1. Evaluate the integrals.

(a) (6 pts) $\displaystyle\int \frac{xe^{x^2}}{e^{x^2}-3}\,dx$

$$u = e^{x^2} - 3$$
$$du = 2xe^{x^2}\,dx$$
$$dx = \frac{1}{2xe^{x^2}}\,du$$

$$\int \frac{\cancel{xe^{x^2}}}{u}\,\frac{1}{\cancel{2xe^{x^2}}}\,du$$
$$= \frac{1}{2}\int \frac{1}{u}\,du$$
$$= \frac{1}{2}\ln|u| + C$$
$$= \boxed{\frac{1}{2}\ln\left|e^{x^2}-3\right| + C}$$

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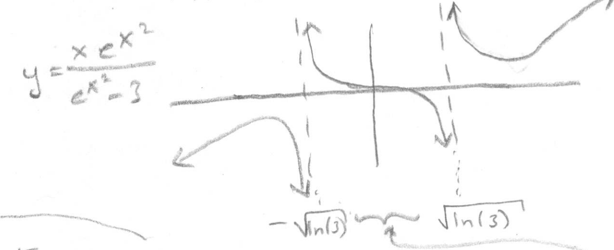


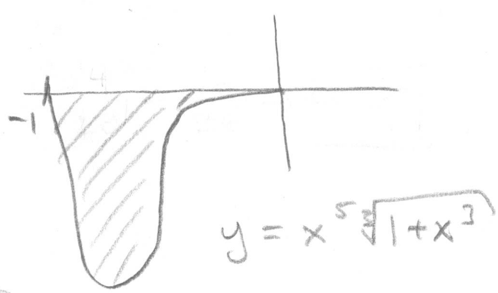
FOR THIS TO MAKE SENSE OVER THE ENTIRE DOMAIN, THE ABSOLUTE VALUE IS CRUCIAL. WITHOUT THEM IT WOULDN'T GIVE THE ANSWER WHEN $e^{x^2}-3$ IS NEGATIVE WHICH IS HERE

(b) (6 pts) $\displaystyle\int_{-1}^{0} x^5\sqrt[3]{1+x^3}\,dx$

$$u = 1+x^3, \quad x^3 = u-1$$
$$du = 3x^2\,dx$$
$$dx = \frac{1}{3x^2}\,du$$

$$\int_0^1 x^{\cancel{5}\,3}\, u^{1/3}\,\frac{1}{3\cancel{x^2}}\,du$$
$$= \frac{1}{3}\int_0^1 (u-1)\,u^{1/3}\,du$$
$$= \frac{1}{3}\int_0^1 u^{4/3} - u^{1/3}\,du$$
$$= \frac{1}{3}\left(\frac{3}{7}u^{7/3} - \frac{3}{4}u^{4/3}\right)\Big|_0^1$$
$$= \left(\frac{1}{7} - \frac{1}{4}\right) - 0$$
$$= \frac{4}{28} - \frac{7}{28} = \boxed{-\frac{3}{28}}$$

ASIDE:



2. (a) (5 pts) Find the exact value of $\lim_{n\to\infty}\sum_{i=1}^{n} \frac{2/n}{1+2i/n}$

$\Delta x = \frac{b-a}{n} = \frac{2}{n} \Rightarrow b-a = 2 \Rightarrow b = a+2$

$x_i = a + i\Delta x = 1 + \frac{2i}{n} \Rightarrow a = 1$, $b = 3$ (or could use $a=0$, or any other constant)

$\int_1^3 \frac{1}{x}\,dx = \ln|x|\Big|_1^3 = \ln(3) - \ln(1) = \boxed{\ln(3)}$

or $\int_0^2 \frac{1}{1+x}\,dx$ $\quad t = 1+x$, $dt = dx$

$\int_1^3 \frac{1}{t}\,dt = \cdots = \ln(3)$

(b) (7 pts) $\int_0^{\pi} |6\cos(x/2) - 3|\,dx$

ONLY SOL'N IN INTERVAL

$6\cos(x/2) - 3 \overset{?}{=} 0 \Rightarrow \cos(x/2) = 1/2 \Rightarrow x/2 = \pi/3 \Rightarrow x = \frac{2\pi}{3}$

$\int 6\cos(x/2) - 3\,dx = 6\int \cos(x/2)\,dx - \int 3\,dx$ $\quad u = x/2$, $du = \frac{1}{2}dx$, $dx = 2du$

$= 6\int \cos(u)\,2du - \int 3\,dx$

$= 12\sin(u) - 3x + C$

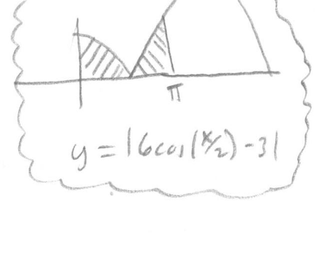
$= 12\sin(x/2) - 3x + C$

$\int_0^{2\pi/3} 6\cos(x/2) - 3\,dx = 12\sin(x/2) - 3x\Big|_0^{2\pi/3} = \left(12\sin(\pi/3) - 3\left(\frac{2\pi}{3}\right)\right) - (0)$

$= \frac{12\sqrt{3}}{2} - 2\pi = 6\sqrt{3} - 2\pi$ (which is positive)

$\int_{2\pi/3}^{\pi} 6\cos(x/2) - 3\,dx = 12\sin(x/2) - 3x\Big|_{2\pi/3}^{\pi} = (12\sin(\pi/2) - 3\pi) - (6\sqrt{3} - 2\pi)$

$= (12 - 3\pi) - (6\sqrt{3} - 2\pi) = 12 - 6\sqrt{3} - \pi$ (which is negative)



$\int_0^{\pi} |6\cos(x/2) - 3|\,dx = (6\sqrt{3} - 2\pi) - (12 - 6\sqrt{3} - \pi)$

$= \boxed{12\sqrt{3} - \pi - 12} \approx 5.64302$

3. (a) (5 pts) Find all positive critical values for the function $f(t) = \int_{3t}^{\frac{1}{3}t^2} \frac{\sqrt{x-1}}{x}\,dx$

FTOC PART I

$$f'(t) = \frac{\sqrt{\frac{1}{3}t^2 - 1}}{\frac{1}{3}t^{\cancel{2}}}\,\frac{2}{3}t - \frac{\sqrt{3t-1}}{3t}\,3 \stackrel{?}{=} 0$$

for $t>0$

$$\Rightarrow \frac{2\sqrt{\frac{1}{3}t^2-1}}{t} - \frac{\sqrt{3t-1}}{t} = 0$$

$$\Rightarrow 2\sqrt{\tfrac{1}{3}t^2-1} = \sqrt{3t-1}$$

$$\Rightarrow 4\left(\tfrac{1}{3}t^2-1\right) = 3t-1$$

$$\Rightarrow \tfrac{4}{3}t^2 - 3t - 3 \stackrel{?}{=} 0$$

$$4t^2 - 9t - 9 = 0$$

$$(4t+3)(t-3) = 0 \Rightarrow \boxed{t=3}$$

$t>0$

(b) (3 pts) Let $f(x)$ be a continuous function and evaluate the limit $\lim_{t\to a} \frac{\int_a^t f(x)dx}{t-a}$.

THIS IS AN INDETERMINANT $\frac{0}{0}$ FORM.
WE CAN USE L'Hôpital's rule

$$\lim_{t\to a} \frac{\int_a^t f(x)dx}{t-a} = \lim_{t\to a} \frac{\frac{d}{dt}\left(\int_a^t f(x)dx\right)}{\frac{d}{dt}(t-a)} = \lim_{t\to a} \frac{f(t)}{1}$$

$$= \boxed{f(a)}$$

because f is continuous

(c) (6 pts) Circle true or false for each of the following statements:

i. $\int_1^3 xf(x^2)dx = \int_1^3 \frac{1}{2} f(u)du$ TRUE $\boxed{\text{FALSE}}$

$u = x^2$
$du = 2x\,dx$
$\int_1^9 f(u)\frac{1}{2}du$

SHOULD BE 9

ii. Approximating $\int_1^3 x^2\,dx$ using left-endpoints and 3-subdivisions

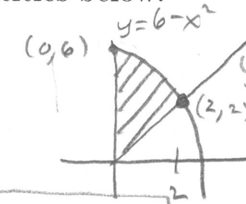
we obtain $L_3 = \frac{2}{3} + \left(\frac{5}{3}\right)^2 \frac{2}{3} + \left(\frac{7}{3}\right)^2 \frac{2}{3}$ $\boxed{\text{TRUE}}$ FALSE

iii. $\frac{d}{dx}\left(x \underbrace{\int_a^b f(t)dt}_{\text{CONSTANT}}\right) = \int_a^b f(t)dt$ $\boxed{\text{TRUE}}$ FALSE

4. (a) (7 pts) Consider the region R bounded between $y = x$ and $y = 6 - x^2$, and bounded on the left by the y-axis. (So the region is all within the first quadrant).

Set up definite integrals (DO NOT EVALUATE) for each of the quantities below:

i. The area of R:



with respect to x

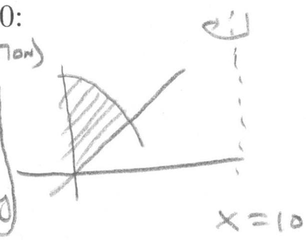
$$\boxed{\int_0^2 6-x^2-x\,dx}$$

or

with respect to y:

$$\boxed{\int_0^2 y\,dy + \int_2^6 \sqrt{6-y}\,dy}$$

ii. The volume of the solid obtained by rotating R the vertical line $x = 10$:



with respect to x (SHELL)

$$\boxed{\int_0^2 2\pi(10-x)(6-x^2-x)\,dx}$$

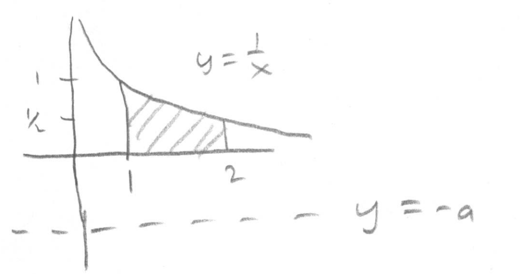
or

with respect to y (CROSS-SECTION)

$$\boxed{\int_0^2 \pi(10)^2-\pi(10-y)^2dy + \int_2^6 \pi(10)^2-\pi(10-\sqrt{6-y})^2dy}$$

(b) (7 points) Let R be the region below the curve $y = \dfrac{1}{x}$, above the x-axis, and between the vertical lines $x = 1$ and $x = 2$. The region R is rotated around the horizontal line $y = -a$ where a is positive and the resulting volume is 100 cubic units. Find the value of a.

with respect to x (CROSS-SECTIONS)



$$\int_1^2 \pi\left(\tfrac{1}{x}+a\right)^2-\pi(a^2)\,dx$$

$$\pi\int_1^2 \frac{1}{x^2}+\frac{2a}{x}+\cancel{a^2}-\cancel{a^2}\,dx$$

$$=\pi\left[-\tfrac{1}{x}+2a\ln|x|\,\Big|_1^2\right]$$

$$=\pi\left[\left(-\tfrac{1}{2}+2a\ln(2)\right)-\left(-\tfrac{1}{1}+2\ln(1)\right)\right]$$

$$=\pi\left(2a\ln(2)+\tfrac{1}{2}\right)\overset{?}{=}100$$

$$\Rightarrow\quad 2a\ln(2)+\tfrac{1}{2}=\frac{100}{\pi}$$

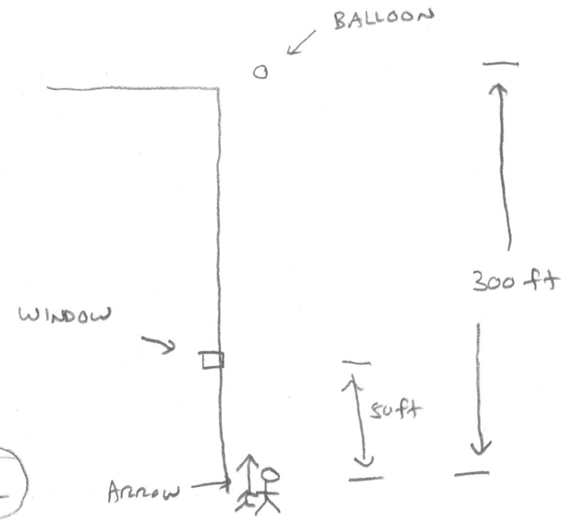
$$\Rightarrow\quad \boxed{a=\frac{\frac{100}{\pi}-\frac{1}{2}}{2\ln(2)}=\frac{200-\pi}{4\pi\ln(2)}}\approx 22.6005$$

WITH RESPECT TO y (SHELL)

$$\boxed{\int_0^{1/2} 2\pi(y+a)(2-1)\,dy + \int_{1/2}^1 2\pi(y+a)\left(\tfrac{1}{y}-1\right)dy \overset{?}{=} 100}$$

5. (8 pts) A water balloon is dropped from the top of a building. You are standing exactly 300 feet directly below the water balloon when it is dropped and you plan to shoot an arrow straight up with an initial velocity of 60 feet/sec. Dr. Loveless' open window is 50 feet above you.

How long after the balloon is dropped should you fire your arrow so that it reaches the balloon precisely when it is outside Dr. Loveless' window? Assume both the balloon and the arrow accelerate at a constant 32 feet/sec^2 downward.



BALLOON

DROPPED

$a(t) = -32 \qquad v(0) = 0 \qquad s(0) = 300$

$\Rightarrow v(t) = -32t + C \Rightarrow C = 0$

$v(t) = -32t$

$\Rightarrow s(t) = -16t^2 + D \Rightarrow D = 300$

$s(t) = -16t^2 + 300$

$-16t^2 + 300 \stackrel{?}{=} 50 \Rightarrow t^2 = \frac{250}{16}$

TIME FOR BALLOON TO REACH THE WINDOW $\rightarrow$ $t = \sqrt{\frac{250}{16}} = \frac{5}{4}\sqrt{10}$ sec

ARROW

$a(t) = -32 \qquad v(0) = 60 \qquad s(0) = 0$

$\Rightarrow v(t) = -32t + C \Rightarrow C = 60$

$\Rightarrow s(t) = -16t^2 + 60t + D \Rightarrow D = 0$

$s(t) = -16t^2 + 60t$

$-16t^2 + 60t \stackrel{?}{=} 50 \Rightarrow -16t^2 + 60t - 50 = 0$

$-8t^2 + 30t - 25 = 0$

$-(4t-5)(2t-5) = 0$

TIME FOR ARROW TO REACH THE WINDOW $\rightarrow$ $\underbrace{t = 5/4 = 1.25 \text{ sec}}_{\text{ON WAY UP}}$ $\qquad t = 5/2 = 2.5$ sec

NEED TO FIRE ARROW EXACTLY $\boxed{\frac{5}{4}\sqrt{10} - 5/4}$ SECONDS AFTER IT IS DROPPED

$\frac{5}{4}(\sqrt{10} - 1) \approx 2.702847$ seconds