Linearizing Inverse Problems and Applications to Tomography and Radiative Transfer

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Introduction

We start with an abstract "inverse problem": Solve

$$\mathcal{A}(f) = h,$$

where $\mathcal{A}$ is a possibly a non-linear map. We are interested in conditions that would imply local uniqueness and Hölder stability.

Assume that A_{f_0} is the differential (the linearization) near some f_0. Clearly, the injectivity of A_{f_0} is of interest. Suppose it is injective. What else is needed?

Main points:

- Injectivity of A_{f_0} alone is not enough for local uniqueness.
- Injectivity of A_{f_0} plus a "reasonable" stability estimate for A_{f_0} does imply local uniqueness and stability.
- Sufficient conditions: $A_{f_0}^* A_{f_0}$ to be an elliptic ΨDO.
- Using analyticity to prove injectivity of A_{f_0} (when possible).
- Examples

Let

$$\mathcal{A} : \mathcal{B}_1 \to \mathcal{B}_2,$$

where $\mathcal{B}_{1,2}$ are Banach spaces.

Definition 1

Weak uniqueness at f_0: $\mathcal{A}(f) = \mathcal{A}(f_0)$ for f near f_0 implies $f = f_0$.
Strong uniqueness at f_0: $\mathcal{A}(f_1) = \mathcal{A}(f_2)$ for $f_{1,2}$ near f_0 implies $f_1 = f_2$.

Definition 2

Conditional Hölder stability:

$$\|f_1 - f_2\|_{\mathcal{B}_1} \le \|\mathcal{A}(f_1) - \mathcal{A}(f_2)\|^{\mu}_{\mathcal{B}_2}, \quad \mu \le 1,$$

for $f_{1,2}$ near f_0 and $f_{1,2} \in \mathcal{K}$, where $\mathcal{K} \subset \mathcal{B}_1$ is compact.

Note: If $\mathcal{A}$ is injective, $\mathcal{A}|_{\mathcal{K}} \to \mathcal{A}(\mathcal{K})$ has a continuous inverse. So we have $\|f_1 - f_2\|_{\mathcal{B}_1} = o\left(\|\mathcal{A}(f_1) - \mathcal{A}(f_2)\|_{\mathcal{B}_2}\right)$. Hölder stability means that "$o = |\cdot|^{\mu}$."

The choice of $\mathcal{K}$ matters. Finitely dimensional $\mathcal{K}$, or $\mathcal{K}$ consisting of analytic functions only is too small.

A finitely dimensional "inverse problem."

Let $\mathcal{A} : \mathbf{R}^n \to \mathbf{R}^m$, $\mathcal{A} \in C^2$. Then

$$\mathcal{A}(x) = \mathcal{A}(x_0) + A_{x_0}(x - x_0) + R_{x_0}(x) \quad \text{with } |R_{x_0}(x)| \le C_{x_0}|x - x_0|^2,$$

for x near x_0. Assume now that A_{x_0} is injective (then $m \ge n$). This immediately implies the estimate

$$|h| \le C|A_{x_0}h|, \quad \forall h \in \mathbf{R}^n.$$

So, **Injectivity implies stability** (of the linear problem) in finite dimensions.

Also, it implies injectivity and stability for the original non-linear problem: If $|x - x_0| \ll 1$, one gets

$$|x - x_0| \le 2C|\mathcal{A}(x) - \mathcal{A}(x_0)|. \tag{1}$$

One can replace $\mathbf{R}^m$ here by an ∞-dim space. In particular, we get (in the original formulation, $\mathcal{A} : \mathcal{B}_1 \to \mathcal{B}_2$), that in $\mathcal{K}$ is finite dimensional, we have trivially Lipshtitz stability for any inverse problem!

In infinite dimensions, injectivity of A_{f_0} does not imply an estimate of the type

$$\|h\|_{\mathcal{B}_1} \le \|A_{f_0}h\|_{\mathcal{B}_2}. \tag{2}$$

We will show below, that a weaker estimate will do, i.e., such an estimate in some norms:

$$\|h\|_{\mathcal{B}_1'} \le C\|A_{f_0}h\|_{\mathcal{B}_2'}, \quad \forall h \in \mathcal{B}_1, \tag{3}$$

with some Banach spaces $\mathcal{B}_2' \subset \mathcal{B}_2$, $\mathcal{B}_1' \supset \mathcal{B}_1$.

Actually, we can always do that (take for example the graph norm). What we really want is to replace C^k or H^s with similar spaces with some loss of regularity but not too big.

Before, that, here is an example showing that without an estimate of the type (3), we may not have local uniqueness of the non-linear problem.

Example

Let ℓ^2 be the Hilbert space of sequences $x = \{x_k\}_{k=1}^{\infty}$ with norm $\|x\|^2 = \sum |x_k|^2$. The Sobolev spaces h^s are defined through the norms $\|x\|_{h^s}^2 = \sum k^{2s}|x_k|^2$. Set

$$\mathcal{A}(x) = Ex - (x, a)x, \tag{4}$$

where

$$a = \{1/k\}, \quad E = \mathrm{diag}\{e^{-k}\}.$$

Clearly, $\mathcal{A}(0) = 0$, the linearization of $\mathcal{A}$ near $x = 0$ is $A_0 = E$, and the latter is an injective map. Moreover, there is an inverse E^{-1} with a dense domain, but E^{-1} is unbounded as an operator from any Sobolev space to any other one, thus (3) does not hold in such spaces.

Since $[\mathcal{A}(x)]_k = e^{-k}x_k - x_k \sum x_m/m$, we get that

$$x^{(k)} = (0, \dots ke^{-k}, 0, \dots),$$

where the non-zero entry is the k-th one, solves $\mathcal{A}(x) = 0$. Therefore, in any neighborhood of 0 in ℓ^2 there are infinitely many solutions of $\mathcal{A}(x) = 0$ despite the fact that the linearization A_0 is injective. Moreover, for any s, in any neighborhood of 0 in h^s, there are still infinitely many solutions. No local uniqueness in this case, neither weak nor strong, in any Sobolev space!

One can still get local uniqueness for $\|x\|_* \ll 1$ by choosing an h^∞ type of norm $\|\cdot\|_*$ with an exponential weight, namely $\|x\|_* = \|E^{-1}x\|$. If we think of $x = \{x_k\}$ as the Fourier coefficients of some 2π-periodic function, then this translates into a neighborhood of the origin that involves certain real analytic functions only. Then $\|\cdot\|_*$ will look like this

$$\|f\|_*^2 = \int e^{2|\xi|}|\hat{f}(\xi)|^2\, d\xi.$$

We think of such topology as "unreasonably restrictive".

Note: We do not claim that a stability estimate of the linear part of the type (3) is a necessary condition for local uniqueness. On can modify the non-linear part above so that we have local uniqueness, for example,

$$\mathcal{A}(f) = E(f + (f,a)f).$$

One can even give an example with Hölder stability and the same unstable linear part! Actually, the differential can even be zero ($\mathcal{A}(x) = x^3$ in $\mathbf{R}$).

Conditions for local uniqueness and stability

Theorem 3 (weak local uniqueness and stability)

Assume that $\mathcal{A}$ has a derivative at f_0 with quadratic estimate on the remainder; Assume (3) with $\mathcal{B}_1 \subset \mathcal{B}_1'$, $\mathcal{B}_2' \subset \mathcal{B}_2$, i.e.,

$$\|h\|_{\mathcal{B}_1'} \leq C\|A_{f_0}h\|_{\mathcal{B}_2'}, \quad \forall h \in \mathcal{B}_1.$$

Assume also that there exist Banach spaces $\mathcal{B}_2'' \subset \mathcal{B}_2'$, $\mathcal{B}_1'' \subset \mathcal{B}_1$ so that $A_{f_0} : \mathcal{B}_1'' \to \mathcal{B}_2''$ and the following interpolation estimates hold

$$\|u\|_{\mathcal{B}_2'} \leq C\|u\|_{\mathcal{B}_2}^{\mu_2}\|u\|_{\mathcal{B}_2''}^{1-\mu_2}, \quad \|h\|_{\mathcal{B}_1} \leq C\|h\|_{\mathcal{B}_1'}^{\mu_1}\|h\|_{\mathcal{B}_1''}^{1-\mu_1} \quad \mu_1, \mu_2 \in (0,1], \quad \mu_1\mu_2 > 1/2.$$

(a) Then for any $K > 0$ there exists $\epsilon > 0$, so that for any f with

$$\|f - f_0\|_{\mathcal{B}_1} \leq \epsilon, \quad \|f\|_{\mathcal{B}_1''} \leq K, \tag{5}$$

one has the conditional stability estimate

$$\|f - f_0\|_{\mathcal{B}_1} \leq C(K)\|\mathcal{A}(f) - \mathcal{A}(f_0)\|_{\mathcal{B}_2}^{\mu_1\mu_2}, \quad C(K) = CK^{2-\mu_1-\mu_2}. \tag{6}$$

In particular, there is a weak local uniqueness near f_0, i.e., if $\mathcal{A}(f) = \mathcal{A}(f_0)$, then $f = f_0$.

Theorem 4 (Strong local uniqueness and stability)

(b) Assume in addition that there is a Banach space $\mathcal{K} \subset \mathcal{B}_1''$ so that (3) holds for f_0 replaced with f close enough to f_0 in $\mathcal{K}$, and $A_f : B_1'' \to B_2''$ is uniformly bounded for such f. Then there exists $\epsilon > 0$, so that for any f_1, f_2 with

$$\|f_1 - f_0\|_{\mathcal{K}} \le \epsilon, \quad \|f_2 - f_0\|_{\mathcal{K}} \le \epsilon, \tag{7}$$

one has the conditional stability estimate

$$\|f_1 - f_2\|_{\mathcal{B}_1} \le C\|\mathcal{A}(f_1) - \mathcal{A}(f_2)\|_{\mathcal{B}_2}^{\mu_1\mu_2}. \tag{8}$$

In particular, there is a strong local uniqueness near f_0, i.e., if $\mathcal{A}(f_1) = \mathcal{A}(f_2)$, then $f_1 = f_2$.

Note:

It seems quite natural that in most cases, one should expect that the constant $C > 0$ in (3):

$$\|h\|_{\mathcal{B}_1'} \leq C\|A_{f_0}h\|_{\mathcal{B}_2'}, \quad \forall h \in \mathcal{B}_1$$

can be chosen uniform when f_0 changes a bit. It is enough to know that A_{f_0} depends continuously on f_0.

There are few possible complications:

- This may require f_0 to be in a space smaller than $\mathcal{B}_1$. In boundary rigidity, f_0 is a metric, and instead of $f_0 \in C^2$, we need at least $f \in C^{2n+3}$ or so at this point.
- If $(\mathcal{B}_1', \mathcal{B}_2') \neq (\mathcal{B}_1, \mathcal{B}_2)$, and $f_0 \mapsto A_{f_0}$ is continuous, one cannot perturb (3). In boundary rigidity,

$$\|f\|_{L^2(M)} \leq C\|N_{g_0}f\|_{\tilde{H}^2(M_1)}, \quad \forall f \in H^1(M), \tag{9}$$

and N_g is of order -1. We could perturb N_g near $g = g_0$ only if we had H^1 instead of $\tilde{H}^2$! The proof is more delicate then.

More remarks

- The assumption
$$\|f\|_{\mathcal{B}_1''} \le K$$
is a typical compactness assumption, at least when $\mathcal{B}_1 = H^s(M)$, $\mathcal{B}_1'' = H^{s''}(M)$, with $s'' > s$, and M is compact.
- Let us say one can choose $\mathcal{B}_2' = \mathcal{B}_2$ at the expense of shrinking further $\mathcal{B}_1'$. Then $\mu_2 = 1$. One can try to take $\mu_1\mu_2 < 1$ arbitrarily close to 1. The price to pay for that is that $\mathcal{B}_1''$ shrinks (we have to assume $\|f\|_{H^{s''}} < K$ with $s'' \gg 1$). The downside is a strong compactness assumption. If $\mathcal{B}_1' = \mathcal{B}_1$, $\mathcal{B}_2' = \mathcal{B}_2$ (i.e., (3) is in the right norms), one gets a non-conditional Lipshitz estimate.
- One can try to choose $\mathcal{B}_2''$ to be very close to $\mathcal{B}_2$ (for example, $s'' > s$ but $s'' - s \ll 1$. This is equivalent to a weak compactness assumption. Then $\mu_1\mu_2 \ll 1$. So, the exponent in the Hölder estimate gets worse.

In other words, the stronger the compactness assumption is, the closer to a Lipshitz estimate we get.

Back to the linearization

So, what conditions, besides injectivity, imply

$$\|h\|_{\mathcal{B}_1} \leq C\|Ah\|_{\mathcal{B}_2}, \quad \forall h \in \mathcal{B}_1, \tag{10}$$

or the weaker estimate

$$\|h\|_{\mathcal{B}_1'} \leq C\|Ah\|_{\mathcal{B}_2'}, \quad \forall h \in \mathcal{B}_1 \tag{11}$$

in "good" norms (spaces) $\mathcal{B}_1'$, $\mathcal{B}_2'$?

The "if and only if" condition for (10) (besides injectivity) is closed range but that is not very helpful.

Let $\mathcal{B}_1 = L^2(M)$ now with M a compact manifold. We replace $A^*_{f_0}A_{f_0}$ by N.

Theorem 5

(a) If N is an elliptic ΨDO of order $-m$, and N is injective on a (closed) subspace $\mathcal{L} \subset L^2(M)$, then (10) holds, i.e.,

$$\|h\|_{L^2(M)} \le C\|Nh\|_{H^m(M)}, \quad \forall f \in \mathcal{L}.$$

Moreover, C can be chosen uniform under small perturbations of N in the topology of Ψ^{-m}.

(b) If N is a hypoelliptic ΨDO with a loss of k derivatives, and N is injective on a (closed) subspace $\mathcal{L} \subset L^2(M)$, then the following version of (11) holds:

$$\|h\|_{L^2(M)} \le C\|Nh\|_{H^{k+m}(M)}, \quad \forall f \in \mathcal{L}.$$

Hypoellipticity here means that $Nf \in H^{m+k} \Longrightarrow f \in L^2$. (If N is elliptic, then it is enough to have $Nf \in H^m$ to reach the same conclusion.)

The uniformity of C in (b) is not clear.

Sketch of the Proof

Since N is elliptic, one can apply a parametrix Q to $Nf = h$ and reduce the problem to solving

$$(Id + K)f = \tilde{h},$$

where K is compact and $\tilde{h} = Qh$.

One can construct Q so that $Id + K = QN$ stays injective. One can always apply $Id + K^*$, so we can assume that K is self-adjoint. Now, since $Id + K$ is injective, then -1 is not an eigenvalue for K, so $(Id + K)^{-1}$ exists. Hence the estimate.

Moreover, this is preserved under a small perturbation of N (that leads to a small perturbation of K).

Example: The weighted X-ray transform

Let $w(x,\theta)$ be smooth, and consider the weighted X-ray transform:

$$I_w f(x,\theta) = \int w(x,\theta) f(x+t\theta)\,dt, \quad x \in \mathbf{R}^n,\ |\theta| = 1.$$

For general $w(x,\theta) \neq 0$, not injective (Boman) even for f compactly supported. If $w = \text{const.}$, or $w(x,\theta) = \exp\left(-\int_{-\infty}^0 a(x+s\theta)\,ds\right)$, (known for the attenuated Radon transform), it is injective with an explicit inversion formula (Bukhgeim, Novikov et al.)

Principal symbol of $I_w^* I_w$:

$$(x,\xi) \mapsto 2\pi \int_{S^{n-1}} |w(x,\theta)|^2 \delta(\xi\cdot\theta)\,d\theta.$$

If $n = 3$, we integrate $|w|^2$ along the grand circle on the sphere in the ξ variable, perpendicular to ξ. For ellipticity, it is enough that

$$\forall (x,\xi),\ \exists \theta \perp \xi \text{ so that } w(x,\theta) \neq 0. \tag{12}$$

In particular, w can vanish on parts of any line, and $I_w^* I^*$ might be still elliptic!

A big set of lines can be thrown away if $n \geq 3$ (just choose $w = 0$ there).

Even if $n = 2$, "half" of the lines are not needed.

Condition (12) can also be written as

$$N^*\text{supp}^0\, w \Supset T^*\Omega, \tag{13}$$

where $\text{supp}^0\, w = \{(x,\xi);\ w(x,\xi) \neq 0\}$, and $\operatorname{supp} f \subset \Omega$. It relates to the Bolker condition but the latter is a bit more restrictive.

Theorem 6

Let w satisfy (13) (for example, $w > 0$) and be real analytic. Then I_w is injective on $L^2(\Omega)$.
Moreover, I_w is injective for a dense open (in C^1) set of weights w's satisfying (13), including real analytic ones. For any such w, the standard stability estimate holds

$$\|f\|_{L^2(\Omega)} \leq C\|I_w^* I_w\|_{L^2(\Omega_1)}, \quad \Omega_1 \Supset \Omega$$

with a locally uniform $C > 0$.

This is a stripped down version of a result by Frigyik, Uhlmann and S. (where arbitrary families of "geodesic like" curves are studied.

Is the ellipticity (besides injectivity) of $N \in \Psi^{-m}$ really needed for the estimate

$$\|h\|_{L^2(\Omega)} \leq C\|Nh\|_{H^m(\Omega_1)}, \quad \forall f \in L^2(\Omega). \tag{14}$$

to be true? Here, $\Omega_1 \Supset \Omega$ as above.

Theorem 7

Yes.

How about the hypoelliptic estimate?

$$\|h\|_{L^2(\Omega)} \leq C\|Nh\|_{H^{m+k}(\Omega_1)}, \quad \forall f \in L^2(\Omega). \tag{15}$$

There are various conditions for hypoellipticity. This situation appears in linearized boundary rigidity, but then there are non ΨDO operators involved; in Doppler tomography (integrals of vector fields) with a non-constant weight.

However, if the full symbol vanishes on an open set, (14) cannot hold regardless of what k is. Another way to say that:

$$\operatorname{supp} \sigma(N) \supset T^*\Omega$$

is a necessary condition for (15) to hold.

Applications to Radiative Transfer: I. An inverse source problem

Let T be our favorite transport operator:

$$Tu(x,\theta) = \theta \cdot \nabla_x u(x,\theta) + \sigma(x,\theta)u(x,\theta) - \int_{S^{n-1}} k(x,\theta,\theta')u(x,\theta')d\theta',$$

$(x,\theta) \in \Omega \times S^{n-1}$. Direct problem: given a source $f(x,\theta)$, find the outgoing flux corresponding to a zero incoming flux. In other words, solve first

$$Tu = f, \quad u|_{\partial_-\Omega} = 0; \tag{16}$$

(if we can), and then measure

$$Xf := u|_{\partial_+\Omega}.$$

We think of this as a perturbed X-ray transform. If $\sigma = 0$, $k = 0$,

$$Xf = \int f(x + t\theta, \theta)\, dt$$

(parametrized accordingly by $(x,\theta) \in \partial_+\Omega$, then $t \le 0$). If $k = 0$, and $\sigma = \sigma(x)$ (no θ), this is the attenuated X-ray transform.

Inverse Source Problem

Given Xf, find f.

If f depends on θ, this is an underdetermined problem with a trivial and non-interesting solution. Namely, if $\sigma = 0$, $k = 0$, then for any $u(x, \theta)$ with $u|_{\partial_-\Omega} = 0$, $u|_{\partial_+\Omega} = h$, with some h fixed,

$$f = \theta \cdot \nabla_x u$$

is a solution to $Xf = h$, and those are all solutions (same if $(\sigma, k) \neq (0, 0)$).

If f depends on x only, or if it has some special structure (polynomial in ξ), then the problem is non-trivial. Assume that $f = f(x)$.

Results by **Sharafutdinov, Bal & Tamasan** ($\sigma = \sigma(x)$, $k = k(x, \theta \cdot \theta') \ll 1$).

To fit this into our framework, compare it to the weighted X-ray transform corresponding to $k = 0$. Then $X =: I_\sigma$, where

$$Xf(x,\theta) = I_\sigma f(x,\theta) := \int E(x+t\theta,\theta)f(x+t\theta)\,dt,$$

where

$$E(x,\theta) = \exp\left(-\int_0^\infty \sigma(x+s\theta,\theta)\,ds\right).$$

Then we compare X^*X and $I_\sigma^* I_\sigma$, acting on functions depending on x only. The operator I_σ is like I_w that we studied before, with a weight $w = E(x,\theta)$. It has a parametrix $v(x,D)|D|$, with $v(x,\xi)$ elliptic and homogeneous in ξ of order 0.

Write

$$X^*X = I_\sigma^* I_\sigma + \mathcal{L}.$$

Then

$$\begin{aligned} v(x,D)|D|X^*X &= v(x,D)|D|I_\sigma^* I_\sigma + v(x,D)|D|\mathcal{L} \\ &= Id + \text{compact operator} + v(x,D)|D|\mathcal{L}. \end{aligned}$$

Question: Is $|D|\mathcal{L}$ compact in $L^2(\Omega)$? In other words, does the contribution of k reduce to a compact term, after applying the parametrix?

This requires the analysis of an operator with singular kernel of the type

$$Bf(x) = \iint \frac{\alpha(x, \widehat{x-z}, \widehat{z-y})}{|x-z|^{n-1}|z-y|^{n-1}} f(y)\, dy\, dz,$$

where $\hat{p} = p/|p|$. We want to show that $B : L^2 \to H^2$.

If there is no $\widehat{z-y}$ there, then B would be a composition of two singular operators, each one of order -1, and we are done.

This is true even in the general case.

Results:

- The direct problem is solvable for generic (σ, k).
- The inverse problem is uniquely solvable for generic (σ, k), i.e., for such (σ, k), the map X is injective.
- For any (σ, k) for which X is injective, we have

$$\|f\|_{L^2(\Omega)} \leq C\|X^*Xf\|_{H^1(\Omega_1)}, \quad (\Omega \Subset \Omega_1)$$

  with $C > 0$ locally uniform w.r.t. (σ, k).

This result is optimal because of Boman's results (there is σ, so that X is not injective for that σ and $k = 0$).

Inverse Problem with Angularly Averaged Measurements [Ian Langmore]

The direct problem:

$$Tu = 0, \quad u|_{\partial_-\Omega} = f_-. \tag{17}$$

Corresponds to a given incoming flux on the boundary and no source inside. We measure the outgoing flux:

$$\mathcal{A}f_- = u|_{\partial_+\Omega}.$$

Then

$$\mathcal{A} : C_0^\infty(\partial_-\Omega) \to \mathcal{D}'(\partial_+\Omega)$$

(it has much better mapping properties), is called the *albedo operator*.

Inverse Problem: Given $\mathcal{A}$, find (σ, k). Thoe most general result: if $\sigma = \sigma(x)$ (no θ), and for k so that the direct problem is solvable, one can solve it explicitly (Choulli & S.).

Inverse Problem with Angularly Averaged Measurements [Ian Langmore]

The averaged measurements problem assumes that we can only measure

$$\mathcal{M}f_-(x) = \int_{\partial_+\Omega} \mathcal{A}f_-(x,\theta)\, d\mu_x(\theta)$$

but we can still probe the medium with signals $f_-(x,\theta)$ depending both on $x \in \partial\Omega$ and θ. Usually, $d\mu_x(\theta) = \theta \cdot \nu(x) d\theta$.

Result [Ian Langmore]:
If $\sigma = \sigma(x)$, $k = k(x)g(x,\theta,\theta')$ with g known, then for generic known (g,μ,σ), one can uniquely recover $k(x) \ll 1$ from knowledge of $\mathcal{M}$. True in particular, for (g,μ,σ) real analytic or close enough to a real analytic triple.

The problem is first linearized, and the linearized map at $k=0$ turns out to be a weighted X-ray transform of $k(x)$ over lines with a non-trivial weight.

Then one can apply the general approach. This also gives a Hölder stability estimate.

Other problems, where this approach works:

- Inverse Backscattering for the Schrödinger equation. Linearization A: a "distorted Fourier transform." Then A^*A is an elliptic ΨDO.
- The boundary rigidity / lens rigidity problem
- Inverse Problem for the hyperbolic DN map

Problems, where this approach does not work:

- EIT. The linearization is a C^∞ smoothing map. No $H^{s_1} \to H^{s_2}$ estimate possible. Nevertheless, we have uniqueness (Sylvester & Uhlmann) for isotropic conductivities but the proof is not through linearization.
- Recovery of an obstacle from the scattering amplitude $a(k,\theta,\omega)$ for a fixed k, θ. The linearization is injective but highly unstable. The known proof for local uniqueness (S. & Uhlmann) does not rely on lineaization.

One way to prove injectivity: using analyticity

If A^*A happens to be an analytic elliptic ΨDOin a neighborhood of M, then one can do the following. Assume that $Nf = 0$ in $M_1 \Supset M$, and f is supported in M. Typically, we start with $Nf = 0$ in M, extend f as zero outside M and show that $Nf = 0$ is preserved in M_1 as well.

Apply a parametrix Q to get $(Id + K)f = 0$. In the analytic calculus, K is analytic regularizing, i.e., it sends any function supported inside M_1 into an analytic function.

Ss, $f = -Kf$ is analytic in M_1, vanishes outside M. Therefore, $f = 0$. So we get that N is injective.

In the example above, $I_w^* I_w$ is an analytic ΨDO, if w is analytic.