$

(b) for a source $S_f(\mathbf{r},\varphi) = S_0(\mathbf{r}) + S_1(\mathbf{r})f(\varphi),\ \ f(\varphi) = \sum_{k=1}^{K}(f_k e^{ik\theta} + \bar{f}_k e^{-ik\theta})$ *we get :*

$$(A + B)S_1 = \varphi_0, \quad \mathcal{H}_1 S_0 + (\sum_{k=2}^{N+1} \bar{f}_{k-1}\mathcal{H}_k - \sum_{k=1}^{N-1} f_{k+1}\bar{\mathcal{H}}_k)S_1 = \varphi_1$$

with $A = \bar{f}_1\mathcal{H}_1 - f_1\bar{\mathcal{H}}_1, \quad B = \sum_{k=2}^{N}(\bar{f}_k\mathcal{H}_k - f_k\bar{\mathcal{H}}_k)$

weakly anisotropic $\rightarrow \{f_k\}$ *s.t.* $\left\| A^{-1}B \right\| < 1$

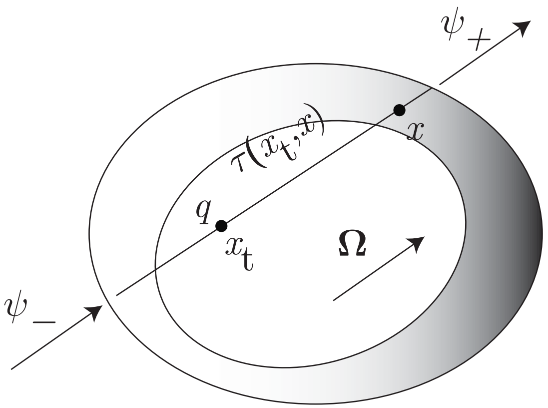
properties of nonradiating sources

$$S_{nr} \rightarrow \psi_{nr}|_{\Gamma} = 0$$

observation: a positive source produces

a strictly positive flux

i) a nonradiating source must change of sign



$$\psi(x) = e^{-\tau(x_t,x)}\psi(x_t) + \int_0^t e^{-\tau(x_{t'},x)} q(x_{t'})dt', \quad \mathbf{r}, \mathbf{r}_t \in \bar{D}$$

$$\mathbf{r}_t = \mathbf{r} - t\mathbf{\Omega}, \quad x_t = (\mathbf{r}_t, E, \mathbf{\Omega})$$

$$\tau(x, x_t) = \int_0^t \Sigma(x_{t'})dt', \quad q = H\psi + S$$

take $x \in \Gamma_+,\ x_t \in \Gamma_+ \rightarrow 0 = \int_0^t e^{-\tau(x_{t'},x)} q(x_{t'})dt' \rightarrow q$ changes of sign

$\rightarrow$ *S changes of sign*

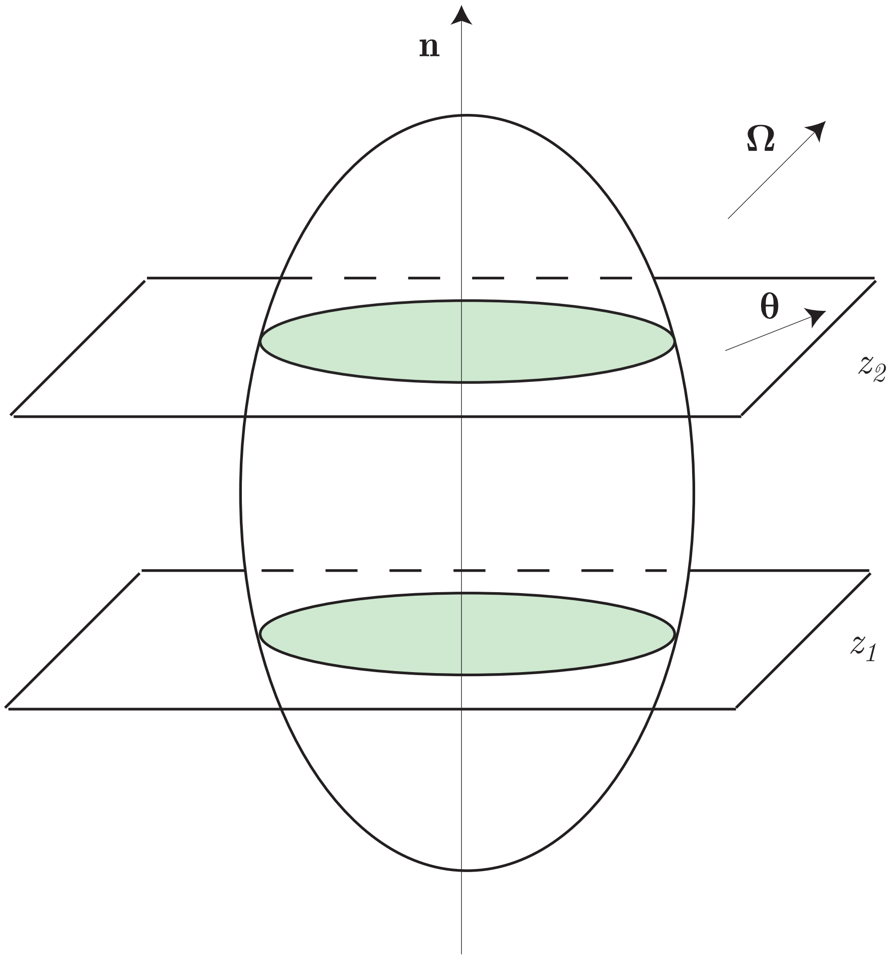
ii) (conjecture) a nonradiating source cannot be isotropic $S(x) = S(\mathbf{r}, E)$

inverse of the 3D AtRT

$$\left.\begin{aligned} L\psi &= S, \quad x \in X \\ \psi &= 0, \quad x \in \Gamma_- \end{aligned}\right\}$$

$$\psi(\mathbf{r},\mathbf{\Omega}),\, S(\mathbf{r}),\, \Sigma(\mathbf{r})$$

$$L = \mathbf{\Omega}\cdot\mathbf{\nabla} + \Sigma$$



for $\mathbf{n}$ *fixed use the inverse of the 2D AtRT on each plane orthogonal to* $\mathbf{n}$

3D energy-dependent results

direct problem with isotropic scattering and sources

$$\left.\begin{aligned} L\psi &= q, \quad x \in X \\ \psi &= 0, \quad x \in \Gamma_- \end{aligned}\right\} \qquad q = H\psi + S$$

for each $E \in \mathcal{E}$ *use the inverse of the 3D AtRT to reconstruct q, then use q to compute the flux from*

$$\psi(x) = \int_0^t e^{-\tau(x_{t'},x)} q(x_{t'})dt', \quad \mathbf{r} \in D, \mathbf{r}_t \in \Gamma_-$$

Now we have $\psi(x)$ *for all* $E \in \mathcal{E}$ *and can calculate* $H\psi$ *and from it we get the source :* $S = q - H\psi$

→ *with isotropic scattering and sources the NISP has a unique isotropic solution* $S(\mathbf{r}, E)$

→ *any other source solution of the problem is of the form* $S' = S + S_{nr}$ *with*

$$\int_{S^2} S_{nr}(\mathbf{r}, E, \mathbf{\Omega})d\mathbf{\Omega} = 0$$

one-group results

(i) Bal & Tamasan (2007) have used a Neumann perturbation technique in conjunction with the 3D AtRT to reconstruct an isotropic source, $S(\mathbf{r}) \in L^2(\mathbb{R}^3)$ *with compact support for the one-group transport equation with weakly anisotropic scattering.*

The reconstruction needs ψ_+ *in all directions* $\mathbf{\Omega}$ *orthogonal to a fix vector* $\mathbf{n}$ *.*

(ii) For a homogeneous half-space in 1D transport, $\psi(z,\mu),\ \mu = \cos\mathbf{\Omega}\cdot\hat{\mathbf{e}}_z$ *,*

$$\left.\begin{array}{ll}(\mu\partial_z + \Sigma)\psi = H\psi + S, & z \in (0,\infty), \mu \in [0,1] \\ \psi(0,\mu) = 0, & \mu \in [-1,0[\end{array}\right\}$$

Larsen (1974) has used Case's singular eigenfunctions together with a Laplace transform to reconstruct an isotropic source from the flux $\psi_+(\mu)$ *exiting a half space with anisotropy of scattering. The solution exists and is unique under the conditions* $\int_0^\infty e^{-\Sigma x} S(x)dx,\ \int_0^\infty e^{-\Sigma x}\psi(x,\mu)dx < \infty$

& $\psi_+(\mu), \mu \in [-1,0[$ *is analytically continuable to the disk* $|z - 1/2| < 1/2$

equivalence relation and regularization (NISP)

the mapping $\pi_+ : L^p \to L^p_+$ *is a continuous morphism (non necessarily onto)*

$$S \mapsto \psi_+$$

$\ker(\pi_+) = \pi_+^{-1}(0) = \{S_{nr}\} =$ *set of nonradiating sources :* $S_{nr} \to \psi\big|_\Gamma = 0$

define the equivalence relation : $S, S' \in L^p, S \Re S'$ for $\psi_+[S] = \psi_+[S']$

a regularization of the NISP consists of giving a rule to identify a unique element in each class

a possibility is to select an isotropic source; with isotropic scattering we know that this source is unique :

$\to$ *with isotropic scattering, the NISP can have at most one isotropic solution*

are there anisotropic sources that are not equivalent to an isotropic one?

anisotropic sources that are not equivalent to an isotropic source

with no scattering in 2D (Bal,2004) the source

$$S_f(\boldsymbol{\rho},\varphi) = S_0(\boldsymbol{\rho}) + S_1(\boldsymbol{\rho})f(\varphi), \int_0^{2\pi} f(\varphi)d\varphi = 0$$

for $f(\varphi)$ *, weakly anisotropic can be uniquely reconstructed from* $\psi_+(x)\ \forall x \in \Gamma_+$

$\rightarrow$ *for* $S_1 \neq 0$ *there is no equivalent isotropic source.*

This result can be extended to energy-dependent in 3D :

$\rightarrow$ *for isotropic scattering and* $f(\mathbf{\Omega}), \int_{S^2} f(\mathbf{\Omega})d\mathbf{\Omega} = 0,$ *weakly anisotropic, the source* $S_f(x) = S_0(\mathbf{r},E) + S_1(\mathbf{r},E)f(\mathbf{\Omega})$ (A) *can be uniquely reconstructed from* $\psi_+(x), \forall x \in \Gamma_+, \mathbf{\Omega}\cdot\mathbf{n} = 0$ *for fixed* $\mathbf{n}$.

Let π *be a plane orthogonal to* $\mathbf{n}$ *and consider the 2D problem with no scattering and source* $q = H\psi + S_f\big|_\pi$ *; q is of the type (A) and can be reconstructed from* $\psi_+\big|_\pi$ *, compute now* $\psi'_+\big|_\pi = \psi_+\big|_\pi - \psi_+[S_1 f|_\pi]$
then from $\psi'_+\big|_\pi$ *for* $\forall E \in \mathcal{E}, \forall\pi \perp \mathbf{n}$ *we can uniquely reconstruct* $S_0(\mathbf{r},E)$

related inverse problems

(i) canonical inverse problem (Zweifel,1999) : with zero source reconstruct
ψ_+ *from* ψ_-

using Caseology, Zweifel proved this reconstruction for the one-group, 1D transport equation with isotropic scattering in a half-space.

observe that the solution of

$$\left.\begin{aligned} (\mu\partial_z + \Sigma)\psi &= H\psi, \quad z \in (0,\infty), \mu \in [-1,1] \\ \psi(0,\mu) &= \psi_-, \qquad\qquad \mu \in [0,1[\end{aligned}\right\}$$

is $\psi = \psi_{unc} + \psi_{col}$ *with* $\psi_{unc}(x,\mu) = H(\mu)e^{-\Sigma x/\mu}\psi_-(\mu)$ *and* ψ_{col} *solution of*

$$\left.\begin{aligned} (\mu\partial_z + \Sigma)\psi_{col} &= H\psi_{col} + S_{fc}, \quad z \in (0,\infty), \mu \in [-1,1] \\ \psi_{col}(0,\mu) &= 0, \qquad\qquad \mu \in [0,1[\end{aligned}\right\}$$

with the isotropic source $S_{fc}(z) = \int_0^1 e^{-\Sigma z/\mu}\psi_-(\mu)d\mu, \; 0 < z < \infty \; (A)$

Now, because $\psi_+[\psi_-] = \psi_+[S_{fc}]$ *,* S_{fc} *can be reconstructed from* ψ_+ *, and the final reconstruction of* ψ_- *is equivalent to the inversion of (A).*

related inverse problems

(ii) the stochastic matrix method is based on solving a 1D finite-slab transport problem in terms of the one-sided boundary (entering and exiting) fluxes:

$$\left.\begin{aligned}(\mu\partial_z + \Sigma)\psi &= H\psi, \quad z \in (L,R), \mu \in [-1,1]\\ \psi(L,\mu) &= \psi_-^L(\mu), \qquad \mu \in [0,1[\\ \psi(L,\mu) &= \psi_+^L(\mu), \qquad \mu \in [-1,0[\end{aligned}\right)$$

the well-posedness of this problem is equivalent to the uniqueness of the following inverse problem

$$\left.\begin{aligned}(\mu\partial_z + \Sigma)\psi &= H\psi, \quad z \in (L,R), \mu \in [-1,1]\\ \psi(L,\mu) &= \psi_-^L(\mu), \qquad \mu \in [0,1[\\ \psi(R,\mu) &= 0 \qquad \mu \in [-1,0[\end{aligned}\right\} \to \text{find } \psi_-^L \text{ from } \psi_+^R$$

write $\psi_+^R = (A+B)\psi_-^L,\ \mu \in [0,1[$ *with*

$$(A\psi_-^L)(z,\mu) = \frac{1}{\mu}\int_L^R e^{-\tau(L,z')/\mu}dz'\int_{-1}^{1}\Sigma_s(z',\mu' \to \mu)\psi(z',\mu)d\mu'$$

$$(B\psi_-^L)(z,\mu) = e^{-\tau(L,R)/\mu}\psi_-^L(\mu)$$

set $g(\mu) = e^{\tau(L,R)/\mu}\psi_+^R(\mu)$ *, then we need to reconstruct* ψ_-^L *from*

$$g = (1 + B^{-1}A)\psi_-^L, \ \mu \in [0,1[$$

let $X = L^1(]0,1[), \ X_w = L^1(]0,1[, w), \ \|f\|_X = \int_0^1 e^{-\tau(L,R)/\mu}|f(\mu)|d\mu$ *and consider*

$$T = 1 + B^{-1}A : X \to X_w$$

$$\psi_-^L \mapsto g$$

the idea is use 'c' to control the size of $B^{-1}A$ (Neumann series),

$$\Sigma_s(z,\mu' \to \mu) = (c\Sigma)(z)h(z,\mu' \to \mu), \int_{-1}^1 h(z,\mu' \to \mu)d\mu = 1$$

so that $\left\|B^{-1}A\right\|_{\mathcal{L}^1(X \to X_w)} < 1$

let $c_{max} = \max_{z\in(L,R)} c(z), \ h_{max} = \max_{z\in(L,R),\mu',\mu\in(-1,1)} h(z,\mu' \to \mu)$

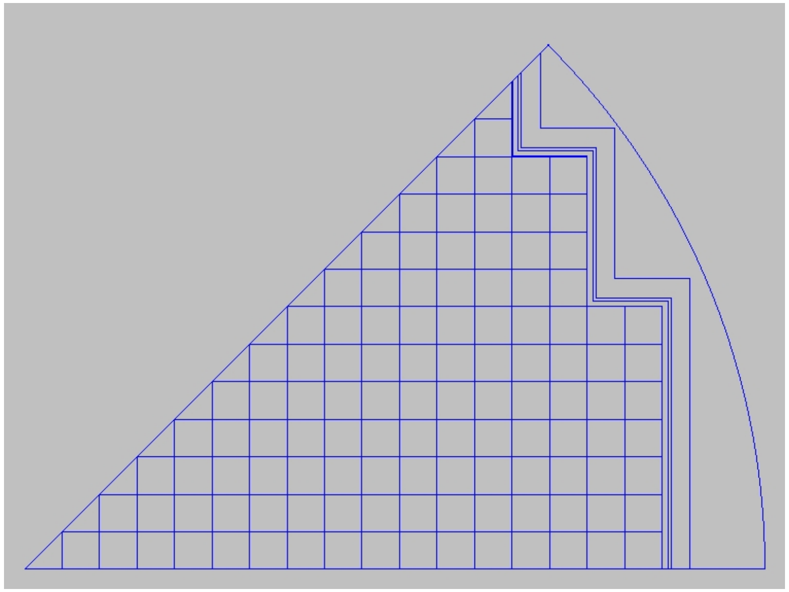
then $\left\|B^{-1}A\right\|_{\mathcal{L}^1(X \to X_w)} \le c_{max}h_{max}[1 - E_2(\tau(L,R))]a$

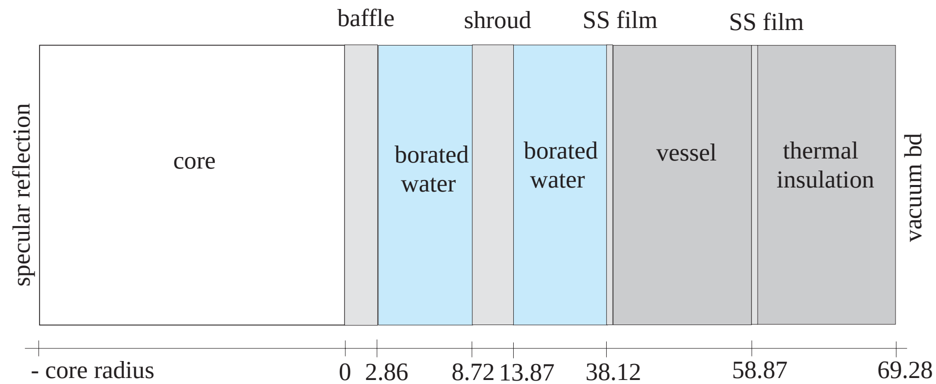
where, with $Y = L^1(]L,R[\times]0,1[)$ *,* $\|\psi\|_Y \le a\left\|\psi_-^L\right\|_X$ *with* a *depending on* c_{max}

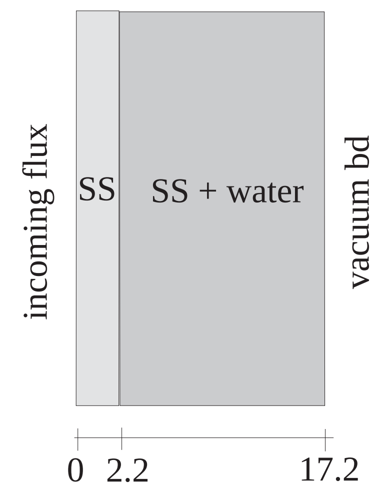
hence, the reconstruction $\psi_+^R \to \psi_-^L$ *if possible for* $c \in [0, c_{max}]$ *with* c_{max} *satisfying*

$$c_{max}h_{max}[1 - E_2(\tau(L,R))]a < 1$$

(iii) reflector homogenization problem :







$$\{\beta^{g\rightarrow g'}\} \longrightarrow \{\Sigma^g, \Sigma_{s,n}^{g\rightarrow g'}\}, \quad \Sigma^g, \Sigma_{s,0}^{g\rightarrow g'} \geq 0$$

$$minimize_{\{\vec{\Sigma}\}_+} \ \mathscr{F}(\vec{\Sigma}) = \sum_{i=1}^{M} \{\beta_i - \beta_i(\vec{\Sigma})\}$$

regularization : $\Sigma_{s,n}^{g\rightarrow g'} = f_n^{g\rightarrow g'} \Sigma_{s,0}^{g\rightarrow g'}$

$$(a)\ {\color{red}\Sigma}^g \geq \sum_g \Sigma_{s,0}^{g\rightarrow g'} 0, \quad (b)\ {\color{red}c}^g \Sigma^g = \sum_g \Sigma_{s,0}^{g\rightarrow g'}, \quad (c)\ {\color{red}\Sigma_a^g} = \Sigma^g - \sum_g \Sigma_{s,0}^{g\rightarrow g'}$$

inverse source methods based on duality

for $f \in W^{p'}, \, g \in W^{p}, \, p \in [1,\infty], \frac{1}{p} + \frac{1}{p'} = 1$

define the scalar products

$$(f,g) = \int_X fg \, d\mathbf{r} \, dE d\mathbf{\Omega}, \quad < f,g >= \int_\Gamma fg \, \mathbf{\Omega} \cdot \mathbf{n}_+ dS dE d\mathbf{\Omega}$$

we have the general duality relation (Green's formula) :

$$(f,Bg) = (B^* f,g) + < f,g > \quad (A)$$

where $B^* = -\mathbf{\Omega} \cdot \mathbf{\nabla} + \Sigma - H^*$ *is the formal adjoint of transport operator B .*

for $g = \psi$ *solution of the transport equation* $\left.\begin{array}{ll} B\psi = S, & x \in X \\ \psi = 0, & x \in \Gamma_- \end{array}\right\}$

(A) yields the 'inverse method'

$$(f,S) = (B^* f,\psi) + < f,\psi >_+$$

for suitable functions $f \in W^{p'}$

$$(f,S) = (B^* f,\psi) + < f,\psi >_+$$

(i) invasive inverse source methods :

select N functions $f \in W^{p'}$ *to reconstruct a source in*

$$F_N = \{\sum_{m,g,k} f_m(\mathbf{r}) g_g(E) h_k(\mathbf{\Omega}), f_m \in P_M, g_g \in \mathscr{E}_G, h_k \in Q_K\}$$

with N unknowns.

If detectors are available in a domain $D_0 \subset D$ *then one can locally reconstruct the source in* D_0 *from the measurements in* D_0 *by using nonradiating sources* $B^* f$ *with support* D_0 *.*

(ii) NISP : here we need to use function such that $B^* f = 0$ *, i.e., homogeneous solutions of the adjoint transport equation.*

inverse method for the NISP in one-group, homogeneous 1D slab transport:
for an isotropic material : $B^{*}f = 0 \rightarrow f(z,\mu) = \psi(z,-\mu)$ with $B\psi = 0$

the generalized solutions of the homogenous transport equation are

$$\psi(z,\mu) = e^{-\Sigma z/\nu}\phi_{\nu}(\mu)$$

where for $\nu \in \sigma = [-1,1] \cup \sigma_d$ *, the* $\phi_{\nu}(\mu)$ *are Case's generalized eigenfunctions*

Case's eigenfunctions are complete for the space of Hölder continuous functions in $[-1,1]$
they satisfy the full-range orthoganality relations $\int_{-1}^{1}(\phi_{\nu}\phi_{\nu'})(\mu)\mu d\mu = N(\nu)\delta(\nu-\nu')$

these generalized eigenfunctions can be use to obtain equations to compute a source $S(z,\mu)$ *from*

$$(f,S) = < f,\psi >_{+}$$

for $f(z,\mu) = e^{-\Sigma z/\nu}\phi_{\nu}(-\mu)$

conclusions

- *the invasive inverse problem has a unique solution*
- *if the source is isotropic and the scattering is isotropic, then there is a unique isotropic solution to NISP*

  (my guess is that this result should be valid with anisotropy of collision, but this remains to be proved)
- *there are anisotropic sources that are not equivalent to an isotropic one*
- *nonradiating sources are non positive*

  are nonradiating sources necessarily anisotropic?

  near the boundary : $\psi \to 0 : \partial_x \psi + \Sigma\psi = H\psi + S \to \partial_x \psi = H\psi + S$
- *the most sensitive approach for security screening applications is to combine a search for specific radiation lines with an inverse source algorithm*

- *for anisotropic scattering in 3D transport one should exploit the large data redundancy of the 3D AtRT for* $\mu_n \in [0,1]$ *with*

$$\mathbf{n}(\mu_n, \phi_n) = \mu_n \hat{\mathbf{e}}_z + \sqrt{1-\mu_n^2}(\hat{\mathbf{e}}_x \cos\phi_n + \hat{\mathbf{e}}_y \sin\phi_n)$$