The Brouwer fixed point theorem

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Suppose that $S$ is a set. A function $f : S \to S$ has a fixed point if there is an element $x \in S$ so that $f(x) = x$. A fixed point theorem is a theorem like this: with some conditions on $S$ or $f$ or both, $f$ must have a fixed point.

**Examples**

- Any contraction from $\mathbb{R}$ to $\mathbb{R}$ has a fixed point.
- The intermediate value theorem implies that every continuous function $f : [0, 1] \to [0, 1]$ has a fixed point.

My topic: the **Brouwer fixed point theorem**, which generalizes both of these.
A function $f : [0, 1] \to [0, 1]$ is a contraction if $f$ contracts distances: for all $x_1, x_2 \in [0, 1],$

$$|x_1 - x_2| > |f(x_1) - f(x_2)|.$$  

**Theorem**

*Any contraction $f : [0, 1] \to [0, 1]$ has a unique fixed point.*

**Outline of proof.**

Pick any $x_0 \in [0, 1]$. Define a sequence $\{x_0, x_1, x_2, \ldots \}$ by

\[
\begin{align*}
x_1 &= f(x_0) \\
x_2 &= f(x_1) = f(f(x_0)) \\
x_3 &= f(x_2) = f(f(f(x_0))) \\
&\vdots
\end{align*}
\]

Then $\lim_{n \to \infty} x_n$ is the (unique) fixed point of $f$. 
Consider $f(x) = \cos x$ on $[0, 1]$. Let $x_0 = 1/5$ and let $x_1 = f(x_0)$. Let $x_2 = f(x_1)$. Let $x_3 = f(x_2)$. Iterate 20 more times... (The actual fixed point is approximately 0.739.)
The same theorem holds in 2 (and higher) dimensions

### Examples

1. Maps
2. Disks

```python
P = circle((0,0), 1, aspect_ratio=1)
s = 0.9  # scale factor
phi = pi/15  # rotation angle
rot = matrix([[cos(phi), -sin(phi)], [sin(phi), cos(phi)]]) # rotation matrix
shift = vector([0, (1-s)/2]) # vertical translation

v = vector((1, 0))
P += point(v, color='red')
v = s*rot*v + shift
v = vector((n(v[0]), n(v[1])))
P += point(v, color='red')
```
Starting with \((1, 0)\):
Starting with $(1, 0)$:
Starting with $(0, -1)$:
Starting with $(-1, 0)$:
The Brouwer fixed point theorem removes the contraction condition.

Theorem

Let \( S \) be the unit interval \([0, 1]\) or the unit square or the unit cube or \ldots Let \( f : S \to S \) be a continuous function. Then \( f \) has a fixed point.

- The fixed point need not be unique.
- There is no obvious limit for finding it: consider rotations or the following example.
Consider $f(x) = 1 - x$ on $[0, 1]$. Let $x_0 = 1/5$, let $x_1 = f(x_0)$, let $x_2 = f(x_1)$, etc. Then

$$x_0 = x_2 = x_4 = \cdots = 1/5,$$

$$x_1 = x_3 = x_5 = \cdots = 4/5.$$
Two subsets $X$ and $Y$ of $\mathbb{R}^2$ are **topologically equivalent** (or **homeomorphic**) if there are continuous bijections $g : X \to Y$ and $g^{-1} : Y \to X$.

### Examples

For example, the following are all topologically equivalent to each other:

- disk of radius 1: $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$
- disk of radius 2
- any disk
- triangle
- quadrilateral
- pentagon
- polygon
Theorem (The Brouwer fixed point theorem in $\mathbb{R}^n$)

Fix an integer $n \geq 0$ and let $D \subset \mathbb{R}^n$ be the unit disk:

$$D = \{x \in \mathbb{R}^n : \|x\| \leq 1\}.$$

Then any continuous function $f : D \to D$ has a fixed point.

Proposition

Suppose that $X$ and $Y$ are topologically equivalent, and that every continuous function $f : X \to X$ has a fixed point. Then every continuous function $h : Y \to Y$ has a fixed point.
Proof of proposition.

Suppose that $X$ and $Y$ are topologically equivalent, and suppose that $h : Y \to Y$ is continuous. Consider

$\begin{array}{c}
  Y \\
  \downarrow g \\
  X \\
  \downarrow g^{-1} \\
  Y \\
  \downarrow g^{-1} \circ h \circ g \\
  X
\end{array}$

Since the bottom function will have a fixed point, so will the top one: if $g^{-1} \circ h \circ g(x) = x$, then $g(x)$ is a fixed point for $h$: applying $g$ to both sides gives $h(g(x)) = g(x)$. 

\[ \square \]
Examples

- contractions
- maps
- salad dressing, cocktails
- the fundamental theorem of algebra
- every real $n \times n$ matrix with all positive entries has a positive eigenvalue
- Hex can’t end in a draw
- existence of Nash equilibrium
Theorem (The Brouwer fixed point theorem in $\mathbb{R}^2$)

Let $D \subset \mathbb{R}^2$ be the unit disk. Then any continuous function $f : D \to D$ has a fixed point.

There are many proofs of the Brouwer fixed point theorem, and I’m going to describe one using a result from combinatorics. First, because of the proposition, we can replace $D$ with anything topologically equivalent to it. We will work with a triangle, instead.
Now for some combinatorics. Suppose \( T \) is a triangle, and suppose it has been subdivided into smaller triangles. Label each vertex with \( A, B, \) or \( C \) according to these rules:

- the vertices of the original triangle all have different labels
- on the side \( AB \), all of the vertices are either \( A \) or \( B \)
- similarly for the other sides
- (no restrictions on the labels of vertices in the middle)

This is called a Sperner labeling. In a Sperner labeling, a triangle is called complete if its vertices have all three labels.
Theorem (Sperner’s lemma)

*At least one triangle in a Sperner labeling is complete.*
Proof of Sperner’s lemma:

- by dimension. In dimension 1: easy.
- In dimension 1: actually prove that there are an odd number of “complete” edges. (Count vertices labeled $A$.)
- In dimension 2: count the number of edges labeled $AB$:

  \[
  x = \#\{\text{triangles labeled } AAB \text{ or } ABB\} \\
  y = \#\{\text{triangles labeled } ABC\}. 
  \]

  Then we get $2x + y$.
- On the other hand, if $i$ is the number of $AB$ edges on the inside and $o$ is the number on the outside, then we get $2i + o$.
- That is, $2x + y = 2i + o$.
- Since $o$ is odd, $y$ must be odd.
Proof of the Brouwer fixed point theorem in $\mathbb{R}^2$:

Let $T$ be a triangle, as above, and suppose $f : T \rightarrow T$ is continuous.

For every point $x$ in $T$, draw an arrow from $x$ to $f(x)$. Chop $T$ into smaller triangles. Label each vertex: label $A$ if the arrow points northeast, label $B$ if the arrow points northwest, label $C$ if any other direction. This produces a Sperner labeling. Therefore there is a complete triangle with vertices $A_1$, $B_1$, $C_1$. 
Chop $T$ into even smaller triangles and find another, smaller, complete triangle $A_2B_2C_2$. Repeat, using smaller and smaller triangles, getting complete triangles $A_nB_nC_n$ for each $n \geq 1$. By a basic fact in topology (compactness of $T$), the set of points $\{A_1, A_2, A_3, \ldots\}$ has a limit point $P$.

Since the triangles get smaller and smaller, $P$ is also a limit point of $\{B_1, B_2, \ldots\}$ and of $\{C_1, C_2, \ldots\}$. So $P$ is a point whose arrow must point northeast, northwest, and south, simultaneously. So the
In higher dimensions: the obvious generalization of Sperner’s lemma holds in $\mathbb{R}^n$ for any $n$, and using it, one can prove the obvious generalization of the Brouwer fixed point theorem.

**Theorem (The Brouwer fixed point theorem in $\mathbb{R}^n$)**

*Fix an integer $n \geq 0$ and let $D \subset \mathbb{R}^n$ be the unit disk:*

$$D = \{ \mathbf{x} \in \mathbb{R}^n : \| \mathbf{x} \| \leq 1 \}.$$

*Then any continuous function $f : D \to D$ has a fixed point. The same is true if $D$ is replaced by any space homeomorphic to it.*