1. (12 pts) Evaluate the following integrals

(a) $\int_1^8 3 + \frac{2}{x^{2/3}}\, dx$ $= \int_1^8 3 + 2x^{-2/3}\, dx$

$= 3x + 2\cdot 3x^{1/3} \Big|_1^8$

$= \left(3(8) + 6(8)^{1/3}\right) - \left(3(1) + 6(1)^{1/3}\right)$

$= 24 + 6\cdot 2 - 9$

$= \boxed{27}$

(b) $\int \frac{x^2}{\cos^2(x^3)}\, dx$

$u = x^3$

$du = 3x^2 dx$

$dx = \frac{1}{3x^2} du$

$\int \frac{\cancel{x^2}}{\cos^2(u)} \frac{1}{3\cancel{x^2}} du$

$\frac{1}{3} \int \frac{1}{\cos^2(u)} du$

$\frac{1}{3} \int \sec^2(u)\, du = \frac{1}{3} \tan(u) + C$

$= \boxed{\frac{1}{3} \tan(x^3) + C}$

2. (12 pts) Evaluate the following integrals

(a) $\int x^3(4-x^2)^6\,dx$

$u = 4 - x^2 \iff x^2 = 4 - u$

$du = -2x\,dx$

$dx = \frac{1}{-2x}\,du$

$\int \cancel{x^3}^{2}\, u^6 \frac{1}{-2\cancel{x}}\,du$

$-\frac{1}{2}\int (4-u)u^6\,du$

$-\frac{1}{2}\int 4u^6 - u^7\,du$

$= -\frac{1}{2}\left(\frac{4}{7}u^7 - \frac{1}{8}u^8\right) + C$

$= \boxed{-\frac{2}{7}(4-x^2)^7 + \frac{1}{16}(4-x^2)^8 + C}$

(b) $\displaystyle\int_{1/4}^{1} \frac{\cos(\pi\sqrt{x})}{\sqrt{x}}\,dx$

$u = \pi\sqrt{x}$

$du = \frac{\pi}{2\sqrt{x}}\,dx$

$dx = \frac{2}{\pi}\sqrt{x}\,du$

$\int_{\pi/2}^{\pi} \frac{\cos(u)}{\cancel{\sqrt{x}}} \frac{2}{\pi}\cancel{\sqrt{x}}\,du$

$\frac{2}{\pi}\int_{\pi/2}^{\pi} \cos(u)\,du$

$\frac{2}{\pi}\left(\sin(u)\Big|_{\pi/2}^{\pi}\right) = \frac{2}{\pi}\left(\sin(\pi) - \sin(\frac{\pi}{2})\right)$

$= \boxed{-\frac{2}{\pi}}$

3. The two parts below are separate unrelated problems.

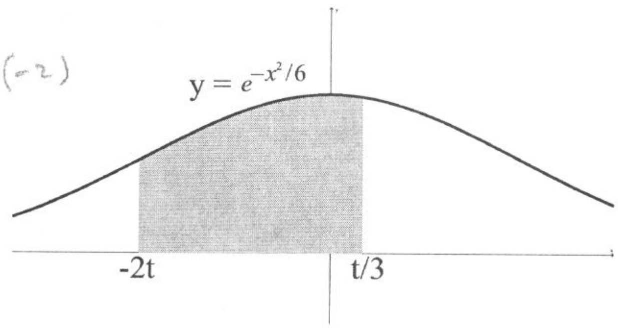
(a) (6 pts) The top of a wall is in the shape of $y = e^{-x^2}$ and the bottom is the x-axis, where x and y are in feet. The wall is being painted in such a way that the area covered at time t minutes is given by

$$A(t) = \int_{-2t}^{\frac{1}{3}t} e^{-\frac{1}{6}x^2}\, dx.$$

Find the rate at which the wall is being painted at $t = 2$ minutes.
That is, find derivative of $A(t)$ at $t = 2$. (Give units)

$$A'(t) = e^{-\frac{1}{6}\left(\frac{1}{3}t\right)^2} \cdot \left(\frac{1}{3}\right) - e^{-\frac{1}{6}(-2t)^2} \cdot (-2)$$

$$A'(t) = \frac{1}{3} e^{-\frac{1}{54}t^2} + 2e^{-\frac{4}{6}t^2}$$



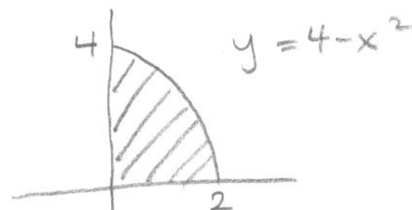
Thus,

$$A'(2) = \frac{1}{3} e^{-\frac{1}{54}(4)} + 2e^{-\frac{2}{3}\cdot(4)}$$

$$= \frac{1}{3} e^{-\frac{2}{27}} + 2e^{-8/3} \ \frac{ft^2}{sec} \approx 0.4485 \ \frac{ft^2}{sec}$$

(b) (6 pts) Use the left-endpoint rule with $n = 3$ subdivisions to approximate the area of the region bounded by $y = 4 - x^2$ in the first quadrant (the first quadrant is where $x \geq 0$ and $y \geq 0$). Write out your work and give your final answer as a decimal to 4 digits after the decimal point.

$$\int_0^2 4 - x^2\, dx$$



$$\Delta x = \frac{2-0}{3} = \frac{2}{3}$$

$$x_0 = 0, \quad x_1 = \frac{2}{3}, \quad x_2 = \frac{4}{3}, \quad x_3 = 2$$

$$\Delta x \left[f(x_0) + f(x_1) + f(x_2) \right]$$

$$= \frac{2}{3}\left[4 - (0)^2 + 4 - \left(\tfrac{2}{3}\right)^2 + 4 - \left(\tfrac{4}{3}\right)^2\right]$$

$$= \frac{2}{3}\left[12 - \frac{4}{9} - \frac{16}{9}\right] = \frac{2}{3}\left[12 - \frac{20}{9}\right] = \frac{2}{3}\cdot\frac{88}{9} = \frac{176}{27}$$

$$= 6.\overline{518} \approx 6.5185$$

ASIDE: ACTUAL VALUE $= \frac{16}{3} = 5.\overline{3}$

4. (a) (6 pts) Find a function $f(x)$ such that $f''(x) = 6x^2 - \sin(x)$, with $f\left(\frac{\pi}{2}\right) = \frac{3\pi}{2}$ and $f'(0) = 4$.

$$f'(x) = 2x^3 + \cos(x) + C$$

$$f'(0) = 4 \implies 2(0)^3 + \cos(0) + C = 4$$

$$1 + C = 4 \implies C = 3$$

$$f(x) = \frac{2}{4}x^4 + \sin(x) + 3x + D$$

$$f(\pi/2) = 3\pi/2 \implies \frac{1}{2}(\pi/2)^4 + \sin(\pi/2) + \frac{3\pi}{2} + D = 3\pi/2$$

$$D = -1 - \frac{\pi^4}{32}$$

$$\boxed{f(x) = \frac{1}{2}x^4 + \sin(x) + 3x - 1 - \frac{\pi^4}{32}}$$

(b) (6 pts) You are standing on top of a tall building exactly 200 meters above your math instructor. You 'accidentally' throw a water balloon straight down. The water balloon lands on your unsuspecting instructor's head after exactly 4 seconds. At what initial velocity did you throw the balloon? (Assume acceleration is a constant -9.8 m/sec^2).

$$a(t) = -9.8$$

$$v(t) = -9.8t + C$$

$$s(t) = -4.9t^2 + Ct + D$$

$$s(0) = 200 \implies D = 200 \implies s(t) = -4.9t^2 + Ct + 200$$

$$s(4) = 0 \implies -4.9(4)^2 + C(4) + 200 = 0$$

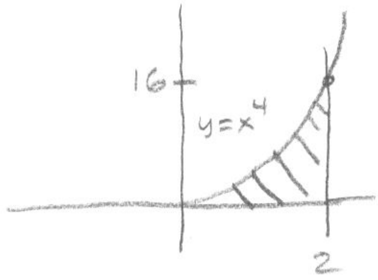
$$4C = 78.4 - 200$$

$$C = \frac{-121.6}{4} = -30.4$$

$$\boxed{v(0) = -30.4 \text{ m/sec}}$$

5. (12 points) Consider the region, R, bounded by the curve $y = x^4$, the **vertical** line $x = 2$, and the x-axis.

(a) (1 pts) Sketch the region R.



(b) (5 pts) Find the value of a, such that the **vertical** line $x = a$ would divide the region R into two regions of equal area.

$$\text{TOTAL} = \int_0^2 x^4\,dx = \frac{1}{5}x^5\Big|_0^2 = \frac{32}{5}$$

$$\int_0^a x^4\,dx \stackrel{?}{=} \frac{1}{2}\cdot\frac{32}{5}$$

$$\frac{1}{5}x^5\Big|_0^a = \frac{16}{5}$$

$$\frac{1}{5}a^5 = \frac{16}{5}$$

$$a^5 = 16$$

$$\boxed{a = (16)^{1/5} \approx 1.741101127}$$

(c) (6 pts) A solid is obtained by rotating the region R around the **horizontal** line $y = -3$. Set up BOTH of the integrals you get from the cylindrical shells and washer methods. (DO NOT EVALUATE)

SHELLS: $\boxed{\int_0^{16} 2\pi(y+3)(2-y^{1/4})\,dy}$

ASIDE: $= \frac{4288\pi}{45}$

WASHERS: $\boxed{\int_0^2 \pi(3+x^4)^2 - \pi(3)^2\,dx}$