EXAM 2 SOLUTIONS

1. Evaluate each of the following integrals.

(a) (5 points) $\displaystyle\int \frac{x^3+1}{x^2-5x}\,dx$

Degree on top is bigger $\Rightarrow$ DIVIDE

$$\begin{array}{r} x+5 \\ x^2-5x \,\overline{)\, x^3+1} \\ -(x^3-5x^2) \\ \hline 5x^2+1 \\ -(5x^2-25x) \\ \hline 25x+1 \end{array}$$

$$\int \frac{x^3+1}{x^2-5x}\,dx = \int x+5+\frac{25x+1}{x^2-5x}\,dx$$

$$= \frac{1}{2}x^2+5x+\int \frac{25x+1}{x(x-5)}\,dx$$

$$\frac{25x+1}{x(x-5)} = \frac{A}{x}+\frac{B}{x-5}$$

$$25x+1 = A(x-5)+Bx$$

$x=0 \Rightarrow 1 = A(-5) \quad A=-\frac{1}{5}$

$x=5 \Rightarrow 126 = B(5) \quad B=\frac{126}{5}$

$$= \frac{1}{2}x^2+5x+\int \frac{-1/5}{x}+\frac{126/5}{x-5}\,dx$$

$$= \boxed{\frac{1}{2}x^2+5x-\frac{1}{5}\ln|x|+\frac{126}{5}\ln|x-5|+C}$$

(b) (5 points) $\displaystyle\int \tan^4(3x)\sec^4(3x)\,dx$ (could start with $u=3x$ but you don't have to)

$$\int \tan^4(3x)\sec^2(3x)\sec^2(3x)\,dx$$

$u=\tan(3x)$

$du = 3\sec^2(3x)\,dx$

$dx = \frac{1}{3\sec^2(3x)}\,du$

$$= \int \tan^4(3x)\left(1+\tan^2(3x)\right)\sec^2(3x)\,dx$$

$$= \int u^4(1+u^2)\,\frac{1}{3}\,du$$

$$= \frac{1}{3}\int u^4+u^6\,du$$

$$= \frac{1}{3}\left[\frac{1}{5}u^5+\frac{1}{7}u^7\right]+C$$

$$= \boxed{\frac{1}{15}\tan^5(3x)+\frac{1}{21}\tan^7(3x)+C}$$

2. Evaluate each of the following integrals.

(a) (5 points) $\displaystyle\int \frac{1}{(x^2+25)^{3/2}}\,dx$

$x = 5\tan(\theta)$

$dx = 5\sec^2(\theta)\,d\theta$

$$= \int \frac{1}{(25\tan^2(\theta)+25)^{3/2}}\, 5\sec^2(\theta)\,d\theta$$

$$= \int \frac{1}{(25\sec^2(\theta))^{3/2}}\, 5\sec^2(\theta)\,d\theta$$

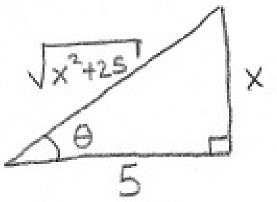
$$= \int \frac{1}{(5\sec(\theta))^3}\, 5\sec^2(\theta)\,d\theta$$

$$= \frac{1}{25}\int \frac{1}{\sec(\theta)}\,d\theta = \frac{1}{25}\int \cos(\theta)\,d\theta$$

$$= \frac{1}{25}\sin(\theta) + C$$

$$= \boxed{\frac{1}{25}\,\frac{x}{\sqrt{x^2+25}} + C}$$

$\tan(\theta) = \frac{x}{5}$



(b) (5 points) $\displaystyle\int_0^\infty xe^{-5x}\,dx$

$u = x \qquad dv = e^{-5x}\,dx$

$du = dx \qquad v = -\frac{1}{5}e^{-5x}$

$$= \lim_{t\to\infty} \int_0^t x e^{-5x}\,dx$$

$$= \lim_{t\to\infty} \left[-\frac{1}{5}xe^{-5x}\Big|_0^t + \int_0^t \frac{1}{5}e^{-5x}\,dx \right]$$

$$= \lim_{t\to\infty} \left[-\frac{1}{5}te^{-5t} + \left(-\frac{1}{25}e^{-5x}\Big|_0^t \right) \right]$$

$$= \lim_{t\to\infty} \left[-\underbrace{\frac{t}{5e^{5t}}}_{\to 0} + -\underbrace{\frac{1}{25e^{5t}}}_{\to 0} + \frac{1}{25} \right]$$

$$= \boxed{\frac{1}{25}}$$

3. Evaluate each of the following integrals.

(a) (5 points) $\int \frac{\cos^3(\ln(x))}{x}\,dx$

$t = \ln(x)$

$dt = \frac{1}{x}dx$

$dx = x\,dt$

$= \int \cos^3(t)\,dt$

$= \int \cos^2(t)\cos(t)\,dt$

$u = \sin(t)$

$du = \cos(t)\,dt$

$= \int (1-\sin^2(t))\cos(t)\,dt$

$= \int 1-u^2\,du$

$= u - \frac{1}{3}u^3 + C$

$= \sin(t) - \frac{1}{3}\sin^3(t) + C$

$= \boxed{\sin(\ln(x)) - \frac{1}{3}\sin^3(\ln(x)) + C}$

(b) (5 points) $\int \frac{x}{\sqrt{40-6x-x^2}}\,dx$

$= \int \frac{x}{\sqrt{40+9-9-6x-x^2}}\,dx$

$x+3 = 7\sin(\theta)$

$dx = 7\cos(\theta)d\theta$

$x = 7\sin(\theta) - 3$

$= \int \frac{x}{\sqrt{49-(x+3)^2}}\,dx$

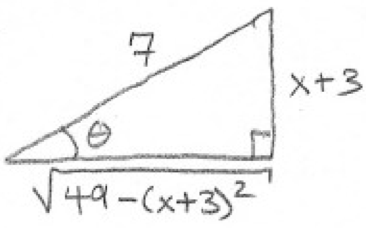
$= \int \frac{7\sin(\theta)-3}{\sqrt{49-49\sin^2(\theta)}}\,7\cos(\theta)d\theta$

$= \int \frac{7\sin(\theta)-3}{\cancel{7}\sqrt{49\cos^2(\theta)}}\,\cancel{7\cos(\theta)}d\theta$

$\sin(\theta) = \frac{x+3}{7}$

$= \int 7\sin(\theta) - 3\,d\theta = -7\cos(\theta) - 3\theta + C$

$= -7\frac{\sqrt{49-(x+3)^2}}{7} - 3\sin^{-1}\left(\frac{x+3}{7}\right) + C$



$= \boxed{-\sqrt{40-6x-x^2} - 3\sin^{-1}\left(\frac{x+3}{7}\right) + C}$

4. (6 points) Use Simpson's Rule with $n = 4$ subintervals to approximate the integral $\int_1^9 \frac{\sin(x)}{x}\, dx$. Write out the correct sum and evaluate all the terms correctly, but you do not have to simplify your answer any further.

$$\Delta x = \frac{9-1}{4} = 2 \qquad x_0 = 1,\ x_1 = 3,\ x_2 = 5,\ x_3 = 7,\ x_4 = 9$$

$$S_4 = \frac{1}{3}\Delta x\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$

$$= \boxed{\frac{2}{3}\left[\frac{\sin(1)}{1} + 4\frac{\sin(3)}{3} + 2\frac{\sin(5)}{5} + 4\frac{\sin(7)}{7} + \frac{\sin(9)}{9}\right]}$$

5. (6 points) Find the average value of $f(x) = \ln(x)$ from $x = 1$ to $x = e$.

$$\frac{1}{e-1}\int_1^e \ln(x)\,dx \qquad u = \ln(x) \quad dv = dx \qquad du = \frac{1}{x}dx \quad v = x$$

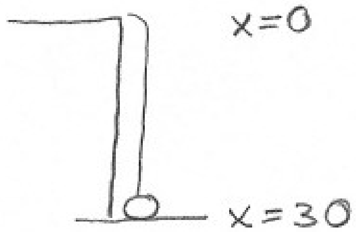
$$\frac{1}{e-1}\left[x\ln(x)\Big|_1^e - \int_1^e x\frac{1}{x}\,dx\right]$$

$$\frac{1}{e-1}\left[(e\ln(e) - 1\ln(1)) - \left(x\Big|_1^e\right)\right]$$

$$\frac{1}{e-1}\left[e - (e-1)\right] = \boxed{\frac{1}{e-1}}$$

6. (8 points) A bag of sand is lifted from the ground to the top of a 30 foot high building at a constant speed with a cable that weighs 2 lb/ft. A small tear in the bag causes sand to slowly pour out. Initially the bag contains 100 pounds of sand, but the sand leaks out at a constant rate and the bag weighs 90 pounds just as it reaches the 30 foot height. How much work is done?
(Hint: Find a linear equation for the force (weight) of the bag of sand at a given height.)

work = work to lift cable
+ work to lift sandbag



work to lift cable

$$= \int_0^{30} 2x\,dx = x^2 \Big|_0^{30} = \boxed{900}$$

work to lift sandbag

$$= \int_0^{30} \text{force}\,dx$$

x = 0 force = 90

x = 30 force = 100

$$\text{force} = mx + b$$

$$m = \frac{100-90}{30-0} = \frac{1}{3}$$

$$b = 90$$

$$\text{force} = \frac{1}{3}x + 90$$

$$= \int_0^{30} \frac{1}{3}x + 90\,dx = \frac{1}{6}x^2 + 90x \Big|_0^{30}$$

$$= \frac{1}{6}900 + 2700 = \boxed{2850}$$

$$\text{Work} = 900 + 2850 = \boxed{3750 \text{ ft-lbs}}$$

Note: you could also take x=0 at the bottom in which case the force equation becomes force $= -\frac{1}{3}x + 100$. (But you get the same value)