1. (10 points) Don't do your work in your head, do it on the page (show me all intermediate steps you use). You do NOT have to simplify your final answer. Put a box around your final answer.

(a) Find $f'(x)$, if $f(x) = (4x^3 - 1)^6 \cdot \left(\frac{x^3}{2} + 7x\right)$ $= (4x^3-1)^6\left(\frac{1}{2}x^3+7x\right)$

$$\boxed{f'(x) = \underbrace{(4x^3-1)^6}_{F}\underbrace{\left(\tfrac{3}{2}x^2+7\right)}_{S'} + \underbrace{6(4x^3-1)^5 \cdot 12x^2}_{F'}\underbrace{\left(\tfrac{1}{2}x^3+7x\right)}_{S}}$$

(b) Find $\frac{dy}{dx}$, if $y = \frac{4x}{5} + 7\sqrt{x^2+4}$. $= \frac{4}{5}x + 7(x^2+4)^{1/2}$

$$\boxed{\frac{dy}{dx} = \frac{4}{5} + \frac{7}{2}(x^2+4)^{-1/2} \cdot 2x} = \frac{4}{5} + \frac{7x}{\sqrt{x^2+4}}$$

(c) The height of a balloon is given by: $h(t) = \frac{t^2 + 4\sqrt{t}}{t^3 + 1}$, where distance is in feet and time is in seconds. Find the instantaneous speed of the balloon at $t = 1$ second. (simplify your numbers and include the units for your final answer).

$$h'(t) = \frac{(t^3+1)(2t+2t^{-1/2}) - (t^2+4t^{1/2})(3t^2)}{(t^3+1)^2}$$

$$h'(1) = \frac{(2)(4) - (5)(3)}{(2)^2} = \frac{8-15}{4} = -\frac{7}{4}$$

$$\boxed{-\frac{7}{4}\ \frac{ft}{sec}} = -1.75$$

2. (6 pts) Assume $f(x) = \frac{45}{x} + 5x$ $\quad = 45x^{-1} + 5x$

(a) Find the second derivative $f''(x)$.

$f'(x) = -45x^{-2} + 5$

$\boxed{f''(x) = 90x^{-3}} = \frac{90}{x^3}$

★ (b) Solve to find all values of x at which the slope of the tangent line to $f(x)$ is 0.

WANT

$-45x^{-2} + 5 \overset{?}{=} 0$

$\Rightarrow -\frac{45}{x^2} = -5$

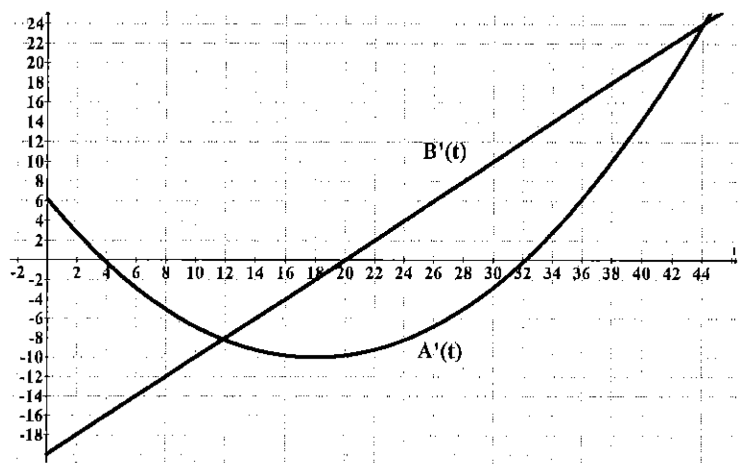
$-45 = -5x^2$

$9 = x^2$

$\boxed{x = \pm 3}$

3. (10 pts)

Two balloons, A and B, start next to each other at 500 feet and are moving vertically straight up and down. Their **rate of ascent** graphs are shown, where t is in minutes and the rate is in feet/minute. Again, these are the graphs of the *derivatives* of the height functions! Use the graph to answer the following questions as accurately as possible.



(a) For each part below, circle which quantity is **bigger**.

i. balloon A height at $t = 0$ or balloon A height at $t = 1$ or They are equal.

ii. balloon A height at $t = 12$ or balloon B height at $t = 12$ or They are equal.

(b) Give *all* times when $A(t)$, the height graph for balloon A, has a horizontal tangent.

SAME

"$A(t)$ has a horiz. tangent"

"$A'(t)$ cross x-axis"

ANSWER: $t =$ 4, 32 min

(c) Find the longest interval over which balloon A is falling and balloon B is rising.

A' negative $\qquad$ B' positive

ANSWER: $t =$ 20 min to $t =$ 32 min

4. (11 pts) Let $f(x) = 3x^2 - 5x + 1$.

(a) Write out and *expand and completely simplify* the formula for

$$\frac{f(x+h) - f(x)}{h}.$$

$$\frac{[3(x+h)^2 - 5(x+h) + 1] - [3x^2 - 5x + 1]}{h}$$

$$= \frac{3(x^2 + 2xh + h^2) - \cancel{5x} - 5h + \cancel{1} - 3x^2 + \cancel{5x} - \cancel{1}}{h}$$

$$= \frac{\cancel{3x^2} + 6xh + 3h^2 - 5h - \cancel{3x^2}}{h}$$

$$= 6x + 3h - 5$$

ANSWER: $\frac{f(x+h)-f(x)}{h} = \boxed{6x + 3h - 5}$

(b) Find the slope of the secant line to $f(x)$ from $x = 2$ to $x = 5$.

TWO WAYS

$\boxed{1}$ $x = 2,\ h = 3$ ABOVE $\Rightarrow$ $6(2) + 3(3) - 5 = 12 + 9 - 5 = \boxed{16}$

OR

$\boxed{2}$ $\frac{f(5) - f(2)}{5 - 2} = \frac{(3(5)^2 - 5(5) + 1) - (3(2)^2 - 5(2) + 1)}{3} = \frac{51 - 3}{3} = \boxed{16}$

ANSWER: secant slope = $\boxed{16}$

(c) Find the equation for the tangent line to $f(x)$ at $x = 2$.

TWO WAYS $\boxed{1}$ $h \to 0 \Rightarrow f'(x) = 6x - 5$

$\boxed{2}$ USE SHORTCUTS $\Rightarrow f'(x) = 6x - 5$

HEIGHT $= f(2) = 3(2)^2 - 5(2) + 1 = 12 - 10 + 1 = 3$

SLOPE $= f'(2) = 6(2) - 5 = 12 - 5 = 7$

$$y = 7(x - 2) + 3 = 7x - 14 + 3 = 7x - 11$$

ANSWER: $\boxed{y = 7(x-2) + 3}$

13

5. (14 pts) You are in charge of marketing a new electronic gadget.
From analyzing the market, you find the demand curve is

$$p = 53 - 2x,$$

where p is price (in dollars per item) and x is in **hundred** items.
From the manufacturer, the total cost function is given by

$$TC(x) = 60 + 5x - 2x^2 + x^3,$$

where x is in hundred items and $TC(x)$ is in hundred dollars. Keep final answers accurate to two digits after the decimal (*i.e.* to the nearest item or nearest dollar).

(a) Find formulas for TR, MR and MC.

$TR(x) = 53x - 2x^2$

$MR(x) = 53 - 4x$

$MC(x) = 5 - 4x + 3x^2$

(b) Find the largest interval on which total revenue is increasing.

TR

FIND WHEN $MR(x) \stackrel{?}{=} 0$

$53 - 4x \stackrel{?}{=} 0 \Rightarrow x = 13.25$

ANSWER: from x= 0 to x = 13.25 hundred items

(c) Find the quantity at which marginal cost is lowest.

FIND WHEN $MC'(x) = -4 + 6x \stackrel{?}{=} 0$

$x = 4/6 = 0.66\ldots$

ANSWER: x = 0.67 hundred items

(d) What selling *price* should you use to maximize profit? (Hint: First, find the quantity that maximizes profit).

$MR(x) \stackrel{?}{=} MC(x)$

$53 - 4x = 5 - 4x + 3x^2$

$\Rightarrow 48 = 3x^2$

$\Rightarrow 16 = x^2$

$\Rightarrow x = \pm 4$

$x = 4$ hundred items

$p = 53 - 2(4) = 45$

ANSWER: selling price = 45 dollars/item