1. (12 points) Compute the following integrals.

(a) $\int \sec^4(x)\tan^3(x)\,dx.$ $= \int \sec^3(x)\tan^2(x)\sec(x)\tan(x)dx$

$= \int \sec^3(x)(\sec^2(x)-1)\sec(x)\tan(x)dx$

$u = \sec(x)$

$du = \sec(x)\tan(x)\,dx$

$= \int u^3(u^2-1)du$

$= \int u^5 - u^3\,du$

$= \frac{1}{6}u^6 - \frac{1}{4}u^4 + C$

$= \boxed{\frac{1}{6}\sec^6(x) - \frac{1}{4}\sec^4(x) + C}$

(b) $\int_1^4 \sqrt{y}\ln(\sqrt{y})\,dy$

$t = \sqrt{y}, \quad t^2 = y$

$2t\,dt = dy$

$y = 1 \Leftrightarrow t = 1$

$y = 4 \Leftrightarrow t = 2$

$\int_1^2 t\ln(t)\,2t\,dt$

$= 2\int_1^2 t^2\ln(t)dt$

$u = \ln(t) \quad dv = t^2dt$

$du = \frac{1}{t}dt \quad v = \frac{1}{3}t^3$

$= 2\left[\frac{1}{3}t^3\ln(t)\Big|_1^2 - \int_1^2 \frac{1}{3}t^2dt\right]$

$= 2\left[\left(\frac{1}{3}2^3\ln(2) - \frac{1}{3}1^3\ln(1)\right) - \frac{1}{9}t^3\Big|_1^2\right]$

$= 2\left[\frac{8}{3}\ln(2) - 0 - \frac{1}{9}(2^3 - 1^3)\right]$

$= 2\left[\frac{8}{3}\ln(2) - \frac{7}{9}\right] = \boxed{\frac{16}{3}\ln(2) - \frac{14}{9}}$

2. (12 points) Compute the following integrals.

(a) $\int \frac{4x-15}{x^3-5x^2}\,dx.$ $= \int \frac{4x-15}{x^2(x-5)}\,dx$

$$\frac{4x-15}{x^2(x-5)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-5}$$

$$4x-15 = Ax(x-5) + B(x-5) + Cx^2$$

$$x=0 \Rightarrow -15 = B(-5) \Rightarrow B=3$$

$$x=5 \Rightarrow 20-15 = C(25) \Rightarrow C = 5/25 = 1/5$$

Expanding $4x-15 = \underbrace{Ax^2} - 5Ax + Bx - 5B + \underbrace{Cx^2}$

$$A + C = 0 \Rightarrow A = -C = -1/5$$

$$= \int \frac{-1/5}{x} + \frac{3}{x^2} + \frac{1/5}{x-5}\,dx$$

$$= \boxed{-\frac{1}{5}\ln|x| - \frac{3}{x} + \frac{1}{5}\ln|x-5| + C}$$

(b) $\int \frac{x}{\sqrt{x^2+8x+25}}\,dx.$

$$x^2+8x+25 = x^2+8x+16-16+25$$
$$= (x+4)^2+9$$

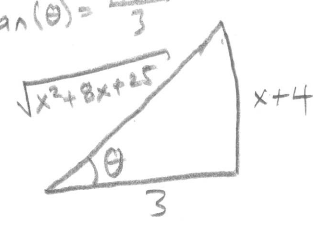
$$\int \frac{x}{\sqrt{(x+4)^2+9}}\,dx$$

$x+4 = 3\tan(\theta)$ $\quad x = 3\tan(\theta)-4$

$dx = 3\sec^2(\theta)\,d\theta$

$$= \int \frac{3\tan(\theta)-4}{\sqrt{9\tan^2(\theta)+9}}\,3\sec^2(\theta)\,d\theta = \int \frac{(3\tan(\theta)-4)}{\cancel{3}\,\cancel{\sec(\theta)}}\,\cancel{3}\sec^{\cancel{2}}(\theta)\,d\theta$$

$\tan(\theta) = \frac{x+4}{3}$



$$= \int 3\sec(\theta)\tan(\theta) - 4\sec(\theta)\,d\theta$$

$$= 3\sec(\theta) - 4\ln|\sec(\theta)+\tan(\theta)| + C$$

$$\boxed{\begin{aligned} &= \cancel{3}\,\frac{\sqrt{x^2+8x+25}}{\cancel{3}} - 4\ln\left|\frac{\sqrt{x^2+8x+25}}{3} + \frac{x+4}{3}\right| + C \\ &= \sqrt{x^2+8x+25} - 4\ln\left|\sqrt{x^2+8x+25} + x + 4\right| + C_1 \end{aligned}}$$

$C_1 = C - 4\ln(3)$

3. **(14 points) Answer the following questions**

(a) (6 pts)Find the average value of $f(x) = \tan^{-1}(3x)$ on the interval $x = 0$ to $x = \frac{1}{3}$.

$$\frac{1}{\frac{1}{3} - 0} \int_0^{1/3} \tan^{-1}(3x)\,dx$$

$t = 3x$
$dt = 3dx$
$dx = \frac{1}{3}dt$
$x = 0 \Rightarrow t = 0$
$x = \frac{1}{3} \Rightarrow t = 1$

$$\cancel{3} \int_0^1 \tan^{-1}(t)\, \frac{1}{\cancel{3}}\, dt$$

$u = \tan^{-1}(t) \qquad dv = dt$
$du = \frac{1}{t^2+1}dt \qquad v = t$

$$= t\tan^{-1}(t)\Big|_0^1 - \int_0^1 \frac{t}{t^2+1}\,dt$$

$$= (1\tan^{-1}(1) - 0\cdot\tan^{-1}(0)) - \int_1^2 \frac{1}{u}\,\frac{1}{2}du$$

$u = t^2+1$
$du = 2t\,dt$
$dt = \frac{1}{2t}du$

$$= \frac{\pi}{4} - \frac{1}{2}\ln|u|\Big|_1^2$$

$$= \pi/4 - \frac{1}{2}(\ln(2) - \ln(1)) = \boxed{\frac{\pi}{4} - \frac{1}{2}\ln(2)}$$

≈ 0.438824573

(b) (8 pts) Consider the arc length of the curve $y = x^3$ from $x = 0$ to $x = 4$.

i. Set up (BUT DO NOT EVALUATE) an integral for this length.

$y' = 3x^2$

$$\text{ARC LENGTH} = \int_0^4 \sqrt{1 + (3x^2)^2}\,dx$$
$$= \boxed{\int_0^4 \sqrt{1 + 9x^4}\,dx}$$

ii. Use Simpson's Method with $n = 4$ subintervals to approximate the value of the arc length.

$$\Delta x = \frac{4-0}{4} = 1$$

$x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 3,\ x_4 = 4$

$$\int_0^4 \sqrt{1+9x^4}\,dx \approx \frac{1}{3}\Delta x\left[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)\right]$$

$$= \frac{1}{3}(1)\left[\sqrt{1+9(0)^4} + 4\sqrt{1+9(1)^4} + 2\sqrt{1+9(2)^4} + 4\sqrt{1+9(3)^4} + \sqrt{1+9(4)^4}\right]$$

$$= \boxed{\frac{1}{3}\left[1 + 4\sqrt{10} + 2\sqrt{145} + 4\sqrt{730} + \sqrt{2305}\right] \approx 64.605588}$$

ACTUAL VALUE
64.67196791

4. (12 points)

(a) Determine if the improper integral $\int_1^{\infty} \frac{\sin\left(\frac{1}{x}\right)}{x^2}\, dx$ converges or diverges. If it diverges, explain why. If it converges, give the value it approaches.

$$\lim_{t\to\infty}\left[\int_1^t \frac{\sin(x^{-1})}{x^2}\, dx\right]$$

$u = x^{-1}$

$du = -x^{-2}dx$

$dx = -x^2 du$

$$= \lim_{t\to\infty}\left[\int_1^{1/t} \frac{\sin(u)}{x^2} \cdot -x^2\, dt\right]$$

$$= \lim_{t\to\infty}\left[\cos(u)\Big|_1^{1/t}\right] = \lim_{t\to\infty}\left[\cos(1/t) - \cos(1)\right]$$

$$= \cos(0) - \cos(1) = \boxed{1 - \cos(1)}$$

$\boxed{\text{CONVERGES}}$

(b) Determine if the improper integral $\int_0^1 x^{-1}\ln(x)\, dx$ converges or diverges. If it diverges, explain why. If it converges, give the value it approaches.

$$\lim_{t\to 0^+}\left[\int_t^1 x^{-1}\ln(x)\,dx\right]$$

$u = \ln(x)$

$du = 1/x\, dx$

$dx = x\, du$

$$\lim_{t\to 0^+}\left[\int_{\ln(t)}^{0} u\, du\right]$$

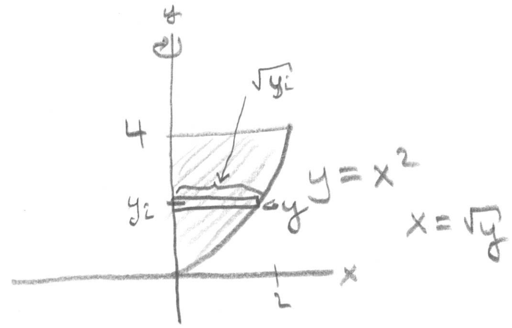
$$\lim_{t\to 0^+}\left[\frac{1}{2}u^2\Big|_{\ln(t)}^{0}\right] = \lim_{t\to 0^+}\left[0 - \frac{1}{2}(\ln(t))^2\right] = -\infty$$

$\boxed{\text{DIVERGES}}$

5. (10 points) Consider the region R in the first quadrant of the xy-plane bounded by $y = x^2$, $y = 4$ and the y-axis ~~for some positive number a.~~ The water in a full tank is in the shape of the solid obtained by rotating R about the y-axis.

Assume all lengths are in meters, so the tank is 4 meters high. And remember the density of water is 1000 kg/m^3 and gravity is 9.8 m/s^2.

Set up and evaluate an integral for the work required to pump all the water to the top of the tank and over the edge.



WORK ON A SLICE $= \underbrace{\underbrace{1000 \cdot 9.8}_{N/m^3} \cdot \underbrace{\pi (\sqrt{y_i})^2 \Delta y}_{m^3}}_{\text{FORCE}} \cdot \underbrace{\underbrace{(4 - y_i)}_{m}}_{\text{DIST}}$

$$\text{WORK} = \int_0^4 9800\,\pi (\sqrt{y})^2 (4-y)\,dy$$

$$= 9800\pi \int_0^4 4y - y^2\,dy$$

$$= 9800\pi \left[2y^2 - \tfrac{1}{3}y^3 \Big|_0^4\right]$$

$$= 9800\pi \left[\left(2\cdot 4^2 - \tfrac{1}{3}4^3\right) - (0)\right]$$

$$= 9800\pi \left[32 - \tfrac{64}{3}\right]$$

$$= 9800\pi \left[\tfrac{96}{3} - \tfrac{64}{3}\right] = \boxed{\frac{9800\pi}{3} \cdot 32 \quad J}$$