Exam 2 Review

Exam 2 covers 1.6, 2.1-2.3, 1.5, 4.1-4.2, and 5.1-5.3. You should know how to do all the homework problems from these sections and you should practice your understanding on several old exams in the exam archive. Make sure you are working through the problems on your own (without looking at solutions or getting help). Put yourself in an exam like setting as you are studying.

Here is a brief summary of things we have done:

- **Finding the equation of a line**: Get two points, find the slope $m = \frac{y_2 - y_1}{x_2 - x_1}$ and write $y = m(x - x_1) + y_1$.
- **Supply-Demand Problems**: Market equilibrium occurs where supply and demand intersect (so you need to solve a system of equations). If price is above market equilibrium, then there is a surplus. And if price is below market equilibrium, then there is a shortage. Know how to solve the system to find market equilibrium and understand the basics of this scenario.
- **Know Function Definitions and How to Use Functional Notation**: You should know how to go from price to TR, how to go from AC to TC, how to go from AVC to VC, how to go from TR/TC to MR/MC, how to go from Distance to ATS, etc.... In other words you need to know the function definitions and how to use them.
- **Remember the Standard Applications We Have Been Discussing All Term**: Maximum profit, break even quantity, break even price (BEP), shut down price (SDP), etc...
- **Know How to Find and Interpret the Vertex of a Parabola**: Once you have a quadratic function $y = ax^2 + bx + c$, then you should know that the vertex occurs at $x = -\frac{b}{(2a)}$. You should also know that if $a > 0$, then the parabola opens upward (so the vertex is a minimum) and if the $a < 0$, then the parabola opens downward (so the vertex is a maximum). You need to be able to interpret what you have to answer various questions.
- **Know How to Solve Equations involving Quadratics**: You need to be comfortable using the quadratic formula. If you are able to get one side to zero and end up with $ax^2 + bx + c = 0$ (so there is only one variable x, and the expression on one side is a quadratic and the other side is zero!), then you can solve by using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{(2a)}$. You need to be able to interpret what you get to answer various questions.
- **Be able to Solve a System of Equations**: Solve for one variable in one equation and substitute into the other equation.
- **Know How to Do Linear Programming**: You need to be able to read information to get constraints and an objective. You need to be able to graph inequalities. You need to be able to **solve a system of equations** to get corners. And you need know how to finish the problem (evaluate the objective at each of the corners).
- **Know the Basics of Working with Exponentials and Logarithms**: Understand what we did in 5.1, 5.2, and 5.3. Evaluating exponentials, working with powers and roots, working with exponentials and logarithms, solving, and using exponential models.

Examples of problems of each type from homework are on the next page. (You need to know all the homework, I just randomly grabbed examples of each type of problem and put them on this review, so that you could remember what those types of problems look like).

- **Finding the equation of a line**.

  1. HW 1.6/5: "The total cost function is linear, and the total cost for 90 sweatshirts is \$4770, whereas the total cost for 280 sweatshirts is \$8380."'
  2. HW 1.6/9: "Assuming that the demand function is linear, write its equation."
  3. HW 1.6/11: "Find the linear functions for supply and demand."
  4. HW 2.3(p2)/2: "You charge \$24 per item for an order of 1 item and \$18 per item for an order of 13 items. If price, p, is a linear function of quantity, q, then find the formula for price."

- **Supply-Demand Problems**:

  1. HW 1.6/7-11: "Is there are shortage or surplus when $p = \$20$? (and by how much)"
  2. HW 1.6/7-11: "Find the market equilibrium"
  3. HW 2.3(pt2)/4-6: "If $p^2 + 8q = 1200$ is the demand and $300 - p^2 + 2q = 0$ is the supply, then find the quantity and price at market equilibrium"
  4. HW 1.5/7: "If supply and demand are given by $p - q = 10$ and $q(2p - 10) = 300$, what is the market equilibrium."

- **Know Function Definitions and How to Use Functional Notation**:

  1. HW 2.2/1 and 2: "Find the average rate of change of $y = 2 + 5x + 0.5x^2$ between $x = 4$ and $x = 6$."
  2. HW 2.2/4-6: "If $f(x) = 5 + x + x^2$, find $\frac{f(x+h)-f(x)}{h}$ and simplify."
  3. HW 2.2/8-9: "If $D(t) = t - 0.025t^2$, find the formula for $ATS(t)$."
  4. HW 2.2/8-9: "If $D(t) = t - 0.025t^2$, find the formula for the average speed over a 5-min interval starting at t minutes. That is find $AS(t) = \frac{D(t+5)-D(t)}{5}$."
  5. HW 2.3(p1)/5: "If price is given by $p = 140 - 0.80x$, then find the formula for revenue $R(x) =$??" Then you are asked to find the price that corresponds to maximum revenue (which is a vertex question).
  6. HW 2.3(p1)/7: "If $VC(q) = \frac{1}{30}q^3 - \frac{3}{10}q^2 + q$ and $FC = 0.8$, then find formulas for $TC(q)$, $AC(q)$, and $AVC(q)$."
  7. HW 2.3(p1)/8: "If $TR(q) = -0.25q^2 + 30q$ and $TC(q) = 17.5q + 100$, give formulas for $MR(q)$, $MC(q)$, $AR(q)$ and $AC(q)$."
  8. HW 2.3(p2)/2: "If price is given by $p = 24.5 - 0.5q$, then find formulas for $TR(q)$ and $MR(q)$."
  9. HW 2.3(p2)/2: "If the average cost per item is a constant \$5 and fixed cost is \$100, then give the formula for $TC(q)$."
  10. HW 2.3(p2)/3: "If $AVC(x) = \frac{4}{9}x + 333$, $FC = 54400$ and price is $p = 2065 - \frac{5}{9}x$, then find formula for $TR(q)$, $VC(q)$, $TC(q)$ and profit."

- **Remember the Standard Applications We Have Been Discussing All Term**:

  1. HW 2.3(p1)/1-3: "If cost and revenue are given by $C(x) = 5600 + 80x + x^2$ and $R(x) = 230x$, find the break-even points." (this also involves solving a quadratic).
  2. HW 2.3(p1)/7: "If $AVC(q) = \frac{1}{30}q^2 - \frac{3}{10}q + 1$, then what is the shutdown price (SDP)?" (This is a vertex question).
  3. HW 2.3(p2)/1: "If $MC(q) = 5q + 3$ and $AC(q) = 2.5q + 3 + \frac{15}{q}$, find the break even price (BEP)?" (You will need to solve $MC = AC$ which will require the quadratic formula).

- **Know How to Find and Interpret the Vertex of a Parabola**:

  1. HW 2.2/7: "If profit is $P(x) = 18x - 0.1x^2 - 50$ dollars, find the quantity that maximizes profit."

2. HW 2.2/10: "If $f(x) = x^2 - 4x + 16$ and $g(x) = -0.5x^2 + 4x + 10$, find the longest interval over which $f(x)$ and $g(x)$ are both increasing."
3. HW 2.3(p1)/4: "Find the maximum value of the revenue function $R(x) = 381x - 0.9x^2$."
4. HW 2.3(p1)/8: "If $TR(q) = -0.25q^2 + 30q$ and $TC(q) = 17.5q + 100$, what quantity maximizes profit?"
5. HW 2.3(p1)/9: "What is the largest value of total revenue?"
6. HW 2.3(p1)/9: "Give the longest interval on which total revenue and profit are both increasing."
7. HW 2.3(p2)/3: "Find the maximum profit and determine the corresponding price per unit that yields this profit."

- **Know How to Solve Equations involving Quadratics**:

1. HW 2.1/1-4: "Solve $\frac{w^2}{8} - \frac{w}{2} - 4 = 0$."
2. HW 2.1/7: "If profit is $P(x) = 135x - 100 - x^2$, what quantity will yield a profit of \$3940?"
3. HW 2.2/10: "If $g(x) = -0.5x^2 + 4x + 10$, find the interval over which $g(x)$ is greater than or equal to 15."
4. HW 2.3(p1)/7: "For what range of quantities is $AVC(q)$ at most \$0.55 per bag?"
5. HW 2.3(p1)/9: "Give the largest quantity at which profit is non negative." (this is asking you to find when profit = 0 and give the larger of the two answers).

- **Be able to Solve a System of Equations**:

1. HW 1.5/1-3: "Solve the system given by (i) $3x + 9y = -3$ and (ii) $2x - 3y = 16$."
2. HW 1.5/4: "Investments are made into two accounts (one at 10% annual interest and one at 11% annual interest). The total amount invested is \$142,000. The interest earned in the first year is \$14,800. How much was initially invested in each account?"
3. HW 1.5/8: "If supply is $2p - q - 20 = 0$ and demand is $(p + 20)(q + 10) = 6300$, find the market equilibrium."

- **Know How to Do Linear Programming**:

1. HW 4.1/2: "Graph the overlapping region given by the constraints $3x + 9y \geq 18$, $2x + 4y \geq 10$, $9x + 3y \geq 15$, $x \geq 0$ and $y \geq 0$."
2. HW 4.1/6-9: These are word problems (they are longer so I won't copy them here), you need to translate to inequalities and graph.
3. HW 4.2/3: "Maximize $f(x, y) = 4x + 5y$ subject to the constraints: $x + y \leq 7$, $2x + y \leq 10$, and $y \leq 6$."
4. HW 4.2/4-7: Again, more word problems to practice.

- **Know the Basics of Working with Exponentials and Logarithms**:

1. HW 5.1&5.2/1: "Evaluate $S(x) = 1400(1.02)^{4x}$ at $x = 8$."
2. HW 5.1&5.2/10: "Solve $8e^{2t-5} = 24$."
3. HW 5.3/1: "Solve $4^{3x} = 56364$."
4. HW 5.3/3: "Solve $83 = 100 - 100e^{-0.04x}$."
5. HW 5.3/8: "The population in a country is 100000 in 1998 and 110585 in 2008. Let $t = 0$ correspond to 1998, and $t = 10$ correspond to 2008. You are told that the population grows according to an exponential function of the form $y = P_0e^{ht}$. Find P_0 and h using the given information, then use the formula to give the population in 2023."
6. HW 5.3/14: "Solve $100 = 500(0.03)^{(0.7^t)}$."

Math 111 Calculation Errors

In helping students with the chapter 2 homework, I have noticed a few recurring calculation mistakes. The vast majority of the students in the class are correctly calculating, but there is a significant number (probably about 30-40%) of the class that is making one or more type of significant calculation error somewhere in their homework. My goal is to address some of these issues here:

Comment 1: Negatives, squaring and notation:
The biggest recurring mistakes I have seen have to do with squaring negatives. Here is a quick example of everything you need to know:

$$5^2 = 25 \quad \text{and} \quad (-5)^2 = 25 \quad \text{and} \quad -5^2 = -25.$$

In other words, if a negative number is being square (for example if you plug a negative into a function that has squaring), then the negative is squared. However, if the negative is on the outside, then you do NOT square the negative (You are just subtracting a value that happens to be squared).
Let's consider the two related mistakes I have seen in the homework:

- Assume you are solving $2x^2 - 8x + 5 = 0$. So you will have $a = 2$, $b = -8$, and $c = 5$ in the QUADRATIC FORMULA. The quadratic formula looks like $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.
My question for you is: If $b = -8$ and you are going to compute b^2 in the quadratic formula, will you get a negative or positive?
Here you are "plugging in" the value -8 for b and you are then squaring.
So you will get $b^2 = (-8)^2 = 64$. It will be positive!!. Here is what it all will look like (I won't finish the calculation, I just wanted to talk about the square): $x = \frac{-(-8)\pm\sqrt{(-8)^2-4(2)(5)}}{2(2)} = \frac{8\pm\sqrt{64-40}}{4}$.
- Assume you are given that $P(x) = -x^2 + 1632x - 51000$ and you find the vertex occurs at $x = 816$ and you need to find the value of the function at this value of x (this is straight from the homework and a commonly asked question in office hours this last week). Okay, let's plug in $x = 816$ and we get $P(816) = -(816)^2 + 1632(816) - 51000$.
My question to you is: Will $-(816)^2$ be negative or positive?
Note that the negative is on the OUTSIDE, it is NOT being squared. So you will still have a negative!! You will get $-(816)^2 + 1632(816) - 51000 = -665856 + 1331712 - 51000 = 614856$.

Comment 2: Unless otherwise specified, **keep ALL your digits until the very end of the problem**.
Here is a big example to explain this:
Assume you are trying to evaluate the function $y = 60x - 4x^2$ at the value $x = \frac{7}{3} = 2.33333\ldots$.
And assume the problem says give a final answer correct to two digits after the decimal.
Here is my question for you: Should you start by rounding to 2.33 or should you wait to round?
You have to keep ALL your digits until the end. Just for comparison see that:

$$\text{INCORRECT WORK: } 60(2.33) - 4(2.33)^2 = 118.0844$$

$$\text{CORRECT WORK: } 60(2.333333\cdots) - 4(2.33333\cdots)^2 = 118.222\cdots$$

The correct final answer rounds to 118.22, if you had rounded at the beginning of the problem you would have gotten 118.08 which is quite a bit off (if this final answer was in hundreds of Items, then that would mean you were off by 14 items, I call that quite significant). Get in the habit of keeping all your units unless otherwise specified! This will be even more important in the last two weeks of the course (if you round too early in those problems, you can get answers that are off by thousands of dollars).

Comment 3: **Remember parentheses when you plug into your calculator!**. Here are two examples of things I have seen typed into calculators this week:

$$6 \div 2 \times 3 = 9 \qquad \text{and} \qquad 6 \div (2 \times 3) = 1.$$

Assume you know that $a = 3$ and $b = -6$ and you are to compute $x = -\frac{b}{2a} = -\frac{(-6)}{2(3)}$. Something like this you should do in your head and you should know the answer is 1. But if you did type it into your calculator, then you MUST put parentheses around the denominator!! You would type it in like the SECOND example above. Let me show you the two main examples that have been coming up in your homework:

- Vertex Formula: $x = -\frac{b}{2a}$ should be typed into your calculator as follows $-b \div (2 \times a)$. (Notice the parentheses).
- Quadratic Formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ should be typed into your calculator as follows:
  $(-b + \sqrt{(b^2 - 4 \times a \times c)}) \div (2 \times a)$ and $(-b - \sqrt{(b^2 - 4 \times a \times c)}) \div (2 \times a)$.
  For the quadratic formula, I showed in class and in review sheets that it is best to NOT plug this all into your calculator at once. Instead it is a good idea to do your work in steps (See my 2.1 review for several fully worked examples of this method):
  Step 1: Compute $b^2 - 4ac$, then take the square root and write this number on a piece of paper.
  Step 2: Compute $2a$, then write this down on a piece of paper.
  Step 3: Now compute $\frac{(-b+\sqrt{b^2-4ac})}{(2a)}$ and $\frac{(-b-\sqrt{b^2-4ac})}{(2a)}$.

Comment 4: **ALWAYS check your final answers!!!** You should know your answer is correct before you type it into webassign. And if you do this, then you will know if you made any one of the silly mistakes mentioned in this review sheet. There are two situations where we have been solving (and in both situations we can easily check our work). Here are those two situations:

- If you are solving $60x - 3x^2 = 5x + 100$, then subtract appropriately to get one side zero. Then use the quadratic formula. When you are all done you will get two values of x.
  Then CHECK YOUR WORK. Take each value of x and plug it back in to the original equation you were trying to solve. You see that $60x - 3x^2$ and $5x + 100$ give the same values. If they do, then you KNOW you are correct. And if they don't, then you KNOW you are incorrect and you can go find your error. Always check in this way!
- If you are trying to solve for q and p in a system like (1) $p^2 + 8q = 1600$ and (2) $600 - p^2 + 10q = 0$. As we have discussed, solve for one variable in one of the equation (for example you could get $8q = 1600 - p^2$, then write $q = 200 - \frac{1}{8}p^2$). Then substitute this into the second equation. After a bit more work, when you are all done, you will have found a value of q and a value of p.
  Then CHECK YOUR WORK. Go back to the original equations. Plug in p and q to $p^2+8q = 1600$ (if you don't get 1600 then you know you are wrong). You also need to check $600 - p^2 + 10q = 0$ (if you don't get zero, then you know you are wrong). If both check out then you KNOW with certainty you are right!

Sections 1.6 Review

Business Functions with Algebra: The first part of this section reviews the business terms, but uses linear functions as opposed to graphs.

1. When a problem says "Profit is linear" or "Total Revenue and Costs are linear", then you know that their functions will take the form $f(x) = mx + b$. (See the previous review sheets for advice about finding the equation of a line).

2. If the market price is a **constant** p dollars/item, then $TR(x) = px$ and $\overline{MR}(x) = p$.
   For example, if each item sells for \$3.50, then $TR(x) = 3x$ and $\overline{MR}(x) = 3$.
   (Recall, graphically TR is a diagonal line with slope 3 and MR is a horizontal line at 3).

3. If the $FC = c$ and the average variable cost per item is a **constant** m dollars/item, then $TC(x) = mx + c$.
   For example, if fixed costs are \$1552 and average variable costs are a constant \$7 per unit,
   then $TC(x) = 7x + 1552$ and $\overline{MC}(x) = 7$.
   (Graphically TC is a line with y-intercept at 1552 and slope of 7, and MC is a horizontal line).

4. If given TR and TC and asked for the 'quantity at which you break even', then you need to solve $TR(x) = TC(x)$ (this is NOT asking about 'break even price', it is asking when profit would be zero for these particular TR and TC functions).

5. If you have the functions for $TR(x)$ and $TC(x)$, then you can get the functions for profit, average cost, average variable cost and average revenue by using the definitions. Just make sure that when you use parantheses and properly distribute when you substitute the functions into the definitions.
   Here is an example: Suppose $TR(x) = 5x$ and $TC(x) = 3x + 10$.

   - Profit $= P(x) = TR(x) - TC(x) = (5x) - (3x + 10) = 5x - 3x - 10 = 2x - 10$
   - Marginal Cost $= \overline{MC}(x) = 3$ (cost to produce next item)
   - Average Cost $= AC(x) = \frac{TC(x)}{x} = \frac{(3x+10)}{x} = 3 + \frac{10}{x}$ (average overall cost per item, including FC)
   - Average Variable Cost $= AVC(x) = \frac{(3x)}{x} = 3$ (average cost per item, excluding FC).
   - Marginal Revenue $= \overline{MR}(x) = 5$ (revenue from next item)
   - Average Revenue $= AR(x) = \frac{(5x)}{x} = 5$ (average overall revenue per item)

   You might have noticed that $MR(x) = AR(x) = 5$ and $MC(x) = AVC(x) = 3$ in this example. That will only happen when working with linear TR and TC functions (the slope of the diagonal is the same as the slope of the tangent in this case). When we work with more complicated functions this won't be the case.

Supply and Demand: We introduced the concepts of supply and demand and we did examples with linear functions.

- A **supply curve** shows the relationship between market price, p, and the quantity, q, of a product the manufacturers are willing to supply for that market price.
  For example, if $(q, p) = (45, 300)$ is a point on the supply curve, then that means that at a market price of 300 dollars/item, the manufactures are willing to produce 45 units.

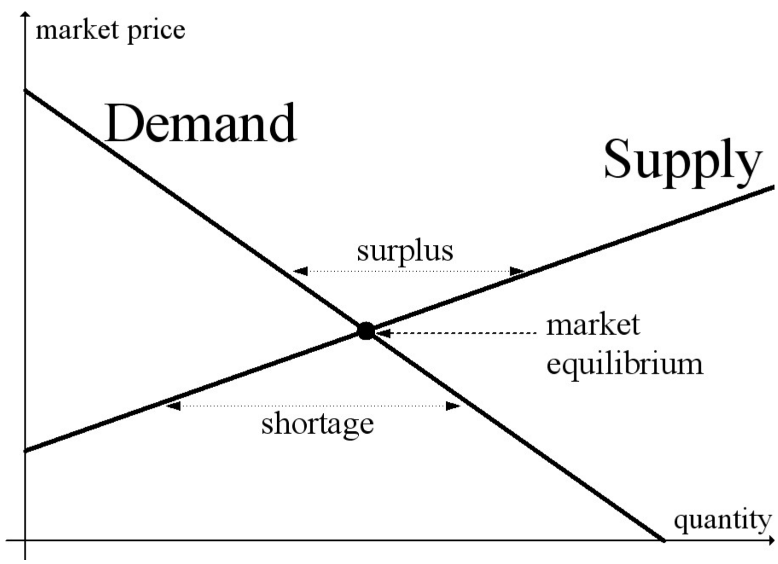
- A **demand curve** shows the relationship between market price, p, and the quantity, q, of a product that consumers will purchase at that price.
  For example, if $(q, p) = (70, 155)$ is a point on the demand curve, then that means that at a market price of 155 dollars/item, consumers will purchase 70 units.

- Some basic properties:

  - (The Law of Supply): The number of quantities supplied will increase as the market price goes up. (In other words, when you draw the supply curve it will go up as you go from left to right).
  - (The Law of Demand): The number of quantities demanded will decrease as the market price goes up. (in other words, when you draw the demand curve it will go down as you go from left to right).

- The quantity and price at which supply and demand intersect is called **market equilibrium**. This gives the price at which the manufacturers and consumers are willing to produce and buy the same number of units.

  - If the market price is greater than market equilibrium price, then there will be a surplus (more items will be produced than sold).
  - If the market price is less than market equilibrium price, then there will be a shortage (more items will be demanded than are produced).



- We did examples where two data points where given for Supply and two data points were given for Demand. Then we found linear functions for Supply and Demand (again, see the 1.1-1.3 for review of how to find equations for lines). Finally we found their intersection to find the market equilibrium.

- Here is a quick example: Suppose supply and demand are linear. You are told that at a price of \$150, the demand will be 50 units and the supply will be 10 units, and at a price of \$300, the demand will be 30 units and the supply with be 60 units. Find market equilibrium.
  Answer on the next page (try the problem before you look at the answer).

Answer to example from previous page:

1. Demand: The given (q, p) points are $(50, 150)$ and $(30, 300)$. Slope $= m = \frac{150-300}{50-30} = \frac{-150}{20} = -7.50$. So the equation for the line is

$$p = -7.50(q - 30) + 300 = -7.5q + 225 + 300 = -7.5q + 525$$

2. Supply: The given (q, p) points are $(10, 150)$ and $(60, 300)$. Slope $= m = \frac{150-300}{10-60} = \frac{-150}{-50} = 3$. So the equation for the line is

$$p = 3(q - 10) + 150 = 3q - 30 + 150 = 3q + 120$$

3. Market equilibrium: We have the same price when

$$-7.5q + 525 = 3q + 120$$

$$405 = 10.5q$$

$$q = \frac{405}{10.5} \approx 38.5724$$

which gives

$$p \approx 3(41.4286) + 120 \approx 235.714.$$

Rounding to the nearest item and nearest cent (this would depend on units of the problem), market equilibrium is at 39 units and a price of \$235.71.

Section 2.3 Review

In this section, we revisit our business applications but now we can use quadratic function and functional notation facts. See the 2.1 review for quadratic facts (*i.e.* solving with the quadratic formula and finding the vertex) and see the 2.2 review for functional notation facts (*i.e.* how to get AR, AC, AVC, MR, and MC from TR and TC).

Applications:

1. **Graph Shape and Vertex Questions**: If a problem asks you about the shape/features of a function, then sketch a picture! (Anything about maximum/minimum or increasing/decreasing) If the function happens to be a quadratic, then you must find and label the **vertex** in order to answer the question. But always sketch a picture before you put your final answer down. Here are examples from homework:
   - Find the maximum revenue ...
     (Sketch a graph of the revenue function, and label the vertex if it's quadratic).
   - Give the longest interval on which total revenue and profit are both increasing.
     (Sketch TR, label the vertex, figure out when it is increasing. Then sketch Profit, label the vertex, figure out when it is increasing.)
   - Find the longest interval on which $f(x) - g(x)$ is increasing.
     (Find the formula for $f(x) - g(x)$, then sketch this new function, label the vertex).
   - Find the maximum profit ...
     (Sketch a graph of the profit function, and label the vertex if it's quadratic).
     Note: You can also find when $MR = MC$.
2. **Solving and Quadratic Equation Questions**: Questions that ask about a particular output value are questions where you must solve an equation. If the equation is quadratic, you can use the quadratic formula. Otherwise, you can solve by getting x by itself on one side. It still is wise to sketch a picture of the function if you don't know how to interpret your answers. Here are examples from homework:
   - Find all values when $g(x) - f(x) = 4$.
     (Set up and solve)
   - For what range of quantities is average variable cost at most \$0.55 per bag?
     (First, solve $AVC(x) = 0.55$, then use a sketch of $AVC(x)$ to figure out what to do with your answers).
   - For what range of quantities is VC less than or equal to TR.
     (First, solve $TR = VC$, then use a sketch of TR and VC to decide what to do with your answers).
   - Find the quantity at which MR exceeds MC by \$4.25.
     (First, give the correct translation into an equation, I'll let you do this. Then solve).
   - Give the largest quantity at which profit is not negative.
     (This is just a fancy way to say, we want to know the largest quantity when profit is greater than or equal to 0. So first solve for when Profit = 0, then sketch a graph to figure out what to do with your numbers).
3. **Special Applications:**
   - The **break even points** are the quantities at which Profit = 0, which is the same as the quantities where $TR(x) = TC(x)$. In these problems you are solving! If it is a quadratic problem, then you will be using the quadratic formula.
   - To find the **quantity at which profit is maximum**:
     METHOD 1: Find the function for Profit. If is is quadratic, then use the vertex formula.
     METHOD 2: Find MR and MC. Then solve MR = MC (if there are multiple intersection points, then you want the answer when it switches from $MR > MC$ to $MR < MC$).
     Note: If a question asks you to find the 'price' that corresponds to maximum revenue or maximum profit, then, at the very end of the problem, you plug the quantity you found into the price formula to get the corresponding price (in such a problem, you would have a price formula at some point earlier in the problem).
   - The **Break Even Price (BEP)** is the lowest y-coordinate of $AC(x)$. It is also the y-coordinate where $AC(x)$ and $MC(x)$ intersect. Typically, this second fact is easier to use. Meaning if you are asked to find the **Break Even Price (BEP)**, then you should solve $AC(x) = MC(x)$ (likely using the quadratic formula). Then plug the value you get for x back into $AC(x)$ or $MC(x)$ to get the y-value (both should give you the same y-value).
   - The **Shutdown Price (SDP)** is the lowest y-coordinate of $AVC(x)$. It is also the y-coordinate where $AVC(x)$ and $MC(x)$ intersect. If $AVC(x)$ is a quadratic, then you can find the y-coordinate of the vertex to get the SDP. Otherwise, you can solve when $AVC(x) = MC(x)$.

OLD EXAM PROBLEMS: The six problems below all come directly from old exams (some midterms and some finals). You should try these completely on your own and you should be able to tell if you are right. Full answers are on the next page, but if you peek you are wasting this great opportunity to practice. Do them on your own first!

1. Suppose you are given $f(x) = x^2 - 10x + 36$ and $g(x) = -0.25x^2 + 4x + 24$.
   (a) Find the values of x at which the graphs cross.
   (b) What is the largest value of $g(x) - f(x)$?
   (c) Find the largest interval on which $f(x)$ and $g(x) - f(x)$ are both increasing.
   (d) Compute and simplify $\frac{f(8+h)-f(8)}{h}$
2. You sell Trinkets and you have the following information:
   - For an order of q Trinkets, the selling price (in dollars per Trinket) is given by $p = -16q + 1000$.
   - Total cost is a **linear** function of quantity q.
   - Fixed costs are \$3484.
   - The total cost to produce 20 Trinkets is \$4004.

   (a) Find formulas for total revenue and total cost.
   (b) For what quantities is profit exactly \$3000.
   (c) What selling price leads to the largest possible profit?
   (d) For what quantity is average cost exactly \$243.75 per Trinket?
   (e) Find formulas for $MR(q)$ and $MC(q)$.
3. You sell calculators. Your fixed costs are \$240, and the **average variable cost** of producing q calculators is given by the function
$$AVC(q) = 0.01q^2 - 0.5q + 26.$$
   (a) Find the **total cost** of producing 30 calculators, $TC(30)$.
   (b) Find a formula for the **average cost** to produce q calculators, $AC(q)$.
   (c) Compute the **shutdown price**.
   (d) For **what value(s)** of q is the average variable cost \$28 per calculator?
4. You make and sell marionettes. The marginal cost and the average cost for producing marionettes are given by the following functions, where q is the number of marionettes and MC and AC are in dollars per marionette.
$$MC(q) = 0.012q^2 - 0.84q + 16.7 \qquad AC(q) = 0.004q^2 - 0.42q + 16.7 + \frac{20}{q}$$
   (a) Find the variable cost of producing 25 marionettes.
   (b) At what quantities does the average variable cost equal \$12.90 per marionette?
   (c) If you sell each marionette for \$26.30, what is your maximum profit?
5. You sell Things. The total cost to produce q **hundred Things** is given by the formula
$$TC(q) = 1.5q^2 + 11.25q + 10 \quad \text{hundred dollars.}$$
   If you sell q **hundred Things**, then selling price is given by the formula
$$p = -3q + 90 \quad \text{dollars per Thing.}$$
   (a) Give the formula in terms of q for total revenue for selling q hundred Things, $TR(q)$.
   (b) Find all quantities at which you break even.
   (c) What is the maximum possible total revenue?
   (d) What is the **selling price** when you maximize **profit**?
6. The weight in ounces of a newborn puppy t days after it's born is given by the function
$$w(t) = 0.5t^2 - 2t + 6.$$
   (a) For what range of time is the puppy's weight at most 5 ounces?
   (b) Compute, simplify and translate the meaning of
$$\frac{w(5+h) - w(5)}{h}.$$

Fully Explained ANSWERS to the old exam questions from previous page:

1. (a) We want to find the values of x when $f(x) = g(x)$.

$x^2 - 10x + 36$	$= -0.25x^2 + 4x + 24$	
$1.25x^2 - 14x + 12$	$= 0$	make one side zero!
x	$= \frac{-(-14) \pm \sqrt{(-14)^2 - 4(1.25)(12)}}{(2(1.25))}$	quad. form. $a = 1.25$, $b = -14$, $c = 12$
x	$= \frac{14 \pm \sqrt{136}}{2.5} \approx \frac{14 \pm 11.661904}{2.5}$	simplifying

Thus, $x \approx \frac{2.338096}{2.5} = 0.9353$ or $x \approx \frac{25.661904}{2.5} = 10.2648$.

(b) First find the function $g(x) - f(x) = (-0.25x^2 + 4x + 24) - (x^2 - 10x + 36) = -0.25x^2 + 4x + 24 - x^2 + 10x - 36$, so $g(x) - f(x) = -1.25x^2 + 14x - 12$.

The questions asks for the largest value of this function. Let's think about what the graph looks like (make a sketch)! The function $y = -1.25x^2 + 14x - 12$ is a parabola that opens downward, so the largest value is at the **vertex**.

The vertex occurs at $x = -\frac{b}{(2a)} = -\frac{14}{2(-1.25)} = \frac{14}{2.5} = 5.6$. And the value of $g(x) - f(x)$ at this location is $g(5.6) - f(5.6) = -1.25(5.6)^2 + 14(5.6) - 12 = 27.2$.

(c) We know the two functions in question $f(x) = x^2 - 10x + 36$ and $g(x) - f(x) = -1.25x^2 + 14x - 12$. The question asks when they both are increasing. Let's think about what the graphs look like (make a sketch)!

- $f(x)$ is a parabola that opens upward. So it is increasing **after its vertex**. The x-coordinate of its vertex is $x = -\frac{-10}{2(1)} = 5$. Thus, $f(x)$ is increasing from $x = 5$ and on (*i.e.* on the interval $x \geq 5$).
- $g(x) - f(x)$ is a parabola that opens downward. So it is increasing **before its vertex**. The x-coordinate of its vertex is $x = -\frac{14}{2(-1.25)} = 5.6$. Hence, $g(x) - f(x)$ is increasing before and up to $x = 5.6$ (*i.e.* on the interval $x \leq 5.6$).

Therefore, they both are increasing on the interval $5 \leq x \leq 5.6$.

(d) A standard algebraic problem of getting the rate formula. Let's go through that algebra again:

$\frac{f(8+h) - f(8)}{h}$	$= \frac{((8+h)^2 - 10(8+h) + 36) - ((8)^2 - 10(8) + 36)}{h}$	using the function def'n
$\frac{f(8+h) - f(8)}{h}$	$= \frac{(64 + 16h + h^2 - 80 - 10h + 36) - (64 - 80 + 36)}{h}$	expanding
$\frac{f(8+h) - f(8)}{h}$	$= \frac{64 + 16h + h^2 - 80 - 10h + 36 - 64 + 80 - 36}{h}$	still expanding, drop the parentheses (dist. negative!)
$\frac{f(8+h) - f(8)}{h}$	$= \frac{16h + h^2 - 10h}{h}$	cancel terms in the numerator
$\frac{f(8+h) - f(8)}{h}$	$= 16 + h - 10 = 6 + h$	simplify

2. (a) Since TR = (Price per item)(Quantity), we have $TR(x) = (-16q + 1000)q = -16q^2 + 1000q$.

The problem says TC is linear, so that means it has the form $TC = m(q - q_1) + y_1$. We are given two points $(q, TC) = (0, 3484)$ and $(q, TC) = (20, 4004)$. Find the slope: $m = \frac{4004 - 3484}{20 - 0} = \frac{520}{20} = 26$ and given the line: $TC(q) = 26(q - 0) + 3484 = 26q + 3484$.

(b) We want to know when profit is \$3000. Let's first find a formula for profit.

$P(q) = TR(q) - TC(q) = (-16q^2 + 1000q) - (26q + 3484) = -16q^2 + 1000q - 26q - 3484 = -16q^2 + 974q - 3484$.

We are asked to solve:

$-16q^2 + 974q - 3484$	$= 3000$	
$-16q^2 + 974q - 6484$	$= 0$	make one side zero!
q	$= \frac{-(974) \pm \sqrt{(974)^2 - 4(-16)(-6484)}}{(2(-16))}$	quad. form. $a = -16$, $b = 974$, $c = -6484$
q	$= \frac{-974 \pm \sqrt{533700}}{-32} \approx \frac{-974 \pm 730.54774}{-32}$	simplifying

Thus, $q \approx \frac{-1704.5477}{-32} = 53.267$ or $x \approx \frac{-243.4523}{-32} = 7.608$.

(c) The question is about when profit is maximum. The profit function is $P(q) = -16q^2 + 974q - 3484$. Let's think about what the graph looks like (make a sketch)! It is a parabola that opens downward, so the maximum occurs at the **vertex**.

The vertex occurs when $q = -\frac{974}{2(-16)} = 30.4375$.

This quantity corresponds to a **price** of $p = -16(30.4375) + 1000 = 531$ dollars per Trinket.

(d) Since $AC(q) = \frac{TC(q)}{q} = \frac{26q + 3484}{q} = 26 + \frac{3484}{q}$, we are being asked to solve the following equation:

$26 + \frac{3484}{q}$	$= 243.75$	
$26q + 3484$	$= 243.75q$	clear denominator! (mult. both sides by q)
3484	$= 217.75q$	this is a linear equation, get x by itself.
$\frac{3484}{217.75} = 16$	$= q$	simplifying

(e) Since $MR(q) = \frac{TR(q+1)-TR(q)}{1}$, we have some algebra to do:

$MR(q)$	$= (-16(q+1)^2 + 1000(q+1)) - (-16q^2 + 1000q)$	using the function def'n
$MR(q)$	$= (-16(q^2+2q+1) + 1000q + 1000) - (-16q^2 + 1000q)$	expanding
$MR(q)$	$= -16q^2 - 32q - 16 + 1000q + 1000 + 16q^2 - 1000q$	drop the parentheses (dist. negative!)
$MR(q)$	$= -32q - 16 + 1000 = -32q + 984$	cancel terms

Since $MC(q) = \frac{TC(q+1)-TC(q)}{1}$, we can do this with algebra (and there is a faster way):

$MC(q)$	$= (26(q+1) + 3484) - (26q + 3484)$	using the function def'n
$MC(q)$	$= (26q + 26 + 3484) - (26q + 3484)$	expanding
$MC(q)$	$= 26q + 26 + 3484 - 26q - 3484 = 26$	drop the parentheses (dist. negative!)

We should have already known this because TC is linear so each additional item costs the same amount to produce (in this case \$26).

3. (a) Since $AVC(q) = \frac{VC(q)}{q}$ and you are told that $AVC(q) = 0.01q^2 - 0.5q + 26$, you know that $\frac{VC(q)}{q} = 0.01q^2 - 0.5q + 26$. Multiplying both sides by q gives $VC(q) = 0.01q^3 - 0.5q^2 + 26q$.
And since $FC = 240$, you know that $TC(q) = 0.01q^3 - 0.5q^2 + 26q + 240$.
The problem asks for the value of $TC(30)$, so you need to compute
$TC(30) = 0.01(30)^3 - 0.5(30)^2 + 26(30) + 240 = \$840$.

(b) Since $AC(q) = \frac{TC(q)}{q}$ and from what I showed in the last part, we have
$AC(q) = \frac{0.01q^3 - 0.5q^2 + 26q + 240}{q} = 0.01q^2 - 0.5q + 26 + \frac{240}{q}$.

(c) The **shutdown price** is the minimum y-value of $AVC(q)$. Think of the graph of $AVC(q)$ (sketch it)! Since $AVC(q) = 0.01q^2 - 0.5q + 26$ it is a quadratic that opens upward. So the minimum occurs at the **vertex**.
We know the vertex occurs at $q = -\frac{-0.5}{2(0.01)} = 25$ and the y value of $AVC(q)$ at the vertex is
$AVC(25) = 19.75$ dollars per item.

(d) We are being asked to solve when $AVC(q) = 28$:

$0.01q^2 - 0.5q + 26$	$= 28$	
$0.01q^2 - 0.5q - 2$	$= 0$	make one side zero!
q	$= \frac{-(-0.5) \pm \sqrt{(-0.5)^2 - 4(0.01)(-2)}}{(2(0.01))}$	quad. form. $a = 0.01$, $b = -0.5$, $c = -2$
q	$= \frac{0.5 \pm \sqrt{0.33}}{0.02} \approx \frac{0.5 \pm 0.574456}{0.02}$	simplifying

Thus, $q \approx \frac{-0.074456}{0.02} = -3.723$ or $q \approx \frac{1.074456}{0.02} = 53.72$. It can't be negative, so $q = 53.72$.

4. (a) Since $AC(q) = \frac{TC(q)}{q}$ and you are given $AC(q) = 0.004q^2 - 0.42q + 16.7 + \frac{20}{q}$, you know that $\frac{TC(q)}{q} = 0.004q^2 - 0.42q + 16.7 + \frac{20}{q}$. Multiplying both sides by q gives: $TC(q) = 0.004q^3 - 0.42q^2 + 16.7q + 20$. Thus, we can see that $FC = TC(0) = 20$ and we can see that $VC(q) = 0.004q^3 - 0.42q^2 + 16.7q$. The question asks for $VC(25) = 0.004(25)^3 - 0.42(25)^2 + 16.7(25) = \$217.50$.

(b) Since $AVC(q) = \frac{VC(q)}{q}$ and using what we did in the previous part we have:
$AVC(q) = \frac{0.004q^3 - 0.42q^2 + 16.7q}{q} = 0.004q^2 - 0.42q + 16.7$. We are asked to find when this function is equal to 12.90:

$0.004q^2 - 0.42q + 16.7$	$= 12.9$	
$0.004q^2 - 0.42q + 3.8$	$= 0$	make one side zero!
q	$= \frac{-(-0.42) \pm \sqrt{(-0.42)^2 - 4(0.004)(3.8)}}{(2(0.004))}$	quad. form. $a = 0.004$, $b = -0.42$, $c = 3.8$
q	$= \frac{0.42 \pm \sqrt{0.1156}}{0.008} = \frac{0.42 \pm 0.34}{0.008}$	simplifying

Thus, $q = \frac{0.08}{0.008} = 10$ or $q = \frac{0.76}{0.008} = 95$.

(c) If the price per item is a constant \$26.30, then you know that $TR(q) = 26.30q$ and $MR(q) = 26.30$. The easiest way to find maximum profit here is to solve when $MR(q) = MC(q)$:

$0.012q^2 - 0.84q + 16.7$	$= 26.30$	
$0.012q^2 - 0.84q - 9.6$	$= 0$	make one side zero!
q	$= \frac{-(-0.84) \pm \sqrt{(-0.84)^2 - 4(0.012)(-9.6)}}{(2(0.012))}$	quad. form. $a = 0.012$, $b = -0.84$, $c = -9.6$
q	$= \frac{0.84 \pm \sqrt{1.1664}}{0.024} = \frac{0.84 \pm 1.08}{0.024}$	simplifying

Thus, $q = \frac{-0.24}{0.024} = -10$ or $q = \frac{1.92}{0.024} = 80$. Note that if you sketch the picture you see that we switch from $MR > MC$ to $MR < MC$ at $q = 80$. So this is the quantity that maximizes profit.
To get the maximum profit, we now need to compute $TR(80) - TC(80)$.

- Since the market price is \$26.30, we have $TR(q) = 26.30q$. So $TR(80) = 26.30 \cdot 80 = \$2104$
- In a previous problem we used the AC formula to deduce the formula for TC. We found $TC(q) = 0.004q^3 - 0.42q^2 + 16.7q + 20$. So $TC(80) = 0.004(80)^3 - 0.42(80)^2 + 16.7(80) + 20 = \$716$.

Therefore, the maximum profit is $P(80) = TR(80) - TC(80) = 2104 - 716 = \$1388$.

5. (a) Since TR = (Price per item)(Quantity), we have $TR(q) = (-3q + 90)q = -3q^2 + 90q$. Note, since q is in hundreds of things and price is in dollars per thing, we get TR in hundreds of dollars.

   (b) The quantities at which you break even are the quantities when profit is equal to zero. This is the same as asking when $TR(q) = TC(q)$:

$1.5q^2 + 11.25q + 10$	$= -3q^2 + 90q$	
$4.5q^2 - 78.75q + 10$	$= 0$	make one side zero!
q	$= \frac{-(-78.75)\pm\sqrt{(-78.75)^2-4(4.5)(10)}}{(2(4.5))}$	quad. form. $a = 4.5$, $b = -78.75$, $c = 10$
q	$= \frac{78.75\pm\sqrt{6021.5625}}{9} \approx \frac{78.75\pm77.59873}{9}$	simplifying

   Thus, $q \approx \frac{1.15127}{9} = 0.13$ or $q \approx \frac{156.34873}{9} = 17.37$ hundred items (you must keep two digits of accuracy since we are in hundreds).

   (c) We already know that $TR(q) = -3q^2 + 90q$. We are asked to find the maximum value of this function. Let's think about what the picture looks like (make a sketch)! The graph of $TR(q)$ is a parabola that opens downward, so the highest point is at the **vertex**.

   The vertex occurs where $q = -\frac{90}{2(-3)} = 15$ hundred Things. And the value of TR at the vertex is $TR(15) = -3(15)^2 + 90(15) = \$675$.

   (d) The question is about when profit is maximum. The profit function is $P(q) = TR(q) - TC(q) = (-3q^2 + 90q) - (1.5q^2 + 11.25q + 10) = -3q^2 + 90q - 1.5q^2 - 11.25q - 10$. So the function for profit is $P(q) = -4.5q^2 + 78.75q - 10$. Let's think about what the graph looks like (make a sketch)! It is a parabola that opens downward, so the maximum occurs at the **vertex**.
   The vertex occurs when $q = -\frac{78.75}{2(-4.5)} = 8.75$ hundred Things.
   This quantity corresponds to a **price** of $p = -3(8.75) + 90 = 63.75$ dollars per Thing.

6. (a) The question is asking when the weight is less than or equal to 5 ounces. First, find when the weight is equal to 5 ounces, then use a sketch of the picture to figure out your answer (note that $w(t)$ is a parabola that opens upward). Here is the algebra:

$0.5t^2 - 2t + 6$	$= 5$	
$0.5t^2 - 2t + 1$	$= 0$	make one side zero!
t	$= \frac{-(-2)\pm\sqrt{(-2)^2-4(0.5)(1)}}{(2(0.5))}$	quad. form. $a = 0.5$, $b = -2$, $c = 1$
t	$= \frac{2\pm\sqrt{2}}{1} \approx 2 \pm 1.4152$	simplifying

   Thus, $t \approx 0.5858$ or $t \approx 3.4142$ days.
   Since $w(t)$ is a parabola that opens upward, it must be below 5 ounces for all the days between these two times. So the puppy's weight is less than or equal to 5 ounces from $t = 0.59$ to $t = 3.41$ days.

   (b) The expression $\frac{w(5+h)-w(5)}{h}$ is the rate of change of the puppy's weight from $t = 5$ days to $t = 5 + h$ days (*i.e.* the rate of change from $t = 5$ days to h days later). The units will be in ounces per day. Now let's do the algebra work:

$\frac{w(5+h)-w(5)}{h}$	$= \frac{(0.5(5+h)^2-2(5+h)+6)-(0.5(5)^2-2(5)+6)}{h}$	using the function def'n
$\frac{w(5+h)-w(5)}{h}$	$= \frac{(0.5(25+10h+h^2)-10-2h+6)-(12.5-10+6)}{h}$	expanding
$\frac{w(5+h)-w(5)}{h}$	$= \frac{12.5+5h+0.5h^2-10-2h+6-12.5+10-6)}{h}$	still expanding, drop the parentheses (dist. negative!)
$\frac{w(5+h)-w(5)}{h}$	$= \frac{5h+0.5h^2-2h}{h}$	cancel terms in the numerator
$\frac{w(5+h)-w(5)}{h}$	$= 5 + 0.5h - 2 = 3 + 0.5h$	simplify

Chapter 2 Problem Overview

I just went through every problem from the 2.1, 2.2, 2.3(part 1), and 2.3 (part 2) homework again. There are only three things we are doing, over and over and over again in these problems.
On the next page, I have organized <u>ALL</u> the major problems from homework into three categories.
But first, let me say a few general things:

If you missed my overview on Wednesday in lecture, then here is how to approach the problems:

STEP 1: Find all related functions. You will have to use the 2.2 functional notation skills!

STEP 2: Make a rough sketch. If the question involves a parabola or line, make a rough sketch (does the parabola open upward or downward?).

STEP 3: Translate the question:

(a) Is it just asking you to evaluate.
(b) Is it asking about the shape/vertex?
(c) Is it asking about solving/quad form?

STEP 4: Check your work! Correctly interpret your work and input your answer. (You might want to reread the original question.

If you are struggling with getting correct answers on the 2.1, 2.2, or 2.3 homework assignment,
then the issue is typically one of these:

a) **Incorrectly translating the question**: I hope this sheet (see the next page) helps you organize your thinking. We really should be able to quickly identify what the question is asking at this point, you have had lots of practice.

b) **Errors in calculation or careless math**: Be organized and write out your steps. Keep **all** your digits until the end of the problem. In class and in review sheets, I have tried to model good behaviors in doing your work. We are doing the same things over and over again, so you need to build a way to be organized and you need to learn how to check your work wherever possible. Remember, you can always check your quadratic formula calculations by putting your final answers into the original question and seeing if they satisfy the equation (so you should know you are correct before you ever even submit into webassign!!!).

c) **Answering the question**: Perhaps the most common error I am seeing is students not answering the given question. For example, if the question asks for "the *price* gives maximum revenue", then you find the quantity that maximized revenue, but at the end you need to find the corresponding *price*! You must read the question and make sure you are doing what it says. Another mistakes has been not using the correct variable (sometimes students are randomly changing the variable, the problem says "t", but they are using "x") or students are incorrectly rounding (if it asks to round to two digits then 345.55555… should round to 345.56, I have seen several students enter 345.55 which is not to the nearest two digits!)

PROBLEM TYPE 1: **Solving an equation**. Most equations we saw in these problems were quadratic so you had to use the quadratic formula (get one side to zero; carefully put your numbers into the quadratic formula, keeping all your digits until the end; then correctly interpret your answer). There were a few that weren't quadratic, they were just linear, in which case you can quickly solve by getting the variable by itself.

"Find the break even points" (For this one you are solving Profit = 0 or TR = TC)
"Find q when AC = 21.50"
"Find q when AR = 16"
"Find q when MR exceeds MC by 6.25" (as we said in class, you are solving MR = MC + 6.25)
"Find when g(x) – f(x) = 4"
"Find when the graphs of g(x) and f(x) cross" (solve g(x) = f(x)).
"Find a range of quantities when AVC ≤ 0.55" (first solve AVC = 0.55)
"Find a range of quantities when VC ≤ TR" (first solve VC = TR)
"Find the largest quantity when Profit is nonnegative." (you want Profit ≥ 0, so first solve Profit = 0).
"Find when g(x) ≥ 15" (first solve g(x) = 15)
"Find when MR = MC"
"Find the y-value of AC when AC = MC (i.e. find BEP)"
(solve AC = MC, then plug *x* into AC to get the y-value)

PROBLEM TYPE 2: **Finding a vertex (or studying the shape of a function).** Write out the function and draw a sketch. If it's a quadratic then find the x-coordinate of the vertex. Then read the question and get the desired value from the x you just found.

"Find the maximum TR" (find the x-coord of the vertex of TR; compute TR at this *x*)
"Find the quantity when profit is maximum (find the x-coord of the vertex and stop)
"Find the maximum profit" (find the x-coord of the vertex of Profit; compute Profit at this *x*)
"Find the price that corresponds to the maximum profit"
(find the x-coord of vertex of Profit; compute the *price* at this *x*)
"Find the smallest y-value of AVC (i.e. find SDP)"
(find the x-coord of the vertex of AVC; compute AVC at this *x*)
"Find the interval when g(x) – f(x) is increasing"
(find the g(x)-f(x) function, sketch the graph, find the vertex)
"Find the interval when both f(x) and g(x) are increasing"
(sketch f(x), find its vertex. And sketch g(x), find its vertex; when do these intervals overlap)
"Find the interval when both TR and Profit are increasing"
(sketch TR, find its vertex. And sketch Profit, find its vertex; when do these intervals overlap)

PROBLEM TYPE 3: **Use definitions and functional notation.** At the beginning of several of the problems you needed to use functional notation to get one function from another. We spend all day in lecture on Monday talking about how to use functional notation, if you missed that, then get the notes from a classmate or check out the 2.2 review sheet. Here is the translation (and then you would use the notation correct):

"Find the formulas for MR"	(TR(q+"one item")-TR(q))/"one item"
"Find the formula for MC"	(TC(q+"one item")-TC(q))/"one item"
"Find the formula for TR"	TR(q) = (price)(quantity) = (price) q
"Find the formula for TC"	TC(q) = VC(q) + FC and TC(q) = q AC(q)
"Find the formula for VC"	VC(q) = TC(q) – FC and VC(q) = q AVC(q)
"Find the formula for AC"	AC(q) = TC(q)/q
"Find the formula for AVC"	AVC(q) = VC(q)/q
"Find the formula for VC"	VC(q) = TC(q) - FC
"Find the formula for ATS"	ATS(t) = D(t) / t
"Find the formula for the average speed from t to t+5"	AS(t) = (D(t+5)-D(t)) / 5
"Find the formula for the average speed from t to t+2"	AS(t) = (D(t+2)-D(t)) / 2
"Find (f(x+h) – f(x))/h"	

Section 2.2 Review

In this section we discussed how to find rates and rate formulas when given a function. You need to be able to work with functional notation well to understand this section. Make sure you check out my Functional notation review (posted earlier in the quarter) for more practice with functional notation.

Rates from Functions:

1. Given a function $f(x)$, recall that we defined:

$$\frac{f(x) - f(0)}{x} = \text{ ‘overall rate of change of } f \text{ from 0 to } x\text{’.}$$

Note in many cases, $f(0) = 0$, and in other cases, we want the slope of the diagonal line. So we also have

$$\frac{f(x)}{x} = \text{ ‘slope of the diagonal line to } f \text{ at } x\text{.’ (Examples: } ATS,\ AR,\ AC,\ AVC)$$

$$\frac{f(b) - f(a)}{b - a} = \text{ rate of change from } a \text{ to } b. \text{ (Examples: } AS,\ MR,\ MC)$$

Intervals can be described in a few different ways, but we always just replace a = START, and b = END. For example,

‘the interval of length h that starts at x’ would be $a = x$ to $b = x + h$.

2. Overall Rate Examples:

$D(t) = 3t - 10t^2$ = Distance	$ATS(t) = \frac{D(t)}{t} = \frac{3t-10t^2}{t} = 3 - 10t$ = Average Trip Speed
$R(x) = 10x + 3x^2$ = Total Revenue	$AR(x) = \frac{TR(x)}{x} = \frac{10x+3x^2}{x} = 10 + 3x$ = Average Revenue
$C(x) = 4 - 7x + 30x^2$ = Total Cost	$AC(x) = \frac{TC(x)}{x} = \frac{4-7x+30x^2}{x} = \frac{4}{x} - 7 + 30x$ = Average Cost
$V(x) = -7x + 30x^2$ = Variable Cost	$AVC(x) = \frac{VC(x)}{x} = \frac{-7x+30x^2}{x} = -7 + 30x$ = Average Variable Cost

3. Incremental Rates: There is more algebraic manipulation in finding an incremental rate. It takes about 5-6 lines of work. But it is always the same steps and with a bit of care and practice you can quickly become good at working with functions in this way. The most common rate we need to find is

$$\frac{f(x+h) - f(x)}{h} = \text{ ‘the rate of change of } f \text{ from } x \text{ to } x + h\text{’}$$

Let’s do a detailed example: Consider the function $f(x) = 5 + 4x + 3x^2$. Let’s find and simplify $\frac{f(x+h)-f(x)}{h}$ for this function.

(a) First, do the substitution and simplify $f(x + h)$.

$f(x+h) = 5 + 4(x + h) + 3(x + h)^2$	
$f(x+h) = 5 + 4x + 4h + 3(x^2 + 2xh + h^2)$	expanding
$f(x+h) = 5 + 4x + 4h + 3x^2 + 6xh + 3h^2$	still expanding

(b) Second, substitute into the full expression and simplify.

$\frac{f(x+h)-f(x)}{h} = \frac{(5+4x+4h+3x^2+6xh+3h^2)-(5+4x+3x^2)}{h}$	using the previous part and the function def’n
$\frac{f(x+h)-f(x)}{h} = \frac{5+4x+4h+3x^2+6xh+3h^2-5-4x-3x^2}{h}$	drop the parentheses and distribute the negative.
$\frac{f(x+h)-f(x)}{h} = \frac{4h+6xh+3h^2}{h}$	cancel terms in the numerator.
$\frac{f(x+h)-f(x)}{h} = 4 + 6x + 3h$	cancel the h from the denominator.

So if $f(x) = 5 + 4x + 3x^2$, then we just found that $\frac{f(x+h)-f(x)}{h} = 4 + 6x + 3h$ is the average rate from x to $x + h$.

Let's see another example:
Assume $R(x) = 42x - x^2$ is the formula for total revenue and assume x is in hundreds of items and revenue is in hundreds of items. Remember, in this situation, that marginal revenue is defined by $MR(x) = \frac{TR(x+0.01)-TR(x)}{0.01}$. Let's compute this new formula using algebra.

1. First, do the substitution and simplify $R(x + 0.01)$.

$R(x+0.01)$	$= 42(x+0.01) - (x+0.01)^2$	
$R(x+0.01)$	$= 42x + 0.42 - (x^2 + 0.02x + 0.0001)$	expanding
$R(x+0.01)$	$= 42x + 0.42 - x^2 - 0.02x - 0.0001$	still expanding

2. Second, substitute into the full expression and simplify.

$\frac{R(x+0.01)-R(x)}{0.01}$	$= \frac{(42x+0.42-x^2-0.02x-0.0001)-(42x-x^2)}{0.01}$	using the previous part and the function def'n
$\frac{R(x+0.01)-R(x)}{0.01}$	$= \frac{42x+0.42-x^2-0.02x-0.0001-42x+x^2}{0.01}$	drop parentheses and distribute negative.
$\frac{R(x+0.01)-R(x)}{0.01}$	$= \frac{0.42-0.02x-0.0001}{0.01}$	cancel terms in the numerator.
$\frac{R(x+0.01)-R(x)}{0.01}$	$= 42 - 2x - 0.01$	dividing by 0.01.
$\frac{R(x+0.01)-R(x)}{0.01}$	$= 41.99 - 2x$	simplify.

So if $R(x) = 42x - x^2$, then we just found that $MR(x) = 41.99 - 2x$.
(Aside just for your own intersest: In Math 112, you'll learn methods to quickly get a good approximation to this formula. Those shortcuts will give an answer of $42 - 2x$ which is very close to what we got. More to come in Math 112.)

Additional Exercises (Try these problems on your own, answers are on the next page):

1. Let $D(t) = 1 + 2t^2$ (t in hours and D in miles). Find and simplify the formula for the average speed over a 2-hour interval starting at t.
2. Let $C(x) = 2 - 3x + x^2$ (x is in Items and C is in dollars). Find formulas for AC and MC.
3. Let $f(x) = 3x + 5x^2$. Find and simplify $\frac{f(x+h)-f(x)}{h}$.
4. Let $g(x) = 3 - 6x$. Find and simplify $\frac{g(x+h)-g(x)}{h}$.

1. We are given $D(t) = 1 + 2t^2$ and we want to find $\frac{D(t+2)-D(t)}{2}$. Here I do all the work together:

$\frac{D(t+2)-D(t)}{2}$	$= \frac{(1+2(t+2)^2)-(1+2t^2)}{2}$	using the function def'n
$\frac{D(t+2)-D(t)}{2}$	$= \frac{(1+2(t^2+4t+4))-(1+2t^2)}{2}$	expanding
$\frac{D(t+2)-D(t)}{2}$	$= \frac{1+2t^2+8t+8-1-2t^2}{2}$	still expanding and dropping the paranthesis (dist. negative!)
$\frac{D(t+2)-D(t)}{2}$	$= \frac{8t+8}{2}$	cancel terms in the numerator
$\frac{D(t+2)-D(t)}{2}$	$= 4t + 4$	simplify

2. We are given $C(x) = 2 - 3x + x^2$ and we want $AC(x) = \frac{TC(x)}{x}$ and $MC(x) = \frac{TC(x+1)-TC(x)}{1}$.

$AC(x)$	$= \frac{2-3x+x^2}{x}$	using the function def'n
$AC(x)$	$= \frac{2}{x} - 3 + x$	simplifying

$MC(x)$	$= \frac{(2-3(x+1)+(x+1)^2)-(2-3x+x^2)}{1}$	using the function def'n
$MC(x)$	$= \frac{(2-3x-3+x^2+2x+1)-(2-3x+x^2)}{1}$	expanding
$MC(x)$	$= \frac{2-3x-3+x^2+2x+1-2+3x-x^2}{1}$	dropping the paranthesis (dist. negative!)
$MC(x)$	$= \frac{-3+2x+1}{1}$	cancel terms in the numerator
$MC(x)$	$= 2x - 2$	simplify

3. We are given $f(x) = 3x + 5x^2$ and we want to find $\frac{f(x+h)-f(x)}{h}$. Here I do all the work together:

$\frac{f(x+h)-f(x)}{h}$	$= \frac{(3(x+h)+5(x+h)^2)-(3x+5x^2)}{h}$	using the function def'n
$\frac{f(x+h)-f(x)}{h}$	$= \frac{(3x+3h+5(x^2+2xh+h^2)-(3x+5x^2)}{h}$	expanding
$\frac{f(x+h)-f(x)}{h}$	$= \frac{3x+3h+5x^2+10xh+5h^2-3x-5x^2}{h}$	still expanding and dropping the paranthesis (dist. negative!)
$\frac{f(x+h)-f(x)}{h}$	$= \frac{3h+10xh+5h^2}{h}$	cancel terms in the numerator
$\frac{f(x+h)-f(x)}{h}$	$= 3 + 10x + 5h$	simplify

4. We are given $g(x) = 3 - 6x$ and we want to find $\frac{g(x+h)-g(x)}{h}$. Here I do all the work together:

$\frac{g(x+h)-g(x)}{h}$	$= \frac{(3-6(x+h))-(3-6x)}{h}$	using the function def'n
$\frac{g(x+h)-g(x)}{h}$	$= \frac{(3-6x-6h)-(3-6x)}{h}$	expanding
$\frac{g(x+h)-g(x)}{h}$	$= \frac{3-6x-6h-3+6x}{h}$	dropping the paranthesis (dist. negative!)
$\frac{g(x+h)-g(x)}{h}$	$= \frac{-6h}{h}$	cancel terms in the numerator
$\frac{g(x+h)-g(x)}{h}$	$= -6$	simplify

Functional Notation Review

You will need to know functional notation well if you want to succeed in this course. Here is a quick review.

Example: Consider the function $f(x) = x^2 - 3x$. Note that

- "f" is the name of the function for later reference.
- "(x)" says that "x" is the input to the function, anything inside the parantheses next to the function name is the input. This is NOT multiplication, the parantheses indicated input.
- "$x^2 - 3x$" is the rule. It says that whatever the input is to the function, but that input in place of all the "x" locations.

Let's input several things into this particular function, just so you can see how it works:

- Input $x = 2$ into the function $f(x)$ and you get $f(2) = (2)^2 - 3(2) = 4 - 6 = -2$.
- Input $x = 10$ into the function $f(x)$ and you get $f(10) = (10)^2 - 3(10) = 100 - 30 = 70$.
- Input $x = w$ into the function $f(x)$ and you get $f(w) = w^2 - 3w$.
- Input $x = BLAH$ into the function $f(x)$ and you get $f(BLAH) = (BLAH)^2 - 3(BLAH)$.
- Input $x = t + h$ into the function $f(x)$ and you get $f(t + h) = (t + h)^2 - 3(t + h)$.

Try the function questions (we are still taking about $f(x) = x^2 - 3x$):

1. What is $f(t)$?
2. What is $f(x + 5)$?
3. What is $f(x) + 5$?
4. What is $\frac{f(x)}{x}$?
5. What is $f(x + h) - f(x)$?

Answers:

1. $f(t) = t^2 - 3t$ (the input to f is t).
2. $f(x + 5) = (x + 5)^2 - 3(x + 5)$ (the input to f is $x + 5$).
3. $f(x) + 5 = x^2 - 3x + 5$ (the input to f is x, then we add 5).
4. $\dfrac{f(x)}{x} = \dfrac{x^2 - 3x}{x}$ (the input to f is x, then we divide by x).
5. $f(x + h) - f(x) = [(x + h)^2 - 3(x + h)] - [x^2 - 3x]$ (the function is used twice, first the input to f is $x + h$ and the second input to f is x. We need to give these expressions then subtract the second from the first). Also note, I like to use brackets, [and], to separate uses of the function, these are the same as parantheses. I just use brackets because paratheneses are used elsewhere.

More Examples Consider these functions:

$g(x) = e^x$	$h(x) = x^3 - \sqrt{x}$	$p(x) = \dfrac{x+1}{x^4}$	$q(x) = \ln(x) - 7$

For these functions, try the questions below:

1. What is $g(5+h) - g(5)$?
2. What is $h(x-3)$?
3. What is $p(b) - p(1)$?
4. What is $q(x+h) - q(x)$?
5. What is $g(x+1) - 2h(x) + q(2) + p(a)$?

Answers:

1. $g(5+h) - g(5) = e^{5+h} - e^5$.
2. $h(x-3) = (x-3)^3 - \sqrt{x-3}$.
3. $p(b) - p(1) = \dfrac{b+1}{b^4} - \dfrac{1+1}{1^4}$
4. $q(x+h) - q(x) = [\ln(x+h) - 7] - [\ln(x) - 7]$
   (note that $\ln(x)$ is another example of a functional notation just with the name "ln")
5. $g(x+1) - 2h(x) + q(2) + p(a) = [e^{x+1}] - 2[x^3 - \sqrt{x}] + [\ln(2) - 7] + [\dfrac{a+1}{a^4}]$.

The Most Important Example for Calculus When we are considering average rates (which lead to instantaneous rates), we will very often see

$$\text{“The average rate of change for } f(x) \text{ from } x \text{ to } x+h\text{”} = \frac{f(x+h) - f(x)}{h}.$$

We will need to be able to write this expression and simplify it. This review has been helping you practice the first step (how to write the expression), the simplification has to do with your algebra skills which you'll practice in homework and quiz section. Let's do a few more functional notation examples for this particular important expressions.

1. For the function $f(x) = x^2$, what is $\dfrac{f(x+h) - f(x)}{h}$?
2. For the function $g(x) = e^x$, what is $\dfrac{g(x+h) - g(x)}{h}$?
3. For the function $p(x) = 4x - 3x^2$, what is $\dfrac{p(x+h) - p(x)}{h}$?

Answers:

1. $\dfrac{f(x+h) - f(x)}{h} = \dfrac{(x+h)^2 - x^2}{h}$.
2. $\dfrac{g(x+h) - g(x)}{h} = \dfrac{e^{x+h} - e^x}{h}$.
3. $\dfrac{p(x+h) - p(x)}{h} = \dfrac{[4(x+h) - 3(x+h)^2] - [4x - 3x^2]}{h}$.

Sections 2.1 Review

We introduce quadratic functions in this section.

Basic Quadratic Facts:

1. A **quadratic equation** is any equation that can be written in the form $ax^2 + bx + c = 0$, where a, b and c are given numbers. (It's important to note that one side is zero and the only variable is x).

2. To solve a quadratic equation:

   (a) Simplify/Clear Denominator: Always clear the denominators and look for easy simplifications (sometimes you can solve by just simplifying).

   (b) Make one side zero: Add/Subtract to get all terms to one side.

   (c) Factor/Quadratic Formula: If you can factor, do it. Otherwise, use the quadratic formula.

   (d) Check your answer

   Here is the quadratic formula:
   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$
   Note that

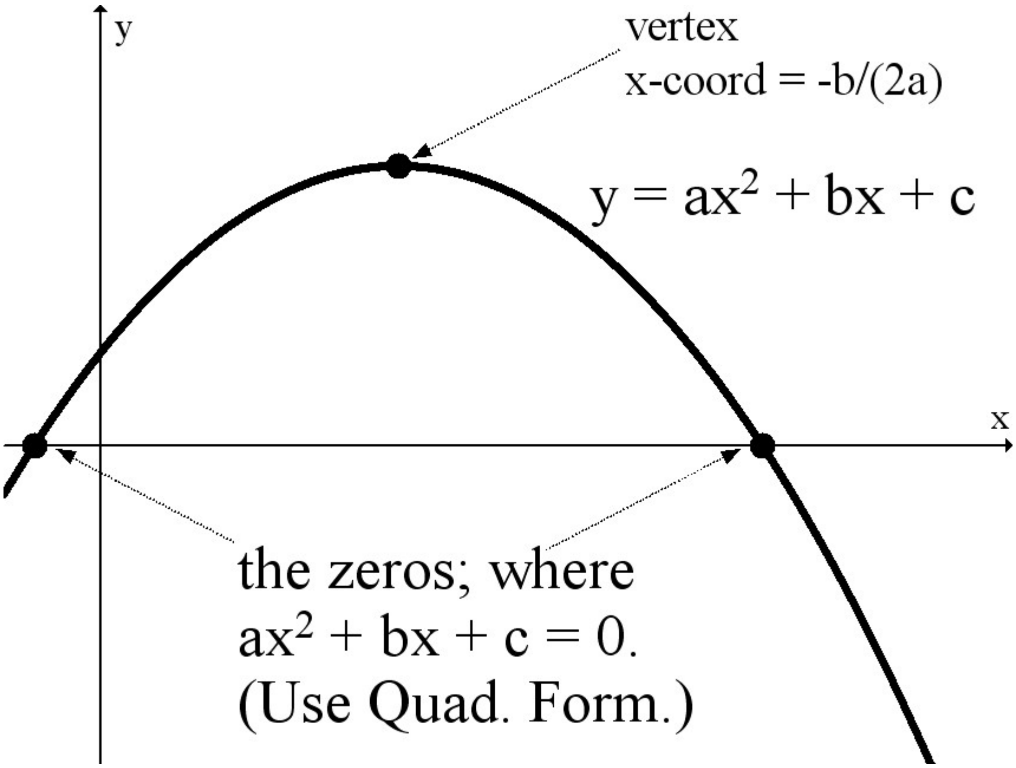
   - if $b^2 - 4ac$ is positive, then there are two solutions.
   - if $b^2 - 4ac$ is zero, then there is one solution.
   - if $b^2 - 4ac$ is negative, then there are no solutions.

3. A **quadratic function** is any function that can be written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are given numbers. (Note that $f(x)$ is the name of the output, so there are two variable here, x is the input and $y = f(x)$ is the output).

4. The graph of the quadratic function $y = ax^2 + bx + c$ is called a parabola. The point where the parabola changes direction is called the vertex. Here are three important facts:

   - If $a > 0$, then the parabola opens upward (it smiles).
   - If $a < 0$, then the parabola opens downward (it frowns).
   - The x-coordinate of the vertex is located at $x = -\frac{b}{(2a)}$.

A picture of a quadratic function, $f(x) = ax^2 + bx + c$. Note that a must be negative (the parabola opens downward).



For extra practice with solving quadratic equations and using the quadatic formula, try these (solutions are online in the full review sheet).

1. $\frac{x^2}{3} + 1 = 5$
2. $(x-1)^2 + 4 = 13$
3. $(3x)^2 - 4 = 10$.
4. $x^2 - 6x + 5 = 0$
5. $x^2 - 7x + 1 = 3 - 2x$
6. $x^2 + 5x + 12 = -10 - 2x^2$

For practice finding the vertex, try these (solutions are online in the full review).
Find the x and y coordinates of the vertex for the following quadratic functions:

1. $y = 3x^2 - 60x + 5$.
2. $y = 46x - 2x^2$.
3. $f(x) = (x^2 + 4x) - (10x - 7)$.

Answers to Solving Quadratic Equation Problems:

1. Answer:

$\frac{x^2}{3}+1$	$=5$	
$\frac{x^2}{3}$	$=4$	subtracted 1 from both sides
x^2	$=12$	multiplied both sides by 3
x	$=\pm\sqrt{12}$	took the square root of both sides (and wrote both answers)

The two answers are $x=\sqrt{12}$ and $x=-\sqrt{12}$.

2. Answer:

$(x-1)^2+4$	$=13$	
$(x-1)^2$	$=9$	subtracted 4 from both sides
$x-1$	$=\pm\sqrt{9}=3$	took the square root of both sides (and wrote both answers)
x	$=1\pm 3$	added 1 to both sides

The two answers are $x=1+3=4$ and $x=1-3=-2$.
Note: You can also expand $(x-1)^2=x^2-2x+1$ at the beginning of the problem and use the quadratic formula. That will give the correct answers (it just will take a bit more work in my opinion).

3. Answer:

$(3x)^2-4$	$=10$	note that $(3x)^2=(3x)(3x)=9x^2$
$9x^2$	$=14$	simplified and added 4 to both sides
x^2	$=\frac{14}{9}$	multiplied both sides by 9
x	$=\pm\sqrt{\frac{14}{9}}$	took the square root of both sides (and wrote both answers)

The two answers are $x=\sqrt{\frac{14}{9}}=\frac{\sqrt{14}}{3}$ and $x=-\sqrt{\frac{14}{9}}=-\frac{\sqrt{14}}{3}$. (You don't have to rewrite your answer in the simplest form as I did)

4. Answer:

Method 1: Factoring

x^2-6x+5	$=0$	
$(x-1)(x-5)$	$=0$	factored (I noticed that $(-1)(-5)=5$ and $(-1)+(-5)=-6$)

Thus, we can conclude $x-1=0$ or $x-5=0$, which gives $x=1$ or $x=5$.

Method 2: Quadratic Formula

x^2-6x+5	$=0$	one side is zero and $a=1$, $b=-6$, $c=5$.
x	$=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(5)}}{(2(1))}$	quadratic formula
x	$=\frac{6\pm\sqrt{36-20}}{2}$	simplifying
x	$=\frac{6\pm\sqrt{16}}{2}$	simplifying
x	$=\frac{6\pm 4}{2}$	simplifying

Thus, we conclude $x=\frac{6-4}{2}=1$ or $x=\frac{6+4}{2}=5$.

5. Answer:

x^2-7x+1	$=3-2x$	make one side zero!
x^2-5x-2	$=0$	subtracted 2 and added $2x$ to both sides
x	$=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(-2)}}{(2(1))}$	quad. form. $a=1$, $b=-5$, $c=-2$
x	$=\frac{5\pm\sqrt{25+8}}{2}$	simplifying
x	$=\frac{5\pm\sqrt{33}}{2}$	simplifying

Thus, we conclude $x=\frac{5-\sqrt{33}}{2}$ or $x=\frac{5+\sqrt{33}}{2}$.

6. Answer:

$x^2+5x+12$	$=-10-2x^2$	make one side zero!
$3x^2+5x+22$	$=0$	added 10 and added $2x^2$ to both sides
x	$=\frac{-(5)\pm\sqrt{(5)^2-4(3)(22)}}{(2(3))}$	quad. form. $a=3$, $b=5$, $c=22$
x	$=\frac{-5\pm\sqrt{25-264}}{6}$	simplifying
x	$=\frac{5\pm\sqrt{-239}}{6}$	negative under radical!!!

Since there is a negative under the radical, we conclude there is no solution.

Note: In the problems where I used the quadratic formula, see how I broke up my use of the formula into steps. (1) Simplify the expression under the radical, (2) simplify the denominator, (3) take the square root, then (4) simplify your answer. You should do the same and you should write out your work on paper. It is very common to make errors when you try to type it all into your calculator at once, so work it in steps and write down your intermediate work!

Answers to Vertex of a Quadratic Function Problems:

1. $y=3x^2-60x+5$.
   The x-coordinate of the vertex is $x=-\frac{(-60)}{2\cdot 3}=\frac{60}{6}=10$.
   The y-coordiante of the vertex is $y=3(10)^2-60(10)+5=300-600+5=-295$.
2. $y=46x-2x^2=-2x^2+46x$.
   The x-coordinate of the vertex is $x=-\frac{46}{2\cdot(-2)}=\frac{46}{4}=11.5$.
   The y-coordiante of the vertex is $y=46(11.5)-2(11.5)^2=529-264.5=264.5$.
3. $f(x)=(x^2+4x)-(10x-7)=x^2-6x+7$.
   The x-coordinate of the vertex is $x=-\frac{(-6)}{2\cdot(1)}=3$.
   The y-coordiante of the vertex is $f(3)=(3)^2-6(3)+7=9-18+7=-2$.

Sections 4.2 Review

The method of **linear programming** is a procedure we will use to optimize (max/min) an *objective* subject to *constraints*. Here is the short version:

- **Step 1: Label the two quantities.**
  Example: "How many pounds of each ... " would mean x = amount of first (in pounds), y = amount of second (in pounds)
- **Step 2: Collect information from the problem.** Make a table of rates for x and y, write out formulas for total amounts, and list any restrictions.
- **Step 3: Give Constraints and Objective.** Write down the inequalities for the constraints and write down the function for the objective. Note that the last sentence typically tells you the objective. For example, "...to minimize cost?" means that the cost function, $C(x, y)$, is the objective you are minimizing. You will only use the objective at the end of the problem.
- **Step 4: Sketch the Feasible Region.** Draw the line and shade the region that corresponds to each of the constraint inequalities. Indicate the overlapping region (this is the feasible region).
- **Step 5: Find all the corners for the overlapping Feasible Region.** You probably will need to do some intersections to do this. Use your picture to figure out which lines you need to intersect.
- **Step 6: Plug each corner into the objective.** The largest output is the maximum and the smallest output is the minimum.

We did four big examples in lecture. Here are a three more for you to try (full answers are on the following pages):

Lawn Care Example: A lawn care company has two types of fertilizer Regular and Deluxe. The profit for Regular is \$0.75 per bag and the profit for Deluxe is \$1.20 per bag. One bag of Regular has 3 pounds of active ingredient and 7 pounds of inert ingredients. One bag of Deluxe has 4 pounds of active ingredients and 6 pounds of inert ingredients. The warehouse has a limit of 8400 pounds of active ingredient and 14100 pounds on inert ingredient.
How many bags of each should we make to maximize profit?

A Boring No Words Question: Find the maximum of the objective $f(x, y) = 14x + 20y$ subject to the constraints: $4x + 6y \leq 1800$, $x \leq 300$, $y \leq 150$, $x \geq 0$, and $y \geq 0$.

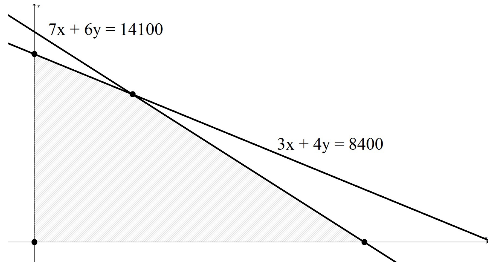
Cookie Company Example: Your company makes two types of chocolate chip cookies. Each bag of 'Chocolate Lite' contains 10 ounces of chocolate chips and 35 ounces of dough. Each bag of 'Chocolate Overload' contains 20 ounces of chocolate chips and 25 ounces of dough. The profit on each bag of Chocolate Lite is \$1.10, while the profit on each bag of Chocolate Overload is \$0.80. Your company current has 1000 ounces of chocolate chips and 2555 ounces of cookie dough.
How many bags of each should you produce to maximize profit?

ANSWER to Lawn Care Example:

Let x = number of bags of Regular and y = number of bags of Deluxe. Here is the collected information:

	x	y	Total formula	Constraints/Objective
Active	3	4	$3x + 4y$	$3x + 4y \leq 8400$
Inert	7	6	$7x + 6y$	$7x + 6y \leq 14100$
Profit	0.75	1.20	0.75x+1.20y	$P(x, y) = 0.75x + 1.20y$ = Objective

Now we get points, draw the lines and shade the feasible region:
$3x + 4y = 8400$ goes through $(0, \frac{8400}{4}) = (0, 2100)$ and $(\frac{8000}{3}, 0) = (2800, 0)$.
$7x + 6y = 14100$ goes through $(0, \frac{14100}{6}) = (0, 2330)$ and $(\frac{141000}{7}, 0) \approx (2014.29, 0)$.
Note that $(0, 0)$ works in both of these constraints so we shade the overlapping 'origin-side' of the lines.



To get the corner at the intersection, we combine: (i) $3x + 4y = 8400$ and (ii) $7x + 6y = 14100$
(i) $3x + 4y = 8400 \quad \Rightarrow \quad 4y = 8400 - 3x \quad \Rightarrow \quad y = 2100 - 0.75x$.
(i) and (ii) $7x+6(2100-0.75x) = 14100 \quad \Rightarrow \quad 7x+12600-4.5x = 14100 \quad \Rightarrow \quad 2.5x = 1500 \quad \Rightarrow \quad x = \frac{1500}{2.5} \approx 600$.
And $y = 2100 - 0.75x = 2100 - 0.75(600) = 1650$

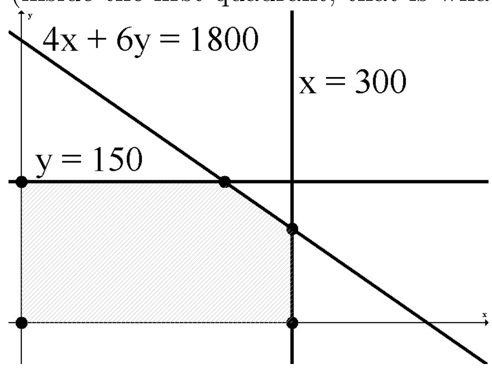
Evaluating the objective at the found corners gives:

$$\begin{aligned} P(0,0) &= 0.75(0) + 1.20(0) &&= 0 \text{ dollars} \\ P(0,2100) &= 0.75(0) + 1.20(2100) &&= 2520 \text{ dollars} \\ P(2014.29,0) &= 0.75(2014.29) + 1.20(0) &&= 1510.71 \text{ dollars} \\ P(600,1650) &= 0.75(600) + 1.20(1650) &&= 2430 \text{ dollars} \end{aligned}$$

Thus, the maximum profit is \$2520 and it occurs when you produce $x = 0$ bags of Regular and $y = 2100$ bags of Deluxe.

ANSWER to A Boring No Words Question:

The problem is already set up for us, so we just need to graph and find corners. First get points, draw the lines and shade the feasible region:
$4x + 6y = 1800$ goes through $(0, \frac{1800}{6}) = (0, 300)$ and $(\frac{1800}{4}, 0) = (450, 0)$.
$y = 150$ is a horizontal lines at $y = 150$.
$x = 300$ is a vertical line at $x = 300$. Note that $(0, 0)$ works in these three constraints so we shade the overlapping 'origin-side' of the lines (inside the first quadrant, that is what $x \geq 0$ and $y \geq 0$ is telling us).



To get the corners at the intersections, we combine.
One corner: (i) $4x + 6y = 1800$ and (ii) $y = 150$
(i) and (ii) $4x + 6(150) = 1800 \Rightarrow 4x + 900 = 1800 \Rightarrow 4x = 900 \Rightarrow x = \frac{900}{4} \approx 225$. So one of the corners is $(225, 150)$.
The other corner: (i) $4x + 6y = 1800$ and (ii) $x = 300$
(i) and (ii) $4(300) + 6y = 1800 \Rightarrow 1200 + 6y = 1800 \Rightarrow 6y = 600 \Rightarrow y = \frac{600}{6} \approx 100$. So the other corner is $(300, 100)$.

Evaluating the objective at the found corners gives:

$$\begin{aligned} f(0,0) &= 14(0) + 20(0) &&= 0 \\ f(0,150) &= 14(0) + 20(150) &&= 3000 \\ f(300,0) &= 14(300) + 20(0) &&= 4200 \\ f(300,100) &= 14(300) + 20(100) &&= 6200 \\ f(225,150) &= 14(225) + 20(150) &&= 6150 \end{aligned}$$

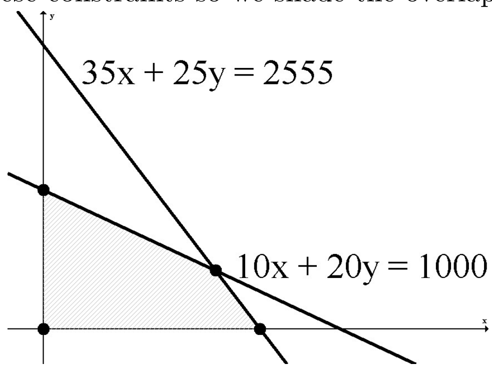
Thus, the maximum is 6200 and it occurs when $x = 300$ and $y = 100$.

ANSWER to Cookie Company Example:

Let x = number of bags of Chocolate Lite and y = number of bags of Chocolate Overload. Here is the collected information:

	x	y	Total formula	Constraints/Objective
Chips	10	20	$10x + 20y$	$10x + 20y \leq 1000$
Dough	35	25	$35x + 25y$	$35x + 25y \leq 2555$
Profit	1.10	0.80	1.10x+0.80y	$P(x, y) = 1.10x + 0.80y =$ Objective

Now we get points, draw the lines and shade the feasible region:
$10x + 20y = 1000$ goes through $(0, \frac{1000}{20}) = (0, 50)$ and $(\frac{1000}{10}, 0) = (100, 0)$.
$35x + 25y = 2555$ goes through $(0, \frac{2555}{25}) = (0, 102.2)$ and $(\frac{2555}{35}, 0) \approx (73, 0)$.
Note that $(0, 0)$ works in both of these constraints so we shade the overlapping 'origin-side' of the lines.



To get the corner at the intersection, we combine: (i) $10x + 20y = 1000$ and (ii) $35x + 25y = 2555$
(i) $10x + 20y = 1000 \quad \Rightarrow \quad 20y = 1000 - 10x \quad \Rightarrow \quad y = 50 - 0.5x$.
(i) and (ii) $35x + 25(50 - 0.5x) = 2555 \quad \Rightarrow \quad 35x + 1250 - 12.5x = 2555 \quad \Rightarrow \quad 22.5x = 1305 \quad \Rightarrow \quad x = \frac{1305}{22.5} \approx 58$.
And $y = 50 - 0.5x = 50 - 0.5(58) = 21$.

Evaluating the objective at the found corners gives:

$$
\begin{aligned}
P(0,0) &= 1.10(0) + 0.80(0) &&= 0 \text{ dollars} \\
P(0,50) &= 1.10(0) + 0.80(50) &&= 40 \text{ dollars} \\
P(73,0) &= 1.10(73) + 0.80(0) &&= 80.30 \text{ dollars} \\
P(58,21) &= 1.10(58) + 0.80(21) &&= 80.60 \text{ dollars}
\end{aligned}
$$

Thus, the maximum profit is \$80.60 and it occurs when you produce $x = 58$ bags of Chocolate Lite and $y = 21$ bags of Chocolate Overload.

Preview of how we will set up the problems in 4.2

Example 1: (From Homework)
Newjet, Inc. manufactures inkjet printers and laser printers. The company has the capacity to make 490 printers per day, and it has 840 hours of labor per day available. It takes 1 hour to make an inkjet printer and 3 hours to make a laser printer. The profits are$90 per inkjet printer and $140 per laser printer. Find the maximum profit.
Answer Set up (how we start):
x = number of inkjet, y = number of laser

	x	y
Hours per item	1	3
Profits per item	90	140

Totals: Total hours = x + 3y, Total profits = 90x + 140y, Total number of printers = x + y.
Constraints: Total number of printers ≤ 490, so x + y ≤ 490.
Total number of hours ≤ 840, so x + 3y ≤ 840.
Objective: Total profits = P(x, y) = 90x + 140y.

Example 2: (From Homework)
A company manufactures two types of electric hedge trimmers, one of which is cordless. The cord-type trimmer requires 4 hours to make, and the cordless model requires 10 hours. The company has only 2000 work hours to use in manufacturing each day, and the packaging department can package only 300 trimmers per day.
Answer Set up (how we start):
x = # of cord-type, y = # of cordless

	x	y
Hours per item	4	10

Totals: Total hours = 4x + 10y, Total number of trimmers = x + y.
Constraints: Total number of trimmers ≤ 300, so x + y ≤ 300.
Total number of hours ≤ 2000, so 4x + 10y ≤ 2000.
Objective: None Given.

Example 3: (From an old exam)
Your company makes two kinds of soda: Regular and Diet. Your total daily production of soda is limited to 1000 gallons. Production requires 2 cup of sugar per gallon of Regular and 1/2 cup of sugar per gallon of Diet. Today, you are limited to 626 cups of sugar. The profit is $1 per gallon of Regular soda and $1.20 per gallon of Diet soda
Answer Set up (how we start):
x = gallons of Regular, y = gallons of Diet

	x	y
Sugar per item	2	0.5
Profit per item	1	1.20

Totals: Total sugar = 2x + 0.5y, Total profits = x + 1.20y, Total gallons = x + y.
Constraints: Total sugar ≤ 626, so 2x + 0.5y ≤ 626.
Total gallons ≤ 1000, so x + y ≤ 1000.
Objective: Total Profits = P(x, y) = x + 1.20y

Example 4 (From an old exam):
Oscar sells boxes of toys and balloons for children's parties. The Standard Box contains 30 toys and 60 balloons. The DeLuxe Box contains 75 toys and 90 balloons. Oscar has just done a complete check of his inventory and has found that he has 20,250 toys and 30,900 balloons currently in stock. He earns a profit of $7 for each Standard Box and $8 for each DeLuxe Box. How many of each Box Oscar should sell in order to maximize profit?
Answer Set up (how we start):
x = boxes of Standard , y = boxes of DeLuxe

	x	y
Toys per item	30	75
Balloons per item	60	90
Profit per item	7	8

Totals: Toys = 30x + 75y, Balloons = 60x + 90y, Profits = 7x + 8y.
Constraints: Toys ≤ 20250, so 30x + 75y ≤ 20250.
Balloons ≤ 30900, so 60x + 90y ≤ 30900.
Objective: Total Profits = P(x, y) = 7x + 8y

IMPORTANT HOMEWORK HINTS FOR NEXT WEEK: Section 1.5 and 4.1 should be very quick. But let me make a few comments based on questions from last year.

Hints for Section 1.5 Homework: Last year, I got two recurring questions about 1.5. Here are some comments pertaining to these issues:

A) **Please don't use the "watch it" in section 1.5**. Last year I had several dozen students asking me to explain the various methods that Webassign showed to solve the problem. In many of the problems the "watch it" uses ways to solve the systems that are way too complicated. Instead do as I will show you in class. For most problems in this class, the easiest way to solve a system is to use substitution (easiest in the sense that, it requires no cleverness). That is, solve for one variable in one of the equations and substitute into the other equation. This will ALWAYS work! So you don't need other methods. If you happen to know how to do other methods that is fine too, but if you can do substitution comfortably, then go ahead and use it. You can see two examples on the posted 1.5 review sheet.

B) **SECTION 1.5 / PROBLEM 6**: The wording is a little different than the other problems, so let me clarify. <u>*You need to compute the total mixed bag cost before you start the problem*</u>. If the price is $3.40 per pound for the mixed bag and it is a total of 70 pounds, then the total money is 3.40*70 = $238.
You are then going to set up equations very much like you did in the other problems. Your set up will look something like: $x + y = 70$ and $2.6x + 5.4y = 238$.
(Your numbers will be different than mine).

Hints for Section 4.1 Homework: This section seemed to go well for most students last year. Once I discuss this section in class, this should be quick to do. But there were a few technical issues last year so let me say a few things to prevent some headache:

A) **Choose solid lines for all your lines** (no rays, no dotted lines, no segment... use only solid lines which are the ones that have arrows in both directions in the menu). And only plot the lines (don't plot points unless it asks you to).

B) **Make sure to graph <u>ALL</u> the lines**. For example if you have

$2x + 5y <= 10$, $x + y <= 6$, $x >= 0$, $y >= 0$

Then **you need to graph all four lines**:

(i) $2x + 5y = 10$, which is *a line through (0,2) and (5,0).*

(ii) $x + y = 6$, which is *a line through (0,6) and (6,0)*.

(iii) $x = 0$, which is the y-axis (*it goes through (0,0) and (0,1)*).

(iv) $y = 0$, which is the x-axis (*it goes through (0,0) and (1,0)*).

C) **Then shade the correct region**. In the example above, (0,0) works in both $x + 5y<=10$ and $x + y <=6$.
So make sure you are shading the overlapping region on the "origin" side of these lines. In the example above you would plot four solid lines, then click the region to shade and that is all you would do (plot nothing else!).

D) **Make sure you don't get the x and y flipped!**

E) **When you are finding an intersection, look at the graph**! If you are intersecting two lines, then you should be working with the equations for those two lines.

F) **If you make a mistake (plotting incorrect lines or plotting extra points or doing anything you didn't need to do in the graph or if you get a red X), then click "Clear all"**. And then redraw the correct lines.

VERY IMPORTANT: In brief, for the 4.1 problems you will do the following:

i) Click on the Solid Extended Line button for all lines (NOT line segments, NOT points, NOT rays)
ii) Type in the points for your line (in the boxes provided after you click on the line button)
iii) Do this again for all your lines.
iv) Shade the region. That is, click the shade button and click in the desired region.

Another Small Note: You have to plot points in the actual window you can see.
For the line $x + 50y = 200$, the x-intercept is (200,0) which may be outside the window.
So pick another point (plug in $x = 100$ and find y and plot that point, or plug in $x = 50$ and find y and plot that point, etc…)

Sections 1.5 and 4.1 Review

In section 4.2 we will discuss the method of linear programming (a method we can use to find maximum and minimum values when we are selling/manufacturing two products). Before we can understand these methods, we need three skills:

- Solving systems of equations.
- Graphing regions given by inequalities.
- Translating a problem into a set of inequalities (we'll practice this in lecture when we discuss 4.2)

Solving Systems of Equations:
All methods involved **combining** the two equations. I never use the words 'set them equal'. In class, I talked about different methods (substitution and adding/subtracting equations), you are welcome to use any correct method, just show your work. In this review sheet, I will just use substitution, it always works and it requires the least amount of cleverness (*i.e.* no tricks needed, everyone can do it).
The method of *substitution* involves solving for one variable in one equation, then substitute into the other equation. Here are two examples:

1. Solve

$$\begin{array}{lrl} \text{(i)} & -3x+2y & =4 \\ \text{(ii)} & 4x+6y & =38 \end{array}$$

   (Note how I label the equations (i) and (ii), I think this is a good strategy to help you stay organized.)

   - Step 1: Solve for one variable from one equation:

   $$\text{(i)} \quad -3x+2y=4 \quad \Rightarrow \quad 2y=4+3x \Rightarrow \quad y=2+1.5x.$$

   - Step 2: Substitute into the other equation:

   $$\text{(i) and (ii)} \quad 4x+6(2+1.5x)=38 \quad \Rightarrow \quad 4x+12+9x=38 \quad \Rightarrow \quad 13x=26 \quad \Rightarrow \quad x=2$$

   And we can go back to either (i) or (ii) (or both) to find y. From (i): $y=2+1.5x=2+1.5(2)=5$. Thus, $(x,y)=(2,5)$ is the solution (and you can check your work, by seeing that this point satisfies both of the original equations).

2. Solve

$$\begin{array}{lrl} \text{(i)} & 2x+5y & =7 \\ \text{(ii)} & 3x-2y & =2 \end{array}$$

   - Step 1: Solve for one variable from one equation:

   $$\text{(i)} \quad 2x+5y=7 \quad \Rightarrow \quad 5y=7-2x \Rightarrow \quad y=1.4-0.4x.$$

   - Step 2: Substitute into the other equation:

   $$\text{(i) and (ii)} \quad 3x-2(1.4-0.4x)=2 \quad \Rightarrow \quad 3x-2.8+0.8x=2 \quad \Rightarrow \quad 3.8x=4.8 \quad \Rightarrow \quad x=\frac{4.8}{3.8}\approx 1.26315789$$

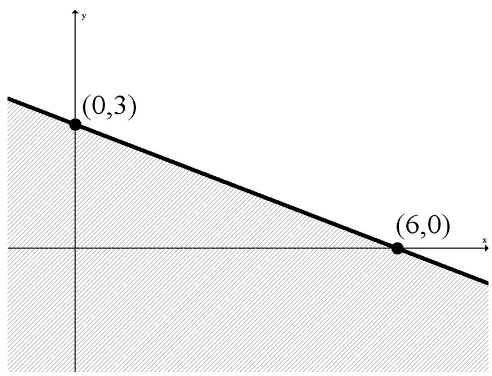
   And we can go back to either (i) or (ii) (or both) to find y.
   From (i): $y=1.4-0.4x=1.4-0.4(1.26315789)=0.89473684$.
   Thus, $(x,y)\approx(1.263, 0.895)$ is the solution (again you can check your work by seeing that this point satisfies both of the original questions).

Graphing Inequalities:
The key step in linear programming is graphing our inequalities. So most of your work in 4.1 and in 4.2 will involve the following skills. Here is a quick summary of the skills we discussed:

1. Given one inequality, graph the line and shade the appropriate side (you can use a sample point to figure out which side to shade).
2. Given multiple inequalities, graph each inequality separately, then shape in the overlapping region.

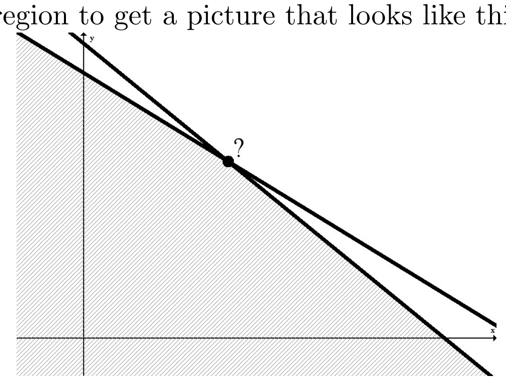
Here are three examples:

1. Graph $2x + 4y \leq 12$
   - Step 1: Graph the line $2x + 4y = 12$. Find two points!
     When $x = 0$, we get $4y = 12$, so $y = 3$. Thus, $(0, 3)$ is a point on the line.
     When $y = 0$, we get $2x = 12$, so $x = 6$. Thus, $(6, 0)$ is a point on the line.
     Now draw this line.
   - Step 2: Take a 'sample point' from one side and put it into the equality to see if you shade that side. The origin $(0, 0)$ is on one side, let's use it as a sample point. In the inequality we get $2(0)+4(0) \leq 12$. Yes, this is true! So we shade the side that includes the point (0,0). Here is the region given by this inequality:



2. Graph $x + y \leq 100$ and $3x + 4y \leq 360$ and shade the overlapping region.
   Do each line separately (like we did in the last example)
   - $x + y = 100$ goes through $(0, 100)$ and $(100, 0)$ and shade the origin side.
   - $3x + 4y = 360$ goes through $(0, 90)$ and $(120, 0)$ and shade the origin side.

   Then shade the overlapping region to get a picture that looks like this:



   Aside: You should know how to find the point I marked with a question mark. It is the intersection point of $x + y = 100$ and $3x + 4y = 360$ (use the methods from the previous page and you should get $x = 40$ and $y = 60$).

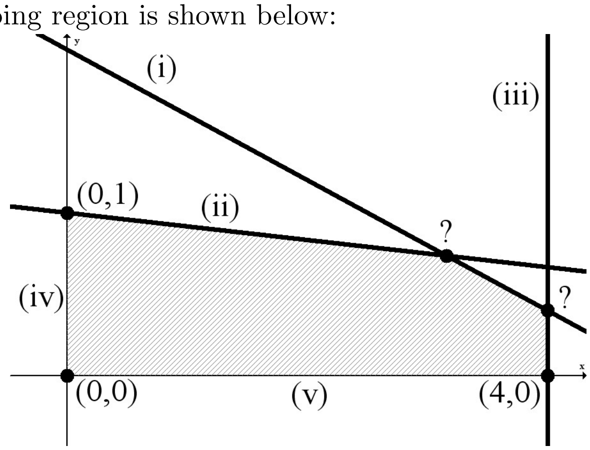
3. Graph the overlapping region given by:

$$\begin{array}{lrl} \text{(i)} & 2x+5y & \le 10 \\ \text{(ii)} & x+12y & \le 12 \\ \text{(iii)} & x & \le 4 \\ \text{(iv)} & x & \ge 0 \\ \text{(v)} & y & \ge 0 \end{array}$$

(a) Step 1: Do each lines separately (like we did in the last example)

- $2x + 5y = 10$ goes through $(0, 2)$ and $(5, 0)$ and shade the origin side.
- $x + 12y = 12$ goes through $(0, 1)$ and $(12, 0)$ and shade the origin side.
- $x = 4$ is a vertical line at $x = 4$ and shade the origin side.
- $x = 0$ is the y-axis and shade to the right of it.
- $y = 0$ is the x-axis and shade above it.

(b) Step 2: The overlapping region is shown below:



Note: In order to do the application problems in 4.2, we need to find the corners of this region. Two of these corners are unknown so far (I have marked them in the picture). To find these, we need to do some intersecting. One is the point of intersection of (i) $2x + 5y = 10$ and (ii) $x + 12y = 12$ and the other is the intersection of (i) $2x + 5y = 10$ and (iii) $x = 4$.

- Intersecting (i) and (ii): Solving for y in (i) gives $5y = 10 - 2x \Rightarrow y = 2 - 0.4x$.
  Combining (i) and (ii) gives

  $$x + 12(2 - 0.4x) = 12 \quad \Rightarrow \quad x + 24 - 4.8x = 12 \quad \Rightarrow \quad -3.8x = -12 \quad \Rightarrow \quad x = \frac{12}{3.8} \approx 3.1578947$$

  And from this we get $y = 2 - 0.4x \approx 2 - 0.4(3.1578947) \approx 0.73684211$.
  So the intersection of (i) and (ii) is at $(x, y) \approx (3.158, 0.737)$.

- Intersecting (i) and (iii):

  $$\text{Combining (i) and (iii):} \quad 2(4) + 5y = 10 \quad \Rightarrow \quad 5y = 2 \quad \Rightarrow \quad y = \frac{2}{5} = 0.4$$

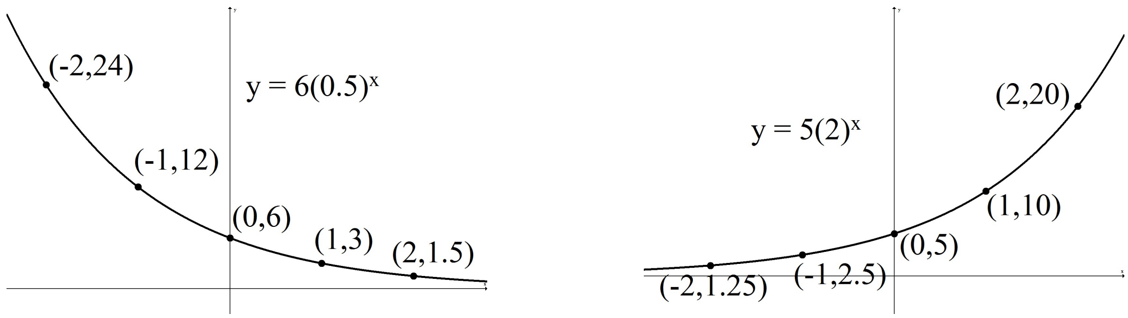
  So the intersection of (i) and (iii) is at $(x, y) \approx (4, 0.4)$.

Sections 5.1-5.3 Review

We introduce exponential and logarithmic functions in this section.

Basic Exponential Facts:

1. An exponential function is any function of the form $f(x) = A(b)^x$.
   - $f(0) = A$ = the y-intercept.
   - b = the base.
   - If $0 < b < 1$, then we have exponential decay and if $b > 1$, then we have exponential growth. Here are examples:



2. The most commonly used based in calculus and applications is $e \approx 2.71828182$ (this is known as Euler's constant). We will see how this comes up when we start discussing compound interest. The graph of the function $y = e^x = (2.71828182)^x$ would look like an exponential growth graph that goes through the point $(0, 1)$.

3. In order to work with exponential functions, you first need to be comfortable with your exponent rules:

Rule	Example	Another Example
$b^0 = 1$	$2^0 = 1$	$e^0 = 1$
$b^x b^y = b^{x+y}$	$x^2 x^3 = x^5$	$e^x e^3 = e^{x+3}$
$(b^x)^y = b^{xy}$	$(x^2)^3 = x^6$	$e^{0.2x} = (e^{0.2})^x$
$b^{-x} = \frac{1}{b^x}$	$2^{-3} = \frac{1}{2^3}$	$e^{-x} = \frac{1}{e^x}$
$b^{1/2} = \sqrt{b}$	$9^{1/2} = \sqrt{9} = 3$	$\sqrt{e^x} = e^{x/2}$
$b^{1/n} = \sqrt[n]{b}$	$8^{1/3} = \sqrt[3]{8} = 2$	$\sqrt[5]{e^x} = e^{x/5}$

Basic Inverse Pairs Facts: The inverse of the exponential function is the logarithm function. More about logarithms on the next page. This a good place to stop and remind you of all your inverse pairs (these are the tools you use to solve equations).

Operation	Inverse	Example
$y = x + a$	$y - a = x$	$x + 3 = 10 \Rightarrow x = 10 - 3 = 7$
$y = x - a$	$y + a = x$	$x - 4 = 13 \Rightarrow x = 13 + 4 = 17$
$y = ax$	$\frac{y}{a} = x$	$4x = 12 \Rightarrow x = \frac{12}{4} = 3$
$y = \frac{x}{a}$	$ay = x$	$\frac{x}{5} = 10 \Rightarrow x = 5 \cdot 10 = 50$
$y = x^a$	$\sqrt[a]{y} = x$ (Include '$\pm$' for even roots)	$x^3 = 8 \Rightarrow x = \sqrt[3]{8} = 2$
$y = \sqrt[a]{x}$	$y^a = x$	$\sqrt[4]{x} = 3 \Rightarrow x = 3^4 = 81$
$y = b^x$	$\log_b y = x$ (You really only need $\ln(x) = \log_e x$)	$e^x = 3 \Rightarrow x = \ln(3) \approx 1.0986$
$y = \log_b x$	$b^y = x$	$\ln x = 3 \Rightarrow x = e^3$

For quadratic equations, we use the quadratic formula. But for all other equations that we can solve in Math 111 and Math 112, we try to get x by itself by using the inverses above (in the correct order).

For example, to solve $(2e^x)^3 + 7 = 15$, we

1. Use the inverse of $() + 7$ (which is $() - 7$) to get $(2e^x)^3 = 15 - 7 = 8$.
2. Use the inverse of $()^3$ (which is $\sqrt[3]{()}$) to get $2e^x = \sqrt[3]{8} = 2$
3. Use the inverse of $2 \cdot ()$ (which is $\frac{()}{2}$) to get $e^x = \frac{2}{2} = 1$.
4. Use the inverse of $e^{()}$ (which is $\ln()$) to get $x = \ln(1) = 0$.

Basic Logarithm Facts:

1. As mentioned on the last page the name we give to the inverses of exponential functions is logarithms.

$$y = b^x \iff \log_b(y) = x.$$

2. Specifically, we will focus on

$$y = e^x \iff \ln(y) = \log_e(y) = x.$$

We only need $\ln(x)$. In fact, $\log_b(x) = \frac{\ln(x)}{\ln(b)}$, so you can always convert to using natural logarithms. You don't even need to know that conversions, you just need to know how to use the natural logarithms rules which are listed below:

Rule	Example
$y = e^x \iff x = \ln(y)$	$20 = e^x$ is the same as $\ln(20) = x$
$\ln(1) = 0$ and $\ln(e) = 1$	
$A = B \Rightarrow \ln(A) = \ln(B)$ (You can take the ln() of both sides)	$2^x = 10 \Rightarrow \ln(2^x) = \ln(10)$
$\ln(ab) = \ln(a) + \ln(b)$	$\ln(3 \cdot 5) = \ln(3) + \ln(5)$
$\ln(b^x) = x\ln(b)$ (Very important, we use this a lot!)	$\ln(2^x) = x\ln(2)$
$\ln(e^x) = x$ and $e^{(\ln(y))} = y$ (They are inverses!)	

3. Solving exponential equations: If we need to solve an equation that involves a variable in an exponent, then we need to do the following:

   (a) Isolate the exponential function: That is get $(b)^x$ by itself.

   (b) Take the natural logarithm of both sides and use the log rule: Remember that $\ln(b^x) = x\ln(b)$.

   (c) Divide by $\ln(b)$. Done!

   Here are two examples

   - Solve $10 + (3)^x = 90$.
     (a) Isolate the exponential: That is subtract 10 to get $(3)^x = 80$.
     (b) Take the ln() of both sides: So we get $\ln(3^x) = \ln(80)$ which becomes $x\ln(3) = \ln(80)$.
     (c) Divide by ln(3) to get: $x = \frac{\ln(80)}{\ln(3)} \approx \frac{4.382026635}{1.098612289} \approx 3.9887$.
   - Solve $30 - 2e^{0.1t} = 20$.
     (a) Isolate the exponential: Subtract 30 to get $-2e^{0.1t} = -10$ and divide by -2 to get $e^{0.1t} = 5$.
     (b) Take the ln() of both sides: So we get $\ln(e^{0.1t}) = \ln(5)$ which becomes $0.1t\ln(e) = \ln(5)$. Since $\ln(e) = 1$, we have $0.1t = \ln(5)$
     (c) Divide by 0.1 to get: $x = \frac{\ln(5)}{0.1} \approx \frac{1.609437912}{0.1} \approx 16.0944$.

Here is a random collection of equations to solve, try them out (solutions below and on the next page):

1. Solve $12 - 5e^{3t} = 3$
2. Solve $2x^3 - 4 = 10$
3. Solve $4\ln(2t+5) = 10$.
4. Solve $\sqrt{x-3} + 7 = 12$.
5. Solve $200000 = 350000(1 - e^{-0.07t})$.
6. Solve $3 - \frac{5}{x} = 1$.
7. Solve $14400 = 1200(7)^{4r}$.
8. Solve $(2^{3x} - 1)^5 + 10 = 40$
9. Solve $x + \frac{2}{x} = 10$.
10. Solve $5 = \frac{60}{e^x+1}$.

Solutions to exercises:

1. Answer:

$12 - 5e^{3t} = 3$	
$-5e^{3t} = -9$	subtracted 12
$e^{3t} = \frac{9}{5}$	divided by -5
$3t = \ln\left(\frac{9}{5}\right)$	take ln() of both sides
$t = \frac{\ln\left(\frac{9}{5}\right)}{3} \approx 0.1959$	divided by 3

2. Answer:

$2x^3 - 4 = 10$	
$2x^3 = 14$	added 4
$x^3 = 7$	divided by 2
$x = (7)^(1/3) \approx 1.9129$	cube root

3. Answer:

$4\ln(2t+5) = 10$	
$\ln(2t+5) = \frac{10}{4} = 2.5$	divided by 4
$2t+5 = e^{2.5}$	exponentiated both sides
$2t = e^{2.5} - 5$	subtracted 5
$t = \frac{e^{2.5}-5}{2} \approx 3.59125$	divided by 2

4. Answer:

$\sqrt{x-3} + 7 = 12$	
$\sqrt{x-3} = 5$	subtract 7
$x - 3 = 25$	square both sides
$x = 28$	add 3

5. Answer:

$350000(1-e^{-0.07t})$	$= 200000$	
$1-e^{-0.07t}$	$= \frac{200000}{350000} \approx 0.571428$	divided by 350000
$-e^{-0.07t}$	≈ -0.428571	subtracted 1
$e^{-0.07t}$	≈ 0.428571	divided by -1
$-0.07t$	$\approx \ln(0.428571) \approx -0.84729786$	take ln() of both sides
t	$\approx \frac{-0.84729786}{-0.07} \approx 12.1043$	divided by -0.07

6. Answer:

$3-\frac{5}{x}$	$= 1$	
$3x-5$	$= x$	multiply both sides by x
$3x$	$= x+5$	add 5
$2x$	$= 5$	subtract x
x	$= \frac{5}{2} = 2.5$	divide by 2

7. Answer:

$1200(7)^{4r}$	$= 14400$	
$(7)^{4r}$	$= \frac{14400}{1200} = 12$	divide by 1200
$\ln(7^{4r})$	$= \ln(12)$	taking ln() of both sides
$4t\ln(7)$	$= \ln(12)$	bringing down exponent
t	$= \frac{\ln(12)}{4\ln(7)} \approx 0.31925$	divide by $4\ln(7)$

8. Answer:

$(2^{3x}-1)^5+10$	$= 40$	
$(2^{3x}-1)^5$	$= 30$	subtract 10
$2^{3x}-1$	$= (30)^{1/5} \approx 1.9743505$	fifth root
2^{3x}	≈ 2.9743505	add 1
$\ln(2^{3x})$	$\approx \ln(2.9743505) \approx 1.09002569$	take ln() of both sides
$3x\ln(2)$	≈ 1.09002569	bring down exponent
x	$\approx \frac{1.09002569}{3\ln(2)} \approx 0.524192$	divide by $3\ln(2)$

9. Answer:

$x+\frac{2}{x}$	$= 10$	
x^2+2	$= 10x$	multiply by x (Quadratic! Make one side zero!)
$x^2-10x+2$	$= 0$	subtract $10x$
x	$= \frac{-(-10)\pm\sqrt{(-10)^2-4(1)(2)}}{2(1)}$	quad. form. $a=1$, $b=-10$, $c=2$
x	$\approx \frac{10\pm 9.5916630566}{2}$	simplifying

We get the two answer $x \approx \frac{10-9.5916630566}{2} \approx 0.204168$ or $x \approx \frac{10+9.5916630566}{2} \approx 9.7958315$.

10. Answer:

5	$= \frac{60}{e^x+1}$	
$5(e^x+1)$	$= 60$	multiply by (e^x+1)
e^x+1	$= \frac{60}{5} = 12$	divide by 5
e^x	$= 11$	subtract 1
x	$= \ln(11) \approx 2.397895$	take ln() of both sides

Solving Equations Brief Overview

One of the main algebraic difficulties students have at this point in the course is solving equations (in almost every question you are solving when the derivatives are equal to zero). There is no way I can cover all of algebra in this short sheet, but let me discuss some common situations and errors.
First, a bit of terminology is necessary:

- A mathematical *expression* is a formula involving numbers and variables (but there is NO equals sign). For example: $5x + 20$ and $e^x + 10\sqrt{x^2 - 1}$ are expressions. We don't solve expressions, that doesn't make sense.
- A *function* is an expression that we can evaluate at a particular variable. In which case, we typically give the function a name (we say we are defining the function). For example: $f(x) = 5x + 20$ or $g(x) = e^x + 10\sqrt{x^2 - 1}$ are function definitions. On the left is a function name and an indication of the input $f(x)$ or $g(x)$ and the equal sign here is being use to tell us the formula we use to compute that rule. We don't solve functions, that doesn't make sense. So even though there is an equal sign, we aren't ever solving $f(x) = 5x + 20$. That is just the definition of a rule for how to compute $f(x)$.
- An *equation* sets two expressions equal to each other. In this case we are trying to solve. For example $5x + 20 = 14$ or $x^2 = 2x + 5$.

Hopefully that is clear. So it only makes sense to be solving an equation if there is:

- An equal sign in each step! (Don't let the equal sign disappear!).
- Expressions (numbers and/or variables) on both sides of the equal sign at each step!

Solving 101

The goal in solving an equation is to get the variable by itself. We do this using a series of inverses. Here are all the basic inverses and how we use them:

The Inverses

Addition/Subtraction:
To solve $x + 5 = 7$, we subtract 5 from both sides to get $x = 2$.
And to solve $x - 3 = 10$, we add 3 to both sides to get $x = 13$.
Multiplication/Division:
To solve $6x = 30$, we divide 6 from both sides to get $x = 5$.
And to solve $\frac{x}{7} = 11$, we multiply 7 to both sides to get $x = 77$.
Powers/Roots:
To solve $x^3 = 10$, we take the cube root of both sides to get $x = 10^{1/3}$.
And to solve $x^{1/5} = 2$, we take the 5th power of both sides to get $x = 2^5$.
Important Notes about powers/roots:

- The bottom of a fractional exponent is a root and the top of a fractional exponent is a power. So for example, $x^{3/4}$ is asking you to take the 3rd power and 4th root of x. So if you are solving $x^{3/4} = 10$, then the inverse would be taking the 3rd root and 4th power, to get $x = 10^{4/3}$.
- When you start with an EVEN power, and you take the root of both sides, you must include $\pm$ to account for both possible solutions. For example, to solve $x^{10} = 3$, you should write $x = \pm 3^{1/10}$.

Exponentials/Logarithms:
To solve $e^x = 7$, we take the natural logarithm of both sides to get $x = \ln(7)$.
And to solve $\ln(x) = 20$, we take the exponential of both sides to get $x = e^{20}$.
Important Note about logs: Using the rules that $\ln(a^b) = b\ln(a)$ and $\ln(ab) = \ln(a) + \ln(b)$, you can solve many more equations that have other bases as well. For example, to solve $3^x = 40$, you take the natural logarithm of both sides to get $\ln(3^x) = \ln(40)$, then rewrite $x\ln(3) = \ln(40)$, then divide to get $x = \ln(40)/\ln(3)$.

General Strategies and Special Cases

1. If there are fractions, clear the denominator. Meaning multiply everything on both sides by the the denominator to get rid of it.
   For example, to solve $\frac{1}{x} - \frac{10}{x^2} = 0$. Start by multiplying by everything by x^2 which gives $x - 10 = 0$, so $x = 10$. Always do this first!!!
2. Get all your x's to the same side from the beginning. Then try to factor, combine or simplify your expressions involving x if you can.
   For example, to solve $x^2 - 3x = 0$, I would look to factor to get $x(x - 3) = 0$, so that my solution is $x = 0$ or $x = 3$.
3. If you end up with a quadratic equation that has all three terms and you can't factor, then you use the quadratic formula.
   For example, to solve $x^2 - 10x + 2 = 0$, I tried to factor but I couldn't quickly see a way to factor so I used $x = \frac{10 \pm \sqrt{100-4(2)}}{2}$.
4. Anything we ask you to solve can be solve by combinations of the simple rules above. Anything beyond these simple rules requires a numerical solution (done with a computer or calculator, which we don't ask you to do).