A function and its Fourier transform cannot both be sharply localized.

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Notation:

$$|E| = \text{Lebesgue measure of } E, \qquad \int = \int_{-\infty}^{\infty}$$

The Fourier transform (on $\mathbb{R}$):

$$\widehat{f}(\xi) = \int e^{-2\pi i \xi x} f(x)\, dx$$
$$f(x) = \int e^{2\pi i x \xi} \widehat{f}(\xi)\, d\xi.$$

Basic facts:

$$\|\widehat{f}\|_\infty \le \|f\|_1, \qquad \|\widehat{f}\|_2 = \|f\|_2,$$
$$\|\widehat{f}\|_q \le \|f\|_p \quad (1 \le p \le 2,\ p^{-1} + q^{-1} = 1),$$
$$\widehat{f'}(\xi) = 2\pi i \xi \widehat{f}(\xi).$$

A *probability distribution function (PDF)* on $\mathbb{R}$ is a function $\rho \ge 0$ with $\int \rho(x)\, dx = 1$. If ρ is a PDF, its *variance* is

$$\mathrm{var}(\rho) = \inf_{a \in \mathbb{R}} \int (x - a)^2 \rho(x)\, dx.$$

Interpretations:

1. Classical: $f(t)$ is the amplitude of a signal at time t. $f(t) = \int e^{2\pi i\omega t} \widehat{f}(\omega)\, d\omega$ expresses f as a superposition of sine waves of different frequencies.

2. Quantum: Suppose $\|f\|_2 = 1$. Then $|f|^2$ and $|\widehat{f}|^2$ are both PDFs. f is the "wave function" of a quantum particle moving on the line, $|f|^2$ is the PDF of its position, and $|\widehat{f}|^2$ is the PDF of its momentum (taking Planck's constant $= 1$).

Heisenberg's Inequality. *If* $f \in L^2(\mathbb{R})$*, then for all* $a, \alpha \in \mathbb{R}$*,*

$$\int (x-a)^2 |f(x)|^2 \, dx \int (\xi - \alpha)^2 |\widehat{f}(\xi)|^2 \, d\xi \geq \frac{\|f\|_2^4}{16\pi^2}. \quad (1)$$

In particular, if $\|f\|^2 = 1$*,*

$$\mathrm{var}(|f|^2)\,\mathrm{var}(|\widehat{f}|^2) \geq 1/16\pi^2.$$

Equality holds in (1) *for a given* a *and* $\alpha \iff f(x) = ce^{2\pi i \alpha x} e^{-b(x-a)^2}$ *for some* $b > 0$*,* $c \in \mathbb{C}$*.*

Proof: By considering $g(x) = e^{-2\pi i\alpha x}f(x+a)$, reduce to the case $a = \alpha = 0$. WLOG, assume that $\int x^2|f(x)|^2\,dx < \infty$ and $\int \xi^2|\widehat{f}(\xi)|^2\,d\xi < \infty$. Note that

$$\int \xi^2|\widehat{f}(\xi)|^2\,d\xi = \frac{1}{4\pi^2}\int |\widehat{f'}(\xi)|^2\,d\xi = \frac{1}{4\pi^2}\|f'\|_2^2, \quad (2)$$

so $f' \in L^2$. Since

$$\frac{d}{dx}(x|f(x)|^2) = |f(x)|^2 + 2\,\mathrm{Re}\,xf(x)\overline{f'(x)},$$

$$-2\,\mathrm{Re}\int_c^d xf(x)\overline{f'(x)}\,dx = -x|f(x)|^2\Big|_c^d + \int_c^d |f(x)|^2\,dx.$$

Let $c \to -\infty$, $d \to \infty$ to get

$$-2\,\mathrm{Re}\int xf(x)\overline{f'(x)}\,dx = \|f\|_2^2,$$

then use Cauchy-Schwarz and (2):

$$\|f\|_2^2 \le 2\|xf\|_2\|f'\|_2 = 4\pi\|xf\|_2\|\xi f'\|_2.$$

Equality holds $\iff$ $f'(x) = bxf(x)$ for some $b \in \mathbb{R}$, which gives $f(x) = ce^{-bx^2}$.

There are generalizations involving other L^p norms. For example, if $1 \le p \le 2$,

$$\|f\|_2^2 \le 2\|xf\|_p\|f'\|_{p'} \le 2\|xf\|_p\|\widehat{f'}\|_p = 4\pi\|xf\|_p\|xf'\|_p.$$

Cowling and Price have obtained general results relating to norms of the form $\|\,|x|^a f\|_p$.

Local Uncertainty Inequality (Faris-Price). *Suppose* $0 < \alpha < \frac{1}{2}$. *There exists* $C_\alpha > 0$ *such that for all* $f \in L^2(\mathbb{R})$ *and all measurable* $E \subset \mathbb{R}$,

$$\int_E |\widehat{f}|^2 \le C_\alpha |E|^{2\alpha} \|\,|x|^\alpha f\|_2^2.$$

Proof: Let $\chi_r(x) = 1$ if $|x| < r$, $\chi_r(x) = 0$ otherwise, and $\chi_r' = 1 - \chi_r$.

$$\begin{aligned}\|\widehat{f}\chi_E\|_2 &\le \|(f\chi_r)\widehat{\ }\chi_E\|_2 + \|(f\chi_r')\widehat{\ }\chi_E\|_2 \\ &\le |E|^{1/2}\|(f\chi_r)\widehat{\ }\|_\infty + \|f\chi_r'\|_2.\end{aligned}$$

Now

$$\begin{aligned}\|(f\chi_r)\widehat{\ }\|_\infty &\le \|f\chi_r\|_1 \\ &\le \left(\int_{-r}^{r} |x|^{-2\alpha}\,dx\right)^{1/2}\left(\int_{-r}^{r} |x|^{2\alpha}|f(x)|^2\,dx\right)^{1/2} \\ &\le C_\alpha r^{(1/2)-\alpha}\|\,|x|^\alpha f\|_2,\end{aligned}$$

and

$$\|f\chi_r'\|_2 \le r^{-\alpha}\|\,|x|^\alpha f\|_2,$$

so

$$\|\chi_E\widehat{f}\|_2 \le (C_\alpha|E|^{1/2}r^{(1/2)-\alpha} + r^{-\alpha})\|\,|x|^\alpha f\|_2.$$

Choose r to minimize this quantity.

Qualitative Uncertainty Principles:

Let $\Sigma(f) = \{x : f(x) \neq 0\}$.

1. $\Sigma(f)$ bounded $\Rightarrow \widehat{f}$ entire $\Rightarrow \mathbb{R} \setminus \Sigma(\widehat{f})$ countable or $f = 0$.

2. If $f \in L^2$ and $f \neq 0$, $|\Sigma(f)|\,|\Sigma(\widehat{f})| \geq 1$.

$$\int |\widehat{f}|^2 \leq |\Sigma(\widehat{f})|\,\|\widehat{f}\|_\infty^2 \leq |\Sigma(\widehat{f})|\,\|f\|_1^2$$
$$\leq |\Sigma(\widehat{f})|\,|\Sigma(f)|\,\|f\|_2^2.$$

3. (Benedicks) If $f \in L^p$ for some $p \geq 1$ and $|\Sigma(f)|\,|\Sigma(\widehat{f})| < \infty$, then $f = 0$.

4. (Hardy) Suppose

$$|f(x)| \leq Ce^{-a\pi x^2}, \qquad |\widehat{f}(\xi)| \leq Ce^{-b\pi\xi^2}$$

for some $a, b, C > 0$. If $ab = 1$ then $f(x) = ce^{-ax^2}$; if $ab > 1$ then $f = 0$.

Definition: A function $f \in L^2(\mathbb{R})$ is *ϵ-concentrated* on a set $A \subset \mathbb{R}$ if $\|f(1-\chi_A)\|_2 \leq \epsilon\|f\|_2$, or equivalently $\|f\chi_A\|_2 \geq \sqrt{1-\epsilon^2}\|f\|_2$.

For $A, B \subset \mathbb{R}$, let

$$P_A f = \chi_A f, \qquad (Q_B f)\widehat{\ } = \chi_B \widehat{f}.$$

Theorem (Donoho-Stark).

a. $\|P_A Q_B f\|_2 \leq |A|^{1/2}|B|^{1/2}\|f\|_2$.

b. *If there exists $f \neq 0$ such that f is ϵ-concentrated on A and $\widehat{f}$ is δ-concentrated on B, then*

$$|A|^{1/2}|B|^{1/2} \geq 1 - \epsilon - \delta.$$

Proof: For (a),

$$\begin{aligned}|P_A Q_B f(x)| &= \chi_A(x) \left| \int \widehat{\chi}_B(x-y) f(y)\, dy \right| \\ &\le \chi_A(x) \|f\|_2 \|\widehat{\chi}_B\|_2 = \chi_A(x) |B|^{1/2} \|f\|_2.\end{aligned}$$

Take L^2 norms of both sides.

For (b), assume $\|f\|_2 = 1$. Then

$$\|P_A(1-Q_B)f\|_2 \le \|(1-Q_B)f\|_2 \le \delta,$$

so

$$\begin{aligned}1-\epsilon-\delta &\le \|f\|_2 - \|f - P_A f\|_2 - \|P_A(1-Q_B)f\|_2 \\ &\le \|P_A Q_B f\|_2 \le |A|^{1/2}|B|^{1/2}.\end{aligned}$$

Landau-Pollak-Slepian Theory:

Take $A = (-\frac{1}{2}T, \frac{1}{2}T)$ and $B = (-\Omega, \Omega)$ and consider

$$S = Q_B P_A Q_B = (P_A Q_B)^*(P_A Q_B).$$

This is a compact self-adjoint operator on L^2, so it has an orthonormal eigenbasis $\{\phi_n\}$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq 0$.

If $Sf = \lambda f$ then

$$\|P_A f\|_2^2 = \|P_A Q_B f\|_2^2 = \langle Q_B P_A Q_B f, f\rangle = \lambda \|f\|_2^2,$$

so f is $\sqrt{1-\lambda}$-concentrated on A and $\widehat{f}$ is 0-concentrated on B.

Theorem: *$\lambda_n \approx 1$ for $n \ll 2\Omega T$, $\lambda_n \approx 0$ for $n \gg 2\Omega T$, and the transition interval has width $O(\log \Omega T)$.*

Eigenfunctions ϕ_n are "prolate spheroidal wave functions."

If ρ is a PDF on $\mathbb{R}$, its *entropy* $E(\rho)$ is

$$E(\rho) = -\int \rho(x) \log \rho(x)\, dx.$$

Proposition: *If* $\operatorname{var}(\rho) < \infty$ *then*

$$E(\rho) \leq \tfrac{1}{2} \log(2\pi e \operatorname{var}(\rho)).$$

Proof: By composing with translations and dilations, reduce to the case where ρ has mean 0 and variance $\operatorname{var}(\rho) = \int x^2\, d\rho(x) = 1$. Let

$$\phi(x) = \sqrt{2\pi}\, e^{x^2/2} \rho(x), \qquad d\gamma(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx,$$

so $\int \phi\, d\gamma = \int \rho\, dx = 1$. By Jensen's inequality,

$$\begin{aligned} 0 &= \left(\int \phi\, d\gamma\right) \log \left(\int \phi\, d\gamma\right) \leq \int \phi \log \phi\, d\gamma \\ &= \int \rho(x)[\tfrac{1}{2} \log 2\pi + \tfrac{1}{2} x^2 + \log \rho(x)]\, dx \\ &= \tfrac{1}{2} \log 2\pi + \tfrac{1}{2} - E(\rho). \end{aligned}$$

Theorem: *If* $\|f\|_2 = 1$ *then*

$$E(|f|^2) + E(|\widehat{f}|^2) \geq 1 - \log 2.$$

Lemma: *If* $\phi(t) \leq \psi(t)$ *for* $t \geq a$ *and* $\phi(a) = \psi(a)$, *then* $\phi'(a) \leq \psi'(a)$.

Applying this to the Hausdorff-Young inequality $\|\widehat{f}\|_q \leq \|f\|_p$ ($q \geq 2$, $p = q/(q-1)$, $a = 2$), we get $E(|f|^2) + E(|\widehat{f}|^2) \geq 0$.

But using Beckner's sharp Hausdorff-Young inequality

$$\|\widehat{f}\|_q \leq p^{1/2p} q^{-1/2q} \|f\|_p,$$

we get the theorem.

Corollary: Heisenberg's inequality.

Corollary: *Suppose* $\int |g|^2 \, d\gamma = 1$. *Let* $Tg(x) = 2^{1/4} e^{-\pi x^2} g(2\sqrt{\pi}\, x)$, *and* $\widetilde{g} = T^{-1}(\widehat{Tg})$. *Then*

$$\int |g|^2 \log |g| \, d\gamma + \int |\widetilde{g}|^2 \log |\widetilde{g}| \, d\gamma \leq \int |\nabla g|^2 \, d\gamma.$$

Without the second term on the left, this is Gross's logarithmic Sobolev inequality.