1. (12 points) Compute:

(a) $\int \tan^{-1}(x)\, dx$. BY PARTS

$u = \tan^{-1}(x) \quad dv = dx$

$du = \frac{1}{x^2+1}\, dx \quad v = x$

$$= x\tan^{-1}(x) - \int \frac{x}{x^2+1}\, dx$$

$$= x\tan^{-1}(x) - \frac{1}{2}\int \frac{1}{u}\, du$$

$u = x^2+1$

$du = 2x\,dx$

$dx = \frac{1}{2x}\, du$

$$= x\tan^{-1}(x) - \frac{1}{2}\ln|u| + C$$

$$= \boxed{x\tan^{-1}(x) - \frac{1}{2}\ln(x^2+1) + C}$$

(b) $\int \frac{x^2-6}{2x^2-x^3}\, dx$. PARTIAL FRACTIONS

$$\frac{x^2-6}{x^2(2-x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{2-x}$$

$$\Rightarrow x^2 - 6 = Ax(2-x) + B(2-x) + Cx^2$$

$x=0 \Rightarrow -6 = 0 + B(2) + 0 \Rightarrow B = -3$

$x=2 \Rightarrow (2)^2 - 6 = 0 + 0 + C(4) \Rightarrow C = -\frac{2}{4} = -\frac{1}{2}$

COMPARING COEFFICIENTS OF x^2: $1 = -A + C \Rightarrow A = C - 1 = -\frac{3}{2}$

NOTE: $\int \frac{1}{2-x}\, dx = -\ln|2-x| + C$

$$= \int \frac{-3/2}{x} + \frac{-3}{x^2} + \frac{-1/2}{2-x}\, dx$$

$$= \boxed{-\frac{3}{2}\ln|x| + \frac{3}{x} + \frac{1}{2}\ln|2-x| + C}$$

2. (12 points) Compute:

(a) $\displaystyle\int \frac{x}{(x^2+4x+13)^{3/2}}\,dx.$

COMPLETE SQUARE & TRIG SUB

$$x^2+4x+4-4+13 = (x+2)^2+9$$

$$= \int \frac{x}{[(x+2)^2+9]^{3/2}}\,dx$$

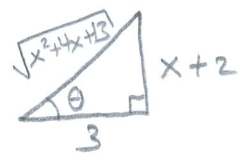
$$x+2 = 3\tan\theta \Rightarrow x = 3\tan\theta - 2$$

$$dx = 3\sec^2\theta\, d\theta$$

$$(x+2)^2+9 = 9\tan^2\theta + 9 = 9\sec^2\theta$$

$$= \int \frac{(3\tan\theta - 2)}{(9\sec^2\theta)^{3/2}}\, 3\sec^2\theta\, d\theta$$

$$= \int \frac{(3\tan\theta-2)}{3^{3\cdot 2}\sec^3\theta}\,\cancel{3}\,\cancel{\sec^2\theta}\,d\theta$$



$$= \frac{1}{9}\int \frac{3\tan\theta}{\sec\theta} - \frac{2}{\sec\theta}\,d\theta$$

$$\cos\theta = \frac{3}{\sqrt{x^2+4x+13}} \qquad \sin\theta = \frac{x+2}{\sqrt{x^2+4x+13}}$$

$$= \frac{1}{9}\int 3\sin\theta - 2\cos\theta\, d\theta$$

$$= -\frac{1}{3}\cos\theta - \frac{2}{9}\sin\theta + C = \boxed{-\frac{1}{\sqrt{x^2+4x+13}} - \frac{2}{9}\frac{(x+2)}{\sqrt{x^2+4x+13}} + C}$$

(b) $\displaystyle\int_0^{\pi/2} \cos^4(x)\sin^3(x)\,dx$

$$u = \cos(x)$$

$$du = -\sin(x)\,dx$$

$$dx = \frac{1}{-\sin(x)}\,du$$

$$\int_0^{\pi/2} \cos^4(x)(1-\cos^2(x))\sin(x)\,dx$$

$$= \int_1^0 u^4(1-u^2)\,\cancel{\sin(x)}\,\frac{1}{-\cancel{\sin(x)}}\,du$$

FLIP

$$= \int_0^1 u^4 - u^6\,du$$

$$= \frac{1}{5}u^5 - \frac{1}{7}u^7\Big|_0^1 = \boxed{\frac{1}{5} - \frac{1}{7} = \frac{7-5}{35} = \frac{2}{35}}$$

3. (13 points)

(a) Compute: $\int \frac{\sqrt{x}}{\sqrt{x}+3}\,dx$.

SUBSTITUTION!

$t=\sqrt{x} \Rightarrow t^2 = x$

$2t\,dt = dx$

$$= \int \frac{t}{t+3}\, 2t\,dt$$

$$= 2\int \frac{t^2}{t+3}\,dt$$

DIVIDE (OR SUBSTITUTE $u=t+3$)

$$\begin{array}{r} t-3 \\ t+3 \overline{)\, t^2 \qquad} \\ -(t^2+3t) \\ \hline -3t \\ -(-3t-9) \\ \hline 9 \end{array}$$

$$= 2\int t-3+\frac{9}{t+3}\,dt$$

$$= 2\left(\tfrac{1}{2}t^2 - 3t + 9\ln|t+3|\right) + C$$

$$= \boxed{x - 6\sqrt{x} + 18\ln(\sqrt{x}+3) + C}$$

(b) Consider the arc length of the curve $f(x) = 30\ln(x)$ from $x=1$ to $x=3$.

i. (2 pts) Set up (DO NOT EVALUATE) an integral for this arc length.

$f'(x) = \frac{30}{x}$

$$\boxed{\text{ARC LENGTH} = \int_1^3 \sqrt{1+\left(\frac{30}{x}\right)^2}\,dx}$$

ii. (5 pts) Use Simpson's rule with $n=4$ to approximate the value of the arc length. (You do not need to simplify your answer, leave it expanded out)

$\Delta x = \frac{3-1}{4} = \frac{1}{2}$, $x_0 = 1$, $x_1 = \frac{3}{2}$, $x_2 = 2$, $x_3 = \frac{5}{2}$, $x_4 = 3$

$$\frac{1}{3}\cdot\frac{1}{2}\left[\sqrt{1+(30)^2} + 4\sqrt{1+(20)^2} + 2\sqrt{1+(15)^2} + 4\sqrt{1+(12)^2} + \sqrt{1+(10)^2}\right]$$

$$= \frac{1}{6}\left[\sqrt{901} + 4\sqrt{401} + 2\sqrt{226} + 4\sqrt{145} + \sqrt{101}\right]$$

$\rightarrow$ APPROX ≈ 33.06657

ASIDE: ACTUAL ≈ 33.0249

4. (13 points)

(a) (7 pts) Evaluate the following **improper** integral: $\displaystyle\int_0^\infty \frac{\sin(\pi e^{-x})}{e^x}\,dx$.

(Give the value if it converges, or show why it diverges).

$$\lim_{t\to\infty} \int_0^t \frac{\sin(\pi e^{-x})}{e^x}\,dx$$

$u = \pi e^{-x}$

$du = -\pi e^{-x}\,dx$

$dx = -\frac{1}{\pi} e^{x}\,du$

$$= \lim_{t\to\infty}\left(\int_{\pi}^{\pi e^{-t}} \sin(u)\cdot\left(-\frac{1}{\pi}\right)du\right)$$

$$= \lim_{t\to\infty}\left(\frac{1}{\pi}\cos(u)\Big|_{\pi}^{\pi e^{-t}}\right)$$

$$= \lim_{t\to\infty}\left(\frac{1}{\pi}\cos(\cancel{\pi e^{-t}}^{\,0}) - \frac{1}{\pi}\cos(\pi)\right)$$

$$= \frac{1}{\pi}\cos(0) - \frac{1}{\pi}(-1) = \boxed{\frac{2}{\pi}}$$

converges

(b) The temperature for a particular object after t minutes is given by $T(t) = 4te^{-2t}$ degrees Celsius. Find the average temperature from $t = 0$ to $t = 3$ minutes.

$$\frac{1}{3-0}\int_0^3 4te^{-2t}\,dt$$

$u = 4t$

$du = 4dt$

$dv = e^{-2t}dt$

$v = -\frac{1}{2}e^{-2t}$

$$= \frac{1}{3}\left[-2te^{-2t}\Big|_0^3 - \int_0^3 -2e^{-2t}\,dt\right]$$

$$= \frac{1}{3}\left[(-6e^{-6} - 0) + -e^{-2t}\Big|_0^3\right]$$

$$= \frac{1}{3}\left[-6e^{-6} + (-e^{-6} - -1)\right]$$

$$\boxed{= \frac{1}{3}\left[1 - 7e^{-6}\right] = \frac{1}{3} - \frac{7}{3e^6} = \frac{1-7e^{-6}}{3}}$$

ASIDE: $\approx 0.32755\,^\circ C$

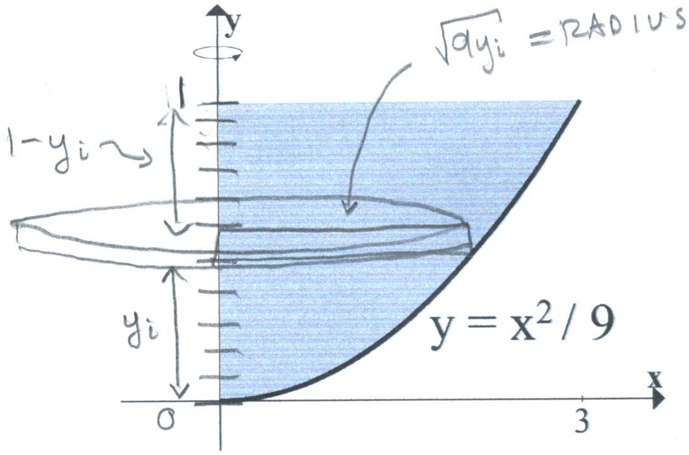
5. (10 points) The portion of the graph $y = \frac{1}{9}x^2$ between $x = 0$ and $x = 3$ is rotated around the y-axis to form a container. The container is full of a liquid that has density 100 lbs/ft^3. Find the work required to pump all of the liquid out over the side of the container. (Distance is measured in feet).

$y = \frac{1}{9}x^2$

$\Rightarrow 9y = x^2$

$\Rightarrow x = \sqrt{9y}$

DIST TO TOP $= 1 - y_i$



BREAK INTO n SUBDIVISIONS

$\Delta y = \frac{1-0}{n} = \frac{1}{n}$

$y_i = \frac{i}{n}$

FOR EACH SUBDIVISION, WE APPROXIMATE

FORCE $\approx \left(100 \frac{\text{lbs}}{\text{ft}^3}\right)\Big(\underbrace{\underbrace{\pi\left(\sqrt{9y_i}\right)^2}_{\text{AREA}} \underbrace{\Delta y}_{\text{THICKNESS}}}_{\text{VOLUME}} \text{ ft}^3\Big) = 100\pi\, 9y_i\, \Delta y$ lbs

DIST TO TOP $\approx 1 - y_i$

TOTAL WORK $= \lim_{n \to \infty} \sum_{i=1}^{n} 100\pi\, 9y_i\, \Delta y\, (1 - y_i)$

$= \int_0^1 100\pi\, 9y\,(1-y)\,dy$

$= 900\pi \int_0^1 y - y^2\,dy$

$= 900\pi \left(\frac{1}{2}y^2 - \frac{1}{3}y^3 \Big|_0^1\right)$

$= 900\pi \left(\frac{1}{2} - \frac{1}{3}\right)$

$= 900\pi \frac{1}{6} = \boxed{150\pi \text{ ft-lbs}}$