1. (12 points) Compute the following integrals.

(a) $\int_1^4 \frac{y\ln(\sqrt{y})}{\sqrt{y}}\,dy$

$t=\sqrt{y}$ $\quad t^2=y$

$2t\,dt=dy$

$y=1 \Leftrightarrow t=1$

$y=4 \Leftrightarrow t=2$

$$\int_1^2 \frac{t^2\ln(t)}{\cancel{t}}\,2\cancel{t}\,dt$$

$$2\int_1^2 t^2\ln(t)\,dt$$

$u=\ln(t)$ $\quad dv=t^2\,dt$

$du=\frac{1}{t}\,dt$ $\quad v=\frac{1}{3}t^3$

$$2\left[\frac{1}{3}t^3\ln(t)\Big|_1^2-\int_1^2\frac{1}{3}t^2\,dt\right]$$

$$2\left[\left(\frac{1}{3}2^3\ln(2)-\underbrace{\frac{1}{3}1^3\ln(1)}_{=0}\right)-\frac{1}{9}\left(t^3\Big|_1^2\right)\right]$$

$$2\left[\frac{8}{3}\ln(2)-\frac{1}{9}(8-1)\right]=\boxed{\frac{16}{3}\ln(2)-\frac{14}{9}}$$

(b) $\int \frac{1}{(x^2-b^2)^{3/2}}\,dx$. ($b$ is a positive constant).

$x=b\sec(\theta)$ $\quad 0\le\theta<\pi/2$

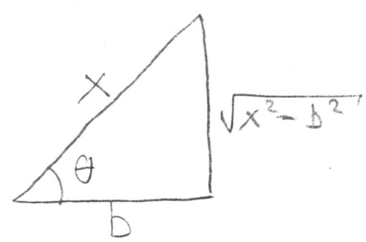
$dx=b\sec(\theta)\tan(\theta)\,d\theta$

$$\int\frac{1}{(b^2\sec^2(\theta)-b^2)^{3/2}}\,b\sec(\theta)\tan(\theta)\,d\theta$$

$\sec(\theta)=\frac{x}{b}$

$$\int\frac{1}{(b^2\tan^2(\theta))^{3/2}}\,b\sec(\theta)\tan(\theta)\,d\theta$$

$$\int\frac{1}{b^3\tan^3(\theta)}\,b\sec(\theta)\tan(\theta)\,d\theta=\frac{1}{b^2}\int\frac{\sec(\theta)}{\tan^2(\theta)}\,d\theta$$



$$\frac{1}{b^2}\int\frac{\cos^{\cancel{2}}(\theta)}{\sin^2(\theta)}\,\frac{1}{\cancel{\cos(\theta)}}\,d\theta=\frac{1}{b^2}\int\frac{\cos(\theta)}{\sin^2(\theta)}\,d\theta$$

$u=\sin(\theta)$

$du=\cos(\theta)\,d\theta$

$$=\frac{1}{b^2}\int\frac{1}{u^2}\,du=-\frac{1}{b^2}\frac{1}{u}+C$$

$$=-\frac{1}{b^2}\frac{1}{\sin(\theta)}+C=\boxed{-\frac{x}{b^2\sqrt{x^2-b^2}}+C}$$

2. (12 points) Compute the following integrals.

(a) $\displaystyle\int \frac{7x+3}{(x-1)(x^2+1)}\,dx.$

$$\frac{7x+3}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow \quad 7x+3 = A(x^2+1) + (Bx+C)(x-1)$$

$x=1$ (OR COVER UP METHOD) $\Rightarrow$ $A = \frac{7(1)+3}{1^2+1} = \frac{10}{2} = 5$ $\quad A=5$

COMPARE COEFFICIENTS $\Rightarrow$ $7x+3 = Ax^2 + A + Bx^2 - Bx + Cx - C$

So: $A+B=0 \Rightarrow B=-A=-5$

and $-B+C=7 \Rightarrow C=7+B=7+{}^{-}5=2$

check: $A-C=3$ ✓

$$\int \frac{5}{x-1} + \frac{-5x+2}{x^2+1}\,dx = \int \frac{5}{x-1}\,dx + \int \frac{-5x}{x^2+1}\,dx + \int \frac{2}{x^2+1}\,dx$$

$$\int \frac{-5\cancel{x}}{u}\,\frac{1}{2\cancel{x}}\,du \qquad \left\{\begin{array}{l} u=x^2+1 \\ du=2x\,dx \\ dx=\frac{1}{2x}\,du \end{array}\right.$$

$$= 5\ln|x-1| - \frac{5}{2}\ln|u| + 2\tan^{-1}(x) + C$$

$$= \boxed{5\ln|x-1| - \frac{5}{2}\ln|x^2+1| + 2\tan^{-1}(x) + C}$$

(b) $\displaystyle\int \frac{x}{\sqrt{x^2+8x+25}}\,dx.$

$$x^2+8x+25 = x^2+8x+16-16+25$$
$$= (x+4)^2+9$$

$$\int \frac{x}{\sqrt{(x+4)^2+9}}\,dx$$

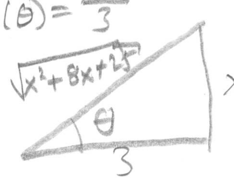
$x+4 = 3\tan(\theta)$ $\qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dx = 3\sec^2(\theta)\,d\theta$

$$\int \frac{3\tan(\theta)-4}{\sqrt{9\tan^2(\theta)+9}}\,3\sec^2(\theta)\,d\theta = \int \frac{(3\tan(\theta)-4)}{\cancel{3}\cancel{\sec}(\theta)}\,\cancel{3}\sec^{\cancel{2}}(\theta)\,d\theta$$

$$= \int 3\tan(\theta)\sec(\theta) - 4\sec(\theta)\,d\theta = 3\sec(\theta) - 4\ln|\sec(\theta)+\tan(\theta)| + C$$

$\tan(\theta) = \frac{x+4}{3}$



$$= 3\,\frac{\sqrt{x^2+8x+25}}{3} - 4\ln\left|\frac{\sqrt{x^2+8x+25}}{3} + \frac{x+4}{3}\right| + C$$

$$\boxed{\begin{aligned} &= \sqrt{x^2+8x+25} - 4\ln\left|\tfrac{1}{3}\left(\sqrt{x^2+8x+25}+x+4\right)\right| + C \\ &= \sqrt{x^2+8x+25} - 4\ln\left|\sqrt{x^2+8x+25}+x+4\right| + D \end{aligned}}$$

$D = C - 4\ln(3)$

3. (12 points)

(a) Using integration by parts and bit of algebraic manipulation, show that for any positive integer $n \geq 2$,

$$\int_0^{\pi/2} \sin^n(x)\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}(x)\,dx.$$

$$\int_0^{\pi/2} \sin^n(x)\,dx \qquad u = \sin^{n-1}(x) \quad dv = \sin(x)dx$$

$$du = (n-1)\sin^{n-2}(x)\cos(x)dx \quad v = -\cos(x)$$

$$= -\cos(x)\sin^{n-1}(x)\Big|_0^{\pi/2} - (n-1)\int_0^{\pi/2} -\sin^{n-2}(x)\cos^2(x)\,dx$$

$$= (\underbrace{-\cos(\tfrac{\pi}{2})\sin^{n-1}(\tfrac{\pi}{2})}_{=0} - \underbrace{-\cos(0)\sin^{n-1}(0)}_{=0}) + (n-1)\int_0^{\pi/2}\sin^{n-2}(x)(1-\sin^2(x))\,dx$$

$$= (n-1)\int_0^{\pi/2}\sin^{n-2}(x)dx - \underbrace{(n-1)\int_0^{\pi/2}\sin^n(x)dx}_{\text{adding back to other side}}$$

$$n\int_0^{\pi/2}\sin^n(x)dx = (n-1)\int_0^{\pi/2}\sin^{n-2}(x)dx$$

$$\boxed{\int_0^{\pi/2}\sin^n(x)dx = \frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}(x)dx}$$

So

$$\int_0^{\pi/2}\sin^n(x) = \frac{n-1}{n}\,\frac{n-3}{n-2}\cdots\frac{1}{2}\int_0^{\pi/2}1\,dx$$

$$= \frac{n-1}{n}\,\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2}$$

n even

(b) Find the length of the curve $f(x) = \int_0^x \sqrt{\left(\sin^2(t) + 2\right)^2 - 1}\; dt$ for $0 \leq x \leq \frac{\pi}{4}$.

$$f'(x) = \sqrt{(\sin^2(x)+2)^2 - 1}$$

$$\text{ARC LENGTH} = \int_0^{\pi/4}\sqrt{1 + [f'(x)]^2}\,dx$$

$$= \int_0^{\pi/4}\sqrt{\cancel{1} + (\sin^2(x)+2)^2 - \cancel{1}}\,dx$$

$$= \int_0^{\pi/4}\sin^2(x) + 2\,dx$$

$$= \int_0^{\pi/4}\frac{1}{2}(1-\cos(2x)) + 2\,dx$$

$$= \int_0^{\pi/4}\frac{5}{2} - \frac{1}{2}\cos(2x)\,dx$$

$$= \frac{5}{2}x - \frac{1}{4}\sin(2x)\Big|_0^{\pi/4} = \left(\frac{5}{2}\,\frac{\pi}{4} - \frac{1}{4}\sin\left(\frac{2\pi}{4}\right)\right) - 0$$

$$= \boxed{\frac{5\pi}{8} - \frac{1}{4}}$$

4. (12 pts) Consider the improper integral $\int_0^1 x^p \ln(x)\, dx$, where p is a constant.

(a) The integral converges if $p > -1$. For $p > -1$, determine the value the integral approaches in terms of p. (Justify your work).

$$\int_0^1 x^p \ln(x)\, dx = \lim_{t\to 0^+} \int_t^1 x^p \ln(x)\, dx$$

$u = \ln(x)$ $dv = x^p dx$

$du = \frac{1}{x}dx$ $v = \frac{1}{p+1}x^{p+1}$

okay because $p \neq -1$

$$= \lim_{t\to 0^+}\left[\frac{1}{p+1}x^{p+1}\ln(x)\Big|_t^1 - \int_t^1 \frac{1}{p+1}x^p\, dx\right]$$

$$= \lim_{t\to 0^+}\left[0 - \frac{1}{p+1}t^{p+1}\ln(t) - \frac{1}{(p+1)^2}x^{p+1}\Big|_t^1\right]$$

$$= \lim_{t\to 0^+}\left[-\frac{1}{p+1}t^{p+1}\ln(t) - \left(\frac{1}{(p+1)^2} - \frac{1}{(p+1)}t^{p+1}\right)\right]$$

$\lim_{t\to 0^+} t^{p+1} = 0$ because $p > -1$ so $p+1 > 0$.

$$\lim_{t\to 0^+} t^{p+1}\ln(t) \overset{0\cdot\infty}{=} \lim_{t\to 0^+}\frac{\ln(t)}{t^{-p-1}} \overset{\frac{\infty}{\infty}}{=} \lim_{t\to 0^+}\frac{1/t}{(-p-1)t^{-p-2}} = \lim_{t\to 0^+}\frac{-1}{p+1}t^{p+1} = 0$$

Thus, $\boxed{\int_0^1 x^p \ln(x)\,dx = -\frac{1}{(p+1)^2}}$ $\boxed{\text{CONVERGES}}$

(b) For $p = -1$, does the integral converge or diverge?
If it converges, give it's value. If it diverges, explain why.

$$\int_0^1 x^{-1}\ln(x)\,dx = \lim_{t\to 0^+}\int_t^1 \frac{\ln(x)}{x}\,dx$$

$u = \ln(x)$

$du = \frac{1}{x}dx$

$$= \lim_{t\to 0^+}\left[\int_{\ln(t)}^0 u\, du\right]$$

$$= \lim_{t\to 0^+}\left[\frac{1}{2}u^2\Big|_{\ln(t)}^0\right]$$

$$= \lim_{t\to 0^+}\left[0 - \frac{1}{2}(\ln(t))^2\right] = -\infty$$

$\boxed{\text{DIVERGES}}$

5. (10 points) You run out of water balloons. So you devise a scheme to dump a bucket of water on your instructor's head instead. Here is your plan:

(a) A tank full of rainwater is outside your dorm. The shape of the tank is described as follows: Consider the region R in the first quadrant of the xy-plane bounded by $y = x^2$, $y = 1$ and the y-axis (lengths are in meters). The full tank is in the shape of the solid obtained by rotating R about the y-axis.

You plan to pump all the water to the top of the tank and over the edge into your bucket.

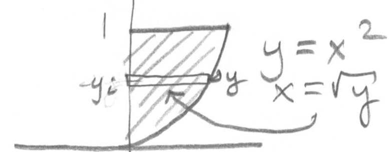
(b) Once all the water is in your bucket. The bucket is lifted by cable to the roof of your dorm (where you will wait for your instructor to walk by). The cable weighs 5 Newtons per meter and the *empty* bucket weighs 100 Newtons. The top of the building is 20 meters high.

Recall the density of water is 1000 kg/m³ and gravity is 9.8 m/s².

Find the total ~~about~~ amount of work done in pumping out the water and lifting the full bucket to the roof of your dorm. (Give your final answer as a decimal in Joules).

PUMPING WATER

$$\sum_{i=1}^{n} \underbrace{\underbrace{1000\cdot 9.8}_{N/m^3} \cdot \underbrace{\pi(\sqrt{y_i})^2 \Delta y}_{m^3}}_{\text{FORCE FOR A SLICE}} \; \underbrace{\underbrace{(1-y_i)}_{m}}_{\text{DIST FOR A SLICE}}$$



WORK TO PUMP WATER $= \int_0^1 9800\pi(\sqrt{y})^2(1-y)\,dy = 9800\pi \int_0^1 y(1-y)\,dy$

$= 9800\pi \int_0^1 y - y^2\,dy = 9800\pi \left[\frac{1}{2}y^2 - \frac{1}{3}y^3 \Big|_0^1\right]$

$= 9800\pi\left[\left(\frac{1}{2}-\frac{1}{3}\right) - 0\right] = \frac{9800}{6}\pi = \frac{4900}{3}\pi$ Joules

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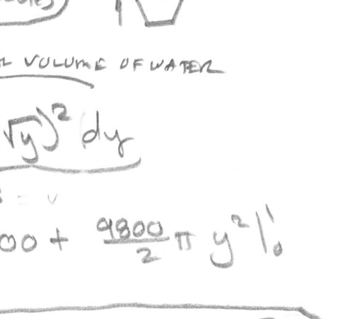
LIFTING CABLE

$$\sum_{i=1}^{n} \underbrace{\underbrace{5}_{N/m}\, \underbrace{\Delta t}_{m}}_{\text{FORCE FOR A SLICE}} \; \underbrace{\underbrace{t_i}_{m}}_{\text{DIST FOR A SLICE}}$$



WORK TO LIFT CABLE ALONE $= \int_0^{20} 5t\,dt = \frac{5}{2}t^2\Big|_0^{20} = \frac{5}{2}400 = 1000$ Joules

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LIFTING BUCKET

$$\text{TOTAL WEIGHT} = \underbrace{100}_{\text{EMPTY WEIGHT OF BUCKET}} + \underbrace{1000\cdot 9.8}_{N/m^3} \underbrace{\int_0^1 \pi(\sqrt{y})^2\,dy}_{m^3 \text{ TOTAL VOLUME OF WATER}}$$

$= 100 + 9800\pi \int_0^1 y\,dy = 100 + \frac{9800}{2}\pi y^2\Big|_0^1$

$= 100 + 4900\pi$ N

WORK = FORCE × DIST $= \underbrace{(100 + 4900\pi)}_{N}\underbrace{(20)}_{m} = 2000 + 98000\pi$ J

$$\text{TOTAL} = \frac{4900}{3}\pi + 1000 + 2000 + 98000\pi \approx \boxed{316{,}007.35 \text{ J}}$$