6.4 Work

This is a review and practice sheet of some of the more common work examples from homework and old exams. The concepts of this section are:

- We define: Work = Force · Distance (for a **constant** force applied through a given distance)
- Given a scenario where the *force*, or the *distance*, **change**, we can still measure work as follows:
  1. Clearly label the scenario. Break it into subdivisions and look at a "typical" subdivision.
  2. Find the pattern for *force* and *distance*.
  3. Integrate: Work $= \int$ Force · Distance .

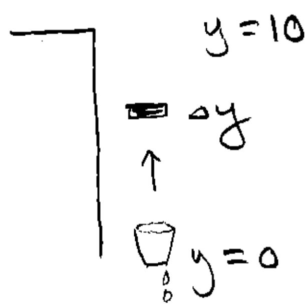
Leaky Bucket: Here are examples where the weight (force) is getting lighter as an object is being lifted. In all these cases, we label, then find a formula for force, and integrate. Since the force chances every instant, then distance each force travels is Δx. So in all these problems:

- Force = $f(x)$ (the pattern we find)
- Distance = dx

Examples:

1. A water bucket is being lifted 10 feet at a constant rate. The bucket is leaking water at a constant rate. It weighs 50 lbs initially and 38 lbs when it gets to the top. How much work is done in lifting the bucket?
2. A sandbag is being lifted 3 meters at a constant rate. The bag is leaking sand at a constant rate. It weighs 21 N initially and 15 N when it gets to the top. How much work is done in lifting the sandbag?
3. A small rocket is blasting off from the ground. As it burns through fuel, the rocket gets lighter. The weight of the rocket when it is x meters off the ground is given by $F(x) = 40 + 50e^{-x/2}$ in Newtons. Find the work done by the rocket in the first 8 meters as it blasts off from the ground.

$\boxed{1}$



when $y=0$, force $= 50$
when $y=10$, force $= 38$
"constant rate" $\Rightarrow$ LINEAR

$\hookrightarrow m = \dfrac{50-38}{0-10} = \dfrac{12}{-10} = -\dfrac{6}{5}$

Force $= f(y) = -\frac{6}{5}(y-0)+50$

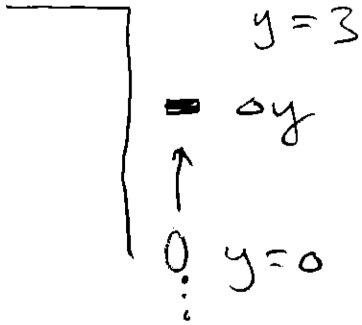
$\boxed{f(y) = -\frac{6}{5}y + 50 \text{ lbs}}$

dist $= \Delta y$

$$\text{Work} = \int_0^{10} \underbrace{-\frac{6}{5}y + 50}_{\text{force}}\, \underbrace{dy}_{\text{dist}} = -\frac{6}{10}y^2 + 50y \Big|_0^{10} = -60 + 500$$

$$= \boxed{440 \text{ ft-lbs}}$$

$\boxed{2}$



when $y=0$, force $= 21$ N
when $y=3$, force $= 15$ N
} constant rate

$$m = \frac{21-15}{0-3} = \frac{6}{-3} = -2$$

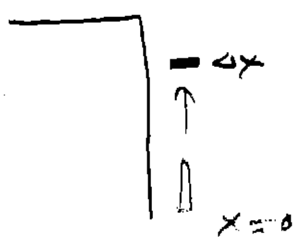
$$\text{force} = f(y) = -2(y-0)+21$$

$$\boxed{f(y) = -2y+21 \text{ N}}$$

$$\text{dist} = \Delta y$$

$$\text{Work} = \int_0^3 -2y+21\,dy = -y^2+21y\Big|_0^3 = -9+63 = \boxed{54 \text{ Joules}}$$

$\boxed{3}$



$$\overset{\text{Force}}{\curvearrowright}\ F(x) = 40+50e^{-\frac{1}{2}x} \text{ Newtons} \leftarrow \text{GIVEN}$$

$$\text{dist} = \Delta x$$

$$\text{Work} = \int_0^8 40+50e^{-\frac{1}{2}x}\,dx$$

$$= 40x - 100e^{-\frac{1}{2}x}\Big|_0^8$$

$$= (320-100e^{-4}) - (0-100)$$

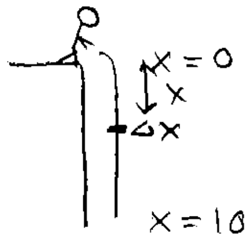
$$= \boxed{420 - \frac{100}{e^4} \text{ Joules}}$$

Lifting a Chain/Cable: Typically, I label the top $x = 0$. If the weight is p lbs/ft, then a small segment of cable with length Δx ft will have a force (weight) of $p\Delta x$ pounds. And if it is being lifted to the top, then the distance will be x.

- Force = pdx
- Distance = formula for the distance moved by a subdivision at x (depends on how you label the problem).

1. A 10 foot chain is hanging over the side of a building. The chain has a density of 3 lbs/ft. How much work is done to pull it to the top of the building?
2. An 8 foot chain weights a total of 120 pounds (so 120/8 =15 lbs/ft) and is hanging over the side of a building. How much work is done to pull it half way up?
3. A 50 foot rope weighs 2 lbs/ft. One end of it has been lifted to a window 15 feet above the ground and the rest is lying coiled on the ground. What is the work needed to pull the whole rope through the window?

$\boxed{1}$

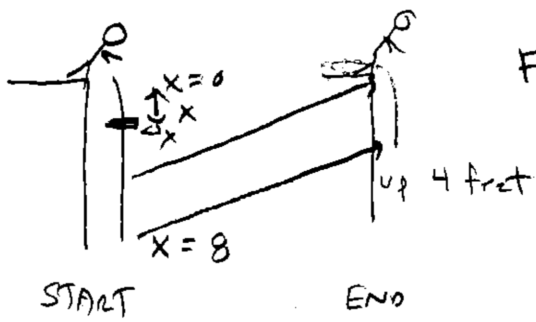


FORCE $= 3\Delta x$ lbs

DIST $= x$ ft

WORK $= \int_0^{10} x \, 3\,dx = \frac{3}{2}x^2 \Big|_0^{10} = \frac{300}{2} = \boxed{150 \text{ ft-lbs}}$

$\boxed{2}$



For $0 \le x \le 4$: FORCE $= 15\Delta x$ lbs

DIST $= x$ ft

WORK $= \int_0^4 x \, 15\,dx = \frac{15}{2}x^2$

$= \frac{15}{2} \cdot 16$

$= 120$ ft-lbs

For $4 \le x \le 8$:

$\left.\begin{array}{l}\text{FORCE} = 15\Delta x \\ \text{DIST} = 4 \text{ ft}\end{array}\right\}$ WORK $= \int_4^8 4 \cdot 15\,dx$

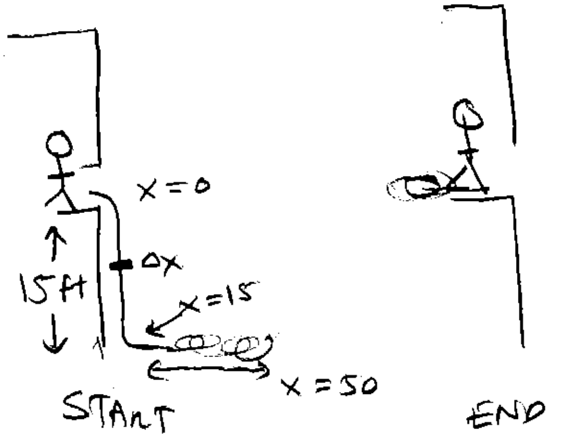
$= 60x \Big|_4^8$

$= 60(8-4) = 240$ ft-lbs

TOTAL $= 120 + 240$

$= \boxed{360 \text{ ft-lbs}}$

$\boxed{3}$



FOR $0 \le x \le 15$:

FORCE $= 2\Delta x$

DIST $= x$

WORK $= \int_0^{15} x \cdot 2 \, dx$

$= x^2 \Big|_0^{15} = 225$ ft-lbs

FOR $15 \le x \le 50$

FORCE $= 2\Delta x$ lbs

DIST $= 15$ ft

WORK $= \int_{15}^{50} 15 \cdot 2 \, dx$

$= 30x \Big|_{15}^{50}$

$= 30(50-15)$

$= 1050$ ft-lbs

TOTAL $= 1050 + 225$

$= \boxed{1275 \text{ ft-lbs}}$

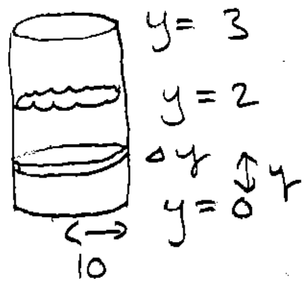
Pumping

Typically, the top of the tank is $y = b$ for some constant b. If the weight is p lbs/ft^3, then a think horizontal subdivision of the liquid with thickness Δy will have force (weight) equal to p(*area of horizontal cross-section*)Δy. And if it is being lifted to the top, then the distance will be $b - y$.

- Force = p(*area of horizontal cross-section*)dy
- Distance = formula for the distance moved by a subdivision at y (typically $b - y$).

1. A small cylindrical pool has a radius of 10 ft, the sides are 3 ft high, and the depth of the water is 2 ft. How much work (in ft-lb) is required to pump all of the water out over the side of the pool? (Water weighs 62.5 lb/ft^3.)
2. A tank has the shape of an open-top hemisphere with radius 10 m that is full of water with density 1000 kg/m^3. Set up an integral which computes the work required to empty the tank by pumping all of the water to the top of the tank. DO NOT EVALUATE THIS INTEGRAL.
3. A conical tank is 4 meters high and has a radius of 1 meter at the top (~~see picture~~). The bottom 3 meters are full of water. How much work is required to pump out all the water over the rim of the tank? (Recall that the mass density of water is 1000kg/m^3 and the acceleration due to gravity is 9.8m/sec^2.)

$\boxed{1}$

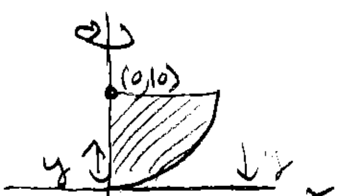


For $0 \le y \le 2$: Force $= \overset{\text{lbs/ft}^3}{62.5} \cdot \pi(10)^2 \cdot \Delta y$ lbs

DIST LIFTED $= 3 - y$ ft

$$\text{Work} = \int_0^2 (3-y)\, 62.5 \cdot \pi(10)^2\, dy$$
$$= 6250\pi \int_0^2 3 - y\, dy$$
$$= 6250\pi \left[3y - \tfrac{1}{2}y^2 \Big|_0^2 \right]$$
$$= 6250\pi\left[(6-2) - 0\right] = \boxed{25{,}000 \text{ ft-lbs}}$$

$\boxed{2}$



$x^2 + (y-10)^2 = 10^2$

$\Rightarrow x = \sqrt{100 - (y-10)^2}$

For $0 \le y \le 10$: Force $= \overset{\text{N/m}^3}{9800} \cdot \pi\left(\sqrt{100-(y-10)^2}\right)^2 \Delta y$ N

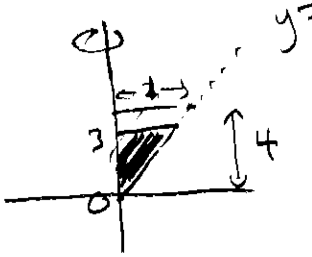
DIST $= 10 - y$

$$\text{Work} = \int_0^{10} (10-y)\, 9800\pi \left(100 - (y-10)^2\right) dy$$

STOP

$$= 9800\pi \int_0^{10} (10-y)(100 - y^2 + 20y - 100)\, dy$$
$$= 9800\pi \int_0^{10} +10y^2 + 200y + y^3 - 20y^2\, dy$$
$$= 9800\pi \left(-\tfrac{30}{3}y^3 + \tfrac{200}{2}y^2 + \tfrac{1}{4}y^4 \Big|_0^{10} \right) =$$
$$= 24{,}500{,}000\pi \text{ Joules}$$

$\boxed{3}$



For $0 \le y \le 3$:

$$\text{FORCE} = 9800 \cdot \pi \left(\tfrac{1}{4}y\right)^2 \Delta y$$

$$\text{DIST} = 4 - y$$

$$\text{WORK} = \int_0^3 (4-y)\, 9800\pi \tfrac{1}{16} y^2 \, dy$$

$$= \frac{9800\pi}{16} \int_0^3 4y^2 - y^3 \, dy$$

$$= \frac{1225}{2}\pi \left(\tfrac{4}{3}y^3 - \tfrac{1}{4}y^4 \Big|_0^3 \right)$$

$$= \frac{1225}{2}\pi \left(36 - \frac{81}{4} \right)$$

$$= \boxed{\frac{77175}{8}\pi \text{ JOULES}}$$