Chapter 5 Review

My reviews and review sheets are not meant to be your only form of studying. It is vital to your success on the exams that you carefully go through and understand ALL the homework problems, worksheets and lecture material. Hopefully this review sheet will remind you of some of the key ideas of these sections.

1. Approximating Areas with Rectangles (Riemann Sums)

   - Understand the way to develop more and more accurate area approximations by using the following development. Here we try to find the area between the x-axis and the graph of $f(x)$ from $x = a$ to $x = b$.

   (a) Subdivide the interval into n sub-intervals of equal width. WIDTH $= \Delta x = \frac{b-a}{n}$.

   (b) Label $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, etc. (So $x_i = a + i\Delta x$).

   (c) Choose a point x_i^* in each sub-interval and plug it into the function to get the height of each rectangle. HEIGHT $= f(x_i^*)$ (For right-endpoints method $x_i^* = x_i$, for left-endpoints $x_i^* = x_{i-1}$, and for midpoints $x_i^* = \bar{x}_i = (x_{i-1} + x_i)/2$.)

   (d) The approximate area is the sum of the areas of each rectangle (WIDTH x HEIGHT).

   $$\text{Approximate Area } = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{i=1}^{n} f(x_i^*)\Delta x \quad \text{so}$$

   $$R_n = \sum_{i=1}^{n} f(x_i)\Delta x, \quad L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x, \quad \text{and} \quad M_n = \sum_{i=1}^{n} f((x_{i-1} + x_i)/2)\Delta x.$$

   (e) The exact area is the value we get in the limit as we put in an infinite number of rectangles:

   $$\text{Exact Area } = \lim_{n \Rightarrow \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

2. The Definite Integral (Riemann Integral)

   - Having a formula for the exact area, we introduce a simpler notation called the *definite integral of $f(x)$ from $x = a$ to $x = b$*:
   $$\int_a^b f(x)dx = \lim_{n \Rightarrow \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

   - The integral represents the "net area". That is:
   $$\int_a^b f(x)dx = \left(\begin{array}{c}\text{area between graph and} \\ \text{above the } x\text{-axis}\end{array}\right) - \left(\begin{array}{c}\text{area between graph and} \\ \text{below the } x\text{-axis}\end{array}\right)$$

   - We did examples of all of these properties in class:
   $$\int_a^b c\,dx = (b - a)c \qquad\qquad\qquad \text{(Integral of a constant)}$$
   $$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx \quad \text{(Breaking up a sum)}$$
   $$\int_a^b cf(x)dx = c\int_a^b f(x)dx \qquad\qquad \text{(Pulling out a constant)}$$
   $$\int_b^a f(x)dx = -\int_a^b f(x)dx \qquad\qquad \text{(Flipping endpoints)}$$

3. The Fundamental Theorem of Calculus

   - Part 1: If f is continuous on $[a, b]$, then
   $$g(x) = \int_a^x f(t)dt \Rightarrow g'(x) = f(x)$$

   - Part 2: If f is continuous on $[a, b]$ and F is an antiderivative of f, then
   $$\int_a^b f(x)dx = F(b) - F(a)$$

- Understand how to use Part 1. You often have to use either (1) *the chain rule* or (2) *breaking up the interval*. (We did examples of all of this situations in lecture).

- We will use Part 2 a lot this quarter. To use it, you (1) *Find an antiderivative*, (2) *Plug the endpoints into the antiderivative*, and (3) *Subtract*.

4. Using FTOC Part 1: There are various functions in the sciences that are defined in terms of integrals. Some examples are: The error function, $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ (from probability), the Fresnel integrals $S(x) = \int_0^x \sin(t^2) dt$ and $C(x) = \int_0^x \cos(t^2) dt$ (from optics), the Sine integral $Si(x) = \int_0^x \frac{\sin(t)}{t} dt$ (from Fourier Series and signal processing), among others. Part 1 of the fundamental theorem, gives us a derivative rule for such functions, for example $S'(x) = \sin(x^2)$. We can use this rule in combination with our other derivative rules, here are some examples:

   (a) (With the chain rule) $\frac{d}{dx} \int_1^{x^3} \sqrt{(t)} dt = 3x^2 \sqrt{x^3}$.

   (b) (Flipping the bounds and using the chain rule) $\frac{d}{dx} \int_{\sin(x)}^1 t^5 dt = \frac{d}{dx} - \int_1^{\sin(x)} t^5 dt = -(\sin(x))^5 \cos(x)$.

   (c) (Separating) $\frac{d}{dx} \int_{\ln(x)}^{x^2} t^2 + t^3 dt = \frac{d}{dx} \left(\int_{\ln(x)}^c t^2 + t^3 dt + \int_c^{x^2} t^2 + t^3 dt \right)$ (where c is a constant)
   $= \frac{d}{dx} \left(- \int_x^{\ln(x)} t^2 + t^3 dt + \int_c^{x^2} t^2 + t^3 dt \right) = -((\ln(x))^2 + (\ln(x))^3)\frac{1}{x} + ((x^2)^2 + (x^2)^3)2x$.

5. The Indefinite Integral

   - The Fundamental Theorem gives the precise way that antiderivatives, derivatives, and areas are related. Thus, if we wish to study general antidifferentiation it makes sense to use the integral notation. So we define the *indefinite integral of $f(x)$* by:

     $$\int f(x)dx = \text{ the general antiderivative of } f(x)$$

     Don't forget to use the '+C' when working with general antiderivatives.

   - *The definite integral of a rate of change of a quantity is the net change of the quantity over the interval.* Understand what this interpretation of the fundamental theorem means an dhow it is used in practice.

   - Understand the difference between net change (displacement) and total change:

     – DISPLACEMENT = net change = $\int_a^b f(t)dt$.
     – TOTAL CHANGE = $\int_a^b |f(t)|dt$.
     – TO COMPUTE TOTAL CHANGE:
       (a) Find the zeros of $f(t)$ (*i.e.* solve $f(t) = 0$).
       (b) Break up the integral into separate integrals for each change zero from the last step.
       (c) Evaluate each integral and take all the areas as positive.

       Here is a simple example: $\int_1^4 |2x - 6|dx$

       (a) FIND ZEROS: $2x - 6 = 0$ implies $x = 3$.
       (b) BREAK UP REGION:

       $* \int_1^3 2x - 6dx = x^2 - 6x|_1^3 = -4$

       $* \int_3^4 2x - 6dx = x^2 - 6x|_3^4 = 1$

       (c) ADD UP AS POSITIVES: $\int_1^4 |2x - 6|dx = 4 + 1 = 5$

6. The Substitution Rule (*u*-substitution)

   - PRACTICE, PRACTICE, PRACTICE!!! The more problems you see the better. It is very important to learn this technique. Ultimately integration is somewhat of an art. It is vital that you learn all the rules of integration well and are willing to flexible, patient and experimental when trying to evaluate an integral.

   - The Substitution Rule is essentially the chain rule in reverse and is stated as follows for the indefinite and definite integrals, respectively:

     – If $u = g(x)$ and $du = g'(x)dx$, then $\int f(g(x))g'(x)dx = \int f(u)du$.

– If $u = g(x)$ and $du = g'(x)dx$, then $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$.

- In practice, you use the substitution rule as follows:

  (a) *Choose the function $u = g(x)$* (This is the art form part, you have to make the choice based on the fact that you see that $g'(x)$ appears elsewhere in the integral. If you choice of u doesn't make the integral simpler, then you have to try a different choice).

  (b) *Compute $du = g'(x)dx$ and compute $g(a)$ and $g(b)$.*

  (c) *Replace $g(x)$ with u, dx with $\frac{du}{g'(x)}$*, and replace the endpoints. Also simplify if you can.

  (d) *If all your x's vanish from the problem, you have successfully done a change of variable (i.e. a substitution)* If not all x's vanish then you either need to simplify more or you need to go back and choose a different u.

- You need to start looking at an integral as a whole. Look for the following:

  (a) *A function and its derivative somewhere in the integral.* For example, if you see x^4 in one part of the function, look for x^3 somewhere else (and perhaps a substitution of $u = x^4$ will work).

  (b) *A function inside another function.* In these situations, it is often a good idea to take $u =$ the inside function.

7. Miscellaneous

- Understand how antiderivatives can be used to solve problems involving acceleration, velocity and distance.