5.5 Substitution

The substitution rule says, if $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du.$$

We often remember this, by writing $du = g'(x)dx$. Here let me discuss some common questions.

1. This is really just the chain rule from differential calculus. Remember that chain rule says that

$$\text{If } \frac{d}{du}(F(u)) = f(u), \text{ then } \frac{d}{dx}(F(g(x))) = f(g(x))g'(x).$$

   In terms of integrals, this says

$$\text{If } F(u) + C = \int f(u)\ du, \text{ then } F(g(x)) + C = \int f(g(x))g'(x)\ dx.$$

   So we really are just undoing the chain rule and the notation $u = g(x)$ and $du = g'(x)dx$ helps us see this in an organized way.

2. In terms of the definition of the derivative remember that

$$\int_a^b f(g(x))g'(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(g(x_i))g'(x_i)\Delta x.$$

   If we want to change the variable to $u = g(x)$. Then we are in fact 'transforming' the function, and interval, into a new coordinates system. Instead of the xy-coordinate system, it will be the uy-coordinate system. So the question becomes how does that effect the rectangles and areas we are computing.

   Graph $u = g(x)$ and look at one of your subdivisions, then label $u_{i-1} = g(x_{i-1})$ and $u_i = g(x_i)$. The slope between these two points on the graph of $u = g(x)$ would be $\frac{u_i - u_{i-1}}{x_i - x_{i-1}} = \frac{\Delta u}{\Delta x}$. If the interval is small, then this slope would be very close to the slope of the tangent line $g'(x_i)$. Thus, we have $\frac{\Delta u}{\Delta x} \approx g'(x_i)$, which we can write as $\Delta u \approx g'(x_i)\Delta x$, with increasing accuracy as Δx gets smaller. (I am just giving a plausibility argument, this is not mathematically rigorous.)

   In any case, going back to the definition of the integral we have

$$\int_a^b f(g(x))g'(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(g(x_i))g'(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(u_i)\Delta u = \int_{g(a)}^{g(b)} f(u)\ du.$$

   This is what we are thinking about when we write $u = g(x)$ and $du = g'(x)dx$.

3. In terms of practicalities, when faced with an integral that isn't in our list of integrals we already know, your job is to pick $u = g(x)$ and compute $du = g'(x)dx$ and HOPE! You hope that making the change gives an integral that is in our list of integrals. Here are common things to try:

   (a) Look for $u =$ 'inside function', and $du =$ 'outside function' dx.

   (b) Look for $u = \ln(x)$, with $\frac{1}{x}$ appearing elsewhere in the problem.

   (c) Look for $u =$ 'denominator' with the derivative of u in the numerator.

   (d) Even if things don't line up perfectly, try $u=$ 'inside function', and if not all the x's cancel, then go back and solve for x in your substitution and see if that helps.