1. (14 pts) Consider the surface $z = x^2 + 2y^2$.

(a) Describe the traces parallel to the given plane (no work needed, just circle your answer).

i. Parallel to the yz-plane (when x is fixed):
[✓ PARABOLAS] CIRCLES ELLIPSES HYPERBOLAS NONE OF THESE

ii. Parallel to the xz-plane (when y is fixed):
[✓ PARABOLAS] CIRCLES ELLIPSES HYPERBOLAS NONE OF THESE

iii. Parallel to the xy-plane (when z is fixed, $z > 0$):
PARABOLAS CIRCLES [✓ ELLIPSES] HYPERBOLAS NONE OF THESE

(b) Clearly circle the name of the surface given by $z = x^2 + 2y^2$:

CONE
PARABOLIC CYLINDER
HYPERBOLIC CYLINDER
[✓ ELLIPTIC PARABOLOID]
SPHERE
CIRCULAR CYLINDER
HYPERBOLOID
HYPERBOLIC PARABOLOID
ELLIPSOID
ELLIPTICAL CYLINDER
CIRCULAR PARABOLOID
NONE OF THESE

(c) A plane, P, is determined by the points P(0,1,7), Q(-3,2,4), and R(1,3,8). A beam of light follows a straight-line path that passed through the point $(0, 1, 4)$ and is orthogonal to the plane, P. Find the two points when the path of the beam of light intersects the surfaces $z = x^2 + 2y^2$.

PLANE: $\vec{PQ} = \langle -3-0, 2-1, 4-7\rangle = \langle -3, 1, -3\rangle$

$\vec{PR} = \langle 1-0, 3-1, 8-7\rangle = \langle 1, 2, 1\rangle$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & -3 \\ 1 & 2 & 1 \end{vmatrix} = \langle 1--6, -3--3, -6-1\rangle = \langle 7, 0, -7\rangle$$

Since the line is orthogonal to the plane, $\langle 7, 0, -7\rangle$ gives a direction vector for the line. For simplicity, let's scale it down to $\vec{v} = \langle 1, 0, -1\rangle$.

LINE: $\langle x, y, z\rangle = \langle 0, 1, 4\rangle + t\langle 1, 0, -1\rangle$ $\boxed{\begin{cases} x = 0 + t \\ y = 1 \\ z = 4 - t \end{cases}}$

INTERSECTION: $z = x^2 + 2y^2 \Rightarrow 4 - t = t^2 + 2(1)^2$

$0 = t^2 + t - 2$

$0 = (t+2)(t-1)$

$t = -2, t = 1$

$t = -2$: $\boxed{(x, y, z) = (-2, 1, 6)}$

$t = 1$: $\boxed{(x, y, z) = (1, 1, 3)}$

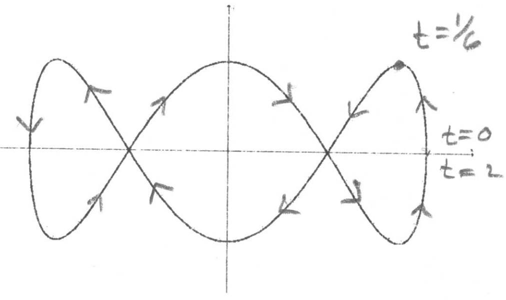
2. (12 points) Olivo is running on a path. His location (x, y) (each in feet) at time t seconds is given by the vector function

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle \cos(\pi t),\ \sin(3\pi t) \rangle .$$

(a) Calculate the following quantities at $t = 1/6$.

- $(x(1/6), y(1/6)) =$ $\left(\cos\left(\frac{\pi}{6}\right), \sin\left(\frac{3\pi}{6}\right)\right)$

$$= \boxed{\left(\frac{\sqrt{3}}{2}, 1\right)}$$



- $\frac{dy}{dx} =$ $\frac{dy/dt}{dx/dt} = \frac{3\pi\cos(3\pi t)}{-\pi\sin(\pi t)} = \frac{-3\cos(3\pi t)}{\sin(\pi t)}$

$$\left.\frac{dy}{dx}\right|_{t=1/6} = \frac{-3\cdot(0)}{1/2} = \boxed{0}$$

- $\frac{d^2y}{dx^2} =$ $\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\left(\frac{(\sin(\pi t))(9\cancel{\pi}\sin(3\pi t)) - (\cancel{\pi}\cos(\pi t))(-3\cos(3\pi t))}{\sin^2(\pi t)}\right)}{-\cancel{\pi}\sin(\pi t)}$

$$= -\frac{(9\sin(\pi t)\sin(3\pi t) + 3\cos(\pi t)\cos(3\pi t))}{\sin^3(\pi t)}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=1/6} = -\frac{(9\cdot(\frac{1}{2})\cdot(1) + 3\cdot(\frac{\sqrt{3}}{2})\cdot(0))}{\left(\frac{1}{2}\right)^3} = -\frac{9}{(\frac{1}{2})^2} = \boxed{-36}$$

concave down ✓

(b) Find Olivo's speed at the first positive time he passes through the point $(x, y) = (\frac{1}{2}, 0)$. (Recall: Speed is the magnitude of the velocity/derivative vector)

$x = \frac{1}{2} \Rightarrow \frac{1}{2} = \cos(\pi t) \Rightarrow \pi t = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ or $\frac{7\pi}{3}$ or . . .

$y = 0 \Rightarrow 0 = \sin(3\pi t) \Rightarrow 3\pi t = 0$ or π or 2π or . . .

$\boxed{t = \frac{1}{3}}$ first positive time both happen.

$$\text{speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

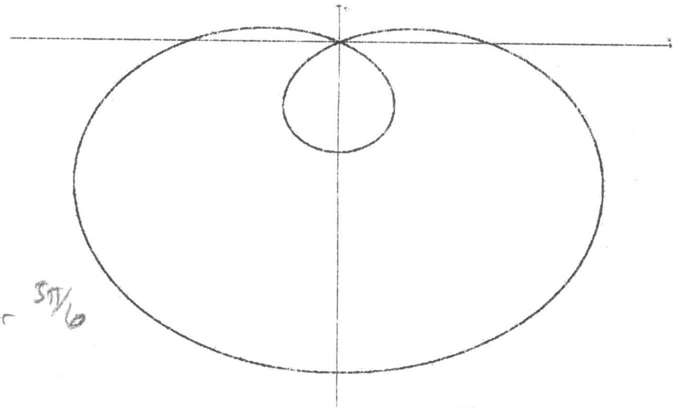
$$= \sqrt{(-\pi\sin(\pi t))^2 + (3\pi\cos(3\pi t))^2}$$

at $t = \frac{1}{3}$

$$\text{speed} = \sqrt{\pi^2\left(\frac{\sqrt{3}}{2}\right)^2 + 9\pi^2(-1)^2} = \pi\sqrt{\frac{3}{4} + 9}$$

$$= \boxed{\pi\sqrt{\frac{39}{4}} = \frac{\pi\sqrt{39}}{2}} \text{ ft/sec.}$$

3. (12 pts) Consider the polar curve given by the equation $r = 3 - 6\sin(\theta)$. The graph of the curve is given below.



(a) The curve intersects the origin at two different values of θ (for $0 \leq \theta < 2\pi$). Find the equations for the tangent lines to the curve at both of these values of θ. Put your answers in the form $y = mx + b$.

$r = 0 \Rightarrow \quad 0 = 3 - 6\sin(\theta)$

$\sin(\theta) = 1/2 \quad \theta = \pi/6$ or $5\pi/6$

$x = r\cos(\theta) = (3 - 6\sin(\theta))\cos(\theta)$

$y = r\sin(\theta) = (3 - 6\sin(\theta))\sin(\theta)$

$$\frac{dy}{dx} = \frac{-6\cos(\theta)\sin(\theta) + (3-6\sin(\theta))\cos(\theta)}{-6\cos(\theta)\cos(\theta) - (3-6\sin(\theta))\sin(\theta)}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \frac{-6\left(\frac{\sqrt{3}}{2}\right)\cdot\frac{1}{2} + 0}{-6\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2} - 0} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$\boxed{y = \frac{1}{\sqrt{3}}x = \frac{\sqrt{3}}{3}x}$ AND $\boxed{y = -\frac{1}{\sqrt{3}}x = -\frac{\sqrt{3}}{3}x}$

(b) Give all four (x, y)-coordinates at which the curve has a horizontal tangent. (Hint: You can get (x, y) without explicitly calculating θ.)

$\frac{dy}{d\theta} = 0 \qquad -6\cos(\theta)\sin(\theta) + (3 - 6\sin(\theta))\cos(\theta) = 0$

$\cos(\theta)\left[3 - 12\sin(\theta)\right] = 0$

① $\cos(\theta) = 0$ or ② $3 - 12\sin(\theta) = 0$

$\theta = \pi/2$, $r = -3$: $\boxed{\begin{aligned} x &= r\cos(\theta) = 0 \\ y &= r\sin(\theta) = -3 \end{aligned}}$

$\theta = 3\pi/2$, $r = 9$: $\boxed{\begin{aligned} x &= r\cos(\theta) = 0 \\ y &= r\sin(\theta) = -9 \end{aligned}}$

$\sin(\theta) = 1/4$

$\cos(\theta) = \pm\sqrt{15}/4$

$r = 3 - 6\,1/4 = 3/2$

$\boxed{\begin{aligned} x &= r\cos(\theta) = \tfrac{3}{2}\tfrac{\sqrt{15}}{4} = \tfrac{3\sqrt{15}}{8} \\ y &= r\sin(\theta) = \tfrac{3}{2}\tfrac{1}{4} = 3/8 \end{aligned}}$

$\boxed{\begin{aligned} x &= -\tfrac{3\sqrt{15}}{8} \\ y &= 3/8 \end{aligned}}$

4. (12 points) The motion of a particular fly in three-dimensions is described by the vector position function $\mathbf{r}(t) = \langle t^2, 3t+6, -2t^2 \rangle$.

(a) Find the curvature at $t = 0$

$\vec{r}\,'(t) = \langle 2t, 3, -4t \rangle$ $\qquad$ $\vec{r}\,''(t) = \langle 2, 0, -4 \rangle$

$\vec{r}\,'(0) = \langle 0, 3, 0 \rangle$ $\qquad$ $\vec{r}\,''(0) = \langle 2, 0, -4 \rangle$

$$\vec{r}\,'(0) \times \vec{r}\,''(0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3 & 0 \\ 2 & 0 & -4 \end{vmatrix} = \langle -12 - 0, 0, -6 \rangle = \langle 12, 0, -6 \rangle$$

$$K(0) = \frac{|\vec{r}\,'(0) \times \vec{r}\,''(0)|}{|\vec{r}\,'(0)|^3} = \frac{\sqrt{12^2 + 0^2 + 6^2}}{\sqrt{0^2+3^2+0^2}^{\,3}} = \frac{\sqrt{144+36}}{3^3}$$

$$= \boxed{\frac{\sqrt{180}}{27} = \frac{\sqrt{36 \cdot 5}}{27} = \frac{2\sqrt{5}}{9}}$$

(b) Find all points on the curve at which the tangent line at that point also travels through the origin.

TANGENT LINE: $\langle x, y, z \rangle = \langle t^2, 3t+6, -2t^2 \rangle + u \langle 2t, 3, -4t \rangle$

want sol'ns to

ⓘ $t^2 + 2ut = 0 \Rightarrow t(t+2u) = 0 \Rightarrow t = 0$ or $t = -2u$

(ii) $3t + 6 + 3u = 0 \Rightarrow u = -t - 2 \Rightarrow u = -2$ or $u = 2u - 2$

$u = 2$

So $t = -4$

(iii) $-2t^2 - 4ut = 0 \Rightarrow -2t(t+2u) = 0$

check

$t = 0$ ✓
$u = -2$

$t = -4$ ✓
$u = 2$

TWO VALUES OF t:

$t = 0$: $\boxed{(x, y, z) = (0, 6, 0)}$ $\rightarrow$ tangent line $\langle x, y, z \rangle = \langle 0, 6, 0 \rangle + u \langle 0, 3, 0 \rangle$

$t = -4$: $\boxed{(x, y, z) = (16, -6, -32)}$ $\qquad$ $\langle x, y, z \rangle = \langle 16, -6, -32 \rangle + u \langle -8, 3, 16 \rangle$