Calculus 1 Max/Min Material That We Will Generalize Today

From 4.1, 4.3 and 4.7

1. We defined a **critical number** of $f(x)$ on domain D to be any number $x = c$ on the domain D such that either
   (a) $f'(c) = 0$, or
   (b) $f'(c)$ does not exist.

2. *The Second Derivative Test*:
   If $x = c$ is a critical point of $f(x)$, then
   (a) If $f''(c) > 0$, then $x = c$ corresponds to a local minimum of f.
   (b) If $f''(c) < 0$, then $x = c$ corresponds to a local maximum of f.
   (c) If $f''(c) = 0$ or if $f''(c)$ does not exist, then the test is inconclusive (we can't conclude if $x = c$ is a local max/min or neither based on this information alone).

3. We discussed that any continuous function on a closed interval must have an absolute (global) maximum and an absolute (global) minimum on that interval (The Extreme Value Theorem). And we discovered that all absolute max/min occur either at a critical number or at an endpoint. This gave us the absolute max/min method:

   To find the absolute max/min of a continuous function $f(x)$ on a closed interval:
   (a) Find the critical numbers.
   (b) Evaluate $f(x)$ at the critical numbers.
   (c) Evaluate $f(x)$ at the endpoints.
   Among these evaluations is our answer.
   The biggest output = the absolute max and it occurs at the corresponding x value(s).
   The smallest output = the absolute min and it occurs at the corresponding x value(s).

4. In applied problems, we had to set up the function to optimize. Here are things I always suggest:
   (a) VISUALIZE/LABEL: Draw a good picture and label **everything** with variables.
   (b) WHAT IS GIVEN?: Write down all the given **constraints**.
   (c) WHAT TO OPTIMIZE?: Write down a formula for that quantity. Then, using the given facts, find a **one variable function** for the quantity that you want to optimize.
   (d) DOMAIN? Over what interval does the problem make sense
   (e) USE CALCULUS: Find the critical numbers. And check the endpoints.
   (f) JUSTIFY/VERIFY: Make sure you actually did find the a max or min as desired.

Multivariable Max/Min

From 14.7

1. We define a **critical point** of $f(x,y)$ on domain D to be any point $(x,y) = (a,b)$ on the domain D such that either
   (a) $f_x(a,b) = 0$ AND $f_y(a,b) = 0$ (both simultaneously), or
   (b) $f_x(a,b)$ does not exist or $f_y(a,b)$ does not exist.

2. *The Second Derivative Test*:
   If (a,b) is a critical point of $f(x,y)$, then define
   $$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2.$$
   (a) If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.
   (b) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.
   (c) If $D(a,b) < 0$, then (a,b) is a saddle point.
   (d) If $D(a,b) = 0$, then the test is inconclusive (use a contour map).

3. The absolute max/min over a closed region occurs either at a critical point or a boundary point. This gives us the absolute max/min method:
   (a) Find the critical points.
   (b) Over each distinct boundary curve.
      i. Find an equation involving x and y that describes the boundary.
      ii. Substitute the boundary curve equation into the function to get a one variable function for z.
      iii. Optimize this one variable function using calculus 1 methods. This will give you 'critical numbers' along each boundary
   (c) Evaluate $f(x,y)$ at all the critical points inside the region.
   (d) Evaluate $f(x,y)$ at all the critical numbers and endpoints on each boundary.
   Among these evaluations is our answer.
   The biggest output = the absolute max.
   The smallest output = the absolute min.

4. In applied problems, we have to set up the function to optimize. Here, again, are things I always suggest:
   (a) VISUALIZE/LABEL: Draw a good picture and label **everything** with variables.
   (b) WHAT IS GIVEN?: Write down all the given **constraints**.
   (c) WHAT TO OPTIMIZE?: Write down a formula for that quantity. Then, using the given facts, find a **two variable function** for the quantity that you want to optimize.
   (d) DOMAIN? Over what region does the problem make sense
   (e) USE CALCULUS: Find the critical points. And check the boundary.
   (f) JUSTIFY/VERIFY: Make sure you actually did find the a max or min as desired.