New results about discrete Laplace operator and Dirichlet-to-Neumann map of a graph

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April 21

Abstract

A function (conductivity) on the edges of a graph induces the Dirichlet-to-Neumann map between boundary values of harmonic functions (potentials and currents). The map can be calculated as a Schur complement of the Kirchhoff matrix of the graph.

It is an amazing fact that the square of the non-local Dirichlet-to-Neumann map on the unit disc equals minus the local(!) Laplace operator on the boundary circle. To answer a question of Gunther Uhlmann and to establish a new connection between discrete and continuous Dirichlet-to-Neumann maps and for the approximations I construct a finite and an infinite graphs which Dirichlet-to-Neumann map (full matrix) has the similar property, its square is a tridiagonal (sparse) matrix. The construction gives a new continued fraction identity. It is interesting to consider the geometric and probabilistic (trajectories of the random walk) consequences of this localizing identity unifying discrete and continuous equations for potentials.

The inverse conductivity problem is to find the conductivity from the Dirichlet-to-Neumann map. On an example of a lattice graph we will show that the map from logarithm of conductivity to the certain logarithms of the determinants of the submatrices of the Dirichlet-to-Neumann map is linear(!) and so the solution of the inverse problem is reduced to solution of the system of linear equations that arise from disjoint paths in the graph. I conjecture that the method generalizes to planar and three dimensional graphs and also to the continuous case.

The eigenvalues of the harmonic continuation from the boundary to the boundary of a graph can be directly measured from the its Dirichlet-to-Neumann map implying new equations for conductivity in terms of the Dirichlet-to-Neumann map for the inverse problems. I will show that operator of analytic (harmonic) continuation on a lattice graph has a positive spectrum. I use a theorem about positivity of eigenvalues of totally positive matrices. I conjecture that by approximation the similar result holds in continuous case on a plane.