Basic Precalculus Toolbox

Often the difficulties students run into in calculus stems from precalculus deficiencies. To excel in calculus you must be strong in algebra, functions, and mathematical models. Algebra skills and your understanding of functions come from years of mathematical training in middle school and high school, so no review I give will be complete. Thus, on the following pages, I try to give just a quick rundown of the basic mathematical functions that will play an important role in this course. At any point during the term if you need a bit of review on any of these topics, please stop by my office or chat with your TA in quiz section or go to the math study center. The key is to clear up any misunderstandings immediately.

The Basic Mathematical Models, Functions and Curves:

1. To get a mathematical model for some given scenario, you plug in the known data for x and y and solve for the constants in the model (by combining the equations given by the data points)

2. Lines: $y = m(x - x_1) + y_1$ (or $y = mx + b$).

   - $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} =$ the slope
   - Model for constant rate of change and for the path of movement on a straight line.
   - To get the model, figure out the slope and use a point to get the other constant.

3. Parabolas: $y = ax^2 + bx + c$ (or $y = a(x - h)^2 + k$).

   - $a > 0$ implies the parabola opens upward (*smiles*) and
     $a < 0$ implies the parabola opens downward (*frowns*).
   - The vertex occurs at $x = -\frac{b}{(2a)}$ which can also be found by completing the square to get $(h, k) =$ location of the vertex.
   - Models the path of a projectile under the effects of gravity. Also come up in maximization/minimization problems for area, profit and other topics.

4. Polynomials: $y = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0 =$ a sum of integer powers of x.

   - The highest power is called the *degree* of the polynomial (so a parabola is a second-degree polynomial and a line is a first degree polynomial).
   - The fundamental theorem of algebra says that a polynomial of degree n can have at most n zeros (n locations where it crosses the x-axis). Thus the function switches from increasing to decreasing (or vice versa) fewer than n times.
   - Higher order polynomials appear often as we try to approximate data or a function with more accurate curves.

5. Rational Functions: $y = \frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomials.

   Rational functions are fractions involving two polynomials. They *tend* to have vertical and horizontal asymptotes as we will explore this quarter. They appear as models of asymptotic behavior and as formulas for ratios of two quantities.

6. Exponentials: $y = y_0 b^x$ (or $y = y_0 e^{kx}$)

   - $b = e^k$ is the base (or multiplier) and y_0 is the initial value.
   - Model where *relative* growth rate is constant. That is, the rate of change of population, or savings account, grows at the same percentage rate relative to the current size.

- Logarithms: $x = \ln(y)$ is the inverse of $y = e^x$. (In other words, $e^x = 10$ is the same relationship as $x = \ln(10)$).
  To solve $2(5)^x = 20$, we divide by 2 to get $5^x = 10$, then we use logarithms to get $\ln(5^x) = \ln(10)$, so $x \ln(5) = \ln(10)$, which gives $x = \ln(5)/\ln(10)$.

- Know your exponent and logarithm rules. Most importantly you should know:

$$(x^a)^b = x^{ab} \qquad \ln(a^b) = b\ln(a)$$
$$x^a x^b = x^{a+b} \qquad \ln(ab) = \ln(a+b)$$
$$\frac{x^a}{x^b} = x^{a-b} \qquad \ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$$

7. Circles: $(x - h)^2 + (y - k)^2 = r^2$.

   - (h, k) = the center of the circle and r = the radius.

   - By solving for y, we get Upper Semicircle: $y = k + \sqrt{r^2 - (x-h)^2}$
     Lower Semicircle: $y = k - \sqrt{r^2 - (x-h)^2}$

8. Trig:

   (a) Sine: $y = \sin(x)$
       Inverse: $x = \sin^{-1}(y)$ defined on $-1 \le y \le 1$, $-\pi/2 \le x \le \pi/2$.

   (b) Cosine: $y = \cos(x)$
       Inverse: $x = \cos^{-1}(y)$ defined on $-1 \le y \le 1$, $0 \le x \le \pi$.

   (c) Tangent: $y = \tan(x)$
       Inverse: $x = \tan^{-1}(y)$ defined for all y and $-\pi/2 < x < \pi/2$.

   Recall that when you use $\sin^{-1}(x)$ (or other trig inverses), you only get the principal solution. You need to use the graph and symmetry to get other solutions.

   The trig functions are defined as the ratio of sides of a right triangle with angle θ.

$sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$	$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$	$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin(\theta)}{\cos(\theta)}$
$\csc(\theta) = \frac{1}{\sin(\theta)}$	$\sec(\theta) = \frac{1}{\cos(\theta)}$	$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)}$

You should know all the following basic trigonometric values. Note that you can use these values along with the rules above and the symmetry of locations on a circle to find several other values of the trigonometric functions.

θ	$\sin(\theta)$	$\cos(\theta)$
0	$\sqrt{0/4} = 0$	$\sqrt{4/4} = 1$
$\pi/6$	$\sqrt{1/4} = 1/2$	$\sqrt{3/4} = \sqrt{3}/2$
$\pi/4$	$\sqrt{2/4} = \sqrt{2}/2$	$\sqrt{2/4} = \sqrt{2}/2$
$\pi/3$	$\sqrt{3/4} = \sqrt{3}/2$	$\sqrt{1/4} = 1/2$
$\pi/1$	$\sqrt{4/4} = 1$	$\sqrt{0/4} = 0$

Here are a few of the most used identities:

- $\sin^2(\theta) + \cos^2(\theta) = 1$, (please note how to use this notation).

- $\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$.

- $\sin(\theta) = \sin(\theta + 2\pi n)$ and $\cos(\theta) = \cos(\theta + 2\pi n)$ for $n = 0, \pm1, \pm2, \ldots$

- $\cos(\theta) = \sin(\theta + \frac{\pi}{2})$ and $\sin(\theta) = \cos(\theta - \frac{\pi}{2})$.

9. Circular Motion and Circle Facts:

Here are a few key formulas and relationships when working with radian measure and trigonometric functions (many of these formulas ONLY hold for radian measure):

- Angular Speed $= \omega = \dfrac{\text{change in angle}}{\text{change in time}}$. Linear Speed $= v = \dfrac{\text{change in distance}}{\text{change in time}}$.
- When working with circular motions, the following formulas can be useful (to use these all angle measurements should be in radians):
    - $s = r\theta$, Arc Length $=$ (radius)(angle).
    - $\theta = \omega t$, Angle $=$ (angular speed)(time).
    - $v = \omega r$, Linear Speed $=$ (angular speed)(time).
    - Area of a Pie Wedge $= \frac{1}{2}r^2\theta$.
- Given a circle of radius r with the center as the origin, consider the point that is obtained by rotating by an angle of θ in standard position. Then the (x, y) coordinates of this point are given by $x = r\cos(\theta)$ and $y = r\sin(\theta)$. We obtained this θ in a couple steps to get the general circular motion equations:

$$x = r\cos(\theta_0 + wt) \qquad y = r\sin(\theta_0 + wt),$$

where

    - $r =$ the radius
    - $\theta_0 =$ the angle in standard position that corresponds to $t = 0$.
    - $\omega =$ the angular speed taken as positive for counterclockwise motion and negative for clockwise motion.
    - $t =$ time.

10. The Sine Wave:

The general sinusoidal wave model is: $y = A\sin\left[\frac{2\pi}{B}(x - C)\right] + D$

- $A =$ amplitude $= \dfrac{\text{MAX Y VALUE} - \text{MIN Y VALUE}}{2}$
- $D =$ mean $= \dfrac{\text{MAX Y VALUE} + \text{MIN Y VALUE}}{2}$
- $B =$ period $=$ how long it takes the wave to repeat $=$ distance from peak to peak
- $C =$ phase shift $=$ x-coordinate of a peak $- \frac{B}{4}$
  $=$ an x-coordinate where the graph crosses the mean line and is increasing

11. Moving Functions Around:

Understand how to reflect, shift and dilate known functions.

- Six types of movement (The change is given along with what you actually do to the coordinates):

  (a) Reflect across y-axis: Replace "x" by "$-x$". (Flip signs of x-coordinates)
  (b) Reflect across x-axis: Replace "y" by "$-y$". (Flip signs of y-coordinates)
  (c) Shift horizontally by h: Replace "x" by "$x - h$". (Add h to all x-coordinates)
  (d) Shift vertically by k: Replace "y" by "$y - k$". (Add k to all y-coordinates)
  (e) Dilate horizontally by c: Replace "x" by "cx". (Divide all x-coordinates by c)
  (f) Dilate vertically by d: Replace "y" by "dy". (Divide all y-coordinates by d)

- Here is the recipe to perform movement from a given graph $y = f(x)$. I will illustrate using the example $y = 2f(3x - 4) - 5$

  (a) Label several points in the graph of the known function.
  (b) Move "outside stuff" to the y side: $\frac{1}{2}(y + 5) = f(3x - 4)$. The two cases for order of operations are illustrated here:
  $c(y + d)$: Do the 'c' movement first.
  $ax + b$: Do the 'b' movement first.
  (c) Horizontal movement: For this example you would
  1. Add 4 to x-coordinates.
  2. Divide x-coordinates by 3.
  (d) Vertical movement: For this example you would
  1. Multiply y-coordinates by 2.
  2. Subtract 5 from y-coordinates.
  (e) Plot the new points and draw the resulting graph.

- A function is said to be *even* if $f(x) = f(-x)$, so that reflecting across the y-axis gives the same function (which means it is symmetric about the y axis).

- A function is said to be *odd* if $f(x) = -f(-x)$, so that reflecting across the y-axis, then across the x-axis gives the same function.

12. How to solve an algebraic equation (No all equations can be solved in a closed form way, but when they can be solved, we follow the steps below):

(a) Try to isolate the variable you want to solve for (say x) by using a sequence of inverses.

(b) At each step cancel the 'outside' function on the x-side by applying the inverse of this outside function.

(c) The inverse pairs are:
Addition/Subtraction
Multiplication/Division
Powers/Roots
Exponentials/Logs
Trig/Inverse Trig.

(d) The only circumstance where we don't do this is when you have an equation of the form $ax^2 + bx + c = 0$ in which case we use the quadratic formula (you could also complete the square then use the inverse approach given above).