1. (10 pts) Evaluate the integrals.

(a) $\displaystyle\int \frac{e^{2x}}{(e^x+1)^2+1}\,dx$

$u = e^x + 1 \Rightarrow e^x = u - 1$

$du = e^x\,dx$

$dx = \frac{1}{e^x}\,du$

$$\int \frac{e^{2x}}{u^2+1}\,\frac{1}{e^x}\,du$$

$$\int \frac{u-1}{u^2+1}\,du = \int \frac{u}{u^2+1}\,du - \int \frac{1}{u^2+1}\,du$$

$v = u^2+1$

$dv = 2u\,du$

$du = \frac{1}{2u}\,dv$

$$= \int \frac{\cancel{u}}{v}\,\frac{1}{2\cancel{u}}\,dv - \int \frac{1}{u^2+1}\,du$$

$$= \frac{1}{2}\ln|v| - \tan^{-1}(u) + C$$

$$= \boxed{\frac{1}{2}\ln\left((e^x+1)^2+1\right) - \tan^{-1}(e^x+1) + C}$$

$$\int \frac{\sin(\ln(x+2))}{(2x+4)\cos^2(\ln(x+2))}\,dx$$

(b) $\displaystyle\int \cancel{\frac{\sec(\ln(\sqrt{x}+2))\tan(\ln(\sqrt{x}+2))}{(2x+4)\sqrt{x}}}\,dx$

$u = \ln(x+2)$

$du = \frac{1}{x+2}\,du$

$dx = (x+2)\,du$

$$\int \frac{\sin(u)}{2\cancel{(x+2)}\cos^2(u)}\,\cancel{(x+2)}\,du$$

$$= \frac{1}{2}\int \sec(u)\tan(u)\,du$$

$$= \frac{1}{2}\sec(u) + C$$

$$= \boxed{\frac{1}{2}\sec(\ln(x+2)) + C}$$

OR

$V = \cos(u)$

$dv = -\sin(u)\,du$

$du = \frac{1}{-\sin(u)}\,dv$

$$= \frac{1}{2}\int \frac{\cancel{\sin(u)}}{v^2}\,\frac{1}{-\cancel{\sin(u)}}\,dv$$

$$= -\frac{1}{2}\int v^{-2}\,dv$$

$$= -\frac{1}{2}\,\frac{1}{-1}\,v^{-1} + C$$

$$= \boxed{\frac{1}{2}\,\frac{1}{\cos(\ln(x+2))} + C}$$

SAME

2. (6 pts) Evaluate $\int_1^3 \left|\frac{x^2-4}{x^3}\right| dx$

ZEROS: $x^2-4=0$

$x=\pm 2$

$\int_1^2 \frac{x^2-4}{x^3}\,dx = \int_1^2 \frac{1}{x} - 4x^{-3}\,dx = \ln|x| - \frac{4}{-2}x^{-2}\Big|_1^2$

$= \left(\ln(2) + 2\frac{1}{2^2}\right) - \left(\ln(1) + 2\frac{1}{(1)^2}\right) = \ln(2) + \frac{1}{2} - 2$

$= \ln(2) - 3/2$

$\int_2^3 \frac{x^2-4}{x^3}\,dx = \ln|x| + \frac{2}{x^2}\Big|_2^3 = \left(\ln(3) + \frac{2}{3^2}\right) - \left(\ln(2) + \frac{2}{2^2}\right)$

$= \ln(3) + \frac{2}{9} - \ln(2) - \frac{1}{2}$

$= \ln(3) - \ln(2) - 5/18$

27/18

$\int_1^3 \left|\frac{x^2-4}{x^3}\right| dx = -\left(\ln(2) - 3/2\right) + \ln(3) - \ln(2) - 5/18$

$= \ln(3) - 2\ln(2) + \frac{22}{18}$

$= \boxed{\ln\left(\frac{3}{4}\right) + \frac{11}{9}}$

3. (6 pts) Evaluate the limit by first recognizing an integral that it defines and then evaluate the integral:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(1+\frac{2i}{n}\right) e^{\left(1+\frac{2i}{n}\right)^2}$$

$\Delta x = \frac{b-a}{n} = \frac{2}{n} \Rightarrow b-a=2$

$x_i = a + i\Delta x = 1 + \frac{2i}{n} \Rightarrow a=1$ so $b=3$

$\int_1^3 x e^{x^2}\,dx$

$u = x^2$

$du = 2x\,dx$

$dx = \frac{1}{2x}du$

$x=1 \rightarrow u=1$

$x=3 \rightarrow u=9$

$\frac{1}{2}\int_1^9 e^u\,du$

$\boxed{\frac{1}{2}\left(e^9 - e\right)}$

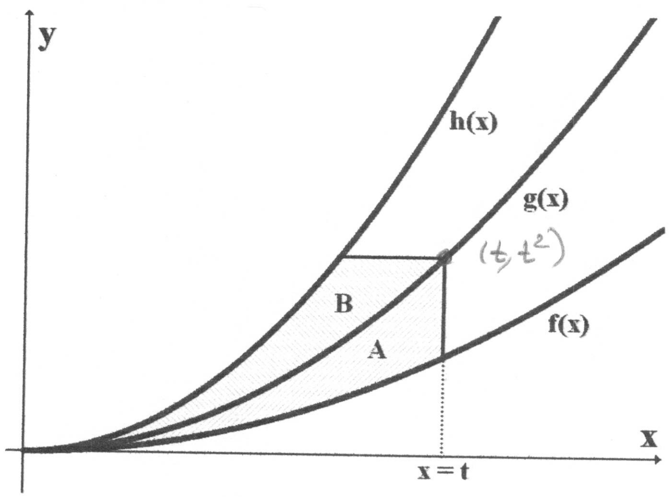
4. (5 pts) Let $f(x) = \int_{2-x}^{x^2} t e^{\sqrt[3]{t}}\,dt$. ~~Find all critical numbers of $f(x)$.~~ Find $f'(1)$

$$f(x) = \int_{2-x}^{a} t e^{\sqrt[3]{t}}\,dt + \int_{a}^{x^2} t e^{\sqrt[3]{t}}\,dt \quad , a \text{ any constant}$$

$$f'(x) = -(2-x)e^{\sqrt[3]{2-x}}(-1) + x^2 e^{\sqrt[3]{x^2}} \cdot 2x$$

$$f'(1) = (2-1)e^1 + 2e^1 = \boxed{3e}$$

5. (10 pts) Let $f(x) = \frac{1}{2}x^2$, $g(x) = x^2$, and $h(x) = ax^2$ for some constant $a > 1$.
A vertical line is drawn at some number $x = t$ and a horizontal ~~line~~ line is drawn through the point where $x = t$ intersects $g(x)$. Let A and B be the regions as shown.
If the areas of A and B are the same for all values of t, what is the value of a?



$y = x^2 \Rightarrow x = \sqrt{y}$

$y = ax^2 \Rightarrow x = \frac{1}{\sqrt{a}}\sqrt{y}$

$$A = \int_0^t x^2 - \frac{1}{2}x^2\,dx$$

$$= \frac{1}{2}\int_0^t x^2\,dx$$

$$= \frac{1}{6}x^3 \Big|_0^t$$

$$\boxed{A = \frac{1}{6}t^3}$$

$$B = \int_0^{t^2} \sqrt{y} - \frac{1}{\sqrt{a}}\sqrt{y}\,dy = \left(1 - \frac{1}{\sqrt{a}}\right)\int_0^{t^2} \sqrt{y}\,dy$$

$$= \left(1 - \frac{1}{\sqrt{a}}\right)\frac{2}{3}y^{3/2}\Big|_0^{t^2}$$

$$\boxed{B = \left(1 - \frac{1}{\sqrt{a}}\right)\frac{2}{3}t^3}$$

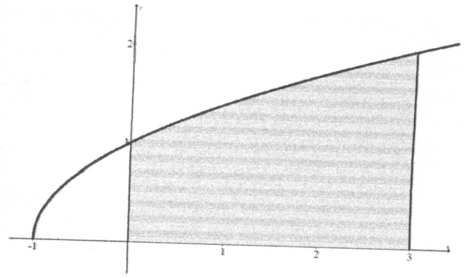
$A = B \Rightarrow \quad \frac{1}{6}\cancel{t^3} = \left(1 - \frac{1}{\sqrt{a}}\right)\frac{2}{3}\cancel{t^3}$

$$\frac{1}{4} = 1 - \frac{1}{\sqrt{a}}$$

$$\frac{1}{\sqrt{a}} = 3/4 \Rightarrow \sqrt{a} = 4/3 \Rightarrow \boxed{a = \frac{16}{9}}$$

6. (13 points)

Consider the region, R, bounded by the curve $y = \sqrt{x+1}$, the x-axis, and between $x = 0$ and $x = 3$. A picture of this region is given at right.



(a) (6 pts) Using both methods (cylindrical shells and cross-sectional slicing), set up two integrals (DO NOT EVALUATE) for the volume of the solid obtained by rotating the region R about the **horizontal line** $y = -3$.

i. Cross-sectional slicing: WASHER

$$\boxed{\int_0^3 \pi\left(\sqrt{x+1}+3\right)^2 - \pi(3)^2\,dx}$$

ii. Cylindrical Shells:

$$\boxed{\int_0^1 2\pi(y+3)\,3\,dy + \int_1^2 2\pi(y+3)\left(3-(y^2-1)\right)dy}$$

(b) (7 pts) Find the volume of the solid obtained by rotating the region R about the **vertical line** ~~[illegible]~~. Set up the integral AND evaluate.

$x=5$

$5-x$

$$\int_0^3 2\pi(5-x)\sqrt{x+1}\,dx$$

$u = x+1$

$du = dx$

$x = u-1$

$$2\pi\int_1^4 [5-(u-1)]u^{1/2}\,du$$

$$2\pi\int_1^4 (6-u)u^{1/2}\,du$$

$$2\pi\int_1^4 6u^{1/2} - u^{3/2}$$

$$2\pi \quad 6\tfrac{2}{3}u^{3/2} - \tfrac{2}{5}u^{5/2}\Big|_1^4$$

$$2\pi\left[\left(4\cdot4^{3/2} - \tfrac{2}{5}4^{5/2}\right) - \left(4(1)^{3/2} - \tfrac{2}{5}(1)^{5/2}\right)\right]$$

$$2\pi\left[32 - \tfrac{64}{5} - 4 + \tfrac{2}{5}\right]$$

$$2\pi\left[28 - \tfrac{62}{5}\right] = 2\pi\left[\tfrac{140}{5} - \tfrac{62}{5}\right] = 2\pi\cdot\tfrac{78}{5} = \boxed{\tfrac{156\pi}{5}}$$

≈ 98.01769

7. (10 pts)You are standing on a bridge over a walkway because you want to drop a small water balloon on your math instructor's head.

You are holding out the balloon exactly 65 feet above the ground. Your instructor is 6 feet tall and is walking at a constant speed of 3 feet/sec. Assume the water balloon falls at a constant acceleration of 32 ft/sec^2. (The picture is not to scale). Let x be the indicated location of the instructor at the instant you release the balloon.

(a) Where should your instructor be located at the instant you **drop** the balloon so that the balloon lands directly on top of his head?

$a(t) = -32$

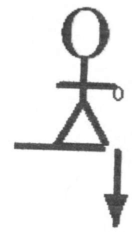
$v(t) = -32t + C$

DROP: $v(0) = 0 \Rightarrow C = 0$

$s(t) = -16t^2 + D$

$s(0) = 65 \Rightarrow D = 65$

$s(t) = -16t^2 + 65$

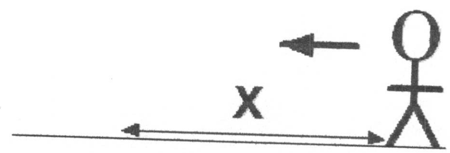


WANT: $-16t^2 + 65 = 6 \Rightarrow t^2 = \frac{59}{16}$

$\boxed{t = \frac{\sqrt{59}}{4}}$

3 ft/sec $\Rightarrow$ $x = 3\,\text{ft/sec} \cdot \frac{\sqrt{59}}{4}\,\text{sec}$

$\boxed{x = \frac{3}{4}\sqrt{59}\ \text{ft}} \approx 5.760859$ ft



(b) You weren't paying attention and now your instructor is at $x = 2$ feet away from the location directly below the water balloon. At what initial velocity downward must you throw the balloon, at that instant, in order for it to land on top of his head?

$a(t) = -32$

$v(t) = -32t + C$

$C = v(0) = ???$

$s(t) = -16t^2 + Ct + D$

$s(0) = 65 \Rightarrow D = 65$

$s(t) = -16t^2 + Ct + 65$

TIME FOR INSTRUCTOR

$2\text{ ft} \cdot \frac{1\text{ sec}}{3\text{ ft}} = \frac{2}{3}$ sec

WANT $s(t) = 6$ when $t = \frac{2}{3}$ sec $\Rightarrow s(\frac{2}{3}) = 6$

$-16\left(\frac{2}{3}\right)^2 + C\left(\frac{2}{3}\right) + 65 = 6$

$-\frac{64}{9} + \frac{2}{3}C = -59 = -\frac{531}{9}$

$\frac{2}{3}C = -\frac{467}{9} \Rightarrow$ $\boxed{C = -\frac{467}{6} = -77.8\overline{3}\ \text{ft/sec} = v(0)}$